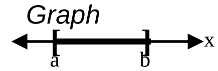
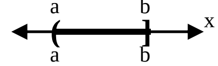
R.2 Inequalities and Interval Notation

In order to simplify matters we want to define a new type of notation for inequalities. This way we can do away with the more bulky set notation. This new notation is called using <u>intervals</u>. There are two types of intervals on the real number line; bounded and unbounded.

Definitions:
Bounded interval- An interval with finite length, i.e. if we subtract the endpoints of the interval we get a real number.
Unbounded interval- Any interval which is not of finite length is unbounded.

Let us proceed to define these intervals by relating them to the better known inequalities.

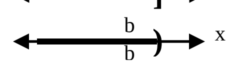
Bounded Intervals

Inequality	*Interval Type*	*Notation*	*Graph*
$a \le x \le b$	Closed	$[a, b]$	
$a < x < b$	Open	(a, b)	
$a \le x < b$	Half-open	$[a, b)$	
$a < x \le b$	Half-open	$(a, b]$	

Note that the lengths of all the intervals above are $b - a$. Which is a real number and thus all the above intervals are bounded by definition.

Lets see the unbounded intervals.

Unbounded Intervals

Inequality	*Interval Type*	*Notation*	*Graph*
$a \le x$	Half-open	$[a, \infty)$	
$a < x$	Open	(a, ∞)	
$x \le b$	Half-open	$(-\infty, b]$	
$x < b$	Open	$(-\infty, b)$	
	Entire Line	$(-\infty, \infty)$	

Notice that when writing in interval notation, we always write our intervals in increasing order. That is, we always have the smaller numbers on the left.

<u>Example 1</u>

Write the following in interval notation

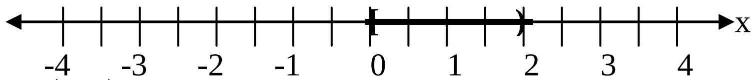
a. $-3 \le x < 1$ b. $0 < x < 2$ c. $x > -3$ d. $x \le 2$

Solution:

a. This is a bounded interval. It may prove helpful to graph the inequality first.

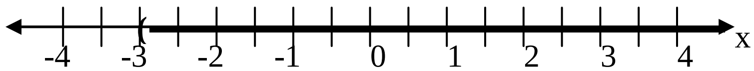
So, as an interval we get $[-3\ 1)$.

b. Again this is a bounded interval. The graph is



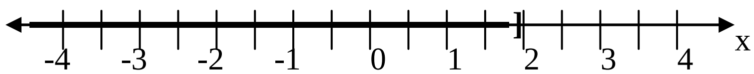
So, we get $(0,\ 2)$.

c. This is an unbounded interval. Graphing we get



Hence our interval is $(-3,\ \infty)$.

d. Finally, we have



Thus, our interval is $(-\infty,\ 2]$.

Now we want to review how to solve an inequality. We start with the properties and rules of inequalities.

Properties and Rules of Inequalities

1. If $a < b$, then $a + c < b + c$ and $a - c < b - c$.
2. If $a < b$, and c is positive, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.
3. If $a < b$, and c is negative, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$.

The rules are similar for <, $\leq$ and $\geq$.

The idea is that we can add or subtract any value on both sides of an inequality symbol and nothing changes and we can multiply or divide any positive value on both sides of an inequality symbol and nothing changes. However, if we multiply or divide any negative value on both sides of an inequality symbol, we must "flip" the inequality symbol.

We use these properties to solve inequalities. Basically, all we have to remember is that we solve them just as we solved equations with the added restriction that any time we multiply or divide by a negative, we have to change the inequality symbol.

Example 2:

Solve and graph. Put your answer in interval notation.

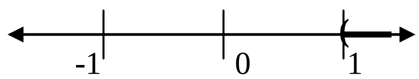
a. $-2x + 3 < 1$ b. $7x + 4 \leq 2x - 6$ c. $3 - 4(x + 2) \leq 6 + 4(2x + 1)$

Solution:

a. So we can simply solve this inequality as we solved equations. We just isolate the x on one side. The only thing we have to remember is that when we have to divide by a negative, we will need to "flip" the inequality symbol. We proceed as follows

$$-2x+3<1$$
$$-3 \quad -3$$
$$\frac{-2x}{-2}<\frac{-2}{-2}$$
$$x>1$$

Now we simply graph and write the answer in interval notation.

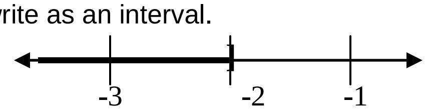


So the solution is $(1,\infty)$.

b. Again, we solve as we did with equations and "flip" the inequality symbol if needed. We get

$$7x+4 \le 2x-6$$
$$5x+4 \le -6 \qquad \text{Subtract 2x on both sides}$$
$$5x \le -10 \qquad \text{Subtract 4 on both sides}$$
$$x \le -2 \qquad \text{Divide by 5 on both sides}$$

So we graph and write as an interval.

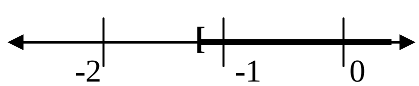


So the solution is $(-\infty,-2]$.

c. Again, we proceed as we did above.

$$3-4(x+2) \le 6+4(2x+1) \qquad \text{Use the distributive property}$$
$$3-4x-8 \le 6+8x+4 \qquad \text{Combine like terms}$$
$$-4x-5 \le 8x+10 \qquad \text{Subtract 8x on both sides and add 5 on both sides}$$
$$-12x \le 15 \qquad \text{Divide by -12 on both sides}$$
$$x \ge -\tfrac{15}{12} \qquad \text{Reduce the fraction}$$
$$x \ge -\tfrac{5}{4}$$

Now we graph



So the solution is $\left[-\frac{5}{4},\infty\right)$.

There is another type of inequality that we need to be able to solve called in <u>compound inequality</u>. Compound inequalities come in several forms. We will only concentrate on three primary types: double inequalities, inequalities containing "and", inequalities containing "or". We will illustrate how to solve these types in the following examples.

Example 3:

Solve and graph. Put your answer in interval notation.

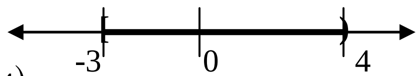
a. $-5 \le 3x + 4 < 16$ b. $-2 < 5 - 4x < 1$

Solution:

a. In this example we have the so called double inequalities. In order to solve these, we want to get the x alone in the middle of the inequality symbols. We do this the same way we did in the previous example. The only difference is whatever we do one part of the inequality, we need to do to all three parts of the inequality. So we proceed as follows

$$\begin{aligned} -5 \le 3x + 4 &< 16 && \text{Subtract 4 everywhere} \\ -9 \le 3x &< 12 && \text{Divide by 3 everywhere} \\ -3 \le x &< 4 \end{aligned}$$

For the graph and solution to this inequality, all we need know is as long as the endpoints are set up in a consistent way (as they are for this one, since –3 is less than 4), then the solution is everything between the endpoints. So we have

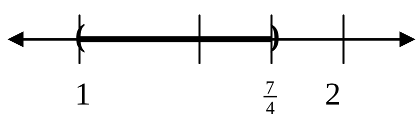


So the solution is $[-3,\ 4)$.

b. Again, we proceed as we did above. Remember, whenever multiply or divide by a negative, we must "flip" the inequality symbol.

$$\begin{aligned} -2 < 5 - 4x &< 1 && \text{Subtract 5 everywhere} \\ -7 < -4x &< -4 && \text{Divide by -4 everywhere} \\ \tfrac{7}{4} > x &> 1 && \text{(Don't forget to "flip" the inequality symbols)} \end{aligned}$$

Now again, the endpoints are consistent because $\frac{7}{4}$ is larger than 1 so the graph is everything between them. This gives



So the solution set is $\left(1,\ \frac{7}{4}\right)$.

Notice that whenever dealing with a double inequality, as long as the endpoints are in the correct order, the inequality will always be all values in between the endpoints.

Now we need to deal with the compound inequalities that have an "and" or an "or". We simply need to remember that when dealing with an "and" we want only the overlapping portion of the graph, when dealing with an "or" we get to keep everything we graph.

Example 4:

Solve and graph. Put your answer in interval notation.

a. $3x + 7 < 10 \textit{ or } 2x - 1 > 5$ b. $2x - 3 \ge 5 \textit{ and } 3x - 1 > 11$

c. $9x - 2 < 7 \textit{ and } 3x - 5 > 10$ d. $3x - 11 \le 4 \textit{ or } 4x + 9 \ge 1$

Solution:

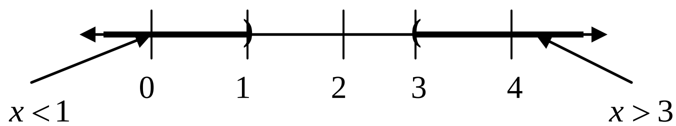
a. First, we can start by solving each of the inequalities separately and deal with the “or” later. We get

$$3x+7<10 \ \textit{or} \ 2x-1>5$$

$$3x<3 \ \ \textit{or} \ 2x>6$$

$$x<1 \ \ \textit{or} \ x>3$$

Now, to graph, we graph each of the inequalities and remember that since we have an “or” we get to keep everything we graph. We get



Since we have two different intervals as part of the solution, we need to use the union symbol to connect them. So the solution set is $(-\infty,1)\cup(3,\infty)$.

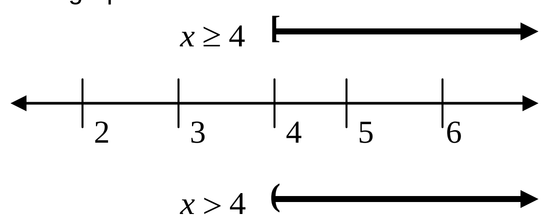
b. Again we start by solving each inequality separately and take care of the “and” later.

$$2x-3\ge 5 \ \textit{and} \ 3x-1>11$$

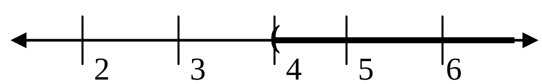
$$2x\ge 8 \ \ \textit{and} \ 3x>12$$

$$x\ge 4 \ \textit{and} \ x>4$$

Since this time we have an “and” we need to just graph the overlapping section. So we start by graphing each inequality individually, above or below the graph, then put the overlap onto the finished graph.



So we can see clearly that the graphs overlap 4 onward. However, at the value of 4, they do not overlap since the bottom piece has a parenthesis, and thus does not contain the value of 4. So our solution is



So we have is $(4,\infty)$.

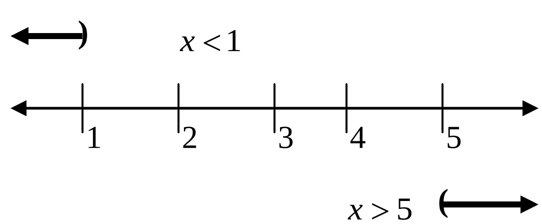
c. We will again proceed as before.

$$9x-2<7 \ \textit{and} \ 3x-5>10$$

$$9x<9 \ \textit{and} \ 3x>15$$

$$x<1 \ \textit{and} \ x>5$$

Since we have an “and” we want only the overlapping section. Graphing individually we get



Since there is no overlapping section, there must be no solution.

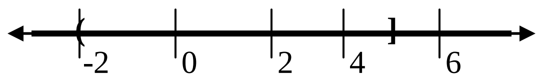
d. Lastly, we will solve as we did above.

$$3x-11\le 4 \text{ or } 4x+9\ge 1$$

$$3x\le 15 \text{ or } 4x\ge -8$$

$$x\le 5 \text{ or } x\ge -2$$

Since we have an "or" we graph both of the inequalities on the same line and we get to keep everything that we graph. We get



Since the entire graph gets covered, we say that the solution set is $(-\infty,\infty)$.

R.2 Exercises

Write the interval notation for the inequality.

1. $x\ge 4$
2. $-4<x<4$
3. $x<-4$
4. $-5<x\le 3$
5. $\frac{3}{2}\ge x>0$
6. $-7\le x<3$
7. $-2\le x$
8. $6>x$
9. $-10<x<-5$
10. $5\ge x\ge -\pi$
11. $x\le -3 \text{ or } x>2$
12. $x\le -1 \text{ or } x>1$

Solve and graph. Put your answer in interval notation.

13. $-3x>6$
14. $\frac{x}{-2}\le -3$
15. $-2-x<7$
16. $4x+3<-1$
17. $7x+4\ge 3x+2$
18. $5x-3\le -3x+2$
19. $5(2x+1)-5>2x$
20. $20-2(x+9)\le 2(x-5)$
21. $10-13(2-x)<5(3x-2)$
22. $6-(2x-1)\ge 5(3x-4)$
23. $6(3-x)\ge 5-(4x-7)$
24. $3-(3-x)\ge 5-2(x-7)$
25. $\frac{3}{8}x+\frac{1}{2}<\frac{1}{4}x+2$
26. $\frac{5}{6}x-\frac{2}{3}<\frac{3}{4}x+2$
27. $\frac{1}{2}x+\frac{2}{5}>\frac{1}{4}x-\frac{2}{3}$
28. $\frac{3}{2}x-\frac{2}{7}>\frac{5}{14}x-\frac{2}{3}$
29. $3(2x-1)-(4x+1)-2(5x-6)\le 8$
30. $3(2x-1)-(5x-4)-(7x+1)\le -3$

31. $-5\le 2x+1<7$
32. $-4<2x-3\le 1$
33. $0<4x+4<7$
34. $-2<3x+7<1$
35. $3<7x-14<31$
36. $-6\le 5x+14\le 24$
37. $-5\le 3x+4<16$
38. $5\le 4x-3\le 21$
39. $0\le 2x-6\le 4$
40. $5<4x-3<21$
41. $0<2x-5<9$
42. $2<3-x<3$
43. $-4<2-3(x+2)<11$
44. $-1<3-(2x-3)<0$
45. $-3<2-\frac{x}{2}\le 1$
46. $-3<x-\frac{3}{2}\le 3$
47. $12\ge \frac{3-x}{2}>1$
48. $1>\frac{x-4}{-3}>-2$

49. $3x+7>-2 \text{ or } 3x+7<-4$
50. $6x+5<11 \text{ or } 3x-1>8$
51. $x+3\ge 6 \text{ and } 2x\ge 8$
52. $3x+1<7 \text{ and } 3x+5\ge -1$

53. $x+4\geq 5$ *and* $2x\geq 6$

54. $9x-2<7$ *or* $3x-5>10$

55. $4x-1>11$ *or* $4x-1\leq -11$

56. $2x-3\leq 5$ *and* $3x-1>11$

57. $6x+5<-1$ *or* $1-2x<7$

58. $9-x\geq 7$ *and* $9-2x<3$

59. $3x+1<7$ *or* $3x+5\geq -1$

60. $5x-3<-18$ *or* $6x-1\geq 17$

61. $3x-1\leq 8$ *and* $6x+5<11$

62. $3x-1<-19$ *or* $2x+4\geq 16$

63. $5x-3\leq -18$ *or* $1-6x<-17$

64. $3x-11<4$ *or* $4x+9\geq 1$

65. $3-7x\leq 31$ *and* $5-4x>1$

66. $8x+2\leq -14$ *and* $4x-2>10$

67. $2x-3>5$ *and* $3x-1>11$

68. $3x-5>10$ *or* $3x-5<-10$

69. $1-4x<-11$ *or* $1-4x>11$

70. $1-3x<16$ *and* $1-4x\geq 5$