Queueing Theory
(Part 2)

M/M/1 Queueing System

M/M/1 Queueing System

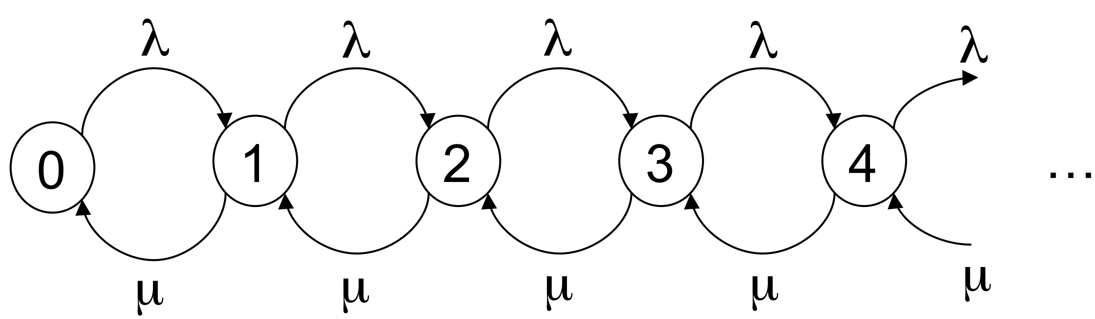
- Simplest queueing system
- Assumptions:
  - Interarrival times are iid, and exponentially distributed. The arrival process is Poisson.
  - Service times are iid, and exponentially distributed.
  - There is one server

- The interarrival times are exponentially distributed with parameter, λ = mean arrival rate
- The service times are exponentially distributed with parameter, μ = mean service rate
- The utilization is $\rho = \lambda / \mu$

M/M/1 Queueing System and Rate Diagram

- The number of customers in the system, N(t), in an M/M/1 queueing system satisfies the assumptions of a birth-death process with
  - $\lambda_n = \lambda$ for all n=0,1,2,…
  - $\mu_n = \mu$ for all n=0,1,2,…
- We require $\lambda < \mu$, that is $\rho < 1$ in order to have a steady state
  - Why?
    If $\rho \geq 1$, $\sum_{n=0}^{\infty} P_n$ won't converge, and the queue will explode (up to $+\infty$)

Rate Diagram



M/M/1 Queueing System

Steady-State Probabilities

Calculate P_n, n = 0, 1, 2, …

Rate In = Rate Out

State 0: $\mu P_1 = \lambda P_0 \quad \Rightarrow \quad P_1 = \frac{\lambda}{\mu} P_0$

State 1: $\lambda P_0 + \mu P_2 = (\lambda + \mu) P_1 \quad \Rightarrow \quad P_2 = \frac{1}{\mu}\left((\lambda+\mu)\frac{\lambda}{\mu}P_0 - \frac{\mu}{\mu}\lambda P_0\right) = \frac{\lambda^2}{\mu^2}P_0$

State n: $P_n = \frac{\lambda\lambda\cdots\lambda}{\mu\mu\cdots\mu}P_0 = \left(\frac{\lambda}{\mu}\right)^n P_0 = \rho^n P_0$

Need $\sum_{n=0}^{\infty} P_n = 1$

$$\sum_{n=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^n P_0 = P_0 \sum_{n=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^n = P_0 \sum_{n=0}^{\infty}\rho^n = P_0\left(\frac{1}{1-\rho}\right) = 1$$

Geometric Series

$$\sum_{n=0}^{\infty}\rho^n = \frac{1}{1-\rho} \quad \text{if } |\rho| < 1$$

For M/M/1 with $\rho < 1$

$P_0 = 1 - \rho$

$P_n = \rho^n (1-\rho) \quad \text{for } n = 0,1,2,\ldots$

M/M/1 Queueing System

L, L_q, W, W_q

$$\frac{d\rho^n}{d\rho} = n\rho^{n-1}$$

Calculate L, L_q, W, W_q

$$\frac{d(1-\rho)^{-1}}{d\rho} = -1(1-\rho)^{-2}(-1)$$

$$\frac{d(1-\rho)^{-1}}{d\rho} = \frac{1}{(1-\rho)^2}$$

$$L = \text{expected number in system } = \sum_{n=0}^{\infty} nP_n = \sum_{n=0}^{\infty} n\rho^n(1-\rho) = \rho(1-\rho)\sum_{n=0}^{\infty} n\rho^{n-1}$$

$$L = \rho(1-\rho)\sum_{n=0}^{\infty}\frac{d\rho^n}{d\rho} = \rho(1-\rho)\frac{d}{d\rho}\sum_{n=0}^{\infty}\rho^n = \rho(1-\rho)\frac{d\left(\frac{1}{1-\rho}\right)}{d\rho} = \frac{\rho(1-\rho)}{(1-\rho)^2} = \frac{\rho}{1-\rho} = \frac{\lambda/\mu}{1-\lambda/\mu} = \frac{\lambda}{\mu-\lambda}$$

$$L_q = \text{expected number in queue } = \sum_{n=1}^{\infty}(n-1)P_n = \sum_{n=1}^{\infty}(n-1)\rho^n(1-\rho) = \sum_{n=1}^{\infty} n\rho^n(1-\rho) - \sum_{n=1}^{\infty}\rho^n(1-\rho)$$

$$L_q = L - (1-P_0) = L - \rho = \frac{\rho}{1-\rho} - \rho = \frac{\rho^2}{1-\rho} = \frac{\lambda^2}{\mu(\mu-\lambda)}$$

$$W = \frac{L}{\lambda} = \frac{\lambda}{(\mu-\lambda)\lambda} = \frac{1}{(\mu-\lambda)}$$

$$W_q = \frac{L_q}{\lambda} = \frac{\lambda^2}{\mu(\mu-\lambda)\lambda} = \frac{\lambda}{\mu(\mu-\lambda)}$$

M/M/1 Queueing System

Distribution of Time in System

ω = waiting time in system

Expected waiting time in system:

$$W = E[\omega] = \frac{1}{\mu - \lambda}$$

Probability that waiting time in system exceeds t:

$$P(\omega > t) = e^{-\mu(1-\rho)t} \quad \text{for } t \geq 0$$

ω_q =waiting time in queue

Expected waiting time in queue:

$$W_q = E[\omega_q] = \frac{\lambda}{\mu(\mu - \lambda)}$$

Probability that waiting time in queue exceeds t:

$$P(\omega_q > t) = \rho P(\omega_q > t) = \rho e^{-\mu(1-\rho)t} \quad \text{for } t \geq 0$$

M/M/1 Example: ER

- Emergency cases arrive independently at random
- Assume arrivals follow a Poisson input process (exponential interarrival times) and that the time spent with the ER doctor is exponentially distributed
- Average arrival rate = 1 patient every ½ hour

  λ= 2 patients / hour

- Average service time = 20 minutes to treat each patient

  μ= 1 patient / 20 minutes = 3 patients / hour

- Utilization

  $\rho = \lambda/\mu=2/3$

- Does this M/M/1 queue reach steady state?

  Yes! $\rho<1$

M/M/1 Example: ER
Questions

In steady state, what is the…

1. probability that the doctor is idle?

2. probability that there are n patients?

3. expected number of patients in the ER?

4. expected number of patients waiting for the doctor?

M/M/1 Example: ER

Questions

In steady state, what is the…

5. expected time in the ER?

6. expected waiting time?

7. probability that there are at least two patients waiting to see the doctor?

8. probability that a patient waits to see the doctor more than 30 minutes?

Car Wash Example

- Consider the following 3 car washes
- Suppose cars arrive according to a Poisson input process and service follows an exponential distribution
- Fill in the following table

	λ	μ	ρ	L	L_q	W	W_q	P_0
Car Wash A	0.1 car/min	0.5 car/min						
Car Wash B	0.1 car/min	0.11 car/min						
Car Wash C	0.1 car/min	0.1 car/min						

What conclusions can you draw from your results?

Car Wash Example

- Consider the following 3 car washes
- Suppose cars arrive according to a Poisson input process and service follows an exponential distribution
- Fill in the following table

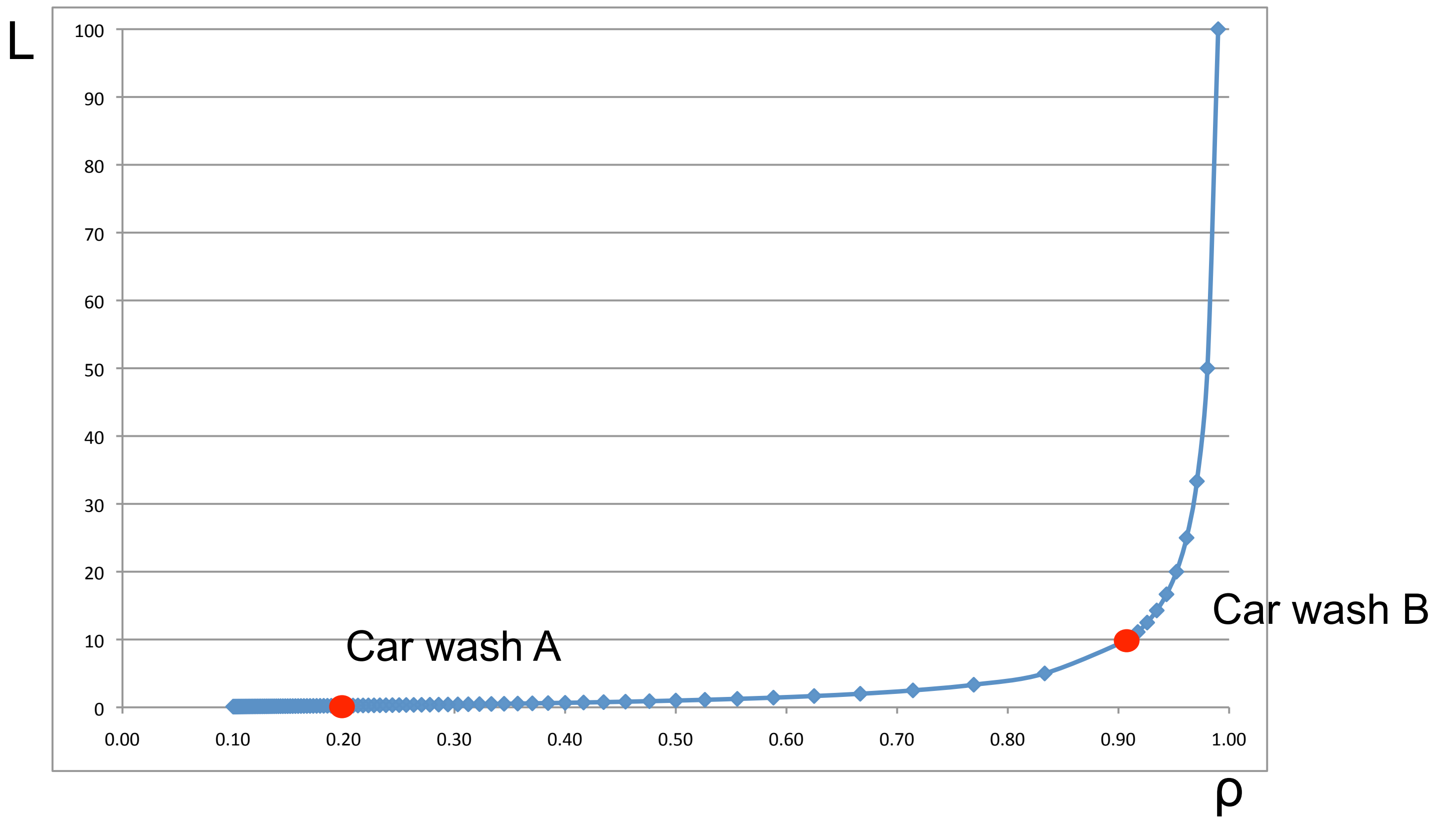
	λ	μ	ρ	L	L_q	W	W_q	P_0
Car Wash A	0.1 car/min	0.5 car/min	0.1/0.5 = 0.2	(0.1)/(0.5-0.1) = 0.25	$(0.1)^2/(0.5)(0.4)$ = 0.05	0.25/0.1 = 2.5 min	0.5/0.1 = 0.5 min	0.4/0.5 = 0.8
Car Wash B	0.1 car/min	0.11 car/min	0.1/1.1 = 0.909	(0.1)/(0.11-0.1) = 10	$(0.1)^2/(0.11)(0.1)$ = 9.09	10/0.1 = 100 min	9.09/0.1 = 90.9 min	0.01/11 = 0.09
Car Wash C	0.1 car/min	0.1 car/min	0.1/0.1 = 1	0.1/(0.1-0.1) = $+\infty$ Not steady state	$+\infty$	$+\infty$	$+\infty$	0/0.1 = 0

What conclusions can you draw from your results?

As $\rho \rightarrow 1$, then L, L_q, W, $W_q \rightarrow \infty$ and $P_0 \rightarrow 0$

Car Wash Example

- M/M/1 with λ=0.1 car/min
- $\rho=\lambda/\mu$
- $L=\lambda/(\mu-\lambda) = \rho/(1-\rho)$



Rule of thumb: never exceed ρ = 0.8, 80% utilization