Robust Optimization:

What Works and What Does Not

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for

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Overview

- Why Optimized Portfolios Are Not Robust?
  - Ex-Post Performance and Multi-period Backtests
- Robust Optimization with SOCP (Second Order Cone Programming)
  - Equivalence to the Quadratic Penalty
- Two-Stage Optimization
  - Make a Portfolio Optimization Process (POP) Robust
- "Less is More"
  - One-period Static Optimization might be Over-Analyzed
- Closing Remarks

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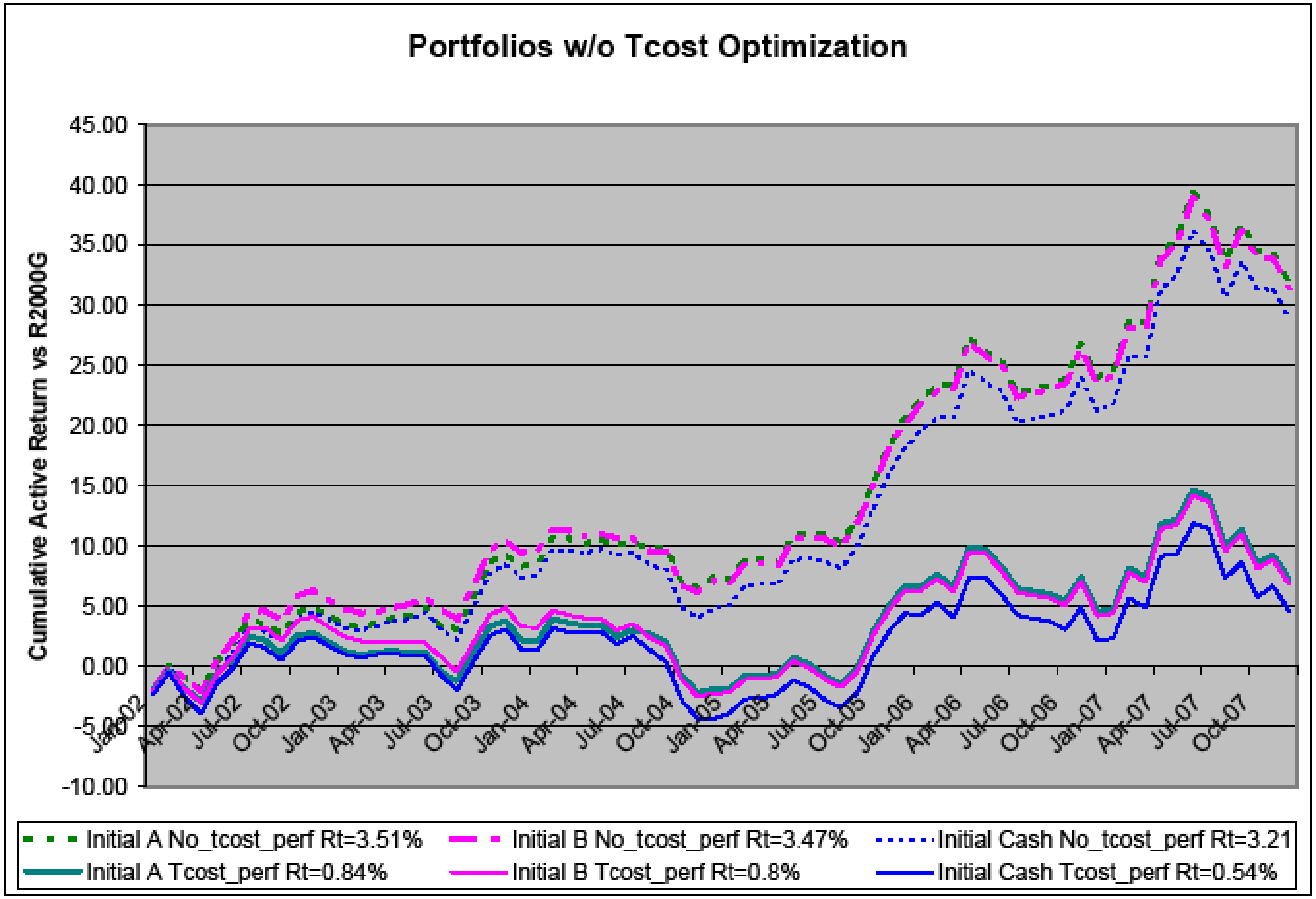
An Example

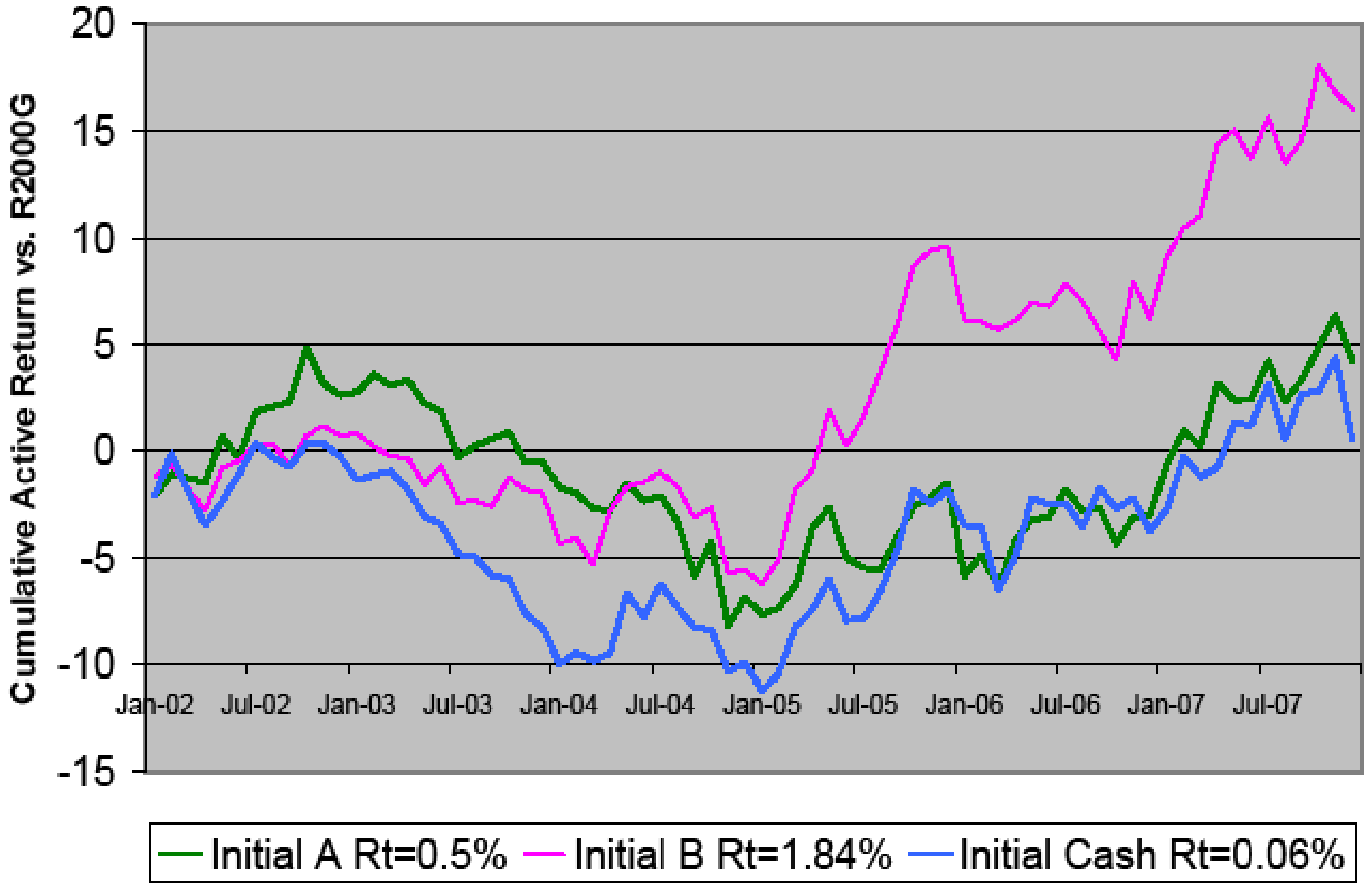
- Long-Only Active Portfolio Benchmarked to Russell 2000 Growth
- Max Alpha (Predicted)
- Tracking Error no more than 4.75%
- Security level active-weight bounds
- Beta neutral to the benchmark

- Tcost (25 bps impact + 3¢/share)
- No more than 200 securities in the portfolio



	Initial Portfolio		
Annualized Performance	A	B	C
Active Returns	**3.67%**	**3.71%**	**3.41%**
Active Returns (after Tcost)	**0.80%**	**0.84%**	**0.54%**
Active Risk	**4.62%**	**4.59%**	**4.74%**
Tcost Impact	**-2.89%**	**-2.89%**	**-2.89%**

Tcost Optimization



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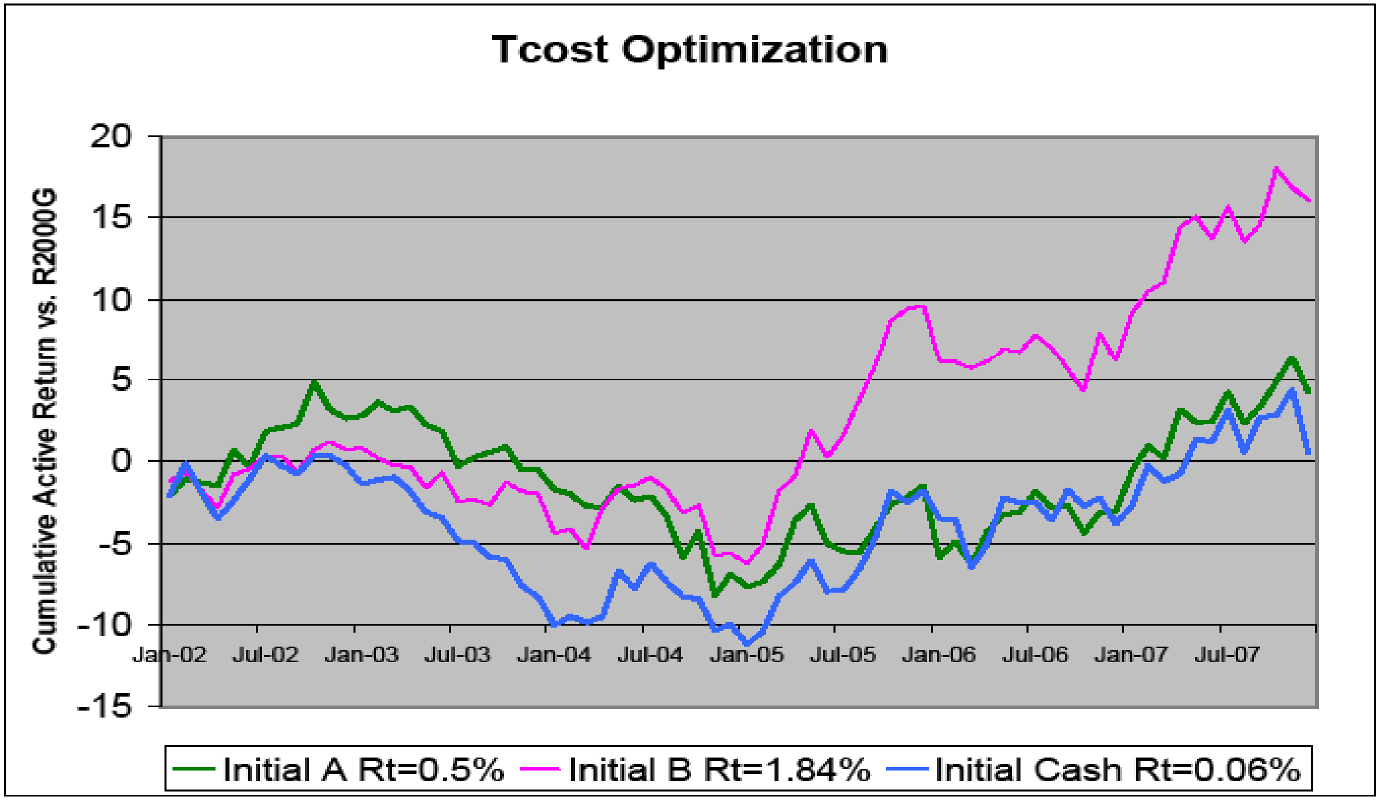

	Initial Portfolio		
Annualized Performance	A	B	C
Active Returns	**0.50%**	**1.84%**	**0.06%**
Active Risk	**4.75%**	**4.81%**	**4.51%**
Tcost Impact	**-0.20%**	**-0.21%**	**-0.26%**

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Does Robust Optimization help?

- **S. Ceria and R. Stubbs. (2006). "Incorporating Estimation Errors into Portfolio Selection: Robust Portfolio Construction." Journal of Asset Management 7. pp. 109-127.**
- **J.H. Lee, D. Stefek and A. Zheleznyak (2006). Robust Portfolio Optimization: A Closer Look. *Barra Research Reports.***
- **F. Fabozzi, P. Kolm, D. Pachamanova and S. Focardi. (2007). Robust Portfolio Optimization. *The Journal of Portfolio Management.***
- **Michaud, R. O. (1989). The Markowitz Optimization Enigma: Is "Optimized" Optimal. *The Financial Analyst Journal V. 45. pp.31-42.***

What is Robust MVO?

A mean-variance optimization (MVO) problem:

α is the expected absolute or excess return of stocks from a universe

$$\begin{array}{lrcl} \max & \alpha' w & & \\ & w'Vw & \leq & \sigma^2 \\ & e'w & = & 1 \\ & w & \geq & 0. \end{array} \qquad (1)$$

Its active counterpart can be written as:

Here, α is the expected active return of stocks from a universe

.

$$\begin{array}{c} \max \quad \alpha' w \\ (w - w^b)' V (w - w^b) \leq \tau^2 \\ e'w = 1 \\ w \geq 0. \end{array} \qquad (2)$$

A very important and well-known property for the active problem is:

$$\alpha' w^b = 0.$$

A benchmark portfolio's alpha is zero! (See Grinold and Kahn (1999)).

What is Robust MVO?

Due to the fact that the true α is not know, and we can only obtain an estimation of the "true" α. Assume

$$\alpha \sim N(\alpha^*, \Omega)$$

a normal distribution of **known** covariance.

For any small probability p, we can construct a $(1-p)$ confidence region for some κ_p as

$$\mathcal{B} = \{\alpha | (\alpha - \hat{\alpha})' \Omega^{-1} (\alpha - \hat{\alpha}) \le (\kappa_p)^2\},$$

where $\hat{\alpha}$ is the estimated expected return of the stocks. The robust optimization tries to optimize the "worst case scenario" out of all possibilities, or over some most likely cases.

$$\begin{aligned} \max \min_{\{\alpha \in \mathcal{B}\}} \quad & \alpha' w \\ w' V w &\le \tau^2 \\ e' w &= 1 \\ w &\ge 0. \end{aligned} \tag{3}$$

What is Robust MVO?

Absolute Return-Risk Robust MVO Formulation

The robust optimization problem can be formulated as,

$$\begin{aligned} \max \quad & \hat{\alpha} w - \kappa_p \sqrt{w' \Omega w} \\ & w' V w \leq \sigma^2 \\ & e' w = 1 \\ & w \geq 0. \end{aligned} \tag{4}$$

This is a second-order cone programming (SOCP)problem instead of a quadratic programming problem (QP) largely due to the appearance of the square root.

Because

$$\begin{aligned} & \min_{\alpha} \{ \alpha' w \mid \quad (\alpha - \hat{\alpha})' \Omega^{-1} (\alpha - \hat{\alpha}) \leq (\kappa_p)^2 \} \\ = \; & \hat{\alpha} w + \kappa_p \min_{x} \{ \; x' u \quad \mid \quad x' x \leq 1 \} \\ = \; & \hat{\alpha} w - \kappa_p \| u \| \\ = \; & \hat{\alpha} w - \kappa_p \sqrt{w' \Omega w}. \end{aligned} \tag{5}$$

$$\max \min_{\{\alpha \in \mathcal{B}\}} \quad \alpha' w = \max \quad \hat{\alpha} w - \kappa_p \sqrt{w' \Omega w}.$$

Our Results

Demystification of the Robust Optimization

$$
\begin{aligned}
\max \quad & \hat{\alpha} w - \kappa_p \sqrt{w' \Omega w} \\
& w' V w \le \sigma^2 \\
& e' w = 1 \\
& w \ge 0.
\end{aligned} \tag{6}
$$

There exists a λ such that the solution of the following problem has an identical solution to the above Robust MVO,

$$
\begin{aligned}
L(\lambda) \quad = \quad \max \quad & \hat{\alpha} w - \lambda w' \Omega w \\
& w' V w \le \sigma^2 \\
& e' w = 1 \\
& w \ge 0.
\end{aligned} \tag{7}
$$

This is a Quadratic Penalty Function problem.

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Our Results

Why Our Claim is true?

$$\begin{array}{lrcl} \max & \hat{\alpha}' w & & \\ & w' V w & \leq & \sigma^2 \\ & e' w & = & 1 \\ & w & \geq & 0. \end{array}$$

It is well-known that the regular MVO can be solved by

$$\begin{array}{lrcl} \max & \hat{\alpha}' w & - & \mu\ w' V w \\ & e' w & = & 1 \\ & w & \geq & 0, \end{array}$$

for some μ. (μ is formally called the Lagrangian Multiplier.)

.

$$\begin{array}{rl} \max & \hat{\alpha} w - \kappa_p \sqrt{w' \Omega w} \\ & w' V w \leq \sigma^2 \\ & e' w = 1 \\ & w \geq 0. \end{array}$$

By the same logic, the above Robust Problem can be solved through

$$\begin{array}{l} \max \quad \hat{\alpha} w \\ \sqrt{w' \Omega w} \leq \theta \quad \text{or} \quad w' \Omega w \leq \theta^2 \\ w' V w \leq \sigma^2 \\ e' w = 1 \\ w \geq 0. \end{array}$$

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Our Results

The Implication of Our Claim

Since V is the risk model, α^* is the expected absolute returns,

$$\alpha \sim N(\alpha^*, V).$$

On account of Axioma's Robust model's assumption $\alpha \sim N(\alpha^*, \Omega)$, we shall conclude that

$$\Omega = \delta V.$$

If this is the case, the Efficient Frontier from the Robust Optimization would be identical to the one from Regular MVO.

.

$$\begin{aligned} L(\lambda) \quad = \quad \max \quad & \hat{\alpha} w - \lambda\delta \; w'Vw \\ & w'Vw \leq \sigma^2 \\ & e'w = 1 \\ & w \geq 0. \end{aligned} \tag{8}$$

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A Big Question

Does Robust MVO Enhance Portfolio Alpha?

Just consider a simpler case where $\Omega = diag(\sigma_i^2)$.

$$\begin{aligned} \max \quad & \hat{\alpha}w - \lambda \left\{ w'\Omega w \right\} \\ & w'Vw \leq \sigma^2 \\ & e'w = 1 \\ & w \geq 0. \end{aligned} \tag{9}$$

$$\begin{aligned} \max \quad & \hat{\alpha}w - \mu \; w'(V + (\lambda/\mu)\Omega)w \\ & e'w = 1 \\ & w \geq 0. \end{aligned} \tag{10}$$

The Robust Optimization suggests use a different risk model that increases the idiosyncratic risks for stocks of which the alphas might be harder to predict.

A Mistake in the Current Robust MVO Formulation

Active Returns/Risks Robust Reformulation

For active returns MVO, the corresponding robust problem is

$$\begin{aligned} \max \min_{\{\alpha \in \mathcal{B}\}} \quad & \alpha' w \\ (w - w^b)' V (w - w^b) & \leq \tau^2 \\ e' w & = 1 \\ w & \geq 0. \end{aligned} \tag{11}$$

Ceria and Stubbs' (2005) from Axioma claimed that the above objective function is equivalent to

$$\max \ \hat{\alpha}(w - w^b) - \kappa_p \sqrt{(w - w^b)' \Omega (w - w^b)}$$

.

Actually, due to the fact that $\alpha' w^b = 0$, their objective is not entirely accurate. We show that a correct form of objectve function should be

A Mistake in the Current Robust MVO Formulation

$$\max \quad \hat{\alpha} w - \kappa_p \sqrt{w' \Omega w - \frac{(w' \Omega w^b)^2}{w^{b\prime} \Omega w^b}},$$

or

$$\begin{aligned} \max \quad & \hat{\alpha} w - \lambda \left(w' \Omega w - \frac{(w' \Omega w^b)^2}{w^{b\prime} \Omega w^b}\right) \\ & (w - w^b)' V (w - w^b) \leq \sigma^2 \\ & e' w = 1 \\ & w \geq 0. \end{aligned} \tag{12}$$

.

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What is the Remedy?

- Robust MVO does not differ much from regular MVO, it is equivalent to the quadratic penalty function method;
- Robust MVO requires to estimate another set of parameters;
- Robust Optimization is a one-period static solution.

Two-Stage Optimization --- A Solution

- Stage 1: Find a path-independent ***"ideal" portfolio***
- Max Alpha

  1a. The Tracking Error upper bound;

  1b. Linear side-constraints; (Such as security-level bounds, factor-bets);

  1c. Upper bound on number of securities (optional).
- Stage 2: Find the ***tradable portfolio***

  Max Utility Function

  same constraints as Stage 1, 1c) shall be included.

  Utility Function = alpha – λ (tracking_err vs. ***ideal portfolio***)^2

  - μ (Tcost against the ***legacy portfolio***)

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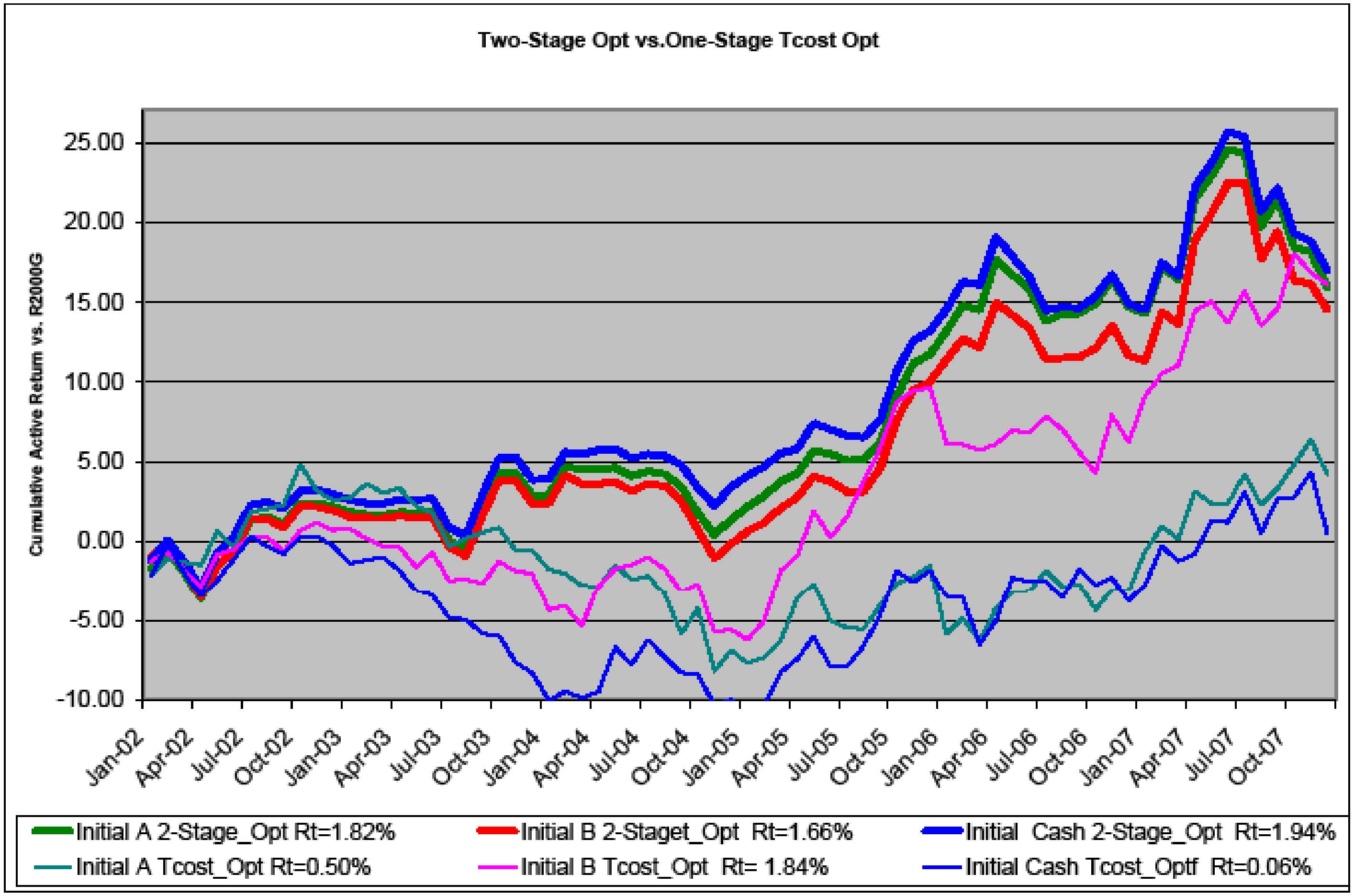
Two-Stage vs. One-Stage



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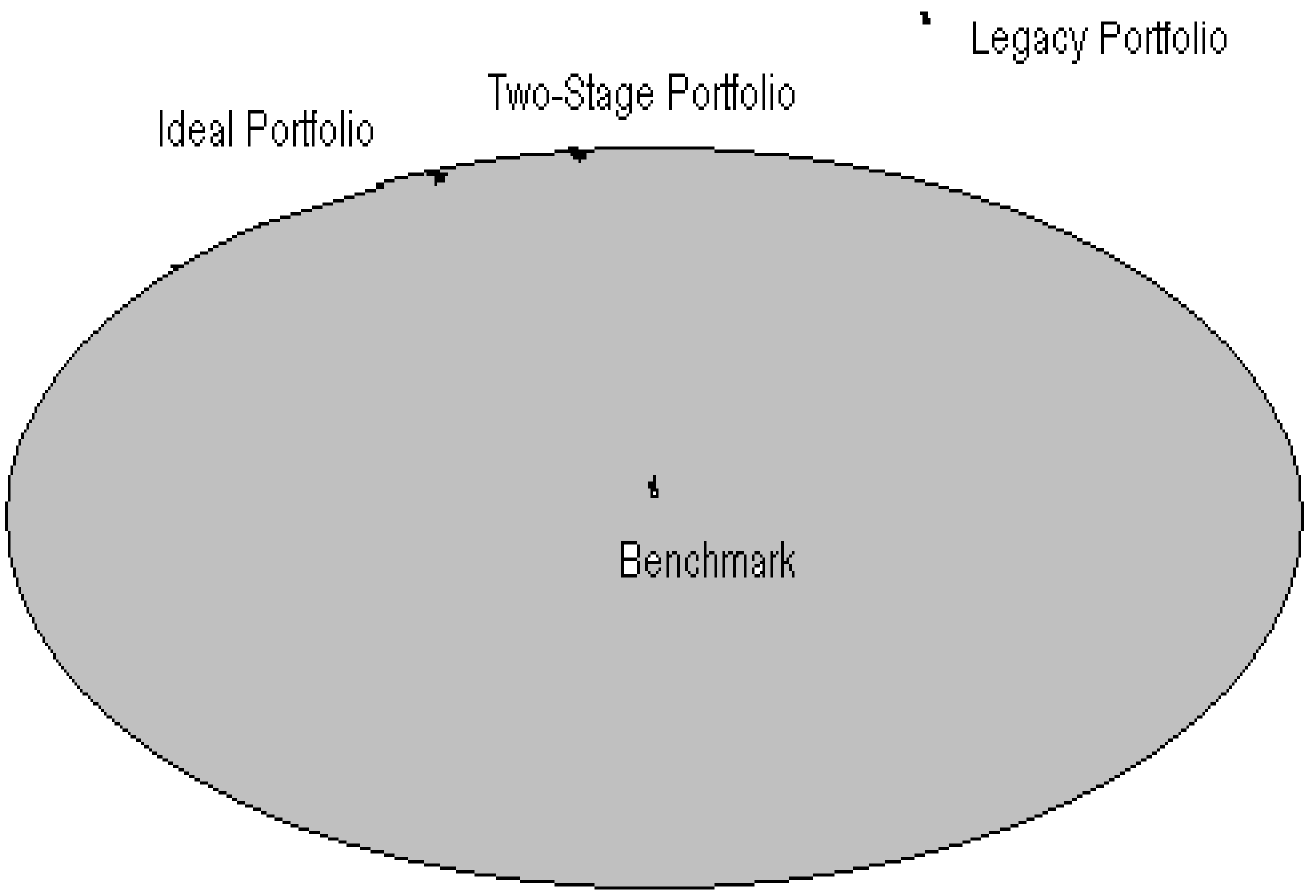
Two-Stage Optimization



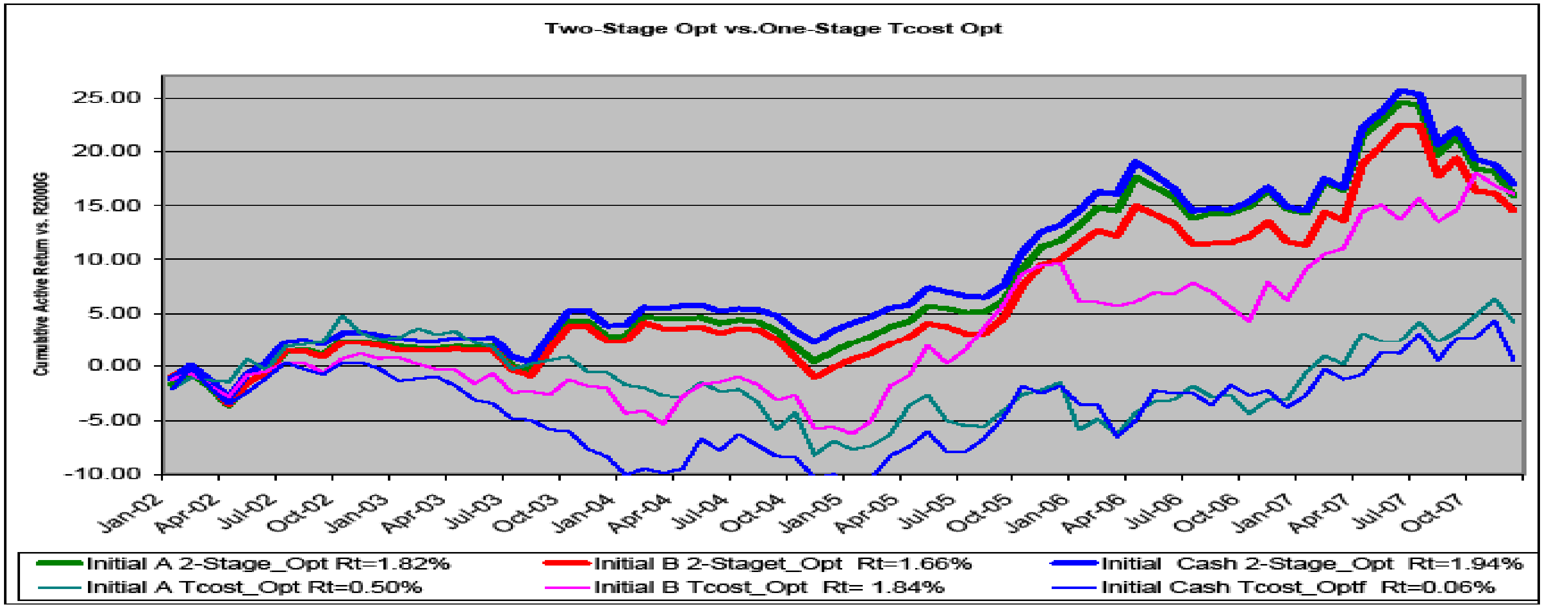
Two-Stage Optimization --- A Solution

- *<u>tradable portfolio</u>*

  =Function (alpha, risk model, bounds, legacy portfolio, λ, μ)

- *How to decide λ and μ is an art.*

Two-Stage Optimization



	Initial Portfolio		
	A	B	C
Active Returns	1.82%	1.66%	1.94%
Active Risk	4.20%	4.28%	4.20%
Tcost Impact	-1.43%	-1.44%	-1.42%

Two-Stage Optimization

	Initial Portfolio		
	A	B	C
Active Returns	1.84%	0.50%	0.06%
Active Risk	4.75%	4.81%	4.51%
Tcost Impact	-0.20%	-0.21%	-0.26%

One-Stage Tcost Optimization

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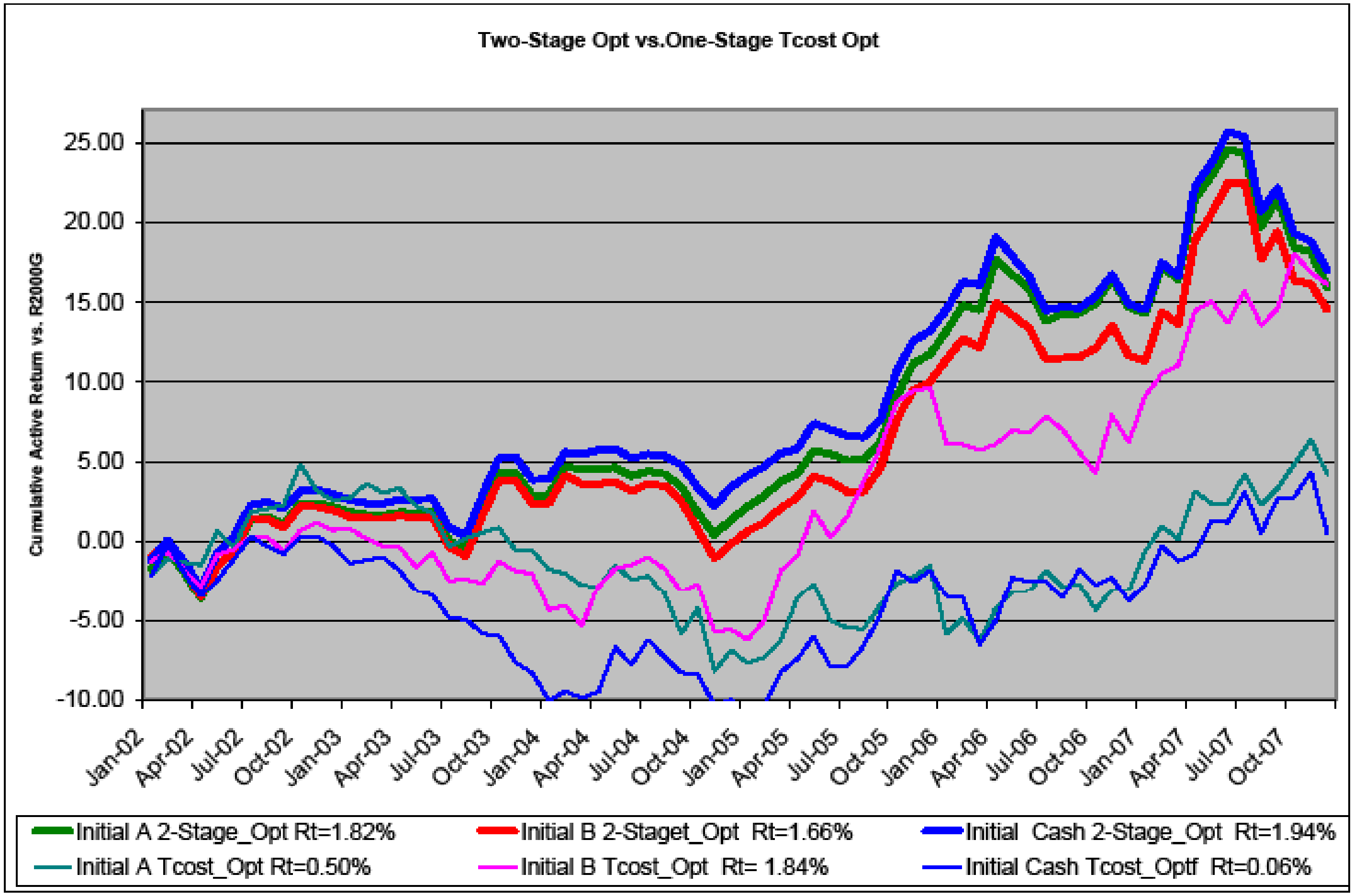
Two-Stage vs. One-Stage



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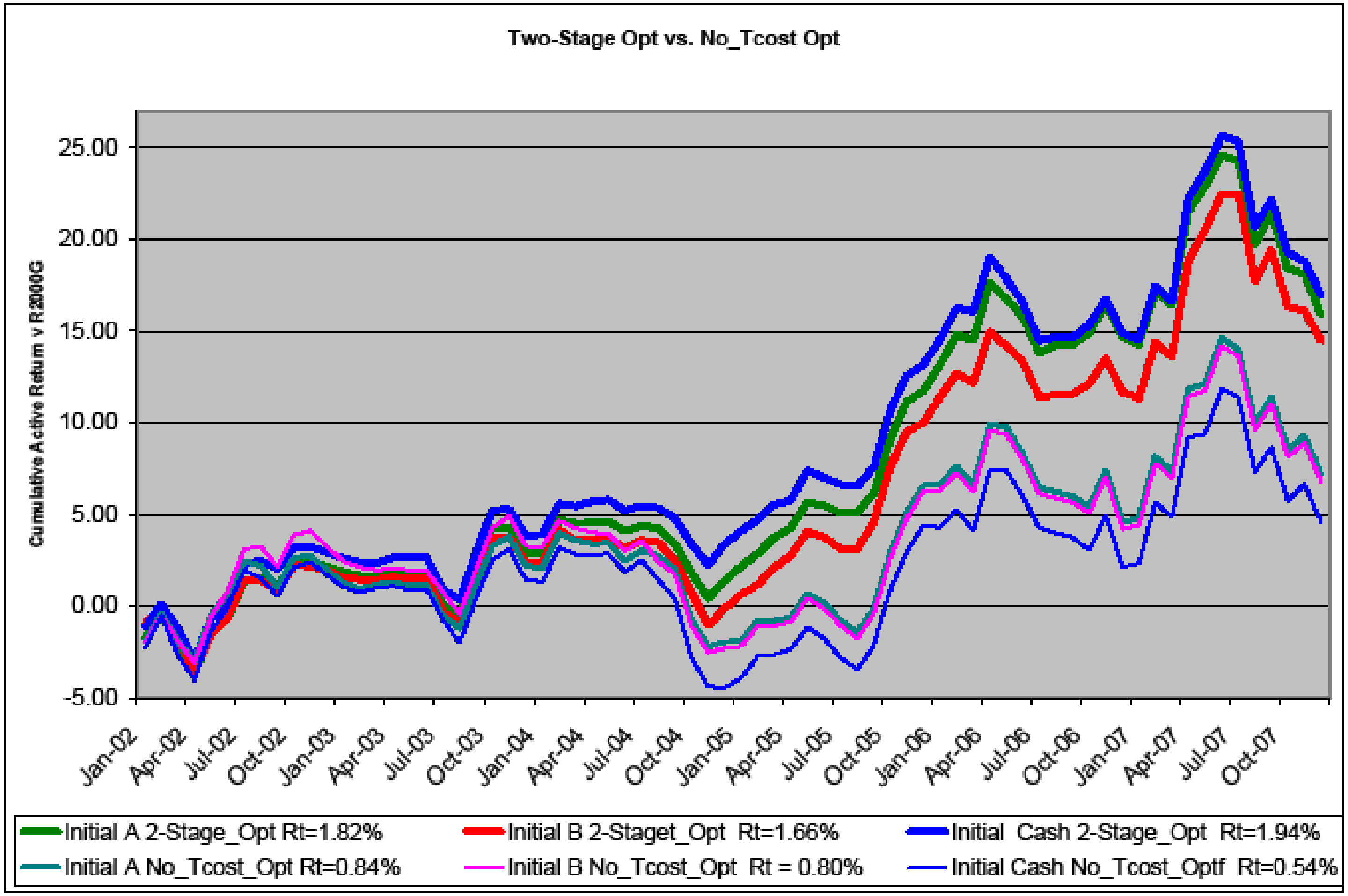
Two-Stage vs. No-Tcost Optimization



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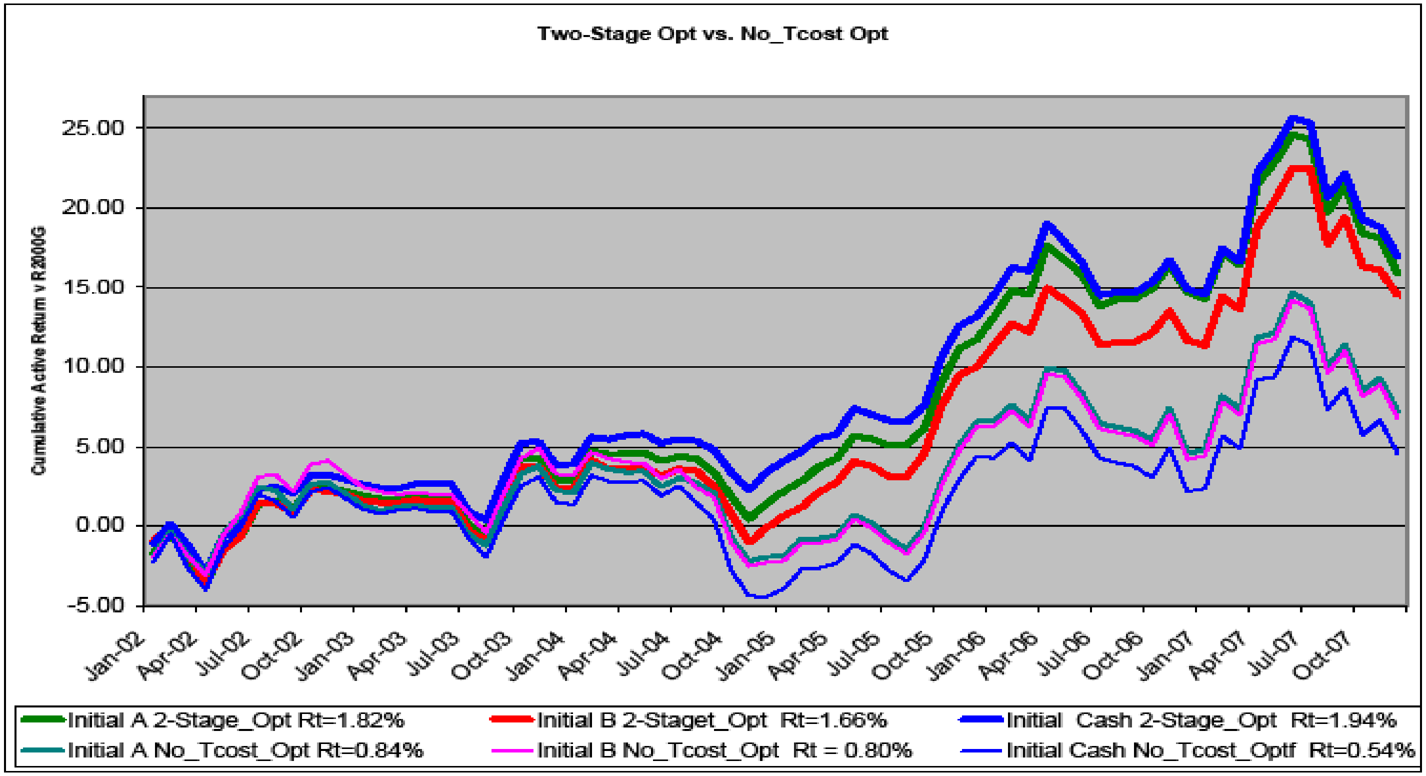

Annualized Performance	Initial Portfolio A	B	C
Active Returns (Two-Stage)	**1.82%**	**1.66%**	**1.94%**
Active Returns (No-Tcost Opt)	**0.80%**	**0.84%**	**0.54%**
Active Risk	**4.57%**	**4.55%**	**4.69%**
Tcost Impact (Two-Stage)	**-1.43%**	**-1.44%**	**-1.42%**
Tcost Impact (No-Tcost Opt)	**-2.89%**	**-2.89%**	**-2.89%**

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Less is More

- It might *not* be a good idea to *over-analyze* the one-period static optimization problems
- How to make a Portfolio Optimization Process (POP) Robust?

1) form a maximization problem that is *concave* if possible;

having a *unique* optimal solution is even better (When the risk model is a positive-definite matrix *and* risk budget is binding).

2) Constraints that limit number of securities in a portfolio may cause the optimization problem *non-concave*, risk-targeting is a ***bad idea.***

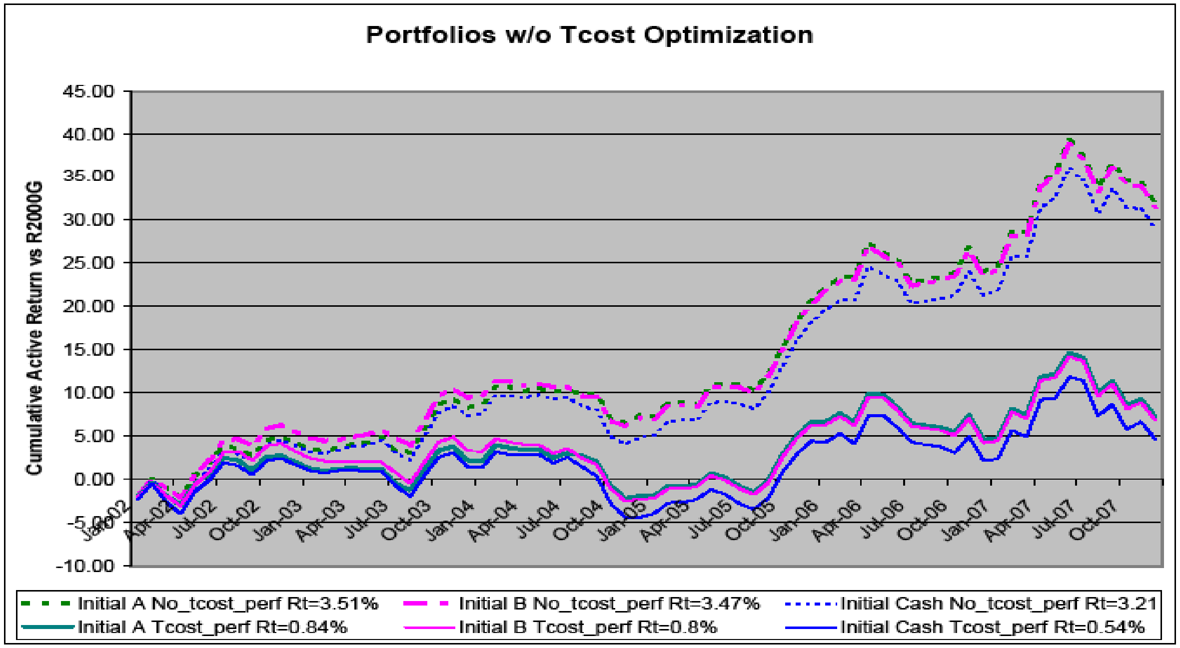
3) Find and understand a series of *path-independent* portfolios. The PMs don't need to trade on these portfolios, but they are better to *know* these portfolios.

4) Including transaction cost and market impact into the POP, or adding turnover constraints may create a series of *path-dependent* portfolios.

Less is More

- **The Portfolio Optimization Process (POP) uses ex-ante data to achieve an ex-post goal, and it is a single-period proxy to a multi-period stochastic problems.**
- **The MVO pioneered by Markowitz was developed originally to trade-off between risks and returns**
- **Currently Optimizer is used as a portfolio construction tool, sometimes, the tool to create a *tradable portfolio*.**

Recommendations



Run a Two-Stage Optimization (It requires an optimizer that can handles the 2nd Benchmark.)

If not, run an optimization ignoring Tcost/Turnover to ensure path-independent.

G. Sofianos, S. Takriti and I. Tierens. (2007) Including Trading Costs in Portfolio Optimization. Equity Execution Strategies, Goldman Sachs.

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Closing Remarks

- **Robust MVO is equivalent to the Quadratic Penalty Function Approach**
- **“ Less is More”**
- **Two-Stage Optimization Enhances Robustness**