RD.2 Rational Exponents

In *sections P2* and *RT1*, we reviewed properties of powers with natural and integral exponents. All of these properties hold for real exponents as well. In this section, we give meaning to expressions with rational exponents, such as $a^{\frac{1}{2}}$, $8^{\frac{1}{3}}$, or $(2x)^{0.54}$, and use the rational exponent notation as an alternative way to write and simplify radical expressions.

Rational Exponents

Observe that $\sqrt{9} = 3 = 3^{2\cdot\frac{1}{2}} = 9^{\frac{1}{2}}$. Similarly, $\sqrt[3]{8} = 2 = 2^{3\cdot\frac{1}{3}} = 8^{\frac{1}{3}}$. This suggests the following generalization:

For any real number a and a natural number $n > 1$, we have

$$\sqrt[n]{a} = a^{\frac{1}{n}}.$$

Notice: The **denominator** of the rational exponent is the **index** of the radical.

Caution! If $a < 0$ and n is an even natural number, then $a^{\frac{1}{n}}$ is <u>not</u> a real number.

Example 1 ► **Converting Radical Notation to Rational Exponent Notation**

Convert each radical to a power with a rational exponent and simplify, if possible. Assume that all variables represent positive real numbers.

a. $\sqrt[6]{16}$ **b.** $\sqrt[3]{27x^3}$ **c.** $\sqrt{\frac{4}{b^6}}$

Solution ► **a.** $\sqrt[6]{16} = 16^{\frac{1}{6}} = (2^4)^{\frac{1}{6}} = 2^{\frac{4}{6}} = \mathbf{2^{\frac{2}{3}}}$

Observation: Expressing numbers as **powers of prime numbers** often allows for further simplification.

b. $\sqrt[3]{27x^3} = (27x^3)^{\frac{1}{3}} = 27^{\frac{1}{3}} \cdot (x^3)^{\frac{1}{3}} = (3^3)^{\frac{1}{3}} \cdot x = \mathbf{3x}$

distribution of exponents; change into a power of a prime number

Note: The above example can also be done as follows:

$$\sqrt[3]{27x^3} = \sqrt[3]{3^3x^3} = (3^3x^3)^{\frac{1}{3}} = \mathbf{3x}$$

c. $\sqrt{\frac{9}{b^6}} = \left(\frac{9}{b^6}\right)^{\frac{1}{2}} = \frac{(3^2)^{\frac{1}{2}}}{(b^6)^{\frac{1}{2}}} = \frac{\mathbf{3}}{\mathbf{b^3}}$, as $b > 0$.

Observation: $\sqrt{a^4} = a^{\frac{4}{2}} = a^2$.

Generally, for any real number $a \neq 0$, natural number $n > 1$, and integral number m, we have

$$\sqrt[n]{a^m} = (a^m)^{\frac{1}{n}} = a^{\frac{m}{n}}$$

Rational exponents are introduced in such a way that they automatically agree with the rules of exponents, as listed in *section RT1*.
Furthermore, the rules of exponents hold not only for rational but also for **real exponents**.

Observe that following the rules of exponents and the commutativity of multiplication, we have

$$\sqrt[n]{a^m} = (a^m)^{\frac{1}{n}} = \left(a^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{a}\right)^m,$$

provided that $\sqrt[n]{a}$ exists.

Example 2 ► **Converting Rational Exponent Notation to the Radical Notation**

Convert each power with a rational exponent to a radical and simplify, if possible.

a. $5^{\frac{3}{4}}$ **b.** $(-27)^{\frac{1}{3}}$ **c.** $3x^{-\frac{2}{5}}$

Solution ►

a. $5^{\frac{3}{4}} = \sqrt[4]{5^3} = \sqrt[4]{125}$

b. $(-27)^{\frac{1}{3}} = \sqrt[3]{-27} = -3$

Notice that $-27^{\frac{1}{3}} = -\sqrt[3]{27} = -3$, so $(-27)^{\frac{1}{3}} = -27^{\frac{1}{3}}$.

However, $(-9)^{\frac{1}{2}} \neq -9^{\frac{1}{2}}$, as $(-9)^{\frac{1}{2}}$ is not a real number while $-9^{\frac{1}{2}} = -\sqrt{9} = -3$.

c. $3x^{-\frac{2}{5}} = \frac{3}{x^{\frac{2}{5}}} = \frac{3}{\sqrt[5]{x^2}}$

Caution: A **negative exponent** indicates a **reciprocal** not a negative number!

Also, the exponent refers to x only, so 3 remains in the numerator.

Observation: If $a^{\frac{m}{n}}$ is a real number, then

$$a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}},$$

provided that $a \neq 0$.

Caution! Make sure to distinguish between a negative exponent and a negative result. Negative exponent leads to a reciprocal of the base. The result can be either positive or negative, depending on the sign of the base. For example,

$$8^{-\frac{1}{3}} = \frac{1}{8^{\frac{1}{3}}} = \frac{1}{2}, \text{ but } (-8)^{-\frac{1}{3}} = \frac{1}{(-8)^{\frac{1}{3}}} = \frac{1}{-2} = -\frac{1}{2} \text{ and } -8^{-\frac{1}{3}} = -\frac{1}{8^{\frac{1}{3}}} = -\frac{1}{2}.$$

Example 3 ► **Applying Rules of Exponents When Working with Rational Exponents**

Simplify each expression. Write your answer with only positive exponents. Assume that all variables represent positive real numbers.

a. $a^{\frac{3}{4}} \cdot 2a^{-\frac{2}{3}}$ **b.** $\frac{4^{\frac{1}{3}}}{4^{\frac{5}{3}}}$ **c.** $\left(x^{\frac{3}{8}} \cdot y^{\frac{5}{2}}\right)^{\frac{4}{3}}$

Solution ►

a. $a^{\frac{3}{4}} \cdot 2a^{-\frac{2}{3}} = 2a^{\frac{3}{4}+\left(-\frac{2}{3}\right)} = 2a^{\frac{9}{12}-\frac{8}{12}} = \boldsymbol{2a^{\frac{1}{12}}}$

b. $\frac{4^{\frac{1}{3}}}{4^{\frac{5}{3}}} = 4^{\frac{1}{3}-\frac{5}{3}} = 4^{-\frac{4}{3}} = \boldsymbol{\frac{1}{4^{\frac{4}{3}}}}$

c. $\left(x^{\frac{3}{8}} \cdot y^{\frac{5}{2}}\right)^{\frac{4}{3}} = x^{\frac{3}{8}\cdot\frac{4}{3}} \cdot y^{\frac{5}{2}\cdot\frac{4}{3}} = \boldsymbol{x^{\frac{1}{2}}y^{\frac{10}{3}}}$

Example 4 ► **Evaluating Powers with Rational Exponents**

Evaluate each power.

a. $64^{-\frac{1}{3}}$ **b.** $\left(-\frac{8}{125}\right)^{\frac{2}{3}}$

Solution ►

a. $64^{-\frac{1}{3}} = (2^6)^{-\frac{1}{3}} = 2^{-2} = \frac{1}{2^2} = \boldsymbol{\frac{1}{4}}$

It is helpful to change the base into a **power of prime number**, if possible.

b. $\left(-\frac{8}{125}\right)^{\frac{2}{3}} = \left(\left(-\frac{2}{5}\right)^3\right)^{\frac{2}{3}} = \left(-\frac{2}{5}\right)^2 = \boldsymbol{\frac{4}{25}}$

Observe that if m in $\sqrt[n]{a^m}$ is a multiple of n, that is if $m = kn$ for some integer k, then

$$\sqrt[n]{a^{kn}} = a^{\frac{kn}{n}} = a^k$$

Example 5 ► **Simplifying Radical Expressions by Converting to Rational Exponents**

Simplify. Assume that all variables represent positive real numbers. Leave your answer in simplified single radical form.

a. $\sqrt[5]{3^{20}}$ **b.** $\sqrt{x} \cdot \sqrt[4]{x^3}$ **c.** $\sqrt[3]{2\sqrt{2}}$

Solution

a. $\sqrt[5]{3^{20}} = (3^{20})^{\frac{1}{5}} = 3^4 = \mathbf{81}$

divide at the exponential level

b. $\sqrt{x} \cdot \sqrt[4]{x^3} = x^{\frac{1}{2}} \cdot x^{\frac{3}{4}} = x^{\frac{1 \cdot 2}{2 \cdot 2} + \frac{3}{4}} = x^{\frac{5}{4}} = x \cdot x^{\frac{1}{4}} = \boldsymbol{x\sqrt[4]{x}}$

add exponents; as $\frac{5}{4} = 1 + \frac{1}{4}$

c. $\sqrt[3]{2\sqrt{2}} = \left(2 \cdot 2^{\frac{1}{2}}\right)^{\frac{1}{3}} = \left(2^{1+\frac{1}{2}}\right)^{\frac{1}{3}} = \left(2^{\frac{3}{2}}\right)^{\frac{1}{3}} = 2^{\frac{1}{2}} = \sqrt{\mathbf{2}}$

This bracket is essential!

Another solution:

$\sqrt[3]{2\sqrt{2}} = 2^{\frac{1}{3}} \cdot \left(2^{\frac{1}{2}}\right)^{\frac{1}{3}} = 2^{\frac{1}{3}} \cdot 2^{\frac{1}{6}} = 2^{\frac{1 \cdot 2}{3 \cdot 2} + \frac{1}{6}} = 2^{\frac{1}{2}} = \sqrt{\mathbf{2}}$

RD.2 Exercises

Concept Check *Match each expression from Column I with the equivalent expression from Column II.*

1.

Column I	Column II
a. $25^{\frac{1}{2}}$	**A.** $\frac{1}{5}$
b. $25^{-\frac{1}{2}}$	**B.** 5
c. $-25^{\frac{3}{2}}$	**C.** -125
d. $-25^{-\frac{1}{2}}$	**D.** not a real number
e. $(-25)^{\frac{1}{2}}$	**E.** $\frac{1}{125}$
f. $25^{-\frac{3}{2}}$	**F.** $-\frac{1}{5}$

2.

Column I	Column II
a. $(-32)^{\frac{2}{5}}$	**A.** 2
b. $-27^{\frac{2}{3}}$	**B.** $\frac{1}{4}$
c. $32^{\frac{1}{5}}$	**C.** -8
d. $32^{-\frac{2}{5}}$	**D.** -9
e. $-4^{\frac{3}{2}}$	**E.** not a real number
f. $(-4)^{\frac{3}{2}}$	**F.** 4

Concept Check *Write the base as a* ***power of a prime number*** *to evaluate each expression, if possible.*

3. $32^{\frac{1}{5}}$ **4.** $27^{\frac{4}{3}}$ **5.** $-49^{\frac{3}{2}}$ **6.** $16^{\frac{3}{4}}$

7. $-100^{-\frac{1}{2}}$ **8.** $125^{-\frac{1}{3}}$ **9.** $\left(\frac{64}{81}\right)^{\frac{3}{4}}$ **10.** $\left(\frac{8}{27}\right)^{-\frac{2}{3}}$

11. $(-36)^{\frac{1}{2}}$ **12.** $(-64)^{\frac{1}{3}}$ **13.** $\left(-\frac{1}{8}\right)^{-\frac{1}{3}}$ **14.** $(-625)^{-\frac{1}{4}}$

Concept Check *Rewrite **with** rational exponents and simplify, if possible. Assume that all variables represent positive real numbers.*

15. $\sqrt{5}$ **16.** $\sqrt[3]{6}$ **17.** $\sqrt{x^6}$ **18.** $\sqrt[5]{y^2}$

19. $\sqrt[3]{64x^6}$ **20.** $\sqrt[3]{16x^2y^3}$ **21.** $\sqrt{\frac{25}{x^5}}$ **22.** $\sqrt[4]{\frac{16}{a^6}}$

Concept Check *Rewrite **without** rational exponents, and simplify, if possible. Assume that all variables represent positive real numbers.*

23. $4^{\frac{5}{2}}$ **24.** $8^{\frac{3}{4}}$ **25.** $x^{\frac{3}{5}}$ **26.** $a^{\frac{7}{3}}$

27. $(-3)^{\frac{2}{3}}$ **28.** $(-2)^{\frac{3}{5}}$ **29.** $2x^{-\frac{1}{2}}$ **30.** $x^{\frac{1}{3}}y^{-\frac{1}{2}}$

Concept Check *Use the **laws of exponents** to simplify. Write the answers with positive exponents. Assume that all variables represent positive real numbers.*

31. $3^{\frac{3}{4}} \cdot 3^{\frac{1}{8}}$ **32.** $x^{\frac{2}{3}} \cdot x^{-\frac{1}{4}}$ **33.** $\frac{2^{\frac{5}{8}}}{2^{-\frac{1}{8}}}$ **34.** $\frac{a^{\frac{1}{3}}}{a^{\frac{2}{3}}}$

35. $\left(5^{\frac{15}{8}}\right)^{\frac{2}{3}}$ **36.** $\left(y^{\frac{2}{3}}\right)^{-\frac{3}{7}}$ **37.** $\left(x^{\frac{3}{8}} \cdot y^{\frac{5}{2}}\right)^{\frac{4}{3}}$ **38.** $\left(a^{-\frac{2}{3}} \cdot b^{\frac{5}{8}}\right)^{-4}$

39. $\left(\frac{y^{-\frac{3}{2}}}{x^{-\frac{5}{3}}}\right)^{\frac{1}{3}}$ **40.** $\left(\frac{a^{-\frac{2}{3}}}{b^{-\frac{5}{6}}}\right)^{\frac{3}{4}}$ **41.** $x^{\frac{2}{3}} \cdot 5x^{-\frac{2}{5}}$ **42.** $x^{\frac{2}{5}} \cdot \left(4x^{-\frac{4}{5}}\right)^{-\frac{1}{4}}$

*Use rational exponents to **simplify**. Write the answer **in radical notation** if appropriate. Assume that all variables represent positive real numbers.*

43. $\sqrt[6]{x^2}$ **44.** $\left(\sqrt[3]{ab}\right)^{15}$ **45.** $\sqrt[6]{y^{-18}}$ **46.** $\sqrt{x^4y^{-6}}$

47. $\sqrt[6]{81}$ **48.** $\sqrt[14]{128}$ **49.** $\sqrt[3]{8y^6}$ **50.** $\sqrt[4]{81p^6}$

51. $\sqrt[3]{(4x^3y)^2}$ **52.** $\sqrt[5]{64(x+1)^{10}}$ **53.** $\sqrt[4]{16x^4y^2}$ **54.** $\sqrt[5]{32a^{10}d^{15}}$

*Use rational exponents to rewrite in a **single radical expression** in a simplified form. Assume that all variables represent positive real numbers.*

55. $\sqrt[3]{5} \cdot \sqrt{5}$ **56.** $\sqrt[3]{2} \cdot \sqrt[4]{3}$ **57.** $\sqrt{a} \cdot \sqrt[3]{3a}$ **58.** $\sqrt[3]{x} \cdot \sqrt[5]{2x}$

59. $\sqrt[6]{x^5} \cdot \sqrt[3]{x^2}$ **60.** $\sqrt[3]{xz} \cdot \sqrt{z}$ **61.** $\frac{\sqrt{x^5}}{\sqrt{x^8}}$ **62.** $\frac{\sqrt[3]{a^5}}{\sqrt{a^3}}$

63. $\frac{\sqrt[3]{8x}}{\sqrt[4]{x^3}}$ **64.** $\sqrt[3]{\sqrt{a}}$ **65.** $\sqrt[4]{\sqrt[3]{xy}}$ **66.** $\sqrt{\sqrt[3]{(3x)^2}}$

67. $\sqrt{\sqrt[3]{\sqrt[4]{x}}}$ **68.** $\sqrt[3]{3\sqrt{3}}$ **69.** $\sqrt[4]{x\sqrt{x}}$ **70.** $\sqrt[3]{2\sqrt{x}}$

Discussion Point

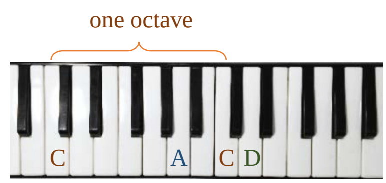
71. Suppose someone claims that $\sqrt[n]{a^n + b^n}$ must equal $a + b$, since, when $a = 1$ and $b = 0$, the two expressions are equal: $\sqrt[n]{a^n + b^n} = \sqrt[n]{1^n + 0^n} = 1 = 1 + 9 = a + b$. Explain why this is faulty reasoning.

Analytic Skills *Solve each problem.*

72. One octave on a piano contains 12 keys (including both the black and white keys). The frequency of each successive key increases by a factor of $2^{\frac{1}{12}}$. For example, middle C is two keys below the first D above it. Therefore, the frequency of this D is

$$2^{\frac{1}{12}} \cdot 2^{\frac{1}{12}} = 2^{\frac{1}{6}} \approx 1.12$$

times the frequency of the middle C.



a. If two tones are one octave apart, how do their frequencies compare?

b. The A tone below middle C has a frequency of 220 cycles per second. Middle C is 3 keys above this A note. Estimate the frequency of middle C.

73. According to one model, an animal's heart rate varies according to its weight. The formula $N(w) = 885w^{-\frac{1}{2}}$ gives an estimate for the average number N of beats per minute for an animal that weighs w pounds. Use the formula to estimate the heart rate for a horse that weighs 800 pounds.

74. Meteorologists can determine the duration of a storm by using the function defined by $T(D) = 0.07D^{\frac{3}{2}}$, where D is the diameter of the storm in miles and T is the time in hours. Find the duration of a storm with a diameter of 16 mi. Round your answer to the nearest tenth of an hour.