Rational Trigonometric Interpolation and Constrained Control of the Interpolant Curves

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ABSTRACT

In the present paper a new method is developed for smooth rational cubic trigonometric interpolation based on values of function which is being interpolated. This rational cubic trigonometric spline is used to constrain the shape of the interpolant such as to force it to be in the given region by selecting suitable parameters. The more important achievement mathematically of this method is that the uniqueness of the interpolating function for the given data would be replaced by uniqueness of the interpolating curve for the given data and selected parameters. Approximation properties have been discussed and confirms that the expected approximation order is $O(h^2)$.

Keywords

Rational cubic trigonometric spline, curve design, error estimation, constrained interpolation, continuity, shape parameters.

1. INTRODUCTION

In Computer aided design and manufacturing processes , it is usually required to generate smooth function passes through given data set. From the last decade demand for more effective tools of computer aided design is high due to increase in model complexity and manufacturing requirements in curve and surface design .Within this content, constrained design has been identified as one of the surface design problems that need to be solved. Spline interpolation has been observed as a powerful tool for curve and surface design.

Among the many generalizations of polynomial splines, the trigonometric splines are of particular theoretical interest and practical importance. The trigonometric B-splines were first introduced by Schoenberg [13]. A study of trigonometric splines has been made by a number of authors, [1,4,5,7,10]. It was found that problems of scattered data interpolation over spherical surfaces can be better handled in terms of accuracy, computational convenience and smoothness of the resulting surface using trigonometric splines. Trigonometric splines have been studied from application point of view in various problems of geometric modeling. Recently trigonometric splines and polynomials have gained very much interest

within CAGD in particular curve designing and in different areas such as electronics or medicine.[15]

Many authors have worked in the area of shape preservation[1,2,3,8,11]. In some manufacturing process the derivative data are not easily available, because of this we have constructed a rational cubic trigonometric spline based on the use of function values only. It is of interest that the shape of the interpolating curves can be controlled just by selection of parameters used in the construction.

The present work is organized as follows: In section 2, construction of a rational cubic trigonometric spline with shape parameters α_i, β_i is given. section 3, deals with the region control of the interpolant curves. The sufficient condition for constraining the interpolating curves to be bounded between two given straight lines is also derived. In section 4 , a numerical example with graphical representation is given to show how the interpolating curve can be controlled in the given region. In section 5, the approximation properties of the interpolant are discussed.

2. A C^1 RATIONAL CUBIC TRIGONOMETRIC SPLINE INTERPOLATION

Let $\{(t_i, f_i), i = 0,1,...,n,n+1\}$ be a given set of data points, such that $a = t_0 < t_1 < ... < t_n < t_{n+1} = b$. The C^1-continuous piecewise cubic trigonometric function based on function values is defined by

$$P(t) = \frac{p_i(t)}{q_i(t)}, \quad i = 0,1,...,n-1, \tag{1}$$

where

$$p_i(t) = (1-\sin\frac{\pi\theta}{2})^3\alpha_i f_i + \sin\frac{\pi\theta}{2}(1-\sin\frac{\pi\theta}{2})(3-\sin\frac{\pi\theta}{2})U_i + \cos\frac{\pi\theta}{2}(1-\cos\frac{\pi\theta}{2})(3-\cos\frac{\pi\theta}{2})V_i + (1-\cos\frac{\pi\theta}{2})^3\beta_i f_{i+1} \tag{2}$$

$$q_i(t) = (1-\sin\frac{\pi\theta}{2})^3\alpha_i + \sin\frac{\pi\theta}{2}(1-\sin\frac{\pi\theta}{2})(3-\sin\frac{\pi\theta}{2}) + \cos\frac{\pi\theta}{2}(1-\cos\frac{\pi\theta}{2})(3-\cos\frac{\pi\theta}{2}) + (1-\cos\frac{\pi\theta}{2})^3\beta_i \tag{3}$$

$$with \quad h_i = t_{i+1} - t_i, \quad \theta(t) = \frac{(t-t_i)}{h_i}, \quad t\varepsilon[t_i, t_{i+1}]$$

and

$$U_i = f_i + \frac{2h_i\alpha_i}{3\pi}\Delta_i$$

$$V_i = f_{i+1} - \frac{2h_i\beta_i}{3\pi}\Delta_{i+1}$$

With $\alpha_i, \beta_i > 0$ and $\Delta_i = \frac{f_{i+1} - f_i}{h_i}$

This rational cubic trigonometric interpolating function satisfies

$$P(t_i) = f_i \quad \text{and} \quad p'(t_i) = \Delta_i.$$

3. SHAPE CONTROL OF THE INTERPOLATION CURVE

For a given straight line g(t) or piecewise linear curve defined on $[t_0, t_n]$ with joints of the partition $\Delta : t_0 < t_1 < ... < t_n < t_{n+1}$ and a data set $\{(t_i, f_i) : i = 0,1,...,n,n+1\}$ with

$$f(t_i) \geq (\leq) g(t_i) \qquad i = 0,1,...,n,n+1,$$

let P(t) be a cubic trigonometric rational interpolating function defined by (1), when $P(t) \geq (\leq) g(t)$ for all $t\varepsilon[t_0, t_n]$,

Suppose

$$P(t) = \frac{p_i(t)}{q_i(t)} \geq g(t) ,$$

which is equivalent to

$$p_i(t) - q_i(t)g(t) = M_i(t) \geq 0. \qquad (4)$$

From (2) and (3) we can write (4) as

$$\begin{aligned}M_i(t) = \{&(1-\sin\frac{\pi\theta}{2})^3\alpha_i f_i + \sin\frac{\pi\theta}{2}(1-\sin\frac{\pi\theta}{2})(3-\sin\frac{\pi\theta}{2})U_i \\ &+ \cos\frac{\pi\theta}{2}(1-\cos\frac{\pi\theta}{2})(3-\cos\frac{\pi\theta}{2})V_i \\ &+ (1-\cos\frac{\pi\theta}{2})^3\beta_i f_{i+1}\} - \{(1-\sin\frac{\pi\theta}{2})^3\alpha_i \\ &+ \sin\frac{\pi\theta}{2}(1-\sin\frac{\pi\theta}{2})(3-\sin\frac{\pi\theta}{2}) + \cos\frac{\pi\theta}{2}(1-\cos\frac{\pi\theta}{2})(3-\cos\frac{\pi\theta}{2}) \\ &+ (1-\cos\frac{\pi\theta}{2})^3\beta_i\}[(1-\theta)g_i + \theta g_{i+1}] \quad \geq \quad 0\end{aligned}$$

where g_i, g_{i+1} represent $g(t_i), g(t_{i+1})$, respectively. Since

$$\begin{aligned}M_i(t) = &(1-\sin\frac{\pi\theta}{2})^3\alpha_i A_i + \sin\frac{\pi\theta}{2}(1-\sin\frac{\pi\theta}{2})(3-\sin\frac{\pi\theta}{2})B_i \\ &+ \cos\frac{\pi\theta}{2}(1-\cos\frac{\pi\theta}{2})(3-\cos\frac{\pi\theta}{2})C_i + (1-\cos\frac{\pi\theta}{2})^3\beta_i D_i \geq 0\end{aligned} \qquad (5)$$

where

$$A_i = (1-\theta)(f_i - g_i) + \theta(f_{i+1} - g_{i+1}) - \theta h_i\Delta_i$$

$$B_i = A_i + \frac{2\alpha_i h_i}{3\pi}\Delta_i$$

$$C_i = D_i - \frac{2h_i\beta_i}{3\pi}\Delta_{i+1}$$

$$D_i = A_i + h_i\Delta_i$$

for $\alpha_i, \beta_i > 0$,and Since $f_i - g_i \geq 0$, for all i, we find a sufficient condition for the rational cubic trigonometric curve $P(t)$ to lie above the straight line $g(t)$ in $[t_i, t_{i+1}]$ given in the following theorem:

Theorem 3.1 Given $\{(t_i, f_i, g_i), \quad i = 0,1,...,n,n+1\}$ with $f_i \geq g_i, i = 0,1,...,n,$ the sufficient condition for the rational cubic trigonometric curve $P(t)$ defined by (1) to lie above the straight line $g(t)$ in $[t_i, t_{i+1}]$ is that the positive parameters α_i, β_i, satisfy

$$\begin{aligned}&A_i \geq 0 \\ &A_i + \frac{2\alpha_i h_i}{3\pi}\Delta_i > 0 \\ &D_i \geq 0 \\ &D_i - \frac{2h_i\beta_i}{3\pi}\Delta_{i+1} \geq 0\end{aligned} \qquad (6)$$

For equally spaced partition Theorem 3.1 has the following corollary:

Corollary 3.1 For the equally spaced partition, the sufficient condition for the rational cubic trigonometric curve $P(t)$ defined by (1) to lie above the straight line $g(t)$ in $[t_i, t_{i+1}]$ is that the positive parameters α_i, β_i satisfy the conditions

$$(1-\theta)(f_i - g_i) + \theta(f_{i+1} - g_{i+1}) - \theta(f_{i+1} - f_i) \geq 0$$

$$(1-\theta)(f_i - g_i) + \theta(f_{i+1} - g_{i+1}) - \theta(f_{i+1} - f_i) + \frac{2\alpha_i}{3\pi}(f_{i+1} - f_i) \geq 0$$

$$(1-\theta)(f_i - g_i) + \theta(f_{i+1} - g_{i+1}) + (1-\theta)(f_{i+1} - f_i) - \frac{2h_i\beta_i}{3\pi}(f_{i+2} - f_{i+1}) \geq 0$$

$$(1-\theta)(f_i - g_i) + \theta(f_{i+1} - g_{i+1}) + (1-\theta)(f_{i+1} - f_i) \geq 0$$

Theorem 3.2 Given $\{(t_i, f_i, g_i), i = 0,1,...,n,n+1\}$ with g_i $f_i \leq g^*{}_i, i = 0,1,...,n+1,$ the sufficient condition for the rational cubic trigonometric curve $P(t)$ defined by (1) to lie above the straight line $g(t)$ and below the straight line $g^*(t)$ in $[t_i, t_{i+1}]$ is that the positive parameters α_i, β_i, satisfy

$$
\begin{aligned}
&(1-\theta)(f_i-g_i)+\theta(f_{i+1}-g_{i+1})-\theta h_i\Delta_i \ge 0\\
&(1-\theta)(f_i-g_i)+\theta(f_{i+1}-g_{i+1})-\theta h_i\Delta_i+\frac{2\alpha_i h_i}{3\pi}\Delta_i \ge 0\\
&(1-\theta)(f_i-g_i)+\theta(f_{i+1}-g_{i+1})-\theta h_i\Delta_i+h_i\Delta_i-\frac{2h_i\beta_i}{3\pi}\Delta_{i+1} \ge 0\\
&(1-\theta)(f_i-g_i)+\theta(f_{i+1}-g_{i+1})-\theta h_i\Delta_i+h_i\Delta_i \ge 0
\end{aligned} \tag{7}
$$

and

$$
\begin{aligned}
&(1-\theta)(g_i^*-f_i)+\theta(g_{i+1}^*-f_{i+1})+\theta h_i\Delta_i \ge 0\\
&(1-\theta)(g_i^*-f_i)+\theta(g_{i+1}^*-f_{i+1})+\theta h_i\Delta_i-\frac{2\alpha_i h_i}{3\pi}\Delta_i \ge 0\\
&(1-\theta)(g_i^*-f_i)+\theta(g_{i+1}^*-f_{i+1})+\theta h_i\Delta_i+h_i\Delta_i+\frac{2h_i\beta_i}{3\pi}\Delta_{i+1} \ge 0\\
&(1-\theta)(g_i^*-f_i)+\theta(g_{i+1}^*-f_{i+1})+\theta h_i\Delta_i+h_i\Delta_i \ge 0
\end{aligned} \tag{8}
$$

4. Numerical Example

The interpolating and constraining data are given in Table 1 and the parameters and α_i *and* β_i $(i=1,2,3)$ are given in Table 2 . It is to test that both the given interpolating and constraining data and the parameters satisfy the relationship (4) and the constraint inequalities (7), (8) so the interpolating curve $P(t)$ defined by (1) must be bounded between $g^*(t)$ and $g(t)$, which can be written as

$$
g^*(t)=\begin{cases}\frac{-8}{50}t+0.107 & ,0\le t\le 0.5\\ \frac{6}{50}t-0.033 & ,0.5\le t\le 1.0\\ \frac{-4}{50}t+0.167 & ,1.0\le t\le 1.5;\end{cases}
$$

$$
g(t)=\begin{cases}-\frac{8}{50}t+0.093 & ,0\le t\le 0.5\\ \frac{6}{50}t-0.047 & ,0.5\le t\le 1.0\\ -\frac{4}{50}t+0.153 & ,1.0\le t\le 1.5;\end{cases}
$$

Fig. 1 shows that the interpolating curve $P(t)$ is bounded between $g^*(t)$ and $g(t)$.

Table 1 The interpolating data and the constraining data

t_i	i	$g^*(t_i)$	$f(t_i)$	$g(t_i)$
0.0	1	0.107	0.1	0.093
0.5	2	0.027	0.02	0.013
1.0	3	0.087	0.08	0.073
1.5	4	0.047	0.04	0.033
2.0	5	0.067	0.06	0.053

Table 2 The parameters α_i and β_i for rational trigonometric interpolation

i	α_i	β_i
1	0.001123	0.0011423
2	0.001555	0.00124
3	0.001510	0.0011905

$$
R[f]=f(t)-P(t)=\int_{t_i}^{t_{i+1}} f^{(2)}(\tau)R_t[(t-\tau)_+]d\tau,\quad t\in[t_i,t_{i+1}] \tag{9}
$$

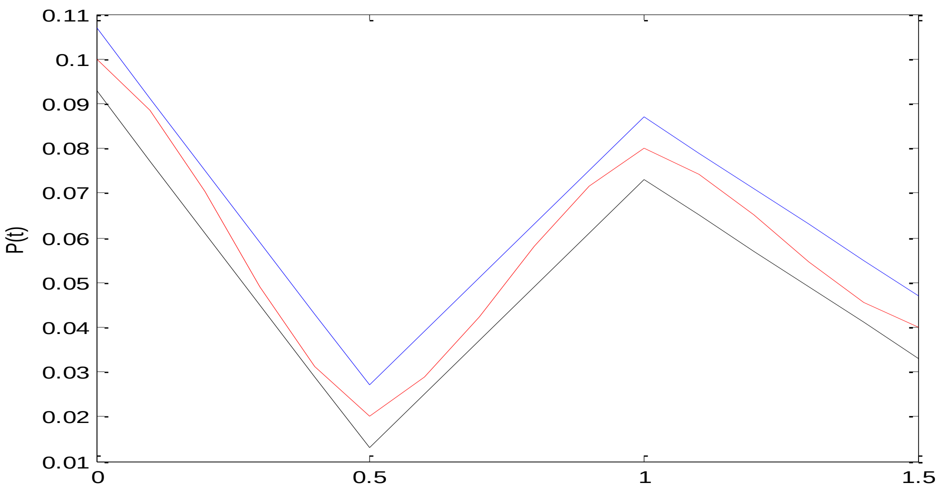

Figure 1: graph of $g^*(t), P(t)$ and $g(t)$ for data in Table 1 and 2 .

5. APPROXIMATION PROPERTIES OF THE DEFINED INTERPOLATION

To estimate error of the rational trigonometric interpolating function defined by (1), since the interpolation is local, without loss of generality, consider the error in the subinterval $[t_i, t_{i+1}]$. When $f(t)\varepsilon C^1[t_0, t_n]$ and P(t) is the rational trigonometric spline interpolating function of $f(t)$ in $[t_i, t_{i+1}]$. It is easy to see that this type of interpolation is exact for $f(t)$, the polynomial being interpolated,in which the degree is no more than 1. Consider the case when the knots are equally spaced,namely, $h_i = h = \frac{t_n - t_0}{n}$ for all $i = 1,2,...,n,$ using the Peano-Kernel Theorem in Schultz [14] gives the following

where

$$R_t[(t-\tau)_+] = \begin{cases} p(\tau) & t_i < \tau < t \\ q(\tau) & t < \tau < t_{i+1} \\ r(\tau) & t_{i+1} < \tau < t_{i+2} \end{cases}$$

where

$$\begin{aligned} p(\tau) = (t-\tau) - \{[(t_{i+1}-\tau)[\sin\frac{\pi\theta}{2}(1-\sin\frac{\pi\theta}{2})(3-\sin\frac{\pi\theta}{2})\frac{2\alpha_i}{3\pi} \\ +\cos\frac{\pi\theta}{2}(1-\cos\frac{\pi\theta}{2})(3-\cos\frac{\pi\theta}{2}) + (1-\cos\frac{\pi\theta}{2})^3\beta_i] \\ -\frac{2h_i\beta_i}{3\pi}\cos\frac{\pi\theta}{2}(3-\cos\frac{\pi\theta}{2})]/N\} \quad t_i < \tau < t; \end{aligned} \tag{10}$$

$$\begin{aligned} q(\tau) = -\{[(t_{i+1}-\tau)[\sin\frac{\pi\theta}{2}(1-\sin\frac{\pi\theta}{2})(3-\sin\frac{\pi\theta}{2})\frac{2\alpha_i}{3\pi} \\ +\cos\frac{\pi\theta}{2}(1-\cos\frac{\pi\theta}{2})(3-\cos\frac{\pi\theta}{2}) + (1-\cos\frac{\pi\theta}{2})^3\beta_i] \\ -\frac{2h_i\beta_i}{3\pi}\cos\frac{\pi\theta}{2}(1-\cos\frac{\pi\theta}{2})(3-\cos\frac{\pi\theta}{2})]/N\} \\ t_i < \tau < t; \end{aligned} \tag{11}$$

$$\begin{aligned} r(\tau) = \{[\frac{(2\beta_i)}{3\pi}\cos\frac{\pi\theta}{2}(1-\cos\frac{\pi\theta}{2})(3-\cos\frac{\pi\theta}{2})(t_{i+2}-\tau)]/N\} \\ t_{i+1} < \tau < t_{i+2}; \end{aligned} \tag{12}$$

where

$$\begin{aligned} N = (1-\sin\frac{\pi\theta}{2})^3\alpha_i + \sin\frac{\pi\theta}{2}(1-\sin\frac{\pi\theta}{2})(3-\sin\frac{\pi\theta}{2}) \\ +\cos\frac{\pi\theta}{2}(1-\cos\frac{\pi\theta}{2})(3-\cos\frac{\pi\theta}{2}) + (1-\cos\frac{\pi\theta}{2})^3\beta_i]\} \end{aligned}$$

Then

$$\begin{aligned} \| R[f] \| = \| f(t) - P(t) \| \\ \le \| f^2(t) \| [\int_{t_i}^{t} |p(\tau)|\,d\tau + \int_{t}^{t_{i+1}} |q(\tau)|\,d\tau + \int_{t_{i+1}}^{t_{i+2}} |r(\tau)|\,d\tau] \end{aligned} \tag{13}$$

for $\lambda \le 1$ $r(\tau) \ge 0$ for all $\tau\varepsilon[t_{i+1}, t_{i+2}]$ it may be seen that

$$\int_{t_{i+1}}^{t_{i+2}} |r(\tau)|\,d\tau = h^2 W_1$$

where

$$W_1 = \frac{1}{2}\{\frac{\frac{2\beta_i}{3\pi}\cos\frac{\pi\theta}{2}(1-\cos\frac{\pi\theta}{2})(3-\cos\frac{\pi\theta}{2})}{M}\}$$

with

$$\begin{aligned} M = \frac{2\alpha_i}{3\pi}\sin\frac{\pi\theta}{2}(1-\sin\frac{\pi\theta}{2})(3-\sin\frac{\pi\theta}{2}) \\ +\cos\frac{\pi\theta}{2}(1-\cos\frac{\pi\theta}{2})(3-\cos\frac{\pi\theta}{2}) + (1-\cos\frac{\pi\theta}{2})^3\beta_i \end{aligned}$$

for $q(\tau)$,since

$$q(t_{i+1}) = \frac{\frac{2h_i\beta_i}{3\pi}\cos\frac{\pi\theta}{2}(1-\cos\frac{\pi\theta}{2})(3-\cos\frac{\pi\theta}{2})}{N} \ge 0 \tag{15}$$

$$q(t) = -h\frac{(1-\theta)M - \frac{2\beta_i}{3\pi}\cos\frac{\pi\theta}{2}(1-\cos\frac{\pi\theta}{2})(3-\cos\frac{\pi\theta}{2})}{N} \le 0$$

there is a zero point τ^* of q(τ) in $[t_i, t_{i+1}]$ given by

$$\tau^* = t_{i+1} - \frac{\frac{2h_i\beta_i}{3\pi}\cos\frac{\pi\theta}{2}(1-\cos\frac{\pi\theta}{2})(3-\cos\frac{\pi\theta}{2})}{M}$$

$$\int_{t}^{t_{i+1}} |q(\tau)|\,d\tau = \int_{t}^{\tau^*} -q(\tau)\,d\tau + \int_{\tau^*}^{t_{i+1}} q(\tau)\,d\tau$$

$$= h^2[W_2]$$

where $W_2 = w_1 + w_2$

with

$$w_1 = M\{\frac{(\theta-1)^2}{2} - [\frac{\frac{2\beta_i}{3\pi}\cos\frac{\pi\theta}{2}(1-\cos\frac{\pi\theta}{2})(3-\cos\frac{\pi\theta}{2})}{M}]^2\}$$

$$w_2 = \frac{2\beta_i}{3\pi}\cos\frac{\pi\theta}{2}(1-\cos\frac{\pi\theta}{2})(3-\cos\frac{\pi\theta}{2})[(\theta-1) + \frac{2\frac{2\beta_i}{3\pi}\cos\frac{\pi\theta}{2}(1-\cos\frac{\pi\theta}{2})(3-\cos\frac{\pi\theta}{2})}{M}]$$

Since $p(\tau) = (t-\tau) + q(\tau)$

and $p(t_i) = (t-t_i) + q(t_i) = S \geq 0$

where $S = \frac{L}{N}$

with

$$L = h\{\theta(1-\sin\frac{\pi\theta}{2})^3\alpha_i + \sin\frac{\pi\theta}{2}(1-\sin\frac{\pi\theta}{2})(3-\sin\frac{\pi\theta}{2})[\theta - \frac{2\alpha_i}{3\pi}] + \cos\frac{\pi\theta}{2}(1-\cos\frac{\pi\theta}{2})\ (3-\cos\frac{\pi\theta}{2})[\theta-1+\frac{2\beta_i}{3\pi}] + (1-\cos\frac{\pi\theta}{2})^3\beta_i[\theta-1]$$

$p(t) = q(t) \leq 0$ and the root τ_* of $p(\tau)$ in $[t_i, t]$ is

$$\tau_* = t_{i+1} - h\frac{(\theta-1) + \frac{2\beta_i}{3\pi}\cos\frac{\pi\theta}{2}(1-\cos\frac{\pi\theta}{2})(3-\cos\frac{\pi\theta}{2})}{M-1}$$

$$\int_{t_i}^{t} |\,p(\tau)\,|\,\mathrm{d}\tau = \int_{t_i}^{\tau_*} p(\tau)\mathrm{d}\tau + \int_{\tau_*}^{t} -p(\tau)\mathrm{d}\tau$$

$$= h^2 W_3$$

where

$$W_3 = [X_1 + \frac{M}{N}X_2 + \frac{2\beta_i}{3\pi N}\cos\frac{\pi\theta}{2}(1-\cos\frac{\pi\theta}{2})(3-\cos\frac{\pi\theta}{2})X_3$$

with

$$X_1 = \{\frac{\theta^2}{2} - [(1-\theta) - \frac{(\theta-1) + \frac{2\beta_i}{3\pi}\cos\frac{\pi\theta}{2}(1-\cos\frac{\pi\theta}{2})(3-\cos\frac{\pi\theta}{2})}{M-1}]^2\}$$

$$X_2 = \{[\frac{(\theta-1) + \frac{2\beta_i}{3\pi}\cos\frac{\pi\theta}{2}(1-\cos\frac{\pi\theta}{2})(3-\cos\frac{\pi\theta}{2})}{M-1}]^2 - \frac{1}{2}((\theta-1)^2+1)\}$$

$$X_3 = \{2[1 - \frac{\theta - 1 + \frac{2\beta_i}{3\pi}\cos\frac{\pi\theta}{2}(1-\cos\frac{\pi\theta}{2})(3-\cos\frac{\pi\theta}{2})}{M-1}] - \theta\}$$

Thus,

$$\|R[f]\| = \|f(t) - P(t)\|$$

$$\leq \|f^2(t)\|\ [\int_{t_i}^{t} |\,p(\tau)\,|\,\mathrm{d}\tau + \int_{t}^{t_{i+1}} |\,q(\tau)\,|\,\mathrm{d}\tau + \int_{t_{i+1}}^{t_{i+2}} |\,r(\tau)\,|\,\mathrm{d}\tau]$$

$$= h^2 \|f^2(t)\|[W_1 + W_2 + W_3]$$

6. CONCLUSION

The construction of a rational cubic trigonometric spline with shape parameters α_i, β_i is given.. The sufficient conditions for constraining the interpolating curves to be bounded between two given straight lines is derived and the approximation properties of the interpolant has been discussed. This concept may have an extensive application in surface generation

7. ACKNOWLEDGEMENT

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