REAL ANALYSIS NOTES

Math 401

Bridgewater State University

Review of Basic Proof Techniques

Recall that

- A **statement** is a sentence which has a true or false value
- An **implication** is a statement of the type `**If P then Q'** where P and Q are two given statements.

Address the following:

1. Give an example of a sentence which is not a statement.

2. Give an example of a sentence which is a statement.

3. Given an example of a statement which is an implication

Direct Proof

In order to prove an implication of the type if P then Q holds by a direct proof, we assume that P holds, and use the fact that P is true to derive that Q is true as well.

Your notes here

Exercise

Prove (using the direct proof method) that if n is an odd integer, then

$$4n^3 + 2n - 1 \text{ is odd.}$$

Exercise

Let n be a natural number. Prove that if $n + \frac{1}{n} < 2$ then $n^2 + \frac{1}{n^2} < 4$.

Proof by contrapositive

We recall that the contrapositive of the implication if P then Q is the statement if not Q then not P. Recall also that an implication and its contrapositive are logically equivalent. In other words, an implication and its contrapositive have the same truth value. Thus, an implication is true if and only if its contrapositive is true as well. Now, a proof by contrapositive is simply a direct proof of the contrapositive of the given statement. Now, let us consider our example: If x and y are two consecutive integers then $x + y$ is odd. We shall now prove that this statement is true by using a proof by contrapositive.

Your notes here

Exercise

Let x be an integer. Prove that if $5x - 7$ is even then x is odd.

Exercise

Let x be an integer. Prove that if $5x - 7$ is odd, then $9x + 2$ is even.

Proof by contradiction

In order to prove that an implication of the type if P then Q is true by contradiction, we assume that P and the negation of Q both hold, and we derive a contradiction as a consequence of our assumption.

Notes

Exercise

Prove that if $(n+1)^2 - 1$ is even then n is even.

Exercise

Prove that $2n^2 + n$ is odd if and only if $\cos\dfrac{n\pi}{2}$ is even

Warmup Exercises (try these at home)

1. Write a direct proof for the following statement: for every integer x and for every integer y, if x is odd and y is odd then xy is odd.
2. Write a proof by contradiction for the following statement: for every integer x and for every integer y, if x is odd and y is odd then xy is odd.
3. Prove by contradiction that $\sqrt{2}$ is irrational. (Hint: assume that $\sqrt{2}$ is rational.)
4. Prove that for every integer x, $x+4$ is odd if and only if $x+7$ is even. (This is a biconditional statement: you must prove that if $x+4$ is odd then $x+7$ is even and if $x+7$ is even then $x+4$ is odd.)

Chapter 1

The set of Natural Numbers

We denote the set $\{1, 2, 3, \ldots\}$ of all *natural numbers* by $\mathbb{N}$. Elements of $\mathbb{N}$ will also be called *positive integers*. Each natural number n has a successor, namely $n+1$. Thus the successor of 2 is 3, and 37 is the successor of 36. You will probably agree that the following properties of $\mathbb{N}$ are obvious; at least the first four are.

N1. 1 belongs to $\mathbb{N}$.
N2. If n belongs to $\mathbb{N}$, then its successor $n+1$ belongs to $\mathbb{N}$.
N3. 1 is not the successor of any element in $\mathbb{N}$.
N4. If n and m in $\mathbb{N}$ have the same successor, then $n = m$.
N5. A subset of $\mathbb{N}$ which contains 1, and which contains $n+1$ whenever it contains n, must equal $\mathbb{N}$.

(Taken from Elementary Analysis
by Kenneth A. Ross)

Exercise

Appealing to Peano's postulates prove the following

1. The number four is not equal to the number one.
2. The number five is not equal to the number four.

Question

Can we derive Axiom N5 from Axiom N1 through Axiom N4?

Answer here

Axiom N5 is the basis of *mathematical induction*. Let $P_1, P_2, P_3, \ldots$ be a list of statements or propositions that may or may not be true. The principle of mathematical induction asserts that all the statements $P_1, P_2, P_3, \ldots$ are true provided

(I_1) P_1 **is true,**
(I_2) P_{n+1} is true whenever P_n is true.

We will refer to (I_1), i.e., the fact that P_1 is true, as the *basis for induction* and we will refer to (I_2) as the *induction step*. For a sound proof based on mathematical induction, properties (I_1) and (I_2) must both be verified. In practice, (I_1) will be easy to check.

(Taken from Elementary Analysis by Kenneth A. Ross)

<u>Your notes here</u>

Exercise 1

(Taken from Elementary Analysis by Kenneth A. Ross)

Prove $1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$ for natural numbers n.

Exercise 2

(Taken from Elementary Analysis by Kenneth A. Ross)

All numbers of the form $7^n - 2^n$ are divisible by 5.

Exercise 3

(Taken from Elementary Analysis by Kenneth A. Ross)

Show that $|\sin nx| \leq n|\sin x|$ for all natural numbers n and all real numbers x.

Section 2

The Set of Rational Numbers

The set of all **integers** is defined as

$$\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}.$$

The set of all **rational numbers** is given by

$$\mathbb{Q} = \left\{\frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0\right\}.$$

Example 1 *Explain why* $\frac{2}{3} \in \mathbb{Q}$.

Example 2 *Explain why* $\sqrt{2} \notin \mathbb{Q}$.

Problem 3 1. *Can you find other examples of numbers which are not rational. List them here and explain your work*

2. *Prove that the number*

$$0.23232323\cdots$$

is a rational number.

3. *Prove that*

$$0.123123123\cdots$$

is a rational number.

Algebraic Numbers
Taken from Elementary Analysis, Kenneth A. Ross

A number is called an *algebraic number* if it satisfies a polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

where the coefficients $a_0, a_1, \ldots, a_n$ are integers, $a_n \neq 0$ and $n \geq 1$.

Exercise 1. Can you explain why any **rational** number is an **algebraic number**?

Exercise 2. Prove that $\sqrt{\sqrt{2}}$ is an algebraic number.

Exercise 3. Prove that

$$\sqrt{\frac{1-2\sqrt{3}}{5}}$$

is an algebraic number.

Rational Zeros Theorem
Taken from Elementary Analysis, Kenneth A. Ross

Suppose that $a_0, a_1, \ldots, a_n$ are integers and that r is a rational number satisfying the polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0 \qquad (1)$$

where $n \geq 1$, $a_n \neq 0$ and $a_0 \neq 0$. Write $r = \frac{p}{q}$ where p, q are integers having no common factors and $q \neq 0$. Then q divides a_n and p divides a_0.

Proof

Problem 4 *Consider the polynomial equation*

$$x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0 = 0$$

where the coefficients of the polynomials are integers and c_0 *is not equal to zero. Any rational solution must be an integer that divides* c_0. *Can you explain why this is the case?*

Answer

Exercise 5 *Prove that* $\sqrt{5} \notin \mathbb{Q}$.

Exercise 6 *Prove that* $\sqrt{2+\sqrt[3]{5}}$ *is not a rational number.*

Section 3
The Set $\mathbb{R}$ of all Real Numbers

- We equip the real numbers with two operations: addition and multiplication.
- The following sets of axioms will be exploited to derive fundamental well-known facts about real numbers.

Field structure

The set $\mathbb{Q}$ endowed with addition and multiplication forms a structure known as a **field**. We shall list below the axioms that turns the rationals together with addition and multiplication into a field

A1. $a + (b + c) = (a + b) + c$ for all a, b, c.
A2. $a + b = b + a$ for all a, b.
A3. $a + 0 = a$ for all a.
A4. For each a, there is an element $-a$ such that $a + (-a) = 0$.
M1. $a(bc) = (ab)c$ for all a, b, c.
M2. $ab = ba$ for all a, b.
M3. $a \cdot 1 = a$ for all a.
M4. For each $a \neq 0$, there is an element a^{-1} such that $aa^{-1} = 1$.
DL $a(b + c) = ab + ac$ for all a, b, c.

The Theory of Calculus, Kenneth Ross (Taken from Elementary Analysis by Kenneth A. Ross)

The set of rationals also has an **ordering structure** which is described

Order Structure

via the following axioms

O1. Given a and b, either $a \leq b$ or $b \leq a$.
O2. If $a \leq b$ and $b \leq a$, then $a = b$.
O3. If $a \leq b$ and $b \leq c$, then $a \leq c$.
O4. If $a \leq b$, then $a + c \leq b + c$.
O5. If $a \leq b$ and $0 \leq c$, then $ac \leq bc$.

The Theory of Calculus, Kenneth Ross (Taken from Elementary Analysis by Kenneth A. Ross)

The set of real numbers together with the axioms above forms a structure which we call an **ordered field**. The first set of axioms gives us a field structure, and the second set of axioms provides an ordering of the real numbers.

Let b be a real number satisfying the equation $a + b = 0$. Then b is unique, we call it the **opposite** of a and it is denoted $-a$.

Exercise Using the axioms above, prove the uniqueness of the opposite of a real number.

Let a be a nonzero real number. The real number b satisfying the equation $ba = 1$ is called the **inverse** of a. This number is unique and is denoted by $\frac{1}{a}$ or a^{-1}.

Exercise Using the axioms above, prove the uniqueness of the opposite of a if a is a nonzero real number.

We say that a number a is **positive** if $a > 0$. Next, we say that $a > b$ if and only if $a - b$ is positive. If $a < b$ or $a = b$ we say that a is less than or equal to b and we write $a \leq b$.

The following results are taken from your textbook

3.1 Theorem.
The following are consequences of the field properties:

(i) $a + c = b + c$ *implies* $a = b$;
(ii) $a \cdot 0 = 0$ *for all* a;
(iii) $(-a)b = -ab$ *for all* a, b;
(iv) $(-a)(-b) = ab$ *for all* a, b;
(v) $ac = bc$ *and* $c \neq 0$ *imply* $a = b$;
(vi) $ab = 0$ *implies either* $a = 0$ *or* $b = 0$;
for $a, b, c \in \mathbb{R}$.

3.2 Theorem.
The following are consequences of the properties of an ordered field:

(i) *if* $a \leq b$, *then* $-b \leq -a$;
(ii) *if* $a \leq b$ *and* $c \leq 0$, *then* $bc \leq ac$;
(iii) *if* $0 \leq a$ *and* $0 \leq b$, *then* $0 \leq ab$;
(iv) $0 \leq a^2$ *for all* a;
(v) $0 < 1$;
(vi) *if* $0 < a$, *then* $0 < a^{-1}$;
(vii) *if* $0 < a < b$, *then* $0 < b^{-1} < a^{-1}$;
for $a, b, c \in \mathbb{R}$.

The Theory of Calculus, Kenneth Ross (Taken from Elementary Analysis by Kenneth A. Ross)

Proof of Theorem 3.1 (i)

Proof of Theorem 3.1 (ii)

Proof of Theorem 3.1 (iii)

Proof of Theorem 3.2 (i)

Proof of Theorem 3.2 (ii)

Proof of Theorem 3.2 (iii)

Proof of Theorem 3.2 (iv)

Proof of Theorem 3.2 (v)

Proof of Theorem 3.2 (vi)

Proof of Theorem 3.2 (vii)

3.3 Definition.
We define

$$|a| = a \quad \text{if} \quad a \geq 0 \qquad \text{and} \qquad |a| = -a \quad \text{if} \quad a \leq 0.$$

$|a|$ is called the *absolute value of* a.

3.4 Definition.
For numbers a and b we define $\text{dist}(a, b) = |a - b|$; $\text{dist}(a, b)$ represents the *distance between* a *and* b.

Taken from the Theory of Calculus, Kenneth Ross

Notes

Taken from Theory of Calculus, Kenneth Ross

3.5 Theorem.

(i) $|a| \geq 0$ *for all* $a \in \mathbb{R}$.

(ii) $|ab| = |a| \cdot |b|$ *for all* $a, b \in \mathbb{R}$.

(iii) $|a + b| \leq |a| + |b|$ *for all* $a, b \in \mathbb{R}$.

3.6 Corollary.
$\text{dist}(a, c) \leq \text{dist}(a, b) + \text{dist}(b, c)$ *for all* $a, b, c \in \mathbb{R}$.

3.7 Triangle Inequality.
$|a + b| \leq |a| + |b|$ *for all* a, b.

Proof of Theorem 3.5

Warm-up Exercises (attempt these exercises at home)

Show that $||a| - |b|| \leq |a - b|$, for all $a, b \in \mathbb{R}$. (Taken from Elementary Analysis by Kenneth A. Ross)

Show that for every $M > 0$, $|a| < M$ if and only if $-M < a < M$.

(Taken from Elementary Analysis by Kenneth A. Ross)

Show that if $a \leq b + \varepsilon$ for every $\varepsilon > 0$, then $a \leq b$.

(Taken from Elementary Analysis by Kenneth A. Ross)

Section 4
The Completeness Axiom

Definition 1 *Let S be a non-empty subset of $\mathbb{R}$*

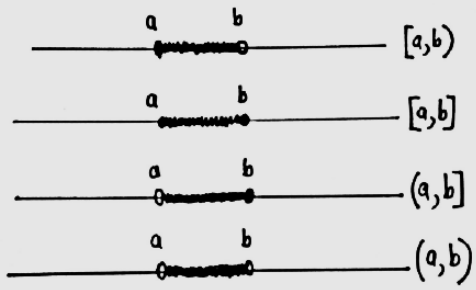
1. *We say that $M = \max S$ (the* ***maximum*** *of S) if*
   (a) $M \in S$
   (b) *For any $s \in S$*
   $$s \leq M$$
2. *We say that $m = \min S$ (the* ***minimum*** *of S) if*
   (a) $m \in S$
   (b) *For any $s \in S$*
   $$m \leq s$$

Exercise

1. Give an example of a subset of $\mathbb{R}$ which has a maximum and a minimum.
2. Give an example of a subset of $\mathbb{R}$ which as a maximum and no minimum.
3. Give an example of a subset of $\mathbb{R}$ which as no maximum but has a minimum.
4. Given an example of a subset of $\mathbb{R}$ which has no maximum and no minimum.

We recall that

$$[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$$
$$[a,b) = \{x \in \mathbb{R} : a \leq x < b\}$$
$$(a,b) = \{x \in \mathbb{R} : a < x < b\}$$
$$(a,b] = \{x \in \mathbb{R} : a < x \leq b\}$$



Write the following set using interval notation

$$A = \left((1,2] \cup \left(\frac{1}{2},3\right)\right) \cap (-\infty,0].$$

Write the following set using interval notation

$$\bigcap_{n=1}^{\infty}[-\frac{1}{n},1+\frac{1}{n}]$$

Example 2 *Complete the following tables and justify your answer*

Set S	$\{1, 2, \cdots, 10\}$	$\left(1, \sqrt{2}\right]$	$\left(1, \sqrt{2}\right)$	$\left[1, \sqrt{2}\right]$	$\left[0, \sqrt{2}\right] \cap \mathbb{Q}$	$\left\{n^{(-1)^n} : n \in \mathbb{N}\right\}$
max S						
min S						

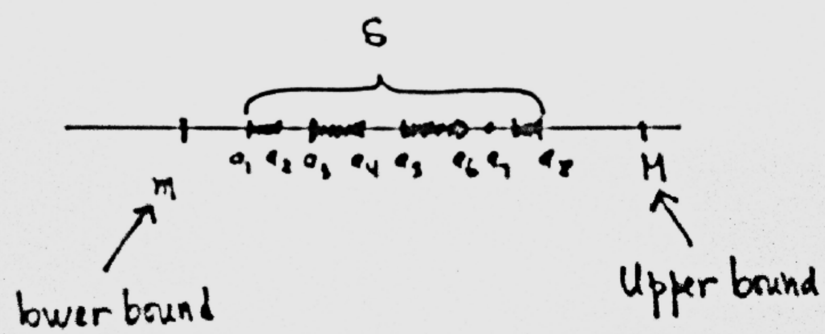
Notes here

Definition 4 *Let S be a non-empty subset of the reals.*

1. *If there exists a real number M such that for any s in S, $s \leq M$ we say that the set S is* ***bounded*** *from above.*

2. *If there exists a real number m such that for any s in the set S, $m \leq s$ we say that the set S is* ***bounded from below***.

3. *We say that a set S is* ***bounded*** *if the set is bounded below and above by some real numbers m and M respectively. More precisely, a set S is bounded if there exist real numbers m and M such that $S \subset [m, M]$.*



Notes here

Example 5 *Decide if the following sets are bounded or not.*

1. $S = [1, \sqrt{2}] \cap \mathbb{Q}$
2. $S = \mathbb{N}$.
3. *The rationals*
4. $\left\{ (n)^{(-1)^n} : n \in \mathbb{N} \right\}$

Answer here

Definition 6 *(supremum) Let S be a non-empty subset of the reals. If S is bounded above and has a least upper bound, then we call it the* ***supremum*** *of S and it is denoted by* $\sup S$

Definition 7 *(infimum) Let S be a non-empty subset of the reals. If S is bounded below and has a greatest lower bound, then we call this number the* ***infimum*** *of S and it is denoted by* $\inf S$

Example 8 *Complete the following tables and justify your answer*

Set S	$\{1, 2, \cdots, 10\}$	$\left(1, \sqrt{2}\right]$	$\left(1, \sqrt{2}\right)$	$\left[1, \sqrt{2}\right]$	$\left[0, \sqrt{2}\right] \cap \mathbb{Q}$	$\left\{n^{(-1)^n} : n \in \mathbb{N}\right\}$
sup S						
inf S						

Completeness Axiom Every nonempty subset S of the reals that is bounded above has a least upperbound. In other words, the supremum of S exists and is a real number.

Example 9 *Using the **Completeness Axiom** prove that every nonempty subset S of the real which is bounded below has a greatest lower bound:* $\inf S$.

Proof

Theorem 10 *(Archimedean Property) If a, b are positive numbers then there exists a positive integer n such that $na > b$.*

Proof

Theorem 11 *(Denseness of $\mathbb{Q}$) Let a, b be two real numbers such that a is less than b then there is a rational number r such that $a < r < b$.*

Proof

Chapter 2

Section 7 Limits of a Sequence

A Sequence is a function whose domain is a subset of the integers or a subset of the set of natural numbers. We shall generally, regard sequences as functions defined over the natural numbers. As such, we shall adopt the following notation for sequences $(s_n)_{n\in\mathbb{N}}$ or $(s_n)_{n=1}^{\infty}$

Question

Write down an example of a sequence

As a starting point, let us consider the following toy example. Let $(s_n)_{n\in\mathbb{N}}$ be a sequence such that

$$s_n = \frac{n+1}{n}.$$

Calculating the values of

$$\frac{n+1}{n} \text{ and } \left|\frac{n+1}{n} - 1\right|,$$

for a few natural numbers n, we obtain the following table

n	$\frac{n+1}{n}$	$\left\lvert\frac{n+1}{n} - 1\right\rvert$
1	2	1
101	$102/101 = 1.0099$	0.00990099
201	$202/201 = 1.00498$	0.00497512
301	$302/301 = 1.00332$	0.00332226
401	$402/401 = 1.00249$	0.00249377
501	$502/501 = 1.002$	0.00199601
601	$602/601 = 1.00166$	0.00166389
701	$702/701 = 1.00143$	0.00142653
801	$802/801 = 1.00125$	0.00124844

Question

What do you observe? Base on your observation are you able to make a conjecture?

n	$\frac{n+1}{n}$	$\left\lvert\frac{n+1}{n}-1\right\rvert$
1	2	1
101	$102/101 = 1.0099$	0.00990099
201	$202/201 = 1.00498$	0.00497512
301	$302/301 = 1.00332$	0.00332226
401	$402/401 = 1.00249$	0.00249377
501	$502/501 = 1.002$	0.00199601
601	$602/601 = 1.00166$	0.00166389
701	$702/701 = 1.00143$	0.00142653
801	$802/801 = 1.00125$	0.00124844

. seems to suggest that as n is getting larger, the corresponding quantity

$$\frac{n+1}{n}$$

is getting closer to 1, and

$$\left\lvert\frac{n+1}{n}-1\right\rvert$$

is getting closer to zero. This is a rather imprecise and intuitive description of what we are observing. Moreover, this table does not tell us anything about the terms of the sequence when n is greater than 801. So, we are not actually certain that our description is correct. Through a series our questions, we will exploit this particular example to capture in a very precise way, the concept of **convergence** of a sequence.

Problem 1 *What does the quantity* $\left\lvert\frac{n+1}{n}-1\right\rvert$ *represent?*

Answer

Problem 2 *For which natural numbers n is the following true:* $\left\lvert\frac{n+1}{n}-1\right\rvert < \frac{1}{10}$?

Answer

Problem 3 *For which natural numbers n is the following true:* $\left|\frac{n+1}{n} - 1\right| < \frac{1}{100}$?

Answer

Problem 4 *In general, given any* $\epsilon > 0$, *is it possible to find a natural number n such that* $\left|\frac{n+1}{n} - 1\right| < \epsilon$.

Answer

Problem 5 *Without solving an inequality, can you now find at least one natural number n such that the distance between* s_n *and* 1 *is less than* $\frac{2}{99}$?

Answer

Problem 6 *Is it safe to assert that given any positive number* $\epsilon > 0$ *there exists a number* $N > 0$ *such that if* $n > N$ *then*

$$\left|\frac{n+1}{n} - 1\right| < \epsilon?$$

Explain your answer

Answer

Problem 7

1. *Can we conclude that as n gets large, the* ***distance*** *between* $s_n = \frac{n+1}{n}$ *and* 1 *becomes arbitrarily small?*

2. *We are interested in capturing the concept that as n gets large, the distance between the terms* s_n *and s becomes arbitrarily small. In other words, we would like to turn this imprecise statement into a formal and rigorous statement. Write down how you would formalize this concept. Do not be discouraged if you struggle with this, it took almost two centuries of grappling with the concept before a precise definition was formulated.*

Answer

Problem 8 *Let*

$$s_n = \frac{2n-3}{5n+1}.$$

Part i *Given any natural number is it possible to find a positive number N such that the distance between* $\frac{2n-3}{5n+1}$ *and* $\frac{2}{5}$ *is less than* ϵ *whenever* $n > N$*?*

Part ii *Can the distance between* s_n *and* $2/5$ *be made arbitrarily small?*

Answer

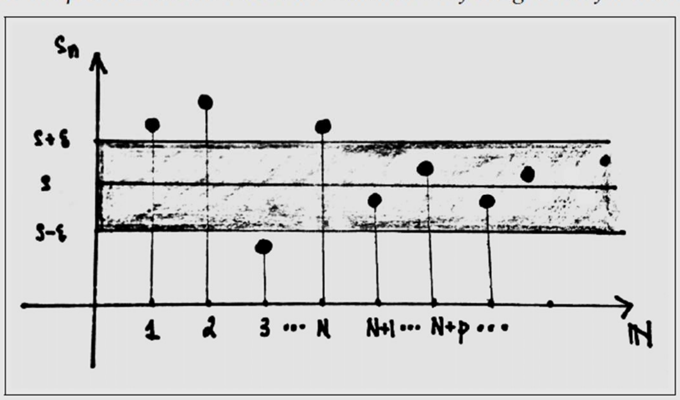
Definition 9 *(Taken from Elementary Analysis, Kenneth Ross)*

> A sequence (s_n) of real numbers is said to *converge* to the real number s provided that
>
> for each $\epsilon > 0$ there exists a number N such that
> $n > N$ implies $|s_n - s| < \epsilon$.
>
> If (s_n) converges to s, we will write $\lim_{n\to\infty} s_n = s$, or $s_n \to s$. The number s is called the *limit* of the sequence (s_n). A sequence that does not converge to some real number is said to *diverge*.

Problem 10 *Answer the following questions*

1. *Explain why the number N stated in the definition above can be taken to be a natural number?*

2. *Is the picture below an accurate illustration of the given definition?*



Problem 11 *Prove that the sequence* $\left((-1)^n\right)_{n\in\mathbb{N}}$ *is **not** convergent.*

Answer

Practicing epsilon-delta proofs

Exercise 1 *Using the ϵ-N definition, give a formal proof for the following statement*

$$\lim_{n\to\infty} \frac{2n+5}{n} = 2.$$

Scrap work

Proof

Exercise 2 *(Your turn) Using the ϵ-N definition, give a formal proof for the following statement*

$$\lim_{n\to\infty} \frac{5n+2}{7n} = \frac{5}{7}.$$

Scrap work

Proof

Exercise 3 *Using the ϵ-N definition, give a formal proof for the following statement*

$$\lim_{n\to\infty} \frac{2n+1}{5n+2} = \frac{2}{5}.$$

Scrap work

Proof

Exercise 4 *(Your turn) Using the ϵ-N definition, give a formal proof for the following statement*

$$\lim_{n\to\infty} \frac{5n+7}{3n+1} = \frac{5}{3}.$$

Scrap work

Proof

Exercise 5 *Using the* ϵ*-N definition, give a formal proof for the following statement*

$$\lim_{n\to\infty} \frac{n^2+2n+1}{n^2+n+2} = 1.$$

Scrap work

Proof

Exercise 6 *(Your turn) Using the ϵ-N definition, give a formal proof for the following statement*

$$\lim_{n\to\infty} \frac{2n^2+3n+2}{3n^2+7n+1} = \frac{2}{3}.$$

Scrap work

Proof

Section 8

Writing Correct ϵ-N Proofs

Exercise 1 *Using the ϵ-N definition, give a formal proof for the following statement*

$$\lim_{n\to\infty} \frac{2n+5}{n} = 2.$$

Scrap work

Proof

Exercise 2 *(Your turn) Using the ϵ-N definition, give a formal proof for the following statement*

$$\lim_{n\to\infty} \frac{5n+2}{7n} = \frac{5}{7}.$$

Scrap work

Proof

Exercise 3 *Using the ϵ-N definition, give a formal proof for the following statement*

$$\lim_{n\to\infty} \frac{2n+1}{5n+2} = \frac{2}{5}.$$

Scrap work

Proof

Exercise 4 *(Your turn) Using the ϵ-N definition, give a formal proof for the following statement*

$$\lim_{n \to \infty} \frac{5n+7}{3n+1} = \frac{5}{3}.$$

Scrap work

Proof

Exercise 5 *Using the ϵ-N definition, give a formal proof for the following statement*

$$\lim_{n\to\infty} \frac{n^2+2n+1}{n^2+n+2} = 1.$$

Scrap work

Proof

Exercise 6 *(Your turn) Using the ϵ-N definition, give a formal proof for the following statement*

$$\lim_{n \to \infty} \frac{2n^2 + 3n + 2}{3n^2 + 7n + 1} = \frac{2}{3}.$$

Scrap work

Proof

Chapter 8 A Discussion about Proofs

Example 1 *Prove that*

$$\lim \frac{4n^3 + 3n}{n^3 - 6} = 4.$$

Scrap work

Proof

Example 2 *(Your turn) Prove that*

$$\lim \frac{7n^5 + n^2}{9n^5 - 1} = \frac{7}{9}.$$

Scrap work

Proof

Example 3 *Let* $(s_n)_{n \in \mathbb{N}}$ *be a sequence of nonnegative real numbers such that* $\lim s_n = s$. *Prove that*

$$\lim \sqrt{s_n} = \sqrt{s}.$$

Scrap work

Proof

Example 4 *Let* $(s_n)_{n\in\mathbb{N}}$ *be a sequence in* $\mathbb{R}$*. Prove that*

$$\lim s_n = 0 \textit{ if and only if } \lim |s_n| = 0.$$

<u>Proof</u>

Section 9: Limit Theorems for Sequences

A sequence $(s_n)_{n\in\mathbb{N}}$ of real numbers is said to be bounded if the set

$$\{s_n : n \in \mathbb{N}\}$$

is a bounded set. In other words, there exists a constant M such that

$$|s_n| \leq M \text{ for all } n \in \mathbb{N}.$$

Exercise 1 *Let $(s_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that*

$$s_n = \frac{n^3+1}{n^3}$$

1. *Prove that $(s_n)_{n\in\mathbb{N}}$ is a convergent sequence.*
2. *Prove that $(s_n)_{n\in\mathbb{N}}$ is a bounded sequence.*

Exercise 2 *Can you come up with an example of a convergent sequence which is not bounded?*

Theorem 3 *Convergent sequences are bounded.*

Theorem 4 *If* $(s_n)_{n\in\mathbb{N}}$ *is convergent to* s *and* $(t_n)_{n\in\mathbb{N}}$ *converges to* t*, prove that*

$$\lim(s_n + t_n) = s + t.$$

Theorem 5 *If* $(s_n)_{n\in\mathbb{N}}$ *is convergent to* s *and* $(t_n)_{n\in\mathbb{N}}$ *converges to* t*, prove that*

$$\lim (s_n \cdot t_n) = s \cdot t.$$

Theorem 6 *If* $(s_n)_{n \in \mathbb{N}}$ *is convergent to* s*, if* $s_n \neq 0$ *for all* n *and if* $s \neq 0$*, then* $(1/s_n)_{n \in \mathbb{N}}$ *converges to* $1/s$*.*

Theorem 7 *Suppose that $(s_n)_{n\in\mathbb{N}}$ is convergent to s and $(t_n)_{n\in\mathbb{N}}$ converges to t. If $s_n \neq 0$ for all n and if $s \neq 0$, then*

$$\lim \left(\frac{t_n}{s_n}\right) = \frac{t}{s}.$$

Example 8 *Prove the following results*

1. *Assume that $p > 0$*
$$\lim_{n\to\infty} \left(\frac{1}{n^p}\right) = 0.$$
2. *Assume that $|a| < 1$. Then*
$$\lim_{n\to\infty} (a^n) = 0.$$
3. $\lim_{n\to\infty} \left(n^{1/n}\right) = 1$
4. *Assume that $a > 0$. Then*
$$\lim_{n\to\infty} \left(a^{1/n}\right) = 1.$$

Example 9 *Let* $\lim_{n\to\infty} n^2 + 1$.

1. *Let* $M = 100$. *Find a positive number* N *such that if* $n > N$ *then* $n^2 + 1 > 100$
2. *Let* $M = 1000$. *Find a positive number* N *such that if* $n > N$ *then* $n^2 + 1 > 1000$.
3. *Let* $M > 0$. *Find a positive number* N *such that if* $n > N$ *then* $n^2 + 1 > M$.

Definition 10 *For a sequence* $(s_n)_{n\in\mathbb{N}}$, *we write that*

$$\lim s_n = \infty$$

provided that for each $M > 0$ *there is a number* N *(which may depend on* M*) such that if* $n > N$ *then* $s_n > M$.

Definition 11 *For a sequence* $(s_n)_{n\in\mathbb{N}}$, *we write that*

$$\lim s_n = -\infty$$

provided that for each $M < 0$ *there is a number* N *(which may depend on* M*) such that if* $n > N$ *then* $s_n < M$.

Example 12 *Prove formally that*

$$\lim \frac{n^5+1}{n+2} = \infty.$$

Example 13 *(your turn) Prove formally that*

$$\lim \frac{n^4+1}{n^3+5} = \infty.$$

Theorem 14 *Let* $(s_n)_{n\in\mathbb{N}}$ *and* $(t_n)_{n\in\mathbb{N}}$ *be sequences such that*

$$\lim s_n = \lim t_n = \infty.$$

Prove that

$$\lim (s_n t_n) = \infty$$

Chapter 10 Monotone Sequences and Cauchy Sequences

- A sequence (s_n) of real numbers is called a **nondecreasing** sequence if

$$s_n \leq s_{n+1} \text{ for all } n$$

- A sequence (s_n) of real numbers is called a **nonincreasing** sequence if

$$s_n \geq s_{n+1} \text{ for all } n$$

- A sequence that is nondecreasing or nonincreasing will be called a **monotone sequence** or a **monotonic sequence.**

Notes

Theorem 1 *All bounded monotone sequences are convergent.*

Proof

Theorem 2 1. *If (s_n) is an unbounded nondecreasing sequence, then*

$$\lim s_n = \infty$$

2. *If (s_n) is an unbounded nonincreasing sequence, then*

$$\lim s_n = -\infty$$

Corollary 3 *If (s_n) is a monotone sequence, then the sequence either converges, diverges to $\pm\infty$. Thus $\lim s_n$ is always meaningful.*

Limsup and Liminf

Let (s_n) be a sequence of real numbers.

- The limsup of (s_n) is defined as

$$\limsup s_n = \lim_{N\to\infty} \left(\sup \{s_n : n > N\}\right)$$

- The liminf of (s_n) is defined as

$$\liminf s_n = \lim_{N\to\infty} \left(\inf \{s_n : n > N\}\right)$$

Notes

Example 4 *Let* (s_n) *be a sequence of real numbers such that*

$$s_n = \cos\left(\frac{2n\pi}{3}\right)$$

Find $\limsup s_n$ *and* $\liminf s_n$

Solutions

Theorem 6 *Let (s_n) be a sequence of real numbers.*

1. *If* $\lim s_n$ *is defined, then*

$$\liminf s_n = \lim s_n = \limsup s_n.$$

2. *If*

$$\liminf s_n = \limsup s_n$$

then $\lim s_n$ *is defined and*

$$\liminf s_n = \lim s_n = \limsup s_n.$$

Homework (try to solve this problems on your own)

The sequence $(x_n)_{n\in N}$ *is given as follows.*

a) $x_n = (-1)^{n+1}\left(3 + \frac{2}{n}\right)$; **b)** $x_n = 1 + \frac{n}{n+2}\cos\frac{n\pi}{2}$.

Determine $\inf\{x_n |\ n \in N\}$, $\sup\{x_n |\ n \in N\}$, $\liminf_{n\to\infty} x_n$ *and* $\limsup_{n\to\infty} x_n$, *and then compare them.*

Solutions (without details)

a) $\inf\{x_n |\ n \in N\} = -4$, $\liminf_{n\to\infty} x_n = -3$, $\limsup_{n\to\infty} x_n = 3$, $\sup\{x_n |\ n \in N\} = 5$.

b) $\inf\{x_n |\ n \in N\} = \liminf_{n\to\infty} x_n = 0$, $\sup\{x_n |\ n \in N\} = \limsup_{n\to\infty} x_n = 2$.

$$\limsup_{n\to\infty} f_n = -\liminf_{n\to\infty}(-f_n)$$

Solutions (without details)

Let us denote

$$\limsup_{n\to\infty} f_n = L, \quad L \in \mathbf{R}. \tag{3.17}$$

Then for every $\varepsilon > 0$, there are

- infinitely many terms f_n such that $f_n > L - \varepsilon$;
- at most finitely many terms f_n such that $f_n > L + \varepsilon$. (3.18)

So from relations (3.18) it follows that for every $\varepsilon > 0$ there are

- infinitely many terms $-f_n$ such that $-f_n < -L + \varepsilon$;
- at most finitely many terms $-f_n$ such that $-f_n < -L - \varepsilon$.

The terms $-f_n$ belong to the sequence $(-f_n)_{n\in\mathbf{N}}$. Thus

$$\liminf_{n\to\infty}(-f_n) = -L.$$

Definition 9 *A sequence* (s_n) *of real numbers is called a Cauchy sequence if for each* $\epsilon > 0$ *there exists a number* N *such that if* $m, n > N$ *then*

$$|s_n - s_m| < \epsilon.$$

Example 10 *Prove that*

$$\left(\frac{1}{n}\right)_n$$

is a Cauchy sequence.

Solutions

Lemma 11 *Convergent sequences are Cauchy sequences.*

Proof

Lemma 12 *Cauchy sequences are bounded*

Proof

Theorem 13 *A sequence is a convergent sequence if and only if it is a Cauchy sequence.*

Proof

Section 11 Subsequences

Let $(s_n)_{n\in\mathbb{N}}$ is a sequence such that

$$s_n = \frac{n}{n+1}.$$

Now, let

$$k : \mathbb{N} \to \mathbb{N}$$

be an increasing function such that

$$k(n) = k_n = n^2 + n.$$

Then

$$s_{k(n)} = s_{k_n} = \frac{k_n}{k_n + 1} = \frac{n^2 + n}{n^2 + n + 1}.$$

and

$$\begin{aligned} s_{k_1} &= \frac{1^2 + 1}{1^2 + 1 + 1} = \frac{2}{3} \\ s_{k_2} &= \frac{2^2 + 2}{2^2 + 2 + 1} = \frac{6}{7} \\ &\vdots \end{aligned}$$

We say that the new sequence

$$(s_{k_n})_{n\in\mathbb{N}}$$

is a subsequence of the given sequence.

Notes here

Definition 1 *Suppose that* $(s_n)_{n\in\mathbb{N}}$ *is a sequence. A subsequence of this sequence is a sequence of the form* $(t_k)_{k\in\mathbb{N}}$ *where for each* k *there is a positive integer* n_k *such that*

$$n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$$

and

$$t_k = s_{n_k}.$$

So (t_k) *is just a selection of some (possibly all) of the* s_n's *taken in order.*

Notes here

Give an example of a sequence with two distinct subsequences. Justify your answer

Theorem 3 *If* $(s_n)_{n\in\mathbb{N}}$ *is a convergent sequence, then every subsequence converges to the same limit.*

The following are fundamental results in real Analysis

Theorem 4 *Every subsequence $(s_n)_{n\in\mathbb{N}}$ has a monotonic subsequence.*

Theorem 5 *Let $(s_n)_{n\in\mathbb{N}}$ is a sequence. There exists a monotonic subsequence whose limit is $\limsup s_n$ and there exists a monotonic subsequence whose limit is $\liminf s_n$*

Theorem 6 *(Bolzano-Weierstrass Theorem) Every bounded sequence has a convergent subsequence.*

Example 7 *Let $(s_n)_{n\in\mathbb{N}}$ is a sequence such that*

$$s_n = \frac{n}{n+1} \cdot \sin\left(\frac{2n\pi}{5}\right)$$

1. *Prove that $(s_n)_{n\in\mathbb{N}}$ is a bounded sequence*
2. *Find a subsequence of s_n which is monotonic*
3. *Find a convergent subsequence of $(s_n)_{n\in\mathbb{N}}$*

Exercise 1 *Prove the Bolzano-Weierstrass Theorem*

Proof

Definition 2 *Let (s_n) be a sequence of real numbers. A subsequential limit is any real number or symbol ∞ or $-\infty$ that is the limit of some subsequence of (s_n).*

Example 3 *Find a subsequence limit of (s_n) where* $s_n = \dfrac{n+1}{n}\cos\left(\dfrac{2\pi n}{3}\right)$

<u>Solutions</u>

Theorem 5 *Let* (s_n) *be a sequence of real numbers, and let* S *be the set of subsequential limits of* (s_n)

1. S *is nonempty*
2. $\sup S = \limsup s_n$ *and* $\inf S = \liminf s_n$
3. $\lim s_n$ *exists if and only if* S *has exactly one element, namely* $\lim s_n$.

Proof

Chapter 12 (Limsup and Liminf)

<u>Exercise</u> Write down the definition of Limsup s_n and Liminf s_n

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<u>Exercise</u> Prove that $\liminf s_n = -\limsup(-s_n)$

Exercise Let (s_n) be a sequence whose terms is given by

$$s_n = 1 + (-1)^n + \frac{1}{2^n}.$$

(a) Find $\limsup s_n$.

(b) Find $\liminf s_n$.

<u>Theorem</u> If $(s_n)_{n\in\mathbb{N}}$ is convergent to a positive real numbers s and if $(t_n)_{n\in\mathbb{N}}$ is any sequence then

$$\limsup (s_n t_n) = s \cdot \limsup t_n .$$

<u>Exercise</u> Let $(s_n)_{n\in\mathbb{N}}$ be a bounded sequence and let k be a nonnegative real number.

1. Prove that $\limsup k s_n = k \cdot \limsup s_n$
2. What happens if k is a negative number?

Homework

(1) Give an example of a sequence satisfying the following conditions

- $\limsup s_n = c_1 \in \mathbb{R}$
- $\liminf s_n = c_2 \in \mathbb{R}$
- $c_1 \neq c_2$

---

(2) Find an example of a sequence such that the set

$$\{n \in \mathbb{N} : \liminf s_n \leq s_n \leq \limsup s_n\}$$

is an empty set.

---

3) Find an example of a sequence $(s_n)_{n\in\mathbb{N}}$ such that the set $\{n \in \mathbb{N} : s_n > \limsup s_n\}$ is infinite.

---

4) Prove that $\limsup |s_n| = 0 \iff \lim s_n = 0$.

---

5) Give an example of a sequence satisfying the following:

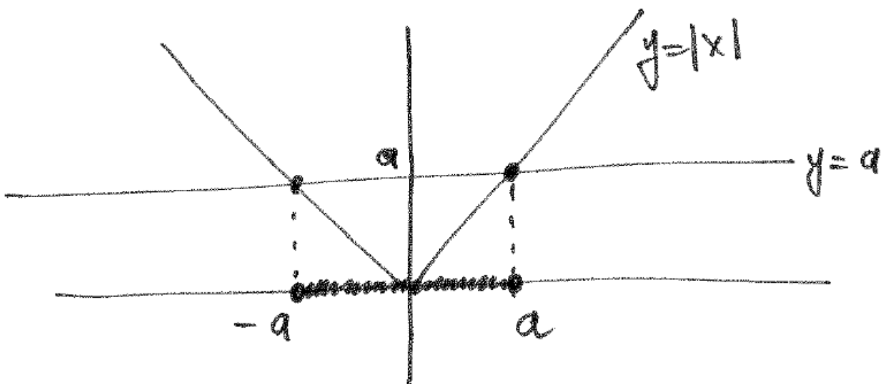
- $(s_n)_{n\in\mathbb{N}}$ is not constant
- $\liminf s_n < \limsup s_n$
- There is a subsequence of $(s_n)_n$: $(s_{n_k})_{k\in\mathbb{N}}$ such that

$$\liminf s_n < \lim_{k\to\infty} s_{n_k} < \limsup s_n$$

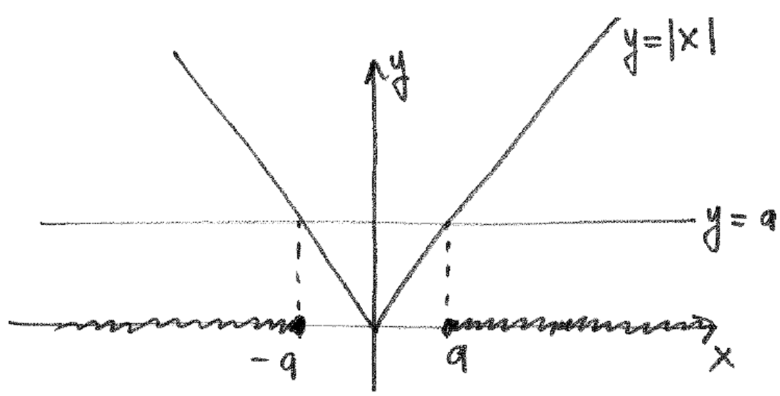
(Before Section 17) Preparing for Continuity

Assume $a > 0$

Let us consider the inequality $|x| \leq a$.



Using the graph above solve the inequality $|x| \leq a$



Using the graph above solve the inequality $|x| \geq a$

In summary,

$|x| \leq a$ iff $x \in [-a, a]$

$|x| \geq a$ iff $x \in (-\infty, -a] \cup [a, \infty)$

Next, let $y = f(x)$, where f is a real-valued function

Exercise

Let $f(x) = x^2 - 3$.

Describe $|f(x)|$ as a piecewise function.

Ex let $f(x) = x+2$.

Prove that for any $\varepsilon > 0$, it is possible to find a positive number δ such that

if $|x-1| < \delta$ then $|f(x) - f(1)| < \varepsilon$.

Ex let $f(x) = x^2$

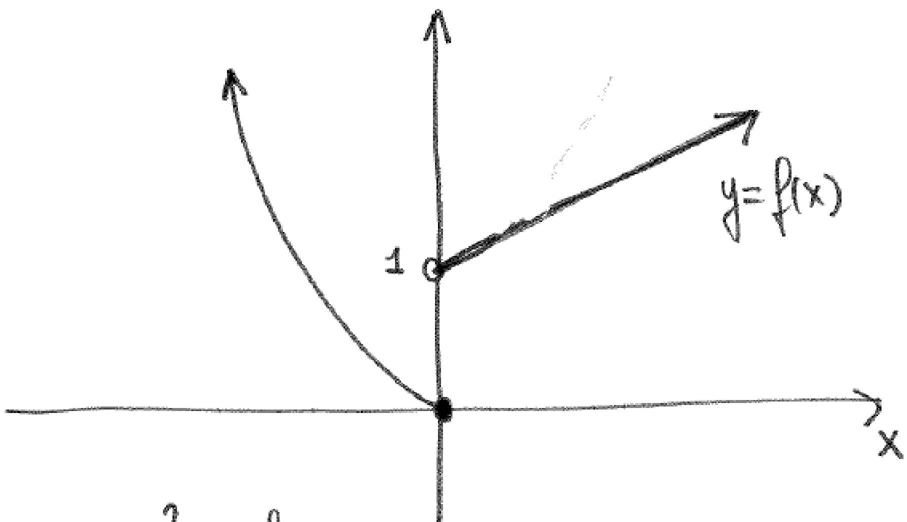
Find $\delta > 0$ such that if $|x-1| < \delta$ then $|x^2 - 1| < \frac{1}{2}$

Ex Let $f(x) = x^2$.

1) Find $\delta > 0$ such that if $|x-1| < \delta$ then $|x^2-1| < \frac{1}{3}$

b) Given any positive number ε, find $\delta > 0$ such that if $|x-1| < \delta$ then $|x^2-1| < \varepsilon$.

Ex Consider the following function



$$f(x) = \begin{cases} x^2 & \text{if } x \leq 0 \\ x+1 & \text{if } x > 0 \end{cases}$$

Given any $\varepsilon > 0$, is it possible to find $\delta > 0$ such that if $|x| < \delta$ then $|f(x)| < \varepsilon$?

Section 17

Continuity

Let f be a real-valued function

- The domain of f is the set of all real numbers x such that $f(x)$ makes sense (or is defined).
- The range of f is the set of all real numbers of the type $f(x)$ such that x is in the domain of f.

$$\text{dom}(f) = \{ x \in \mathbb{R} \text{ such that } f(x) \text{ makes sense} \}$$

$$\text{Ran}(f) = \{ f(x) \text{ such that } x \in \text{dom}(f) \}$$

Exercise Find the domain and range of the following function

$$f(x) = \frac{x}{(x-1)(x+2)}$$

Solution

Definition Let f be a real-valued function whose domain is a subset of $\mathbb{R}$. The function f is continuous at x_0 in $dom(f)$ if for every sequence (x_n) $\{x_n \in dom(f)\}$ converging to x_0 $\lim_{n \to \infty} f(x_n) = f(x_0)$.

If f is continuous at each pt of a set $S \subseteq dom(f)$, then we say that f is continuous on S. The function f is said to be continuous if it is continuous on its domain.

Theorem Let f be a real-valued function whose domain is a subset of $\mathbb{R}$. Then f is continuous at $x_0 \in dom(f)$ if and only if

$\forall \varepsilon > 0$ $\exists \delta > 0$ such that $x \in dom(f)$ and $|x - x_0| < \delta$ imply $|f(x) - f(x_0)| < \varepsilon$.

Proof

Exercise

Let $f(x) = ax^2 + 1$, $x \in \mathbb{R}$.

Prove that f is continuous on $\mathbb{R}$ by

(a) Using the definition.

(b) Using the ε-δ Theorem.

Solution

Exercise

Let $f(x) = 3x^2+2$. Show that f is continuous using the ε-δ theorem

---

Solution

<u>Theorem</u> Let f be a real-valued function with domain $\mathrm{dom}(f) \subseteq \mathbb{R}$. If f is continuous at x_0 in $\mathrm{dom}(f)$ then $|f|$ and kf, $k \in \mathbb{R}$ are continuous at x_0.

Proof

<u>Theorem</u> Let f, g be real-valued functions that are cont at x_0 in $\mathbb{R}$. Then

(i) $f+g$ is continuous at x_0.

(ii) fg is continuous at x_0.

(iii) f/g is continuous at x_0 if $g(x_0) \neq 0$.

Proof

Section 18 Properties of continuous functions

April 23, 2016

- Let $f : D \to \mathbb{R}$ be a real-valued function where D is a subset of the reals.
- We say that f is a **bounded function** if there exists a real number M such that

$$|f(x)| \leq M$$

  for any $x \in D$.

Example 1 *Answer the following questions*

1. *Give an example of a bounded real-valued function.*

2. *Given an example of an unbounded real-valued function.*

Theorem 2 *Let $f : [a, b] \to \mathbb{R}$ be a continuous real-valued function. The following holds true*

1. *f is a bounded function*
2. *f assumes its maximum and minimum on $[a, b]$*

Proof of Theorem 2

Example 3 *Give an example which supports that the theorem above fails if we replace* $[a,b]$ *by an open interval.*

Theorem 4 ***(Intermediate Value Theorem)*** *If* f *is a continuous real-valued function on* $I \subseteq \mathbb{R}$ *then if* $a, b \in I, a < b$ *and* y *lies between* $f(a), f(b)$ *then there is at least one* $x \in (a,b)$ *such that* $f(x) = y$.

Proof of Theorem 4

Exercise 5 *Show that a polynomial of odd degree with real coefficients has at least one real zero.*

Proof of Exercise 5

Exercise 6 *Let* f *be a continuous function which maps* $[0,1]$ *into* $[0,1]$. *Show that* f *has a fixed point. In other words, there exists* $x_0 \in [a,b]$ *such that* $f(x_0) = x_0$

Proof of Exercise 6