CEE 271: Applied Mechanics II, Dynamics – Lecture 21: Ch.16, Sec.1–4 –

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Date: ____________________

PLANAR RIGID BODY MOTION: TRANSLATION and ROTATION

Today's objectives: Students will be able to

1. Analyze the kinematics of a rigid body undergoing ***planar translation*** or ***rotation*** about a fixed axis.

In-class activities:

- Reading Quiz
- Applications
- Types of Rigid-Body Motion
- Planar Translation
- Rotation About a Fixed Axis
- Concept Quiz
- Group Problem Solving
- Attention Quiz

READING QUIZ

1. If a rigid body is in translation only, the velocity at points A and B on the rigid body ________.
   (a) are usually different
   (b) are always the same
   (c) depend on their position
   (d) depend on their relative position
   ANS: __
2. If a rigid body is rotating with a constant angular velocity (ω) about a fixed axis, the velocity vector at point P ($\boldsymbol{r}_p$) is ________.
   (a) $\boldsymbol{\omega} \times \boldsymbol{r}_p$
   (b) $\boldsymbol{r}_p \times \boldsymbol{\omega}$
   (c) $d\boldsymbol{r}_p/dt$
   (d) All of the above.
   ANS: __

APPLICATIONS

- Passengers on this amusement ride are subjected to ***curvilinear*** translation since the vehicle moves in a circular path but they always remains ***upright***.
- If the angular motion of the rotating arms is known, how can we determine the velocity and acceleration experienced by the passengers? Why would we want to know these values?
- Does each passenger feel the ***same*** acceleration?

APPLICATIONS(continued)

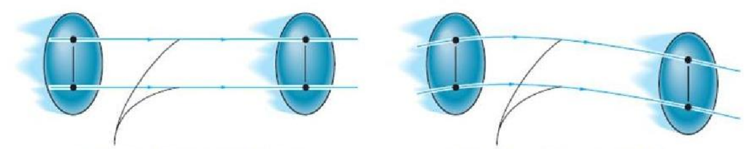
- Gears, pulleys and cams, which rotate about fixed axes, are often used in machinery to generate motion and transmit forces. The ***angular*** motion of these components must be understood to properly design the system.
- To do this design, we need to relate the angular motions of contacting bodies that rotate about different fixed axes. How is this different than the analyses we did in earlier chapters?

RIGID BODY MOTION (Section 16.1)

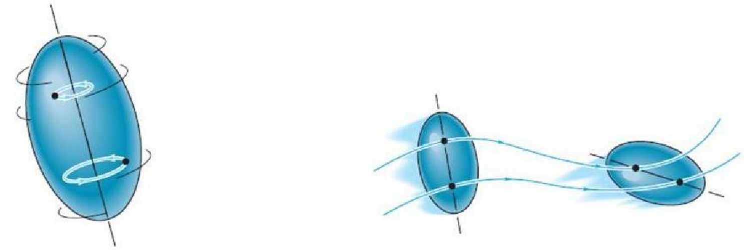
- There are cases where an object ***cannot*** be treated as a particle. In these cases the ***size*** or ***shape*** of the body must be considered. ***Rotation*** of the body about its center of mass requires a different approach.
- For example, in the design of gears, cams, and links in machinery or mechanisms, rotation of the body is an important aspect in the analysis of motion.
- We will now start to study ***rigid body motion***. The analysis will be limited to ***planar motion***.
- A body is said to undergo planar motion when all parts of the body move along paths equidistant from a fixed plane.

PLANAR RIGID BODY MOTION

- There are three types of planar rigid body motion.

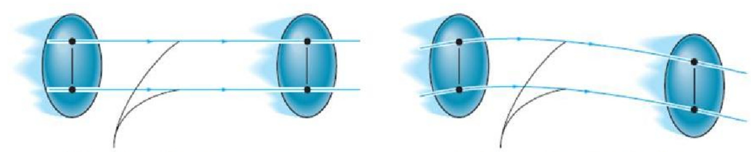


Path of rectilinear translation   Path of curvilinear translation



Rotation about a fixed axis   General plane motion

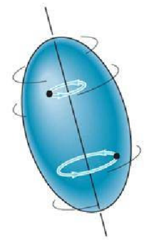
PLANAR RIGID BODY MOTION (continued)



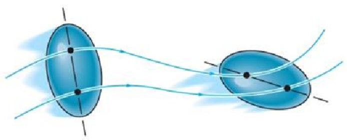
Path of rectilinear translation   Path of curvilinear translation

A. ***Translation***: Translation occurs if every line segment on the body remains parallel to its original direction during the motion. When all points move along straight lines, the motion is called ***rectilinear*** translation. When the paths of motion are curved lines, the motion is called ***curvilinear*** translation.

PLANAR RIGID BODY MOTION (continued)



Rotation about a fixed axis



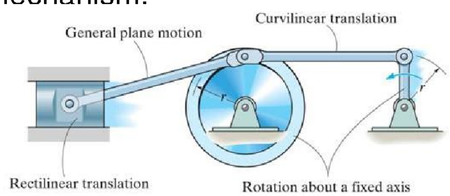
General plane motion

B. ***Rotation about a fixed axis***: In this case, all the particles of the body, ***except*** those on the axis of rotation, move along ***circular paths*** in planes perpendicular to the axis of rotation.

C. ***General plane motion***: In this case, the body undergoes both ***translation and rotation***. Translation occurs within a plane and rotation occurs about an axis perpendicular to this plane.

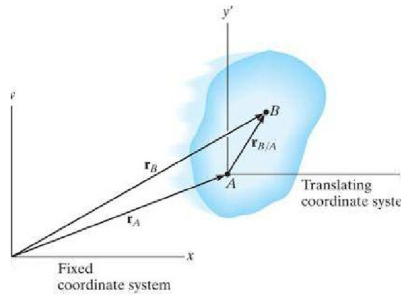
PLANAR RIGID BODY MOTION (continued)

An example of bodies undergoing the three types of motion is shown in this mechanism.



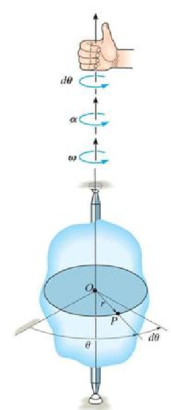
1. The wheel and crank undergo ***rotation about a fixed axis***. In this case, both axes of rotation are at the location of the pins and perpendicular to the plane of the figure.
2. The piston undergoes ***rectilinear translation*** since it is constrained to slide in a straight line.
3. The connecting rod (right) undergoes ***curvilinear translation***, since it will remain horizontal as it moves along a circular path.
4. The connecting rod (left) undergoes ***general plane motion***, as it will both translate and rotate.

RIGID-BODY MOTION: TRANSLATION (Section 16.2)

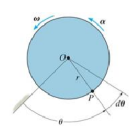


- The positions of two points A and B on a translating body can be related by
$$\boldsymbol{r}_B = \boldsymbol{r}_A + \boldsymbol{r}_{B/A}$$
where $\boldsymbol{r}_A$ and $\boldsymbol{r}_B$ are the absolute position vectors defined from the fixed $x-y$ coordinate system, and $\boldsymbol{r}_{B/A}$ is the relative-position vector between B and A. The velocity at B is
$$\boldsymbol{v}_B = \boldsymbol{v}_A + \frac{d\boldsymbol{r}_{B/A}}{dt}$$
- Now ____________ since $\boldsymbol{r}_{B/A}$ is constant.
- So, $\boldsymbol{v}_B = \boldsymbol{v}_A$, and by following similar logic, $\boldsymbol{a}_B = \boldsymbol{a}_A$.
- Note, all points in a rigid body subjected to translation move with the ____ velocity and acceleration.

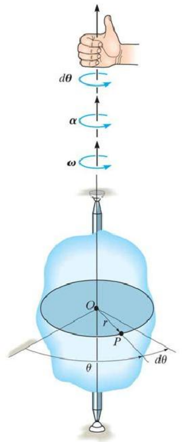
RIGID-BODY MOTION: ROTATION ABOUT A FIXED AXIS (Section 16.3)



- When a body rotates about a fixed axis, any point P in the body travels along a ***circular*** path. The angular position of P is defined by $\boldsymbol{\theta}$.
- The change in angular position, $d\boldsymbol{\theta}$, is called the angular displacement, with units of either radians or revolutions: 1 revolution = (2π) radians
- ***Angular velocity***, ω, is obtained by taking the time derivative of angular displacement:
$$\boldsymbol{\omega} = \frac{d\boldsymbol{\theta}}{dt} \quad (\mathrm{rad/s})$$
- Similarly, ***angular acceleration*** is
$$\boldsymbol{\alpha} = \frac{d^2\boldsymbol{\theta}}{dt^2} = \frac{d\boldsymbol{\omega}}{dt} \quad \text{or} \quad \alpha = \omega(\frac{d\omega}{d\theta}) \quad \mathrm{rad/s^2}$$



RIGID-BODY MOTION: ROTATION ABOUT A FIXED AXIS (continued)

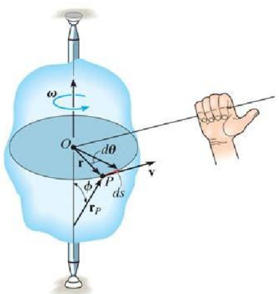


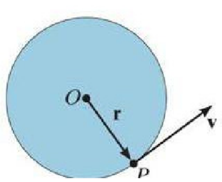
- If the angular acceleration of the body is ***constant***, $\alpha = \alpha_C$, the equations for angular velocity and acceleration can be integrated to yield the set of ***algebraic*** equations below.

$$
\begin{aligned}
\omega &= \omega_0 + \alpha_C t &(1)\\
\theta &= \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha_C t^2 &(2)\\
\omega^2 &= \omega_0^2 + 2\alpha_C(\theta - \theta_0) &(3)
\end{aligned}
$$

where θ_0 and ω_0 are the initial values of the body's angular position and angular velocity. Note these equations are very similar to the constant acceleration relations developed for the ***rectilinear*** motion of a particle.

RIGID-BODY ROTATION: VELOCITY OF POINT P



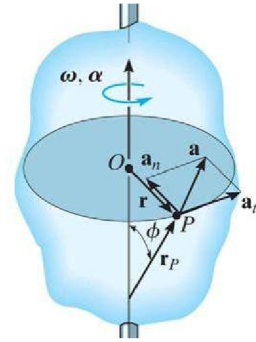


- The magnitude of the velocity of P is equal to ωr (the text provides the derivation). The velocity's direction is tangent to the circular path of P.
- In the ***vector*** formulation, the magnitude and direction of $\boldsymbol{v}$ can be determined from the cross product of $\boldsymbol{\omega}$ and $\boldsymbol{r}_p$. Here $\boldsymbol{r}_p$ is a vector from any point on the axis of rotation to P.

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}_p = \boldsymbol{\omega} \times \boldsymbol{r}$$

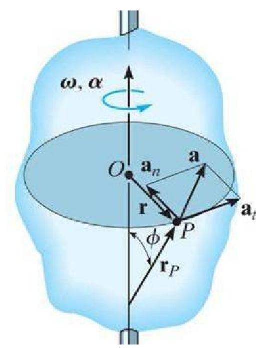
- The direction of $\boldsymbol{v}$ is determined by the right-hand rule.

RIGID-BODY ROTATION: ACCELERATION OF POINT P



- The acceleration of P is expressed in terms of its ***normal*** (a_n) and ***tangential*** (a_t) components. In scalar form, these are $a_t = \alpha r$ and $a_n = \omega^2 r$.
- The ***tangential*** component, a_t, represents the time rate of change in the velocity's ***magnitude***. It is directed tangent to the path of motion.
- The ***normal*** component, a_n, represents the time rate of change in the velocity's ***direction***. It is directed toward the ***center*** of the circular path.

RIGID-BODY ROTATION: ACCELERATION OF POINT P (continued)



- Using the vector formulation, the acceleration of P can also be defined by differentiating the velocity.

$$
\begin{aligned}
\boldsymbol{a} &= \frac{d\boldsymbol{v}}{dt} = \frac{d\boldsymbol{\omega}}{dt} \times \boldsymbol{r}_P + \boldsymbol{\omega} \times \frac{d\boldsymbol{r}_P}{dt}\\
&=
\end{aligned}
$$

- It can be shown that this equation reduces to

$$\boldsymbol{a} = \qquad = \boldsymbol{a}_t + \boldsymbol{a}_n$$

- The ***magnitude*** of the acceleration vector is

$$a = \sqrt{(a_t)^2 + (a_n)^2}$$

ROTATION ABOUT A FIXED AXIS: PROCEDURE

- Establish a ***sign convention*** along the axis of rotation.
- If a relationship is known between any ***two*** of the variables (α, ω, θ, or t), the other variables can be determined from the equations:

$$\omega = \frac{d\theta}{dt}, \quad \alpha = \frac{d\omega}{dt}, \quad \alpha d\theta = \omega d\omega$$

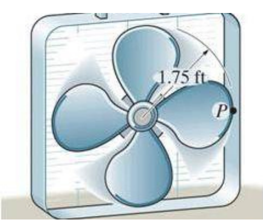
- If α is ***constant***, use the equations for constant angular acceleration.
- To determine the ***motion of a point***, the scalar equations

$$v = \omega r,\ a_t = \alpha r,\ a_n = \omega^2 r \text{, and } a = \sqrt{(a_t)^2 + (a_n)^2}$$

can be used.
- Alternatively, the vector form of the equations can be used (with $\mathbf{i}, \mathbf{j}, \mathbf{k}$ components).

$$\begin{aligned} \boldsymbol{v} &= \boldsymbol{\omega} \times \boldsymbol{r}_P = \boldsymbol{\omega} \times \boldsymbol{r} \\ \boldsymbol{a} &= \boldsymbol{a}_t + \boldsymbol{a}_n = \boldsymbol{\alpha} \times \boldsymbol{r}_P + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_P) = \boldsymbol{\alpha} \times \boldsymbol{r} - \omega^2 \boldsymbol{r} \end{aligned}$$

EXAMPLE



- Given: The motor gives the blade an angular acceleration $\alpha = 20e^{-0.6t}\mathrm{rad/s^2}$, where t is in seconds. The initial conditions are that when $t = 0$, the blade is at rest.
- Find: The velocity and acceleration of the tip P of one of the blades when $t = 3\,\mathrm{sec}$. How many revolutions has the blade turned in $3\,\mathrm{sec}$?
- Plan:
  1. Determine the angular velocity and displacement of the blade using kinematics of angular motion.
  2. The magnitudes of the velocity and acceleration of point P can be determined from the scalar equations of motion for a point on a rotating body. Why scalar?

EXAMPLE (Solution)

1. Since the ***angular acceleration*** is given as a function of time, $\alpha = 20e^{-0.6t}\,\mathrm{rad/s^2}$, the angular velocity and displacement can be found by integration.

$$\omega = \omega_0 + \int_0^t \alpha dt = 0 + \int_0^t 20e^{-0.6t} dt = \frac{100}{3}\left(1 - e^{-0.6t}\right) \quad (4)$$

when $t = 3\,\mathrm{sec}$,

Angular displacement

$$\begin{aligned} \theta &= \theta_0 + \int_0^t \omega dt = 0 + \int_0^t \frac{100}{3}\left(1 - e^{-0.6t}\right) dt \quad (5) \\ &= \frac{100}{3}\left[t - \frac{1}{0.6}\left(1 - e^{-0.6t}\right)\right] \end{aligned}$$

when $t = 3\,\mathrm{sec}$, $\qquad = 8.53\,\mathrm{rev}$.

Also , when $t = 3\,\mathrm{sec}$, $\alpha = 20e^{-0.6(3)} =$

EXAMPLE (continued)

2. ***The velocity of point*** P on the the fan, at a radius of 1.75 ft, is determined as

$$v_P = \omega r = (\qquad)(1.75) = 48.685\,\mathrm{ft/s}$$

The ***normal*** and ***tangential*** components of acceleration of point P are calculated as

$$\begin{aligned} a_n &= \omega^2 r = (\qquad)^2(1.75) = \qquad \mathrm{ft/s^2} \\ a_t &= \alpha r = (3.306)(1.75) = \qquad \mathrm{ft/s^2} \end{aligned}$$

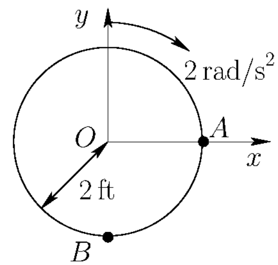
The ***magnitude*** of the acceleration of P is determined by

$$\begin{aligned} a_P &= \sqrt{(a_n)^2 + (a_t)^2} = \sqrt{(1354.41)^2 + (5.786)^2} \\ &= \end{aligned}$$

CONCEPT QUIZ

1. A disk is rotating clockwise at $4\,\mathrm{rad/s}$. If it is subjected to a constant angular acceleration of $2\,\mathrm{rad/s^2}$, determine the acceleration at B.

(a) $(+4\mathbf{i} + 32\mathbf{j})\,\mathrm{ft/s^2}$
(b) $(+4\mathbf{i} - 32\mathbf{j})\,\mathrm{ft/s^2}$
(c) $(-4\mathbf{i} + 32\mathbf{j})\,\mathrm{ft/s^2}$
(d) $(-4\mathbf{i} - 32\mathbf{j})\,\mathrm{ft/s^2}$

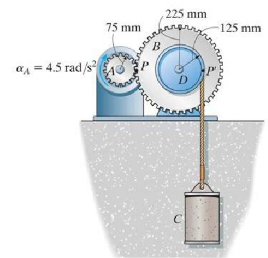


ANS: __

CONCEPT QUIZ (continued)

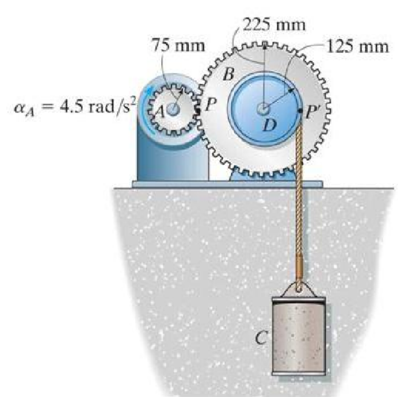
2. A Frisbee is thrown and curves to the right. It is experiencing
   (a) rectilinear translation.
   (b) curvilinear translation.
   (c) pure rotation.
   (d) general plane motion.
   ANS: __

GROUP PROBLEM SOLVING



- Given: Starting from rest when gear A is given a constant angular acceleration, $\alpha_A = 4.5\,\mathrm{rad/s^2}$. The cord is wrapped around pulley D which is rigidly attached to gear B.
- Find: The velocity of cylinder C and the distance it travels in 3 seconds.
- Plan:
  1. The angular acceleration of gear B (and pulley D) can be related to α_A.
  2. The acceleration of cylinder C can be determined by using the equations for motion of a point on a rotating body since $(a_t)_D$ at point P is the same as a_C.
  3. The velocity and distance of C can be found by using the constant acceleration equations.

GROUP PROBLEM SOLVING (Solution)



1. Gear A and B will have the ***same*** speed and tangential component of acceleration at the point where ***they mesh***. Thus,

$$a_t = \alpha_A r_A = \alpha_B r_B \quad (6)$$
$$(4.5)(75) = \alpha_B (225) \quad (7)$$
$$\alpha_B = 1.5\mathrm{rad/s^2} \quad (8)$$

Since gear B and pulley D turn together, $\alpha_D = \alpha_B =$

GROUP PROBLEM SOLVING(continued)

2. Assuming the cord attached to pulley D is inextensible and does not slip, the velocity and acceleration of cylinder C will be the same as the velocity and tangential component of acceleration along the pulley D:

$$a_C = (a_t)_D = \alpha_D r_D = (1.5)(0.125) =$$

3. Since α_A is constant, α_D and α_C will be constant. The constant acceleration equation for rectilinear motion can be used to determine the velocity and displacement of cylinder C when $t = 3\,\text{s}$ ($s_0 = v_0 = 0$):

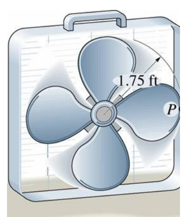
$$v_c = v_0 + a_C t = 0 + 0.1875(3) = \quad \uparrow \qquad (9)$$

$$s_c = s_0 + v_0 t + (0.5)a_C t^2$$

$$= 0 + 0 + (0.5)0.1875(3)^2 = \qquad (10)$$

ATTENTION QUIZ

1. The fan blades suddenly experience an angular acceleration of 2 rad/s^2. If the blades are rotating with an initial angular velocity of 4 rad/s, determine the speed of point P when the blades have turned 2 revolutions (when ω = 8.14 rad/s).



(a) 14.2 ft/s
(b) 17.7 ft/s
(c) 23.1 ft/s
(d) 26.7 ft/s
ANS: __

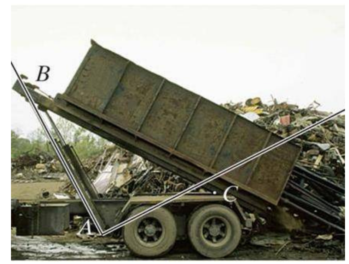
2. Determine the magnitude of the acceleration at P when the blades have turned the 2 revolutions.

(a) 0 ft/s^2
(b) 3.5 ft/s^2
(c) 115.95 ft/s^2
(d) 116 ft/s ^{2}
ANS: __

ABSOLUTE MOTION ANALYSIS

Today's objectives: Students will be able to

1. Determine the velocity and acceleration of a rigid body undergoing general plane motion using an ***absolute motion*** analysis.



In-class activities:

- Reading Quiz
- Applications
- ***General Plane*** Motion
- Concept Quiz
- Group Problem Solving
- Attention Quiz

READING QUIZ

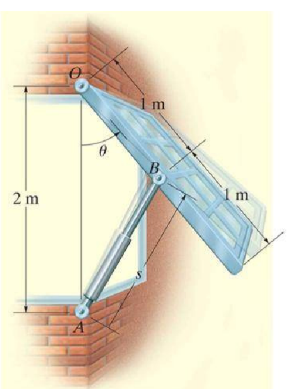
1. A body subjected to general plane motion undergoes a/an
   (a) translation.
   (b) rotation.
   (c) simultaneous translation and rotation.
   (d) out-of-plane movement.
   ANS: __

2. In general plane motion, if the rigid body is represented by a slab, the slab rotates
   (a) about an axis perpendicular to the plane.
   (b) about an axis parallel to the plane.
   (c) about an axis lying in the plane.
   (d) None of the above.
   ANS: __

APPLICATIONS

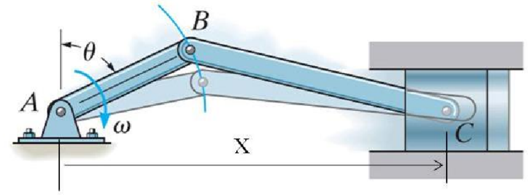
- The dumping bin on the truck rotates about a fixed axis passing through the pin at A. It is operated by the extension of the ***hydraulic cylinder BC***.
- The angular position of the bin can be specified using the ***angular*** position coordinate , and the ***position of point C*** on the bin is specified using the coordinate .
- As a part of the design process for the truck, an engineer had to relate the ***translational velocity*** at which the hydraulic cylinder extends and the resulting ***angular velocity*** of the bin.

APPLICATIONS(continued)



- The large window is opened using a hydraulic cylinder AB.
- The position B of the hydraulic cylinder rod is related to the angular position, θ, of the window.
- A designer has to relate the ***translational*** velocity at B of the hydraulic cylinder and the ***angular*** velocity and acceleration of the window. How would you go about the task?

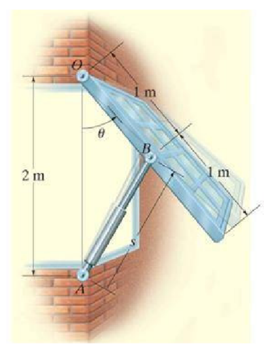
APPLICATIONS(continued)



- The position of the piston, x, can be defined as a function of the angular position of the crank, θ. By differentiating x with respect to time, the velocity of the piston can be related to the angular velocity, ω, of the crank. This is necessary when designing an engine.
- The stroke of the piston is defined as the total distance moved by the piston as the crank angle varies from 0 to $180°$. How does the length of crank AB affect the stroke?

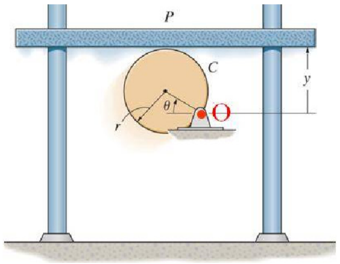
ABSOLUTE MOTION ANALYSIS (Section 16.4)

- The figure below shows the window using a hydraulic cylinder AB.



- The ***absolute motion analysis method*** relates the position of a point, B, on a rigid body undergoing rectilinear motion to the angular position, θ, of a line contained in the body.
- Once a relationship in the form of is established, the velocity and acceleration of point B are obtained in terms of the angular velocity and angular acceleration of the rigid body by taking the ***first*** and ***second time-derivatives*** of the position function.
- Usually the ***chain*** rule must be used when taking the derivatives of the position coordinate equation.

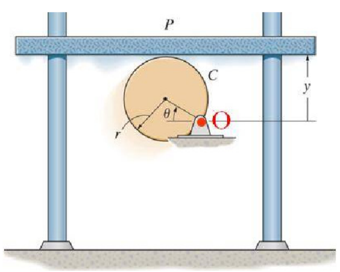
EXAMPLE I



- Given: The platform P is constrained to move vertically by the smooth vertical guides. The cam C rotates with a <u>constant</u> angular velocity, ω.
- Find: The velocity and acceleration of platform P as a function of the angle θ of cam C.
- Plan : Use the fixed reference point O and define the position of the platform, P, in terms of the parameter θ. Take successive time derivatives of the position equation to find the velocity and acceleration.

EXAMPLE I(continued)

- Solution:



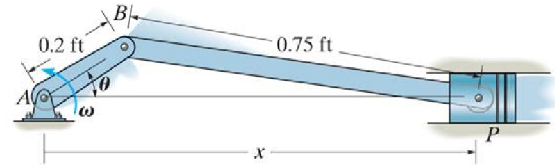
- By geometry, $y = r + r \sin\theta$
- By differentiating with respect to time,

$$v_P = r\cos\theta(\dot{\theta}) =$$

- Note that the cam rotates with a constant angular velocity, ω.
- Differentiating v_P to find the acceleration,

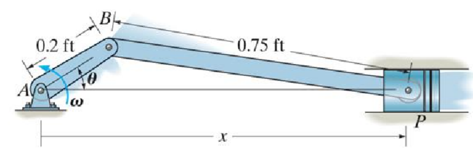
$$a_P = \frac{d(r\omega\cos\theta)}{dt} = r\omega(-\sin\theta)(\omega) =$$

EXAMPLE II



- Given: Crank AB rotates at a constant velocity of $\omega = 150\,\mathrm{rad/s}$.
- Find: The velocity of point P when $\theta = 30°$.
- Plan: Define x as a function of θ and differentiate with respect to time.

EXAMPLE II(continued)



- Solution:

$$\begin{aligned} x_P &= 0.2\cos\theta + \sqrt{(0.75)^2 - (0.2\sin\theta)^2} && (11) \\ v_P &= -0.2\omega\sin\theta + (0.5)[(0.75)^2 - (0.2\sin\theta)^2]^{-0.5} \\ & \quad \times(-2)(0.2\sin\theta)(0.2\cos\theta)\omega \\ &= -0.2\omega\sin\theta - \frac{[0.5(0.2)^2\sin 2\theta\omega]}{\sqrt{(0.75)^2 - (0.2\sin\theta)^2}} && (12) \end{aligned}$$

- At $\theta = 30°$, $\omega = 150\,\mathrm{rad/s}$ and

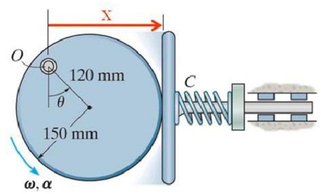
$$v_P =$$

CHECK YOUR UNDERSTANDING QUIZ

1. The position, s, is given as a function of angular position, θ, as $s = 10\sin 2\theta$. The velocity, v, is
   (a) $20\cos 2\theta$
   (b) $20\sin 2\theta$
   (c) $20\omega\cos 2\theta$
   (d) $20\omega\sin 2\theta$
   ANS: __
2. If $s = 10\sin 2\theta$, the acceleration, a, is
   (a) $20\alpha\sin 2\theta$
   (b) $20\alpha\cos 2\theta - 40\omega^2\sin 2\theta$
   (c) $20\alpha\cos 2\theta$
   (d) $-40\alpha\sin 2\theta$
   ANS: __

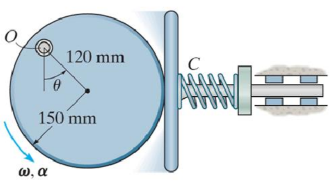
GROUP PROBLEM SOLVING (continued)

- Solution:



- Relate x, the distance between O and the plate, to θ.
  $$x = 120\sin\theta + 150(\text{ mm})$$
- Take time derivatives of the position to find the velocity and acceleration.
- $v_C = dx/dt = 120\cos\theta(\dot{\theta}) = \qquad (\text{mm/s})$
  $a_C = dv_C/dt = \qquad (\text{mm/s}^2)$
- When $\theta = 30°, \omega = 4\,\text{rad/s}$, and $\alpha = 2\text{rad/s}^2$. Substituting,
  $v_C = 120(4)\cos 30° = \qquad \text{mm/s}(\rightarrow)$
  $a_C = 120(2)\cos 30° + 120(4^2)(-\sin 30°) = \qquad \text{mm/s}^2(\leftarrow)$

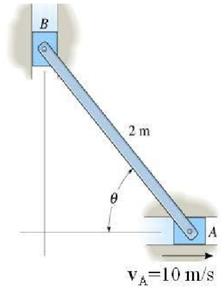
GROUP PROBLEM SOLVING



- Given: The circular cam is rotating about the fixed point O with $\omega = 4\,\text{rad/s}$, $\alpha = 2\,\text{rad/s}^2$ with the dimensions shown.
- Find: The velocity and acceleration of the plate when $\theta = 30°$.
- Plan: Set the coordinate x to be the distance between O and the plate. Relate x to the angular position, θ. Then take time derivatives of the position to find the velocity and acceleration.

ATTENTION QUIZ

1. The sliders shown below are confined to move in the horizontal and vertical slots. If $v_A = 10\,\text{m/s}$, determine the connecting bar's angular velocity when $\theta = 30°$.



(a) 10rad/s ↺
(b) 10rad/s ↻
(c) 8.7rad/s ↻
(d) 8.7rad/s ↺
ANS: __