Summer MA 15200 Lesson 4 Section P.3

I Square Roots

If $b^2 = a$, then b is a square root of a.
If a is a nonnegative real number, the nonnegative number b such that $b^2 = a$, denoted by $b = \sqrt{a}$, is the **principal** square root of a.

radical sign $\longrightarrow \sqrt{a} \longleftarrow$ radicand

radical expression

Ex 1: Evaluate each. If not real, write 'not real'.

$a)\ -\sqrt{81}$

$b)\ \sqrt{\dfrac{25}{36}}$

$c)\ \sqrt{36+64}$

$d)\ \sqrt{36}+\sqrt{64}$

$e)\ \sqrt{-49}$

Many times students believe that $\sqrt{a^2} = a$. However, the principal square root is always positive. Examine the following.

$\sqrt{8^2} = \sqrt{64} = 8$

$\sqrt{(-8)^2} = \sqrt{64} = 8$, not -8

$-\sqrt{8^2} = -\sqrt{64} = -8$

$\sqrt{-8^2} = \sqrt{-64}$, which is not real

> In general:
> $$\sqrt{a^2} = |a|$$
> Therefore, we will always assume that variables represent positive numbers in order to avoid using absolute value signs.

II Other Types of Roots

$\sqrt[n]{a} = b$ means that $b^n = a$ If n is even, then a and b must be positive. If n is odd, a and b can be any real numbers.

index

$$\sqrt[n]{a}$$

> If no index is written, the root is assumed to be a square root.

Ex 2: Evaluate each. If not real, write ‘not real’.

$a)$ $\sqrt[3]{-125}$

$b)$ $\sqrt[4]{-81}$

$c)$ $\sqrt[6]{64}$

$d)$ $\sqrt[3]{\dfrac{27}{8}}$

$e)$ $\sqrt{0.04}$

$f)$ $\sqrt[5]{-32}$

III The Product and Quotient Rules of Radicals

> If all expressions represent real numbers,
>
> $$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab} \text{ and } \sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$$
>
> $$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}} \text{ and } \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \qquad (b \neq 0)$$

Note: These properties are for multiplication and division. Similar statements are not true for addition or subtraction. ($\sqrt[n]{a+b} \neq \sqrt[n]{a} + \sqrt[n]{b}$, for example)

Ex 3: Use the product or quotient rules of radicals (if you can) to write as one radical. Simplify, if possible.

$a)$ $\sqrt{3} \cdot \sqrt{10} =$

$b)$ $\dfrac{\sqrt[3]{54}}{\sqrt[3]{2}} =$

$c)$ $\sqrt{5} \cdot \sqrt[3]{2} =$

IV Simplifying Square Root Radicals

A square root is simplified when its radicand has no factors other than 1 that are perfect squares.

Remember: $\sqrt{a^2} = a$, if a is assumed to be positive. We will assume all variables represent positive values.

Ex 4: Use factoring and the product (and/or quotient)rule to simplify each.

$a)\quad \sqrt{18x^3} =$

$b)\quad \sqrt{81a^7b^5} =$

$c)\quad \sqrt{32x} \cdot \sqrt{2x^5} =$

$d)\quad \dfrac{\sqrt{44m^3n^8}}{\sqrt{11mn^5}} =$

V Addition and Subtraction of Square Roots

Two or more square roots can be combined if they have the same radicand. Such radicals are called **like radicals.** Sometime one or more radical must be simplified in order to combine.

Ex 5: Simplify and combine where possible.

$a)\quad \sqrt{32} + \sqrt{162} =$

$b)\quad 3a\sqrt{3a} - \sqrt{48a^3} =$

$c)\quad 4\sqrt{6a^3} - 3a\sqrt{54a} =$

VI Rationalizing Denominators

The process of rewriting a square root radical expression as an equivalent expression in which the denominator no longer contains any radicals is called **rationalizing the denominator.**

- First, simplify any radicals.
- Secondly, multiply the numerator and denominator by the radical factor that remains.

Ex 6: Simplify by rationalizing the denominator.

$a)\quad \dfrac{2}{\sqrt{5}} =$

$b)\quad \sqrt{\dfrac{3}{8}} =$

$c)\quad \dfrac{12}{\sqrt{3m^3}} =$

VII Conjugates

Radical expressions that involve the sum and difference of the same two terms are called **conjugates.** Examples are $\sqrt{2}+\sqrt{5}$ and $\sqrt{2}-\sqrt{5}$ or $3-\sqrt{x}$ and $3+\sqrt{x}$.
The product of two conjugates will contain no radicals!

$$\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right)=\left(\sqrt{a}\right)^2-\left(\sqrt{b}\right)^2=a-b$$

In radical expressions with a binomial (two terms) in the denominator, to rationalize the denominator, multiply numerator and denominator by the conjugate of the denominator.

Ex 7: Rationalize and simplify each.

$a)\quad \dfrac{2}{\sqrt{3}-\sqrt{5}} =$

$b)\quad \dfrac{\sqrt{3}}{\sqrt{8}+1} =$

VIII Rational Exponents

Examine: $\left(\sqrt{3}\right)^2 = 3$ and $\left(3^{\frac{1}{2}}\right)^2 = 3^1 = 3$ Since both $\sqrt{3}$ and $3^{\frac{1}{2}}$ squared equal 3, they must be equivalent.

Definition of $a^{\frac{1}{n}}$

If $\sqrt[n]{a}$ represents a real number, where $n \geq 2$ is an integer, then $a^{\frac{1}{n}} = \sqrt[n]{a}$.

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

The denominator of the rational exponent becomes the index of the radical.

> The textbook and online homework may use a regular fraction bar for a rational exponent or a slash fraction bar.
>
> $a^{\frac{1}{n}} = a^{1/n}$

Ex 8: Evaluate each, if it exists.

$a)$ $9^{\frac{1}{2}} =$

$b)$ $125^{\frac{1}{3}} =$

$c)$ $\left(\dfrac{81}{16}\right)^{-\frac{1}{4}} =$

$d)$ $-1^{\frac{1}{8}} =$

$e)$ $(-4)^{\frac{1}{2}} =$

Examine: $\begin{cases} a^{\frac{3}{4}} = \left(a^{\frac{1}{4}}\right)^3 = \left(\sqrt[4]{a}\right)^3 \text{ and } a^{\frac{3}{4}} = \left(a^3\right)^{\frac{1}{4}} = \sqrt[4]{a^3} \\ \text{Therefore: } a^{\frac{3}{4}} = \left(\sqrt[4]{a}\right)^3 \text{ or } \sqrt[4]{a^3} \end{cases}$

Definition of $a^{\frac{m}{n}}$

If $\sqrt[n]{a}$ represents a real number and $\dfrac{m}{n}$ is a positive rational number, $n \geq 2$, then $a^{\frac{m}{n}} = \sqrt[n]{a^m}$ *or* $\left(\sqrt[n]{a}\right)^m$. It can be evaluated or simplified by finding the power first, then the root or by finding the root first, then the power. Because you will not have a calculator on quizzes or your first exam, I recommend finding the root first, then raise to the exponent power.

The numerator is the exponent.

$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m \text{ or } \sqrt[n]{a^m}$$

The denominator is the index.

Ex 9: Evaluate, if possible.

a) $36^{\frac{3}{2}} =$

b) $8^{\frac{4}{3}} =$

c) $16^{\frac{3}{4}} =$

Ex 10: Evaluate, if possible.

a) $4^{-\frac{3}{2}} =$

b) $(-32)^{-\frac{2}{5}} =$

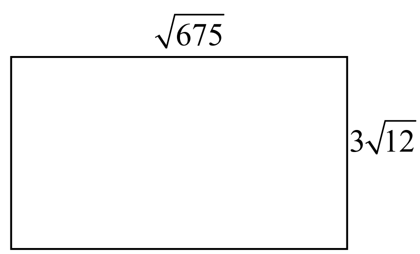
Ex 11: Use the properties of exponents to simplify.

a) $\left(\dfrac{-8}{a^3b^6}\right)^{-\frac{2}{3}} =$

b) $\dfrac{9^{\frac{3}{2}}}{(8x^3)^{-\frac{1}{3}}} =$

c) $(64x^6y^{-12})^{\frac{5}{6}}(x^6y^{-3})^{\frac{2}{3}} =$

Ex 12: A rectangle below has the given width and length. Find the perimeter (using radicals as needed) and the area (using radicals as needed) of this rectangle. Simplify each.



Some mathematical models may be equations that have radical expressions.

Ex 13: Suppose $E = 5\sqrt{x} + 34.1$ models the number of elderly Americans ages 65-84, in millions, for x number of years after 2010. Project the number of Americans ages 65-84, in millions, in 2019 and 2059. Express the increase in number of elderly Americans from 2019 to 2059.