10.4 Motion in space

Fort Pulaski, GA

Let $r(t) = \langle f(t), g(t), h(t) \rangle$ be the position of the object

Length of curve $s(t) = \int_0^t \sqrt{[f'(u)]^2 + [g'(u)]^2 + [h'(u)]^2}\, du$

Speed of object, $s'(t) = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}$

$r'(t) = \langle f'(t), g'(t), h'(t) \rangle$ is the velocity of the object

$r''(t) = \langle f''(t), g''(t), h''(t) \rangle$ is the acceleration of the object

Note: velocity is a vector, speed is a scalar i.e driving at 35 mph is the speed, driving at 35mph going east is the velocity because it has a direction and is therefore a vector.

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Ex. Find the velocity and position of an object at any time t, given its acceleration $a(t) = \langle 6t, 12t + 2, e^t \rangle$, its initial velocity is $v(0) = \langle 2, 0, 1 \rangle$ and its initial position is $r(0) = \langle 0, 3, 5 \rangle$.

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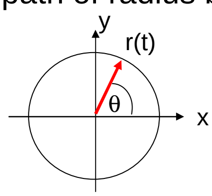
Ex. Find the force acting on an object moving along a circular path of radius b with constant angular speed.



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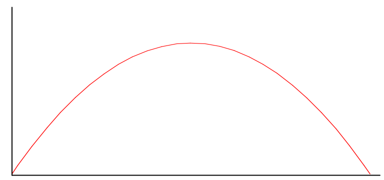
One early use of calculus was to study projectile motion.



In this section we assume ideal projectile motion:

Constant force of gravity in a downward direction

Flat surface

No air resistance (usually)

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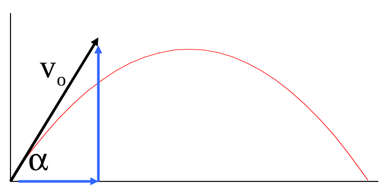


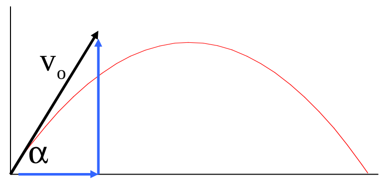
We assume that the projectile is launched from the origin at time t =0 with initial velocity $\mathbf{v}_o$.

$$\text{Let } \mathrm{v}_0 = |\mathbf{v}_0|$$

$$\text{then } \mathbf{v}_0 = (\mathrm{v}_0 \cos\alpha)\,\mathbf{i} + (\mathrm{v}_0 \sin\alpha)\,\mathbf{j}$$

The initial position is: $\mathbf{r}_0 = 0\,\mathbf{i} + 0\,\mathbf{j} = 0$

→



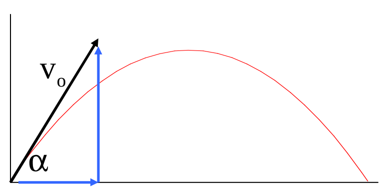
Newton's second law of motion:

$$f = ma$$

$$f = m\frac{d^2r}{dt^2}$$

Vertical acceleration

→



Newton's second law of motion:

$$f = ma$$

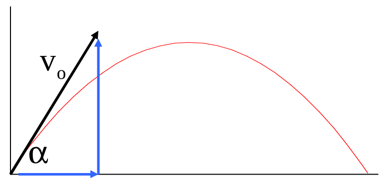
$$f = m\frac{d^2r}{dt^2}$$

The force of gravity is:

$$f = -mg\,\mathbf{j}$$

Force is in the downward direction

→



Newton's second law of motion:

$$f = ma$$

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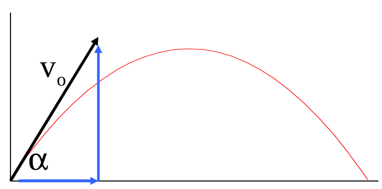
The force of gravity is:

$$f = -mg\ \mathbf{j}$$

$$m\frac{d^2r}{dt^2} = -mg\ \mathbf{j}$$

→



Newton's second law of motion:

$$f = ma$$

$$f = m\frac{d^2r}{dt^2}$$

The force of gravity is:

$$f = -mg\ \mathbf{j}$$

g = 32 ft/s^2 (9.8m/s^2)

$$\cancel{m}\frac{d^2r}{dt^2} = -\cancel{m}g\ \mathbf{j}$$

→

$$\frac{d^2 r}{dt^2} = -g\,\mathbf{j}$$

Initial conditions: $r = r_0 \qquad \frac{dr}{dt} = v_0 \quad \text{when } t = 0$

$$\frac{dr}{dt} = -gt\,\mathbf{j} + v_0$$

$$\mathbf{r} = -\frac{1}{2}gt^2\,\mathbf{j} + v_0 t + \mathbf{r_0}$$

$$\mathbf{r} = -\frac{1}{2}gt^2\,\mathbf{j} \qquad + (v_0\cos\alpha)t\,\mathbf{i} + (v_0\sin\alpha)t\,\mathbf{j} \qquad + 0$$

→

$$\mathbf{r} = -\frac{1}{2}gt^2\,\mathbf{j} \qquad + (v_0\cos\alpha)t\,\mathbf{i} + (v_0\sin\alpha)t\,\mathbf{j} \qquad + 0$$

Vector equation for ideal projectile motion:

$$\mathbf{r} = (v_0\cos\alpha)t\,\mathbf{i} + \left((v_0\sin\alpha)t - \frac{1}{2}gt^2\right)\mathbf{j}$$

→

$$\mathbf{r} = (v_0 \cos\alpha)t\,\mathbf{i} + \left((v_0 \sin\alpha)t - \frac{1}{2}gt^2 \right)\mathbf{j}$$

Vector equation for ideal projectile motion:

$$\mathbf{r} = (v_0 \cos\alpha)t\,\mathbf{i} + \left((v_0 \sin\alpha)t - \frac{1}{2}gt^2 \right)\mathbf{j}$$

Parametric equations for ideal projectile motion:

$$x = (v_0 \cos\alpha)t \qquad y = (v_0 \sin\alpha)t - \frac{1}{2}gt^2$$

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Example 1:
A projectile is fired at 60° and 500 m/sec.
Where will it be 10 seconds later?

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The maximum height of a projectile occurs when the vertical velocity equals zero.

$$\frac{dy}{dt} = v_0\sin\alpha - gt = 0$$

$$\frac{v_0\sin\alpha}{g} = t$$

$$\frac{v_o \sin\alpha}{g} = t$$

time at maximum height

→

The maximum height of a projectile occurs when the vertical velocity equals zero.

$$\frac{dy}{dt} = v_0\sin\alpha - gt = 0$$

$$v_0\sin\alpha - gt = 0$$

$$\boxed{\frac{v_0\sin\alpha}{g} = t}$$

We can substitute this expression into the formula for height to get the maximum height.

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$$y = (v_0 \sin\alpha)t - \frac{1}{2}gt^2$$

$$y_{max} = (v_0 \sin\alpha)\frac{v_0 \sin\alpha}{g} - \frac{1}{2}g\left(\frac{v_0 \sin\alpha}{g}\right)^2$$

$$y_{max} = \frac{(v_0 \sin\alpha)^2}{g} - \frac{(v_0 \sin\alpha)^2}{2g}$$

→

$$y = (v_0 \sin\alpha)t - \frac{1}{2}gt^2$$

$$y_{max} = (v_0 \sin\alpha)\frac{v_0 \sin\alpha}{g} - \frac{1}{2}g\left(\frac{v_0 \sin\alpha}{g}\right)^2$$

$$y_{max} = \frac{(v_0 \sin\alpha)^2}{g} - \frac{(v_0 \sin\alpha)^2}{2g}$$

$$y_{max} = \frac{(v_0 \sin\alpha)^2}{2g}$$

} maximum height

→

When the height is zero:

$$0 = (v_0 \sin\alpha)t - \frac{1}{2}gt^2$$

$$0 = t\left(v_0 \sin\alpha \frac{1}{2}gt \right)$$

time at launch: $t = 0$

→

When the height is zero:

$$0 = (v_0 \sin\alpha)t - \frac{1}{2}gt^2$$

$$0 = t\left(v_0 \sin\alpha - \frac{1}{2}gt \right)$$

time at launch: $t = 0$

$$0 = v_0 \sin\alpha - \frac{1}{2}gt$$

$$v_0 \sin\alpha = \frac{1}{2}gt$$

time at impact (flight time) $\left\{ \quad t = \frac{2v_0 \sin\alpha}{g} \right.$

→

If we take the expression for flight time and substitute it into the equation for x, we can find the range.

$$x = v_0 \cos\alpha \; t$$

$$x = v_0 \cos\alpha \; t \cdot \frac{2 v_0 \cos\alpha}{g}$$

→

If we take the expression for flight time and substitute it into the equation for x, we can find the range.

$$x = v_0 \cos\alpha \; t$$

$$x = v_0 \cos\alpha \cdot \frac{2 v_0 \sin\alpha}{g}$$

$$x = \frac{v_0^2}{g} \cdot 2 \sin\alpha \cos\alpha$$

$$x = \frac{v_0^2}{g} \cdot \sin 2\alpha \quad \Big\} \text{ Range}$$

→

The range is maximum when $\sin(2\alpha)$ is maximum.

$$\sin 2\alpha = 1$$

$$2\alpha = 90^0$$

$$\alpha = 45^0$$

Range is maximum when the launch angle is 45°.

$$x = \frac{v_0^2}{g} \cdot \sin 2\alpha \quad \Big\} \text{ Range}$$

→

If we start with the parametric equations for projectile motion, we can eliminate *t* to get *y* as a function of *x*.

$$x = v_0 \cos\alpha \ t$$

$$y = (v_0 \sin\alpha)t - \frac{1}{2} g t^2$$

$$\frac{x}{v_0 \cos\alpha} = t$$

If we start with the parametric equations for projectile motion, we can eliminate *t* to get *y* as a function of *x*.

$$x = v_0 \cos\alpha \; t \qquad y = (v_0 \sin\alpha) t - \frac{1}{2} g t^2$$

$$\frac{x}{v_0 \cos\alpha} = t \qquad y = (v_0 \sin\alpha) \frac{x}{v_0 \cos\alpha} - \frac{1}{2} g \left(\frac{x}{v_0 \cos\alpha} \right)^2$$

This simplifies to:

$$y = -\left(\frac{g}{2 v_0^2 \cos^2\alpha} \right) x^2 + x \tan\alpha$$

which is the equation of a parabola.

→

If we start somewhere besides the origin, the equations become:

$$x = x_0 + (v_0 \cos\alpha)\, t \qquad y = y_0 + (v_0 \sin\alpha)\, t - \frac{1}{2} g t^2$$

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Example 4:

A baseball is hit from 3 feet above the ground with an initial velocity of 152 ft/sec at an angle of 20^{o} from the horizontal. A gust of wind adds a component of -8.8 ft/sec in the horizontal direction to the initial velocity.

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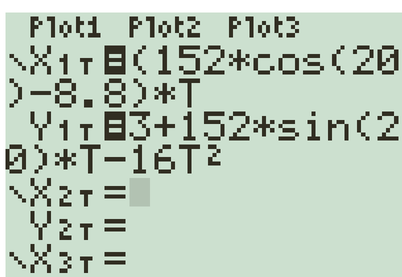
These equations can be graphed on the TI-83 to model the path of the ball:



Note that the calculator is in degrees.

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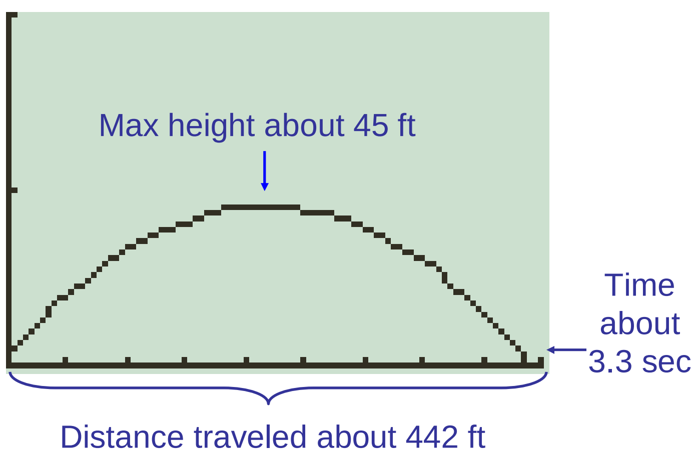
```
WINDOW
 Tmin=0
 Tmax=6
 Tstep=.1
 Xmin=0
 Xmax=450
 Xscl=50
↓Ymin=0
```

```
WINDOW
↑Tstep=.1
 Xmin=0
 Xmax=450
 Xscl=50
 Ymin=0
 Ymax=100
 Yscl=50
```

→

Using the trace function:



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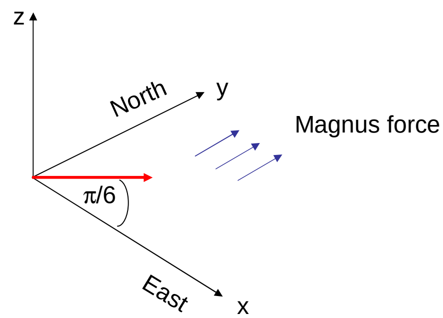
A projectile in 3D

A projectile of mass 1kg is launched ground level toward the east at 200m/s at an angle π/6 to the horizontal. If the spinning of the projectile applies a northerly Magnus force of 2N to the projectile, find the landing location of the projectile and its speed at impact



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