Rational Bounds and the Robust Risk Management of Derivatives

Abstract

The risk management of derivative portfolios is vulnerable to model error. This paper explores risk management strategies based on no-arbitrage bounds, which are independent of any model. In particular, we determine the bounds on the price of a general barrier option given the price of a set of European call options and identify the hedging strategy that enforces the bounds. The strategy puts a floor on the maximum loss that can be incurred by the writer of the barrier option. We show how the strategy can be made dynamic and the floor raised over time. The distribution of hedge errors under the strategy is compared with that under alternative strategies.

JEL Classification: G13, G18

Introduction

The risk management of complex derivative portfolios is vulnerable to error. Institutions make heavy use of sophisticated pricing models to extract information from market prices and to compute optimal hedging strategies. But the hedged portfolio is still subject to substantial risks. The risks result both from errors in the specification of the model itself, and from errors in parameter estimation. The risks are potentially large, and may offset any benefits from hedging. In this paper we argue for the use of robust hedging strategies, based on rational bounds, which are free from model risk. We develop a methodology to identify rational bounds, we apply it to barrier options, and we show that robust hedging of barrier options is indeed effective.

Andersen and Andreasen (2001) and Longstaff, Santa-Clara and Schwartz (2001)) examine the early exercise decision on an American or Bermudan swaption. Both papers show that calibrating a model to a set of apparently closely related instruments – the prices of plain European swaptions - is not sufficient to price a complex claim accurately; the model itself needs to be correctly specified. Backus, Foresi and Zin (1998) illustrate how the use of an incorrectly specified model of the term structure can cause substantial errors even where the model is carefully calibrated to fit market prices. Dumas, Fleming and Whaley (1998) analyse the hedging properties of index options, and show that overfitting can lead to a substantial *increase* in hedging errors as compared with using a simpler model which does not attempt to fit all market prices simultaneously.

These findings pose a dilemma for someone who wanting to hedge a position, who has a large set of closely related instruments which can be used for hedging, and who does not know the true model. On the one hand, it seems highly desirable to make use of all the information in market prices. On the other hand, overfitting an almost surely mis-specified model to the data and refitting it over time does not look a promising way of designing a reliable hedging strategy.

This paper offers a way out of the dilemma. It makes full use of the market prices of the instruments that are available for hedging, so all available information is used. But it does not use a model, so there is no danger of over-fitting. The absence of an underlying model also means that the optimal hedge can be revised, and the position can be hedged dynamically, without any internal inconsistency.

The approach is based on "rational bounds". Merton (1973) argues that a necessary property of any rational option pricing theory is that options be priced so that they are neither dominating nor dominated securities. He identifies bounds on the prices of standard options that must hold in any rational model, and calls them rational bounds. The rational bounds on the value of a call option on a non-dividend paying asset are its intrinsic value (Max [S - PV(K), 0] where K is the exercise price) and the asset price, S.

Rational bounds are associated with hedging strategies. For example, the writer of a call option can limit his liability to the upper bound S by buying the asset, holding it, and delivering it if the call is exercised. The strategy is robust in that it does not depend on any model of asset price dynamics.

The rational bounds on the standard call option are very wide, and the corresponding robust hedges are of limited practical use. As Green and Figlewski (1999) show, delta hedging vanilla options is reasonably effective in reducing risk, while the rational bounds hedge still leaves the writer with substantial risk.

A number of authors have sought to tighten the bounds on the price of a call option. Perrakis and Ryan (1984), Levy (1985), Ritchken (1985) and Bergman, Grundy and Wiener (1996) have shown how to tighten Merton's bounds by imposing restrictions on the behavior of asset prices or on preferences. Cochrane and Saa-Requejo (2000), Bernardo and Ledoit (2000), Carr, Geman and Madan (2001) and Cerny and Hodges (2002) extend Merton (1973)'s strict no arbitrage restriction by requiring that prices be such as to exclude not only arbitrages but also excessively good deals.

We follow a different approach to getting useful bounds. We stay with Merton's strict no-arbitrage conditions, but widen the scope to examine the bounds on a general derivative imposed by multiple hedging instruments. We use these bounds to construct robust hedging strategies. The upper bound on a claim is also the cost of a dominating portfolio (see for example El Karoui and Quenez (1995)). We exploit this to identify robust hedging strategies for complex derivatives that make full use of the range of hedge instruments available.

With the growing number of derivative contracts that are traded, the optimal use of multiple hedging instruments is of great practical interest. It is an area where traditional methods do not work well. The standard approach involves building a pricing model calibrated to the market prices of hedge instruments, and using the model to compute hedge ratios. The approach does not generally identify which contracts to use for hedging, nor how frequently the hedge should be rebalanced. More importantly, the approach generates hedge errors as the portfolio is rebalanced over time. Rebalancing the hedge, which would not require or generate cash if the model were correct, creates gains and losses as trades are executed at market rather than model prices.[1]

It is precisely in this richer environment with multiple hedge instruments that robust hedging comes into its own. In general, the greater the number of hedge instruments the tighter the rational bounds, and the more useful the hedging strategy. The model cannot generate hedge errors because no assumptions are made about how securities are priced. If the hedge portfolio is rebalanced, it can generate only positive cashlows as one dominating portfolio is exchanged for a cheaper one. Furthermore, the bounds analysis identifies the specific instruments to be used for the hedge.

To illustrate the approach, we focus on one particular class of derivative – barrier options – and we take the hedge instruments to be a set of European call options. The problem is rich enough to be interesting. Barrier options are so widespread, and they present such severe hedging problems, that the identification of a robust hedge for this particular class of derivatives is valuable in its own right. We show how the approach can be applied to other types of option.

By focusing on barrier options, we can compare our approach with that of Carr, Ellis and Gupta (1998). Their method assumes "put-call symmetry" under which the implied volatility curve is symmetrical with respect to the strike price. Although the theoretical foundations are very different, both methods use European options to hedge barrier options, and do not involve trading except at inception and when the barrier is breached.

The hedges we identify are totally robust in the sense that the downside is bounded whatever the path of prices. Previous work has used the term robust hedging in a rather weaker sense. Ahn,

[1] Derman, Ergener and Kani (1995) have proposed a static hedging methodology to avoid some of the problems of continuous rebalancing, but this too suffers from dynamic inconsistency.

Muni and Swindle (1997) show how to find hedging strategies which maximize expected utility in the worst case outcome from a restricted set of volatility scenarios. Avellaneda, Levy and Parás (1995) and Lyons (1995) examine super-replication bounds for standard options, where the instantaneous volatility is uncertain but lies between upper and lower bounds. Our approach can be seen as the limiting case of an agent who wants to minimize his loss in the worst possible outcome, without restricting the range of outcomes.

The paper is organised as follows: in the first section we show how rational bounds on the price of a general barrier option, and the corresponding robust hedging strategies, can be found in a zero interest rate world. In Section 2, we apply the model and examine the bounds for a number of simple options including the digital barrier option and up-and-in puts and calls. Section 3 examines the hedging performance of the robust strategies that underpin the bounds. In Section 4 we relax the main assumptions of our basic model, and consider the effect of jumps in the price of the underlying asset, positive interest rates and having only a sparse array of European options for hedging. We also show how the approach generalizes to a variety of other options. The final section concludes.

1. The Basic Model

The problem we address is the following: what bounds can be placed on the price of a derivative security given the price of the underlying asset and the prices of a set of traded instruments which can be used for hedging it? More specifically, what is the least upper bound on the price of a barrier option given prices for the underlying and for all European call options with the same maturity as the barrier option?

In this section we develop a methodology to find rational bounds that exploits the duality between pricing and hedging. We pick a set of price processes that are consistent with the absence of arbitrage and also price the hedge instruments correctly; we search for the process within the set that maximizes the value of the barrier option; we find the corresponding hedging strategy that enforces the upper bound; and we verify that the strategy does indeed dominate the barrier option for all possible price processes, rather than just for the processes within the selected set.

This methodology depends critically on the choice of the set of price processes. If too small a set is selected, the upper bound on the price of the derivative will not be a global upper bound, and the hedging strategy will not dominate the barrier option on all possible paths. If too large a set is selected, the search is hard to implement. In this section, we set out the basic framework, choose a suitable set of price processes, identify the upper bound for an up-and-in barrier option, and extend the result to other types of barrier option.

A. The Set-up

The price of the underlying at time t is denoted by S_t. The exotic option is an up-and-in barrier option which pays $Y(S_T)$ at time T if the price of the underlying breaches a barrier B ($> S_0$) in the period $[0, T]$, and zero otherwise. As will be shown later, the extension to other types of barrier options (down-and-in or up-and-out for example) and also to lower bounds is straightforward.

The time zero price of a European call with strike K and maturity T is $C(K)$. The set of strikes available in the market is denoted by $\boldsymbol{K}$.

We assume:

[A1] There is no arbitrage among the call option prices, the underlying and cash.

[A2] There is a portfolio composed of cash and call options whose cost is finite and which dominates a claim paying $|Y(S_T)|$.

These two assumptions imply the following properties in the dual space of price processes for the underlying:

- *Existence of a Martingale Pricing Measure:* The set of possible martingale processes $\boldsymbol{M}$ for the underlying asset, which exactly price all the European call options, is non-empty (Harrison and Kreps (1979)).
- *Saddle Point Property:* The cost of the cheapest portfolio that dominates the exotic claim is equal to the maximal value of the claim under the processes belonging to $\boldsymbol{M}$.

To obtain useful bounds on the barrier option we make some additional assumptions:

[A3] We restrict the processes $\boldsymbol{M}$ to a subset $\boldsymbol{M_C}$ of processes that are continuous at their first stopping time to the barrier level B. More precisely, at the first stopping time τ for the underlying at or above the barrier, we require that $S_\tau = B$. This ensures that when the underlying first hits the price level B, the agent can buy or sell it at exactly that price.

[A4] The interest rate is zero.

[A5] The set $\boldsymbol{K}$ is the same as the support of the underlying at date T.

We relax all three of these additional assumptions in section 4. In particular we consider what happens if the underlying can jump across the barrier, if the riskless rate of interest is some positive constant and if only a finite set of calls is traded.

B. Price Processes

We want to identify a set of processes $\boldsymbol{P} \subset \boldsymbol{M_C}$ which is large enough to include a global supremum, yet compact enough that the supremum can be found through a simple search. The value of the exotic option depends on the joint distribution of the terminal value and whether the barrier is hit. The precise time at which the barrier is hit is not important. This suggests that we lose little by restricting the family $\boldsymbol{P}$ to jump processes, with at most two jumps. Paths which breach the barrier reach the barrier level in the first jump, and go to their terminal value in the second jump, while paths which do not breach the barrier go to their terminal value in the first jump[2].

To formalize this, we pick two arbitrary times t_1 and t_2 where $0 < t_1 < t_2 < T$, and define the subset $\boldsymbol{P}$ to consist of those processes $P \in \boldsymbol{M_C}$ which satisfy the assumptions described above, and where:

- S_t jumps at most at times t_1 and t_2, and is otherwise constant, and

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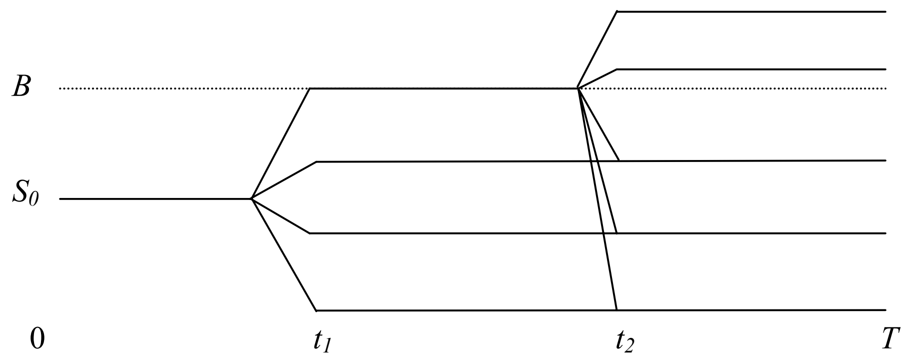
[2] This is more convenient than the alternative approach of partitioning $\boldsymbol{M_C}$ into equivalence classes defined by whether S_t hits B and by the value of S_T. Although the process is a pure jump process, none of the paths jumps through the barrier, so [A3] is satisfied.

- S_t is equal to S_T ($< B$) or to B for all $t \in [t_1, t_2)$.

One such process is illustrated by the following diagram:



Time is on the horizontal axis and price is on the vertical axis. The diagram shows eight different price paths. Five paths jump to the barrier B at time t_1 then jump again at time t_2. Three paths fail to reach the barrier at time t_1 and remain constant thereafter. None of the paths jumps above the barrier at time t_1.

Each process $P \in \boldsymbol{P}$ can be fully characterized by the distribution function

$$F_B(S) = \Pr\{S_{t_1} = B \text{ and } S_T < S\} \quad (1)$$

for the final outcome of paths which hit the barrier. This is because the unconditional distribution function

$$F(S) = \Pr\{S_T < S\} \quad (2)$$

is obtained by differentiating the call option prices with respect to the strike, K, as

$$\begin{aligned} F(S) &= \Pr\{S_T < S\} \\ &= 1 + dC/\mathrm{d}K \end{aligned} \quad (3)$$

because of assumption [A5] (continuum of strikes).

Note that $F_B(S)$ is defined as the total probability for paths through B, rather than as a conditional probability. The limit $F_B(\infty)$ is thus the probability of B being hit.

C. The Main Result

Having identified a feasible function for $F_B(S)$ to characterize a particular process P, the value of the barrier option is determined as:

$$V(F_B) = \int Y(S)\, dF_B(S). \tag{4}$$

Proposition 1*: The process P^* that maximizes the value of the barrier option can be found by a one-dimensional search. The maximal value $V^* = V(F_B^*)$ is the least upper bound on the price of the barrier option. The bound is enforced by a trading strategy that requires trading of options only at the initial date, and trading of the underlying only as and when the barrier is hit.*

Proof:

We have to choose the function $F_B(S)$ to:

maximize $\int Y(S)\, dF_B(S)$ subject to the constraints:

$$\int (S - B) dF_B(S) = 0, \tag{5a}$$

$$0 \leq dF_B(S) \leq dF(S), \text{ with} \tag{5b}$$

$$dF_B(S) = dF(S), \text{ for } S > B. \tag{5c}$$

First note that a solution exists and is bounded. This is because assumptions [A1] and [A2] guarantee a bounded solution within ***M***. The restriction to the subset ***P*** can only change the value of the bound and not its existence.

Rewrite (5), incorporating the martingale constraint (5a) in the objective function:

$$\underset{dF_B(S)}{Max}\; V(F_B) = \int (Y(S) - \lambda(S - B)) dF_B(S), \text{ subject to (5a-c).} \tag{6}$$

The first order conditions characterize the solution as

$$dF_B(S) = \begin{cases} dF(S) & \text{if } Y(S) - \lambda(S-B) \geq 0 \text{ and } S < B, \\ 0 & \text{if } Y(S) - \lambda(S-B) < 0 \text{ and } S < B, \\ dF(S) & \text{if } S \geq B, \end{cases} \tag{7$_U$}$$

with λ being determined by (5a) (this is the one-dimensional search). The maximal value V^* of the exotic under ***P*** is then given by equation (4).

From (4) and (7_U), for a price of V^*, it is possible to buy the European claim with pay-off $h(S_T)$ where:

$$h(S_T) = \begin{cases} Y(S_T) - \lambda(S_T - B) & \text{if } S_T \geq B, \\ Max[Y(S_T) - \lambda(S_T - B), 0] & \text{otherwise.} \end{cases} \qquad (8_U)$$

The exotic claim pays $Y(S_T)$ in the event of the barrier B being reached, and otherwise zero. It can be super-replicated by the following strategy: buy a portfolio of European call options which has a pay-off $h(S_T)$. If the barrier is subsequently hit, immediately purchase λ units of the underlying forward, at zero cost (buying one share and borrowing amount B per unit). This ensures that:

> *if B is not hit:* the hedge portfolio pays $h(S_T)$, which is non-negative, while the exotic pays nothing, and
>
> *if B is hit:* the hedge portfolio pays $h(S_T) + \lambda(S_T - B)$[3], which is never less than $Y(S_T)$, the pay-off to the exotic.

So V^* is not only a feasible price of the barrier option, it is also the cost of a super-replicating strategy. This completes the proof. ■

D. Some Corollaries

Proposition 1 applies directly only to pure up-and-in options – options which pay Y if the barrier is hit and nothing otherwise, and where the barrier B is above the initial asset price. The extension to down-and-in barriers is trivial. It is also straightforward to extend the result to up-and-out options, or barrier options where a rebate is paid if the barrier is not hit. To see this, observe that a general barrier option pays $Y(S_T)$ if the barrier is hit and $Z(S_T)$ if it is not. It is the

[3] We invoke assumption [A3] to ensure that the underlying can be bought at the barrier price B immediately the barrier is breached.

sum of a European claim Z (which can be priced exactly) and an up-and-in option with pay-off Y-Z (which can be bounded by Proposition 1).

Proposition 1 characterizes the upper bound on the option value. The lower bound has a similar characterization in terms of the dearest strategy whose payoff never exceeds that of the barrier option. The extremal process and the initial claim are given by the equations:

$$dF_B(S) = \begin{cases} dF(S) & \text{if } Y(S) - \lambda(S-B) \le 0 \text{ and } S < B, \\ 0 & \text{if } Y(S) - \lambda(S-B) > 0 \text{ and } S < B, \\ dF(S) & \text{if } S \ge B, \end{cases} \tag{7_L}$$

and:

$$h(S_T) = \begin{cases} Y(S_T) - \lambda(S_T - B) & \text{if } S_T \ge B, \\ Min[Y(S_T) - \lambda(S_T - B), 0] & \text{otherwise.} \end{cases} \tag{8_L}$$

The cheapest dominating strategy does not require trading the European options except when the hedge is established. One interesting corollary of this is that the bounds on the price of a barrier option cannot be tightened by allowing intermediate trading of options.

2. Application to Specific Barrier Options

In this section we identify rational bounds on a number of common barrier options. We examine the hedging strategies that enforce those bounds, and investigate the width of those bounds numerically.

A. Upper Bound on the Digital Barrier Option

The simplest barrier option is the Digital Barrier option. It pays \$1 if the barrier B is hit, and zero otherwise. Using the notation above, $Y(S) = 1$. Equation (7_U) implies that the limiting process P_U is:

$$dF_B(S) = \begin{cases} dF(S) & S \ge K_U, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

where $K_U < B$, and λ are chosen such that

$$\int_{K_U}^{\infty}(S-B)\,dF(S)=0 \text{ and } \lambda=-\frac{1}{B-K_U}<0. \tag{10}$$

All paths that hit the barrier end at or above K_U; all other paths end at or below K_U. K_U satisfies equation (10); this is the martingale condition that the expected terminal value of paths terminating at or above K_U equals B. It turns out that this process P_U is also the limiting process for a number of other barrier options.

Equation (8_U) shows that the hedge portfolio h comprises $1/(B–K_U)$ European calls with strike K_U. An agent who is short the digital barrier option, and long the portfolio h, can ensure a non-negative terminal pay-off. If the barrier is not hit, the agent is simply long a call option. If the barrier is hit, the agent sells $1/(B–K_U)$ units of the underlying at the barrier price B. The terminal pay-off is then:

$$\frac{Max(S_T-K_U,0)}{B-K_U}-\frac{S_T-B}{B-K_U}-1=\frac{Max(K_U-S_T,0)}{B-K_U}, \tag{11}$$

which is positive. $1/(B\text{-}K)$ calls with strike K dominate the Digital for any $K < B$. The search for the least upper bound is the (one-dimensional) search for the strike that minimizes the cost of the strategy.

The upper bound is shown in Figure 1. The top panel shows the hedging strategy while the lower panel shows the limiting process. It is easy to check that the cost of the cheapest dominating strategy is the same as the expected value of the barrier option under P_U, and that under P_U the dominating strategy is a replication strategy.

B. Lower Bound on the Digital Barrier Option

To find the upper bound on a barrier option we use equations (7_L) and (8_L) with $Y(S) = 1$. The limiting process P_L is characterized by:

$$dF_B(S)=\begin{cases} dF(S) & \text{if } S\geq B,\\ 0 & \text{if } K_L\leq S<B,\\ dF(S) & \text{if } S<K_L,\end{cases} \tag{12}$$

where K_L is chosen so that $\int_B^\infty + \int_{-\infty}^{K_L} (S-B)dF(S) = 0$. The process is illustrated in Figure 2. All paths that hit the barrier end either above B or below K_L while paths that do not hit the barrier end between K_L and B.

The lower bound equals the cost of the most expensive portfolio that is dominated by the barrier option. Equation (8_L) shows that the portfolio is the European claim:

$$h(S) = \begin{cases} 0 & \text{if } S \in [K_L, B), \\ \dfrac{S-K_L}{B-K_L} & \text{otherwise.} \end{cases} \tag{13}$$

If the barrier is hit, $1/(B-K_L)$ forward contracts are sold. The strategy is illustrated in the top panel of Figure 2.

C. A Numerical Example of the Digital Barrier Option

Figure 3 shows the rational bounds on the price of a 1 year Digital Barrier option when all European options are trading on an implied volatility of 30% and the asset price is \$100. The figure shows the Black-Scholes price of the digital, which lies as expected between the upper and lower bounds. The European Digital option is dominated by the Digital Barrier option, and provides an obvious lower bound. Figure 3 shows that its price is substantially lower than the lower bound we have identified.

The rational bounds are wide. For example when the barrier is at \$120, the rational bounds are \$0.309 - \$0.567, while the Black-Scholes value is \$0.494. Note that an up-and-in digital together with an up-and-out digital with the same barrier make an unconditional digital – a portfolio which pays \$1 whatever. So another interpretation of the graph is to say that, with the barrier at \$120, the split in value between the up-and-in and the up-and-out is 49:51 according to Black-Scholes. However, no arbitrage possibilities arise unless the split is outside the range 31:69 to 57:43.

The lower panel of Figure 3 shows the same data but with prices expressed as implied Black-Scholes volatilities. It shows that even though all European options trade on an implied (Black-

Scholes) volatility of 30%, the digital option with a barrier at $120 could trade on an implied volatility anywhere in the range 19.1% to 37.2% without creating an arbitrage possibility.

The results are only shown for one level of maturity and volatility. But the width of the bounds is insensitive to the parameters chosen; with different parameters, the horizontal axis of Figure 3A would be rescaled, but the shape would not change greatly. In general, if all the European options trade on the same implied volatility, then the maximum difference between the Black-Scholes price of a digital and its upper bound cannot exceed 8.1% of its face value whatever the maturity, the level of volatility or the barrier level. We will make use of this in Section 3.

D. Rational Bounds on Other Barrier Options

The procedure for identifying bounds and hedging strategies can readily be applied to other barrier options. Table I shows the upper bounds not only for the up-and-in Digital, already discussed, but also for up-and-in Calls and Puts; Table II shows the corresponding lower bounds. There are just two types of limiting process that together give the maximum price to the barrier option. They are the P_U and P_L processes[4] which define the upper and lower bounds on the Digital Barrier option. The table characterizes the cheapest strategy that dominates the barrier option. The agent buys the European portfolio *h* and buys λ of the underlying at the barrier.

The bounds on the up-and-in put are shown graphically in Figure 4. As before, we assume a maturity of 1 year and that all European options are trading on an implied volatility of 30%. The current price is $100, and the barrier is at $120. The price of the European put option, the Black-Scholes price and the upper and lower bounds on the price of the barrier option are graphed as a function of the strike. For strikes below $63 ($K_L$) the upper bound is trivial – it is equal to the European price; for strikes below $91 ($K_U$) the lower bound is trivial – it is equal to zero. As the

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[4] Recall that in the P_U process, all paths that hit the barrier have $S_T \geq K_U$ for some K_U, while in the P_L process the paths which hit the barrier have $S_T \geq B$ or $\leq K_L$ for some K_L. The fact that there are only two limiting processes for all the barrier options in the Tables reflects the simple pay-offs we consider. Equations (7_U) and (7_L) show that to get more complex processes, the function $Y(S) - \lambda(S-B)$ needs to have more zeroes.

strike rises the width of the bounds widens, then narrows to zero when the strike is equal to the barrier, before widening out again.

The up-and-out put plus the up-and-in put equal a European put. Figure 4 can be interpreted as showing the range of partitions of the European put value between the two types of barrier option that can exist without creating an arbitrage opportunity.

3. Robust Hedging Using Rational Bounds

In this section we analyse the performance of hedges based on rational bounds. We show how the rational bounds hedge can be improved by rebalancing. We compare its performance with delta hedging, and also with the hedging strategy of Carr, Ellis and Gupta (1998. We focus specifically on the hedging of a digital barrier option.

A. The Rebalanced Rational Bounds Hedge

The performance of the rational bounds hedge can be greatly improved by revising it from time to time. The dominating portfolio which is cheapest initially is likely not to remain cheapest over the life of the barrier option. The agent can exchange the portfolio for a cheaper one, thereby reducing the maximum loss.

We use Monte-Carlo simulation to explore the magnitude of the improvements from revision. The digital option with a face value of $1, a barrier 20% above the spot price, and maturity of 1 year, is written at its Black-Scholes value. The cheapest dominating portfolio is set up. Each period, the probability of hitting the barrier is computed. If the barrier is hit, the hedge is liquidated at the barrier price. Otherwise the hedge is liquidated at the end of the period, and replaced with the cheapest dominating portfolio. At the end the terminal wealth is computed. The simulation is done in a Black-Scholes world where the underlying has constant volatility of 30%, and all European call options trade at Black-Scholes prices. There are five periods per year[5] and the simulation is based on 1,000 runs.

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[5] The results are similar if the period length is shorter and the hedge is revised more frequently.

Figure 5 compares the hedge error with and without revising. Without revision, the distribution of the hedge error is highly skewed. The average hedge error, as expected, is insignificantly different from zero. The maximum loss is $0.073, and this is the actual loss 82% of the time. The rest of the time, the hedge often makes profits that are far larger in magnitude than the maximum loss.

By revising the hedge, the absolute magnitude of the hedge error is greatly reduced. The probability of making a loss falls to 60%, and the maximum loss in the simulations is reduced to $0.062 (though the maximum possible loss remains $0.073). The expected size of loss, conditional on a loss occurring, falls from $0.073 to $0.034.

A rational bounds hedge, whether revised or not, puts a limit on the maximum loss on the position. The simulation shows that revising the hedge reduces the size of the hedge error. While the benefits of revising the hedge will vary according to the barrier option being hedged, the simulation suggests that the gains may be considerable.

B. Comparison with Delta Hedging

It is natural to compare the rational bounds hedge with a delta hedge. While a full comparison is outside the scope of this paper, a somewhat informal argument is sufficient to demonstrate that the rational bounds hedge is likely to compare well with a delta hedge.

If the volatility of the underlying asset is known and constant, if the asset price follows a diffusion process, and if there are no frictions, the barrier option can be hedged perfectly. But volatility is notoriously hard to predict, and is not constant; asset prices do jump. It is instructive to focus on just one source of hedge error: errors in volatility forecasts.

Suppose that the only deviation from the Black-Scholes assumptions is that agents do not know what future volatility will be when they write options. However, immediately the option is written, they get a perfect forecast of future volatility. The option can then be replicated perfectly. In this world, the hedge error is equal to the change in the value of the option when the true volatility is revealed.

This argument suggests that errors in forecasting volatility alone will be sufficient to cause a hedge error equal to the difference between the values of the option using expected and realized volatility. In the example used previously (barrier = 120% of spot, maturity 1 year, all options initially trading on an implied volatility of 30%), the digital barrier option's Black-Scholes price is \$0.494 against an upper bound of \$0.567. If the agent sells the barrier option at the Black-Scholes price, the robust hedge limits the maximum loss to \$0.073.

The upper bound has an implied Black-Scholes volatility of 37% (see bottom panel of Figure 3). So if realized volatility exceeds this level, the *average* loss from a delta hedge will exceed the *maximum* loss from the rational bounds hedge. To put it another way, the delta hedge can incur larger losses than the rational bounds hedge unless it is known that realized volatility will not exceed the implied volatility of the rational bound price. Given the difficulty of forecasting volatility, this makes delta hedging relatively unattractive to someone who puts heavy weight on avoiding large losses.

C. Comparison with Carr, Ellis, Gupta (1998)

It is unsurprising that the delta hedge performs badly against the rational bounds hedge since it is exposed to volatility changes. A fairer comparison is with the strategy of Carr, Ellis and Gupta (1998) (hereafter CEG). Although CEG and rational bounds hedges have very different theoretical bases, they do have common features. Both make use of vanilla options. Both hedges are static, except when the barrier is hit.

CEG make two assumptions. One is similar to our assumption [A3] on continuity at the barrier. The second they call Put-Call Symmetry. Put-call symmetry requires that the implied volatility curve for European options of the same maturity is symmetric about the forward price of the asset. In particular this means that[6]:

$$C_t\left(a\,S_t\right) = a\,P_t\left(S_t\,/\,a\right), \qquad (14)$$

[6] For simplicity, we maintain the assumption that interest rates are zero.

where a is any positive number, S_t is the spot price of the underlying at time t, and $C_t(x)$ and $P_t(x)$ are the prices at time t of call and put options with maturity T and strike x.

The quality of the CEG hedge depends on how well put-call symmetry holds at the time the barrier is hit. The following Proposition shows that for a digital option the hedge error depends only on the slope of the implied volatility curve for the at-the-money strike; this is zero if put-call symmetry holds.

The slope of implied volatility can be estimated from the "risk reversal" which is monitored by some information vendors. It is the difference between the Black-Scholes implied volatility of the out-of-the money call with a delta of +0.25, and the out-of-the-money put with a delta of –0.25. With put-call symmetry the risk reversal should be roughly zero. Proposition 2 relates the error in the CEG hedge to the risk reversal when the barrier is hit.

Proposition 2: *Using a CEG hedge for a digital barrier option, the hedge error depends on the derivative of the implied volatility with respect to the strike for at the money options, at the time the barrier is hit. On a digital with a pay-out of \$1, the error Err is given by:*

$$Err = -\$2n\left(\sigma\sqrt{\tau}/2\right)\sqrt{\tau}\left.\frac{d\sigma}{d\ln K}\right|_{K=S_t}, \tag{15}$$

where n(.) is the standard normal density function, τ is the time to maturity from the first breach of the barrier and $\sigma(K)$ is the implied volatility at strike K. If the barrier is not hit, Err = 0. Err can be expressed as a function of the risk reversal RR and the implied volatility σ at the time of the breach:

$$Err \approx \$0.64\frac{RR}{\sigma}. \tag{16}$$

Proof: in Appendix.

Suppose that we are trying to hedge a digital barrier option with face value of \$1, and that all options are trading on the same implied volatility at the outset. The CEG value and the Black-Scholes value of the digital are identical. As shown in Section 2C, the rational bounds hedge allows the agent to write the digital at the Black-Scholes price and put a floor on his loss of at most \$0.081. Proposition 2 shows that the CEG hedge can only do worse than this if the barrier is hit, and if at the time the barrier is hit RR/σ exceeds 0.081/.64 or 12.7%.

From equation (15) it is clear that CEG works least well for options on equities, equity indices and on illiquid currencies where implied volatility curves are known to exhibit a pronounced slope. It is likely to be much more effective for hedging options on liquid currency pairs. For empirical tests, we therefore focus on the latter.

Figure 6 shows the risk reversal as a proportion of the at-the-money volatility for both the Euro/$ and for the Yen/$ (prior to 1999 the Deutschemark is used in place of the Euro). The data are daily data based on 1 month options and are provided by Citibank. The mean levels of risk reversal (*RR*) are –0.2% and –0.7% respectively, compared with mean at the money volatilities (σ) of 10.8% and 11.9%.

Over the sample period, the ratio generally stays below the critical level of 12.7% for both Euro/$ and Yen/$, implying that (with one possible exception) the worst loss on the CEG hedge would have been no higher than the worst possible loss with the rational bounds hedge. However, the graph also shows many occasions on which the ratio falls below –12.7% for Yen/$. Transposing currency pairs reverses the sign on the risk reversal and converts up-and-in digitals into down-and-in digitals. So the same analysis suggests that a CEG hedge on down-and-in Yen/$ digitals would have frequently incurred losses in excess of those possible under a rational bounds hedge.

The effectiveness of a hedge based on rational bounds depends on the option being hedged, and on the specific market. Hedgers may also differ in the weight they put on extreme adverse out-turns. But the analysis in this section suggests that rational bounds, though wide in absolute terms, support hedging strategies that compare well with alternatives. They also provide a firm limit on maximum loss, and do not require rebalancing.

4. Relaxing the Principal Assumptions

To simplify the presentation, we made three key assumptions: that it is feasible to trade at the barrier [A3], that interest rates are zero [A4], and that there exists a call option for every strike [A5]. In this section of the paper, we examine the effects of relaxing these assumptions in turn. We also show ho the approach can be generalized to apply to options other than barrier options.

A. Jumping through the Barrier

We assumed that prices do not jump through the barrier. This means that trades triggered by the barrier being breached are executed at the barrier price. We now drop the assumption. All we know is that the price immediately after the barrier is breached is at or above the barrier.

Proposition 3: *If jumps are possible, then the least upper bound on the value of the barrier option depends on whether the solution without jumps has $\lambda \leq 0$ or $\lambda > 0$. In the former case the bound is unaffected by the possibility of jumps. In the latter case the bound becomes:*

$$V_J = \int_0^{\infty} h_J(S)dF(S)$$
$$\text{where } h_J(S) = \begin{cases} Y(S) & \text{if } S \geq B, \\ Max[Y(S),0] & \text{otherwise.} \end{cases} \qquad (17)$$

Proof: The possibility of jumps could widen but not narrow the bounds. If $\lambda \leq 0$ then the upper bound is unaffected. The hedging strategy still dominates the barrier option even with jumps since the only impact is that any sale of the underlying which takes place immediately the barrier is breached may now take place at a higher price. If $\lambda > 0$, it is clear that the strategy of buying the claim V_J and not trading dominates the barrier option. V_J is therefore an upper bound. The proof that it is also a feasible price is in the Appendix. ■

For the specific barrier options we have considered (up-and-in digitals, puts and calls) the upper bounds are the same with and without jumps, while the lower bounds are generally much weaker. The lower bound on the Digital barrier option is the European digital, and the lower bound on the down-and-in puts and calls is the price of the European option which coincides with $Y(K)$ on $[B, \infty)$ and is zero elsewhere.

B. Positive Interest Rates

Suppose that the interest rate is some constant $r > 0$[7]. By changing the numeraire from nominal dollars to dollars at time T, the option's maturity date, the zero interest rate assumption can be

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[7] We maintain the assumption that interest rates are non-stochastic. The presence of two sources of uncertainty is beyond the scope of the present paper.

reinstated. $C(K)$ is interpreted as $\exp(rT)$ times the time zero dollar price of a call with strike K and maturity T, and S_t is $\exp(r(T-t))$ times the time t dollar price of the underlying asset. The only significant change is that the barrier B which is constant in nominal terms has the value $B_t = B \exp(r(T-t))$ in the new numeraire. Thus the problem of finding bounds in a positive interest rate, constant barrier level world is transformed into finding bounds in a zero interest rate, changing barrier world.

Proposition 4: *In the presence of positive interest rates, and if the interest rate is sufficiently small, then the upper bound can be found by solving for bounds in a world with zero interest rates and the barrier is either at B (if $\lambda \leq 0$) or B exp(rT) (otherwise). The limiting processes have their first jumps coming as close to maturity or as close to time 0 as possible, with the timing again depending on the sign of λ.*

Proof: in the zero interest rate, constant barrier world, the cheapest strategy that dominates the barrier option is denoted by the pair $(\lambda(B), h(.;B))$, where we now explicitly recognize the dependence on the barrier level. The cost of this strategy is $V^*(B)$. There is a corresponding permissible process $P(B, t_1, t_2)$ under which the model price of the barrier option is $V^*(B)$. The parameters t_1 and t_2 are the times of the two jumps; in the zero interest rate, constant barrier world, the only restriction is that $0 < t_1 < t_2 \leq T$.

Assume that $\lambda(B) \leq 0$, so the dominating strategy involves selling the underlying if and when the barrier is breached. Suppose the trader sells the barrier option and hedges using the strategy $(\lambda(B), h(.;B))$, trading when S_t reaches B_t rather than B. The trader's terminal wealth is, if anything, enhanced by the changing real level of the barrier since the sale is at B_t rather than B, and therefore the agent gets a better price when the barrier is breached. $V^*(B)$ remains an upper bound on the price of the barrier option.

Now consider the process $P(B_t, t, T)$. This is a martingale, and is consistent with the prices of traded options. The price of the barrier option under the process is $V^*(B_t)$ so this is a feasible price. As t tends to T, B_t tends to B and the model price tends to $V^*(B)$. Thus the rational bound on the price of the barrier option remains $V^*(B)$.

If $\lambda(B_0) \geq 0$, the worst case occurs when the barrier is hit early on. In present value terms the underlying is purchased at $B_0 = B \exp(rT)$. In this case, the agent should follow the strategy appropriate to a fixed barrier/zero interest rate world with the barrier at B_0. If the barrier is hit at time $t > 0$, the underlying is bought at B_t which is better than buying at B_0. So $V(B_0)$ is an upper bound on the value of the barrier option. To show it is the least upper bound, consider the process $P(B_t, t, t_2)$, but this time let t tend to 0. This shows that $V(B_0)$ is on the border of the feasible set of prices for the barrier option.

This still leaves the case where $\lambda(B) > 0$ but $\lambda(B_0) < 0$. It not seem to be of practical significance. It does not arise if the interest rate is small enough since as rT tends to zero the two λ's converge. It also does not arise with barrier options such as knock-in or knock-out puts and calls where the function $\lambda(B)$ does not change sign. ■

C. Finite Strikes

So far we have assumed that there is a European call option for every strike K. We now relax that assumption and assume that there is a finite set of strikes $\boldsymbol{K} = \{K_n | \; n = 1, \ldots, N\}$ with corresponding option prices $\boldsymbol{C} = \{C_n | \; n = 1, \ldots, N\}$. For convenience, use $n = 0$ for the zero strike option so $C_0 = S_0$.

We found a family of simple strategies which dominate the barrier option, $(\lambda, h(\lambda))$, indexed by λ. The strategy involves buying the European claim $h(.;\lambda)$ and buying λ units of the underlying at the barrier. This family is defined independently of the prices of traded options. With a continuous set of strikes, we showed that the cheapest dominating strategy was a member of this family. To find the cheapest dominating strategy when the number of strikes is finite, we search for the cheapest portfolio of traded (European) options which dominates $h(.;\lambda)$ for some value of λ. The corresponding strategy clearly dominates the barrier option; we will show that it is the cheapest portfolio which dominates the barrier option.

The problem of finding the cheapest portfolio which dominates one member of the family can be expressed formally as follows: find the parameters $\{\alpha, \beta_0, \ldots, \beta_N, \lambda\}$ which:

$$\begin{aligned} &\text{minimise } \alpha + \sum_{n=0}^{N} \beta_n C_n \\ &\text{subject to } \alpha + \sum_{n=0}^{N} \beta_n [X - K_n]^+ \geq h(X;\lambda) \quad \forall X \geq 0. \end{aligned} \tag{18}$$

Provided that there is some portfolio of traded calls which dominates $h(.;\lambda)$ for some value of λ, a feasible solution exists. The Lagrangian is:

$$\alpha + \sum_{n=0}^{N} \beta_n C_n - \int_0^\infty \left(\alpha + \sum_{n=0}^{N} \beta_n [X - K_n]^+ - h(X;\lambda) \right) d\mu(X) \tag{19}$$

where $d\mu(X)$ is the shadow price of the constraint in (18). $d\mu(X) = 0$ when the constraint is not binding at X, and positive when it is binding. The first order conditions from optimising with respect to α, β_n and λ respectively are:

$$1 = \int_0^\infty d\mu(X) \tag{20a}$$

$$C_n = \int_0^\infty [X - K_n]^+ d\mu(X) \text{ for } n = 0, \ldots, N \tag{20b}$$

$$\lambda^{opt} = \text{Argmin}\left\{ \int_0^\infty h(X;\lambda) d\mu(X) \right\}. \tag{20c}$$

The non-negativity of $d\mu(X)$, together with the fact that it integrates to unity (from (20a)) means that $\mu(X)$ can be interpreted as a cumulative probability function.

We now have a solution to the problem of finding a portfolio that dominates the barrier option. Suppose the cost of the portfolio is V. V is an upper bound on the price of the barrier option. To show it is the least upper bound, we introduce "fictitious securities" into the economy, as in Karatzas, Lehoczky, Shreve and Xu (1991). Options may be introduced as long as their prices do not create any demand to hold them. Take some K for which there is no traded option. Introduce a new option with strike K and price $E^{\mu}\left[(S_T - K)^+\right]$. Since the price of the option equals its shadow price, the solution is unchanged by the addition of this security. Indeed we can add options for every possible value of K in this way. We then have a complete set of options; the cheapest portfolio that dominates a member of the family still costs V. But we now have a complete set of options and the provisions of proposition 1 apply. There is a process P under which the expected value of the barrier option is V. Since this process must be consistent with the original set $\boldsymbol{C}$, this expected value is less than or equal to the upper bound. Thus V must be the least upper bound.

Proposition 5: *The upper bound on the value of a barrier option when only a finite set of European options is traded can be found by a simple two stage process (provided that the set of options is rich enough to bound the pay-off to the barrier option). In the first stage, the task is to find, for each value of λ, the cheapest portfolio of traded options which dominates $h(\lambda)$. The second stage is to identify the cheapest of all these portfolios*[8].

To give some feel for the impact of having a limited set of traded options, we revisit the rational bounds on the Digital barrier option. As before we assume that traded options are trading on an implied volatility of 30%, the maturity of the option is 1 year, the interest rate is zero and the price of the underlying is \$100. Instead of assuming a complete set of calls, we assume that the only traded securities, apart from the underlying, are calls with strikes at \$70, \$80, ..., \$140, \$150. The range of strikes is chosen so that the highest and lowest strike calls have deltas of approximately 10% and 90%.

Figure 7 shows the upper and lower bounds on the Digital barrier option. The middle line and the inner two lines correspond to the Black-Scholes value and the rational bounds with infinitely many strikes as shown in Figure 3. The outer two lines correspond to the bounds with a restricted set of options. The upper bound is virtually unaffected by the paucity of traded options. The optimal hedge itself takes the form of a call option, and the lack of an option at precisely the desired strike has little impact. It is only for barriers which are very close to the spot that the absence of a full set of calls – in this case the absence of very low strike calls – matters.

The lower bound is substantially affected by the paucity of strikes. The optimal hedge, as shown by Figure 2, consists of a short put, a long call, and a long European digital. The digital is hard to replicate using calls with widely spaced strikes.

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[8] It may be worth noting that despite λ being a continuous variable the search is a finite one, because of the linear programming structure of the problem.

D. Generalizing the Approach to Other Derivatives

We have concentrated on barrier options. We have taken the hedging instruments to be the European options with the same maturity as the barrier option. In principle the same approach can be applied to other exotic options, and to other sets of hedging instruments. We now examine some of the practical problems in applying it, and some of the solutions we have found. We retain the assumption throughout that interest rates are non-stochastic, that there is just one underlying asset, that the underlying is traded and that all derivative claims have pay-offs that are uniquely determined by the price path of the underlying asset.

The approach can be characterized in the following way: starting from the set of all possible price processes, the objective is to find the process which maximizes the value of the particular exotic option subject to some constraints. The constraints are that under the process the discounted price of the underlying follows a martingale, and that the model price of each of the hedging securities matches its market price. With the appropriate formal structures in place, and invoking the Martingale Representation Theorem (see for example Huang and Litzenberger (1988) *p.* 231), it can be seen that the price of the exotic under this extreme process is the rational bound.

In practice, the approach is not feasible. The space of all possible processes is too large. It is necessary to discretize the space to make the problem have finite dimensionality. If time takes T values, and prices N values, the number of parameters needed to characterize a process is of the order of N^T. For any realistic values of N and T the search problem is intractable. To find the bounds, the dimension of the space needs to be restricted without excluding processes that generate extreme values.

With barrier options, we restricted ourselves to processes with at most two jumps. A similar approach can be used to find rational bounds on the value of double barrier options. The value of a double barrier option depends in general on the terminal value of the underlying, whether either or both the barriers have been breached, and if so in what order. This can be modelled by a three stage process with only four possible paths in the first two stages. A similar model can be used to bound the value of a single barrier option when the hedging securities include not only European options but also an otherwise identical barrier option with a different barrier level[9].

Look-back options can also be handled. A look-back call gives the holder the right to receive the difference between the maximum price level reached in the period and some fixed strike. It can be viewed as the limit case of a portfolio of up-and-in digital calls with barriers at and above the strike. As Hobson (1998) has shown, the rational bounds on the look-back are identical to the integral of the bounds on the corresponding portfolios of digitals.

Similar simplifications make it possible to bound the value of forward start options and cliquets. Consider a forward straddle that pays the absolute value of the price difference between two future dates (so the claim pays $|S_2 - S_1|$). Take the hedging securities to be the European options that mature on those two dates. Then it is sufficient to model the process as a two stage process; the dimensionality of the problem is N^2.

The principle of using rational bounds to generate robust hedging strategies is of general validity. We have shown how the principle can be applied in practice when the derivative to be hedged is a barrier option and the hedging instruments are all European options. We have indicated how a similar approach can be used in the case of other exotic options and other families of hedging instruments.

5. Conclusions

The risk management of complex derivatives poses a particular challenge. While there are often many closely related instruments that are liquid enough to be used for hedging, standard models are poorly designed to provide effective hedge strategies. We have shown how to exploit the properties of rational bounds in order to design strategies that make use of the range of instruments available and are robust to model specification and model estimation error.

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[9] Details of this approach and results are available from the authors on request.

In this paper, we have focused on one family of exotic options, and have derived specific bounds for barrier options. But the approach can readily be applied to many other types of exotic option including multiple barrier options, look-back options, and cliquets.

The robust hedging strategy based on rational bounds puts a firm floor under possible losses. This is particularly attractive in the context of capital adequacy regulations that focus on the most unfavorable events. Traditional hedging strategies by contrast are less well equipped for the purpose since they are heavily model dependent and can generate heavy losses when model assumptions are violated.

Robust hedging strategies have another interesting feature: after the hedge is established, they require no trading in the derivatives market. This is attractive since in many derivative markets transaction costs are significant and liquidity is not assured. Traditional hedging strategies need to be rebalanced. They require most rebalancing after major market moves; this may be precisely the time model and market prices diverge most widely. By contrast, the robust hedger can choose to rebalance at such times if there is an opportunity to trade one bounding portfolio for a cheaper one, but has no need to do so.

The cheapest super-replicating portfolio varies over time. By revising the portfolio periodically, the hedge performance can be greatly improved while still retaining a firm floor on the maximum size of loss. The choice of hedging strategy will necessarily depend on a multitude of factors: the instrument to be hedged, the available hedge instruments, the costs of transacting, the predictability of asset price dynamics, the preferences of the agent. We have presented evidence to show that the robust hedge compares well with more conventional alternatives such as delta hedging and the CEG static option hedge.

APPENDIX

Proof of Proposition 2

Put-call symmetry implies:

$$C_t\left(S_t\, a\right)/\, a = P_t\left(S_t \,/\, a\right). \qquad \text{(A-1)}$$

Suppose the barrier B is hit at time t. Differentiating (A-1) with respect to a, and evaluating at a = 1 gives:

$$C_t'(B) - C_t(B)\,/\,B + P_t'(B) = 0\,. \qquad \text{(A-2)}$$

The left hand side of (A-2) is the cost at time t of a European claim which pays 1 if $S_T < B$ and -S_T/B if $S_T \geq B$. Under put-call symmetry, the value of such a claim is zero. Consider the strategy:

- at time 0 buy the portfolio with European pay-off (1 + S_T/B) if $S_T > B$ and 0 otherwise;
- if the barrier is hit for the first time at $t \leq T$, buy the portfolio represented by the left hand side of (A-2).

With put-call symmetry the strategy is self-financing. If the barrier is not hit, $S_T < B$ and the strategy has a pay-off of zero. If the barrier is hit, the portfolio has a pay-off of 1 for all S_T. The strategy replicates the binary barrier option exactly.

If put-call symmetry does not hold, the strategy still replicates the binary barrier option perfectly when the barrier is not hit. When it is hit, it generates a hedge error given by:

$$Err = C_t'(B) - C_t(B)\,/\,B + P_t'(B)\,, \qquad \text{(A-3)}$$

where t is the time at which the barrier is hit. If $\sigma(K)$ is the Black-Scholes implied volatility for a call with strike K, then $C(K)$ = BS(S_0, K, T-t, $\sigma(K)$) where BS(.) is the Black-Scholes valuation formula (with zero interest rates). Put-call parity holds so the implied volatility of puts and calls with the same strikes is identical. Differentiating and rearranging gives:

$$Err = 2n\left(\sigma\sqrt{\tau}/2\right)\sqrt{\tau}\left.\frac{d\sigma}{d\ln K}\right|_{K=S_t}, \tag{A-4}$$

where $\tau = T\text{-}t$, and $n(.)$ is the standard normal density.
This is the first part of Proposition 2. To relate this expression to the risk reversal, note that the risk-reversal measures the variation of implied volatility with the delta of a call option. The delta, $\Delta(K)$, is the partial derivative of the Black-Scholes price with respect to the spot, $\partial BS/\partial S$. Differentiate the Black-Scholes equation, recalling that implied volatility is a function of strike:

$$\left.\frac{d\Delta}{d\ln K}\right|_{K=S_t} = n\left(\frac{\sigma\sqrt{\tau}}{2}\right)\left\{-\frac{1}{\sigma\sqrt{\tau}} + \frac{\sqrt{\tau}}{2}\frac{d\sigma}{d\ln K}\right\}. \tag{A-5}$$

Now:

$$\frac{d\sigma}{d\ln K} = \frac{d\sigma}{d\Delta}\frac{d\Delta}{d\ln K}. \tag{A-6}$$

Substitute back from (A-5) and (A-6) into (A-4):

$$Err = -2n^2\left(\sigma\sqrt{\tau}/2\right)\frac{1}{\sigma}\frac{d\sigma}{d\Delta}\left[1 - \frac{\sqrt{\tau}}{2}\frac{d\sigma}{d\Delta}n\left(\sigma\sqrt{\tau}/2\right)\right]^{-1}. \tag{A-7}$$

The risk-reversal is the difference in Black-Scholes implied volatility between options with a delta of 0.25 and 0.75. Interpolating linearly, the derivative $d\sigma/d\Delta$ can be represented by twice the risk reversal (*RR*). When σ and τ are not too large, the denominator of the right hand side of (A-7) is approximately unity, and $n(.)$ can be approximated by $n(0)$. So equation (A-7) can be approximated by:

$$Err \approx .64\frac{RR_t}{\sigma_t}. \tag{A-8}$$

With τ under one year, σ of 30% or less, and with a risk reversal of no more than 10%, the coefficient lies between 0.61 and 0.65.

Proof of Proposition 3

To complete the proof it is necessary to show that there is a process P^* under which the expected pay-off to the barrier option is equal to the cost of the dominating portfolio, V_J. Define the distribution function F_B^* by $dF_B^*(S) = dF(S)$ if $S \geq B$ or $Y(S) > 0$, and 0 otherwise. Define B^* implicitly as $\int (S - B^*) dF_B^*(S) = 0$. The fact that $\lambda > 0$ means that $B^* > B$. Consider a process P^* which would be a member of ***P*** except for the fact that those paths which jump twice go to B^* rather than B at their first jump, and where the distribution function is F_B^*. The process is a martingale. The value of the barrier option under this process is V_J.

Barrier Option Type	Limiting Process	λ	Hedge portfolio, $h(.)$
Digital	P_U	$-\dfrac{1}{B-K_U}$	$\left(\dfrac{S-K_U}{B-K_U}\right)I_{S>K_U}$
Up-and-in Call			
$X \le K_U$	P_U	$-\dfrac{K_U-X}{B-K_U}$	$\dfrac{(B-X)(S-K_U)}{(B-K_U)}I_{S>K_U}$
$X \ge K_U$	P_U	0	$(S-X)I_{S>X}$
Up-and-in Put			
$X \le K_L$	P_L	0	$(X-S)I_{S<X}$
$K_L \le X \le B$	P_L	$-\dfrac{X-K_L}{B-K_L}$	$\dfrac{(B-X)(K_L-S)}{(B-K_L)}I_{S<K_L}+\dfrac{(S-B)(X-K_L)}{(B-K_L)}I_{S>B}$
$B \le X$	P_U	$-\dfrac{X-K_U}{B-K_U}$	$\dfrac{(X-B)(S-K_U)}{(B-K_L)}I_{S>K_U}+(S-X)I_{S>X}$

Table I: Upper Bounds on Up-and-in Barrier Options

The table shows the limiting process which enforces the upper bound for a variety of barrier options. The strike is denoted by X, and the barrier by B. The process P_U is one where paths which hit the barrier have terminal price $S_T \in [K_U, \infty)$, while in P_L paths which hit the barrier have $S_T \in [0, K_L] \cup [B, \infty)$. The strategy which enforces the upper bound is characterised by the pair (λ, $h(.)$). An agent who has written the barrier option for its upper bound price, can hedge by buying the European claim which pays $h(S_T)$, and going long λ forward contracts if and when the barrier is breached. The strategy has zero initial cost, a non-negative pay-off on all possible paths, and a zero pay-off on paths with positive measure under the limiting process. I_x is the indicator function which takes the value 1 if x is true, and zero otherwise.

Barrier Option Type	Limiting Process	λ	Hedge portfolio, $h(S)$
Digital	P_L	$-\dfrac{1}{B-K_L}$	$\left(\dfrac{S-K_L}{B-K_L}\right)\left(I_{S<K_L}+I_{S>B}\right)$
Up-and-in Call			
$X \le K_L$	P_L	$-\dfrac{K_L-X}{B-K_L}$	$(X-S)I_{S<X}+\dfrac{(B-X)(S-K_L)}{(B-K_L)}\left(I_{S<K_L}+I_{S>B}\right)$
$K_L \le X \le B$	P_L	0	$(S-X)I_{S>B}$
$B \le X$	P_L	0	$(S-X)I_{S>X}$
Up-and-in Put			
$X \le K_U$	P_U	0	0
$K_U \le X \le B$	P_U	$-\dfrac{X-K_U}{B-K_U}$	$(S-X)I_{S>X}-\dfrac{(B-X)(S-K_U)}{(B-K_U)}I_{S>K_U}$
$B \le X$	P_L	$-\dfrac{X-K_L}{B-K_L}$	$(S-X)I_{S>X}+\dfrac{(X-B)(S-K_L)}{(B-K_L)}\left(I_{S<K_L}+I_{S>B}\right)$

Table II: Lower Bounds on Up-and-in Barrier Options

The table shows the limiting process which enforces the lower bound for a variety of barrier options. The strike is denoted by X, and the barrier by B. The process P_U is one where paths which hit the barrier have terminal price $S_T \in [K_U, \infty)$, while in P_L paths which hit the barrier have $S_T \in [0, K_L] \cup [B, \infty)$. The strategy which enforces the lower bound is characterised by the pair (λ, $h(.)$). An agent who has bought the barrier option for its lower bound price, can hedge by writing the European claim which pays $h(S_T)$, and going short λ forward contracts if and when the barrier is breached. The strategy has zero initial cost, a non-negative pay-off on all possible paths which are continuous at the barrier, and a zero pay-off on paths with positive measure under the limiting process. I_x is the indicator function which takes the value 1 if x is true, and zero otherwise.

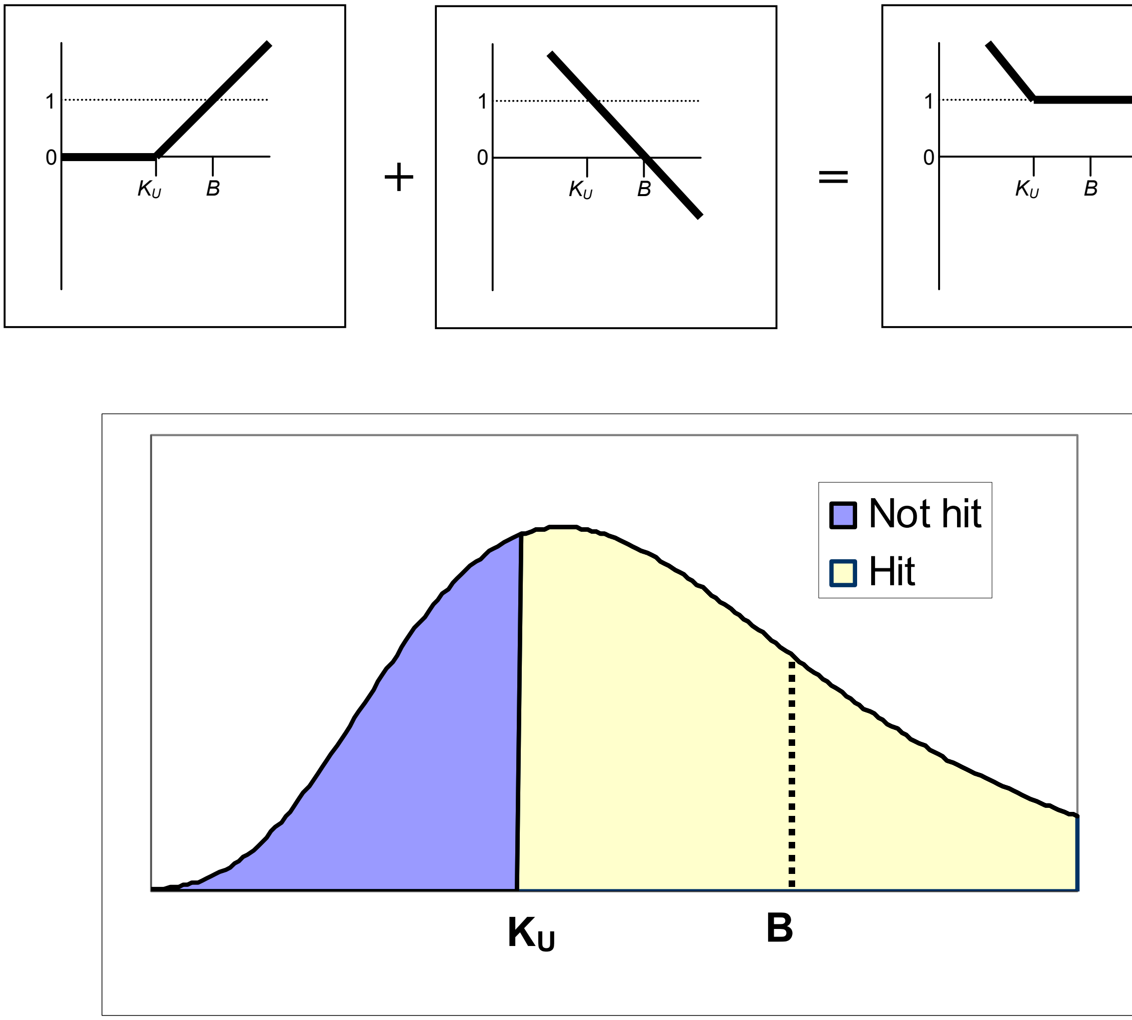


Figure 1: Upper Bound on up-and-in Digital Option

The top panel illustrates the hedging strategy that enforces the upper bound on the up-and-in digital. It shows the initial hedge portfolio. This is also the final portfolio if the barrier is not breached. It is followed by the forward sale of the underlying when the barrier is breached, and the resulting position at maturity if the barrier is breached.

The lower panel shows the distribution of the terminal price in the extreme process under which the digital reaches its maximum value. The paths that hit the barrier B end at K_U or above, while those that fail to hit the barrier end at K_U or below.

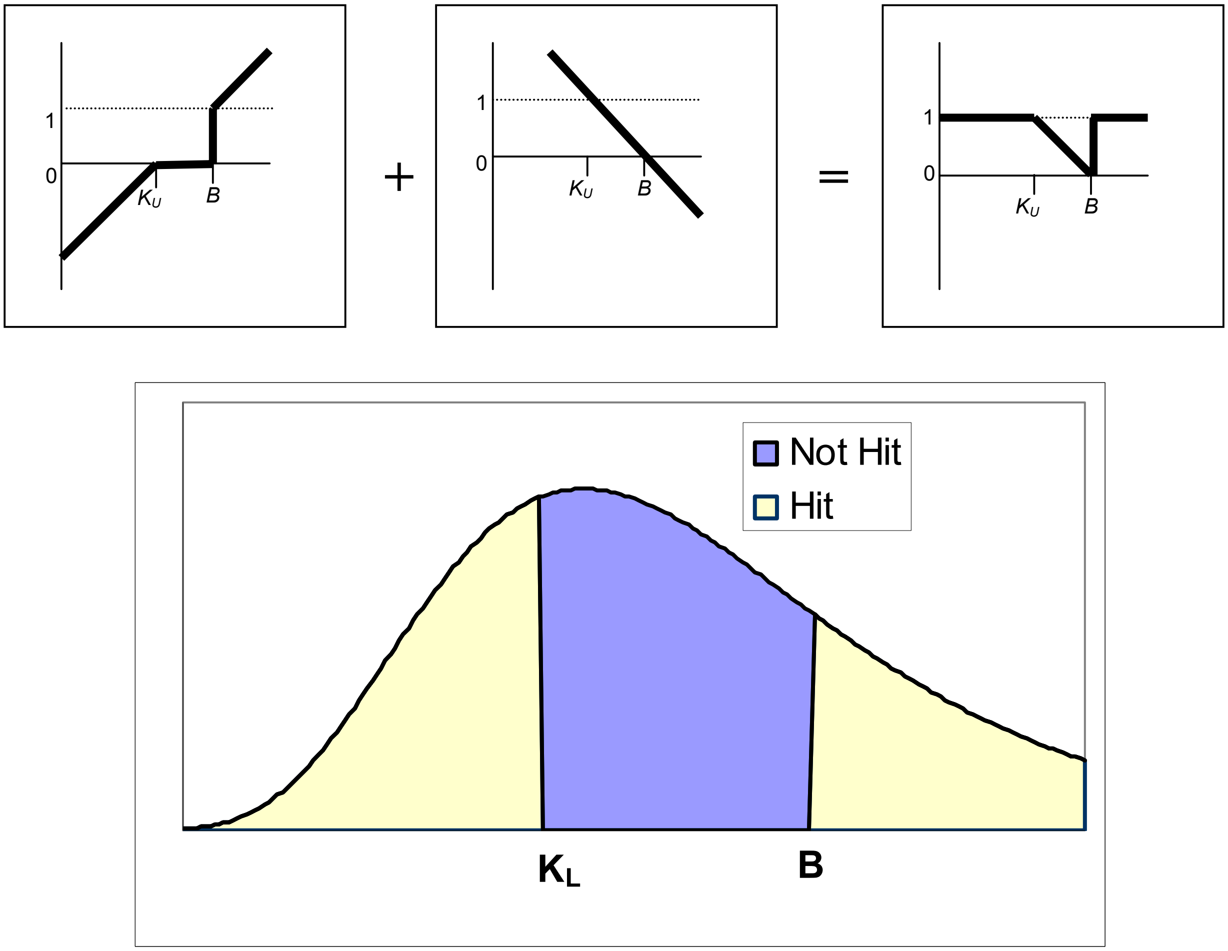


Figure 2: Lower Bound on up-and-in Digital Option

The top panel illustrates the hedging strategy that enforces the lower bound on the up-and-in digital. It shows the initial hedge portfolio. This is also the final portfolio if the barrier is not breached. It is followed by the forward sale of the underlying when the barrier is breached, and the resulting position at maturity if the barrier is breached.

The lower panel shows the distribution of the terminal price in the extreme process under which the digital reaches its minimum value. The paths that hit the barrier B either end above the barrier, or end at K_L or below, while those that fail to hit the barrier end between K_L and B.

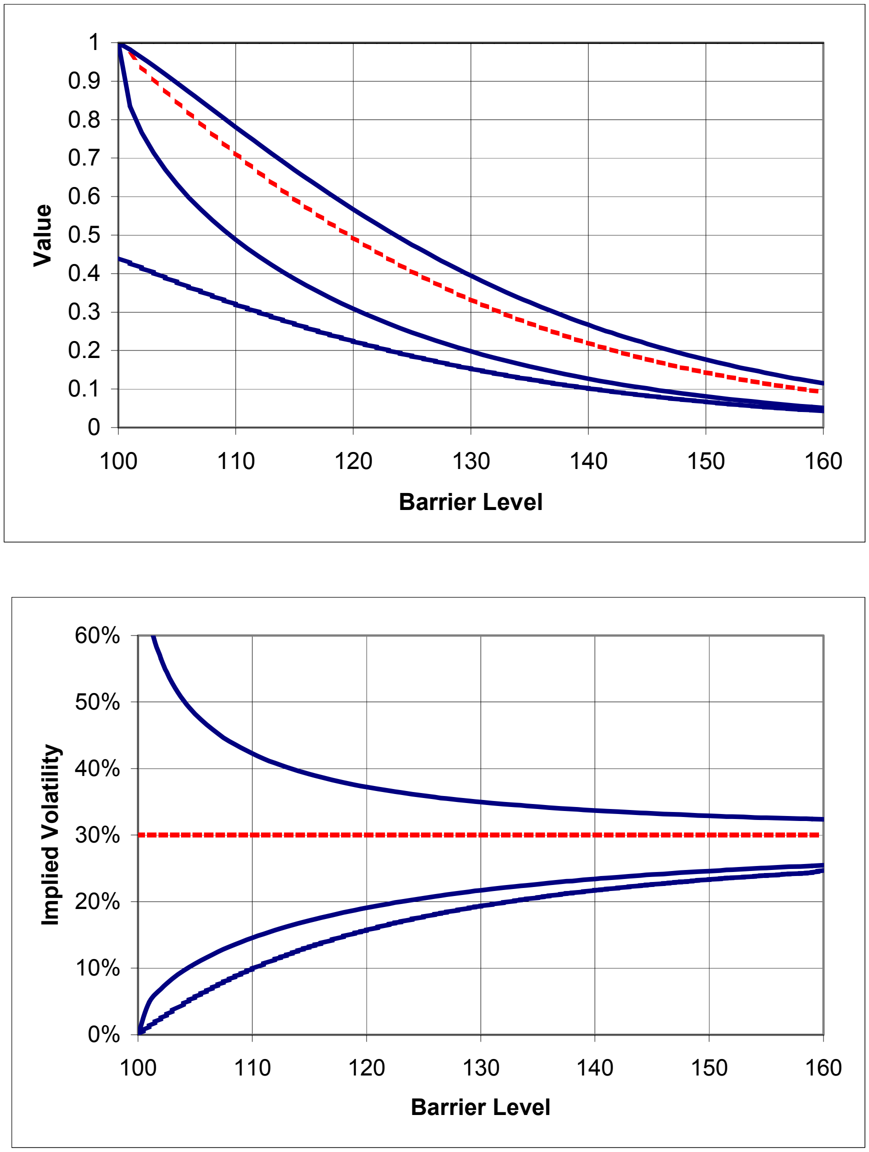


Figure 3: Rational Bounds on an up-and-in Digital

The top panel shows the bounds on the price of a 1 year up-and-in digital option, which pays $1 if the barrier is hit and zero otherwise, as a function of the barrier level. The initial stock price is $100, the maturity is 1 year and all European options trade on an implied volatility of 30%. The interest rate is zero. The dashed line is the Black-Scholes value; the higher solid line is the upper bound while the lower solid lines correspond to the lower bounds with and without jumps.

The lower panel conveys the same information but with prices re-expressed as Black-Scholes implied volatilities.

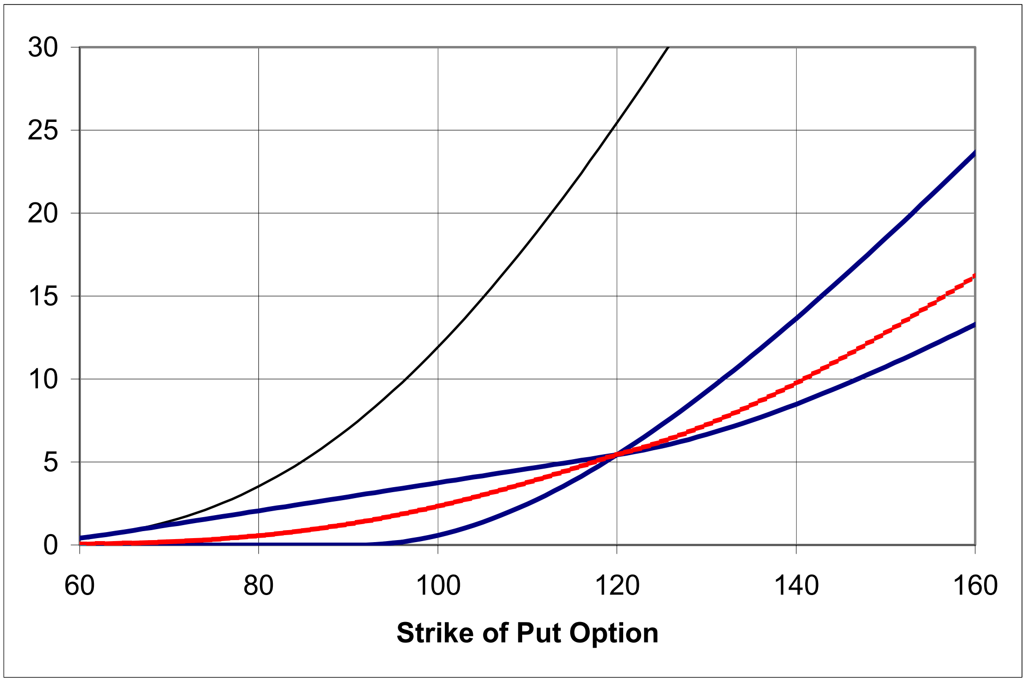


Figure 4: Rational Bounds on the Price of an up-and-in Put

The chart shows the rational bounds on the price of an up-and-in put option as a function of the strike price of the option. The top curve is the Black-Scholes price of the equivalent conventional put option, while the other three lines show the upper bound, the Black-Scholes price and the lower bound of the up-and-in put option. The maturity is 1 year, the initial asset price is \$100, the barrier is set at \$120, and all European options trade initially on an implied Black-Scholes volatility of 30%. Interest rates are zero.

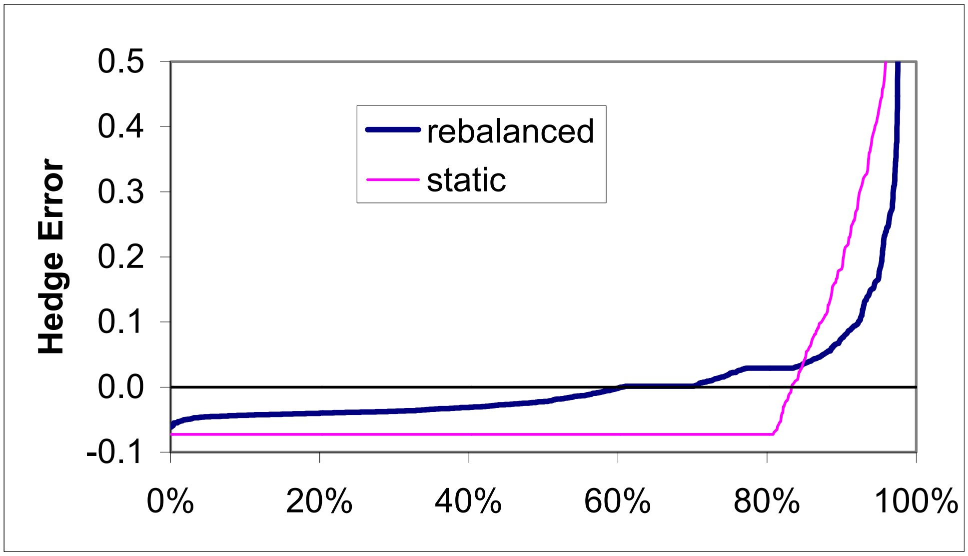
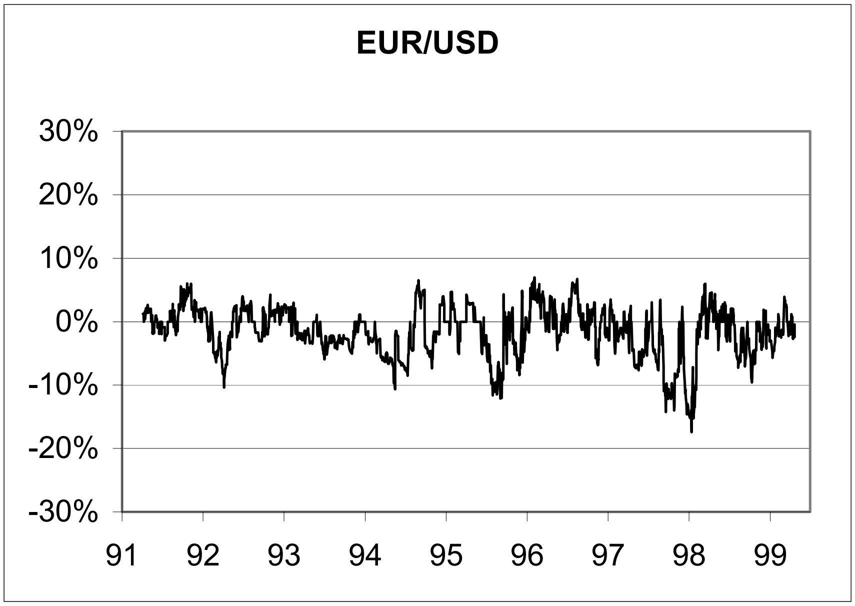


Figure 5: Distribution of Error with Rational Bounds hedge

The chart shows the distribution of the hedge error when the agent sells a one year up-and-in digital option for its Black-Scholes value and hedges using the rational hedge. The lighter line shows the distribution of the error if the hedge is held constant; the heavier corresponds to the case where the hedge is replaced at the end of each period by the cheapest dominating portfolio.

The results are obtained by simulation (1000 simulations, 5 time periods). The underlying has a constant volatility of 30%, and all European options are traded on that implied volatility. The initial stock price is \$100 and the barrier is at \$120. The interest rate is zero.

At the outset, the Black-Scholes price of the barrier option is \$0.4939, while the hedge costs \$0.5669, so limiting the maximum loss to \$0.073. With revision (figures for the unrevised hedge in parentheses) the mean hedge error over the 1000 simulations is \$0.036 (-\$0.007). The standard deviation of the hedge error is \$0.282 (\$0.177), with losses in 59.7% (82.4%) of cases. Conditional on making a loss, the mean hedge error is –\$0.034 (-\$0.073).



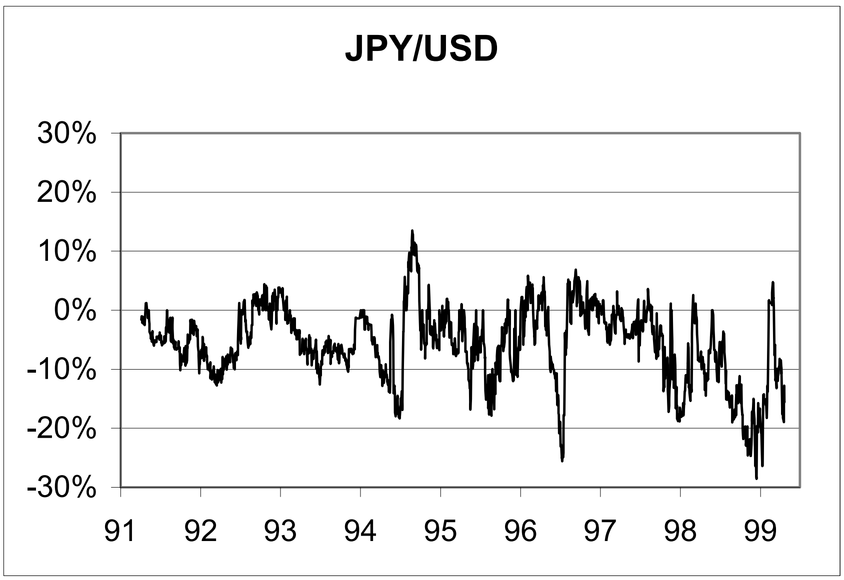


Figure 6: Risk Reversal on Euro/US$ and Yen/US$ 1991-2000

The chart shows the level of the risk reversal as a proportion of the volatility of the at-the-money option for two currency pairs over the last decade. The risk reversal is the difference in implied Black-Scholes volatility for calls with a delta of 0.25 and puts with a delta of –0.25.

The option maturities are for one month, and the data themselves are daily data.

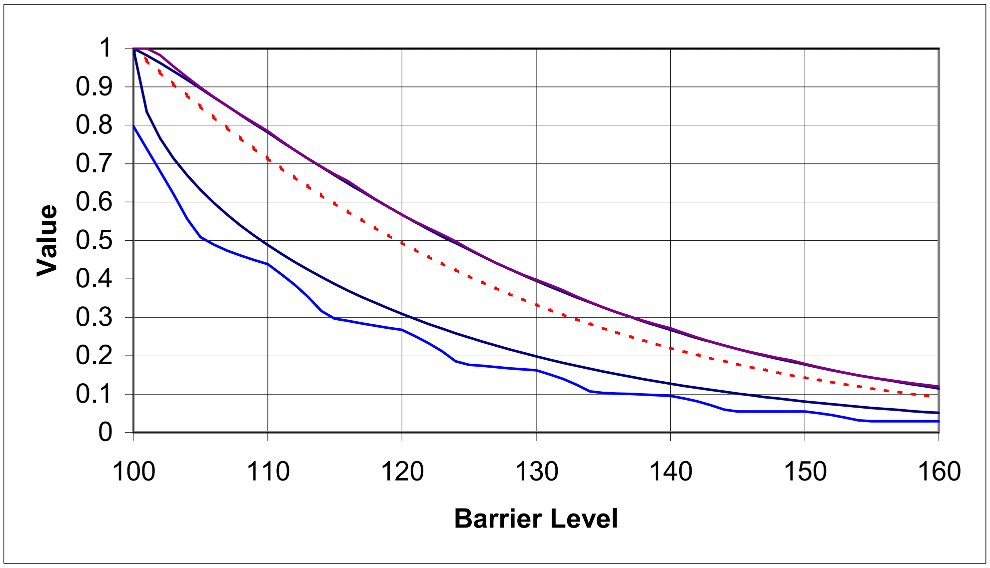


Figure 7: Bounds on the up-and-in Digital with sparse strikes

The chart shows the bounds on the price of a 1 year up-and-in digital option, which pays $1 if the barrier is hit and zero otherwise, as a function of the barrier level. The initial stock price is $100, the maturity is 1 year and all European options trade on an implied volatility of 30%. The interest rate is zero. The dashed line is the Black-Scholes value; the solid lines on either side of the dashed line are the bounds assuming the existence of European options with a continuity of strikes, and correspond to those in Figure 3. The outer two lines assume that the only traded options have strikes in the set {$70, $80, ..., $150}. The upper bound is barely affected by the lack of continuous strikes so the two upper bounds largely coincide.

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