Recent developments in étale cohomology

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MPIM Bonn

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Introduction

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Key points: For any reasonable scheme, have a category $D^b_c(X, \mathbf{Q}_\ell)$ which satisfies a six operations formalism; Lefschetz trace formula; theory of weights for varieties over finite fields.

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Figure: The creation of étale cohomology

It is true that many mathematicians can profitably use étale cohomology as a black box, never looking beyond Freitag-Kiehl or Milne. However, it is **not** a dead subject!

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End of the initial period of development.

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So much for being a dead subject. Is there still anything left to be done?

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MISSING: A natural class of sheaves (with torsion or $\mathbf{Z}_\ell$-coefficients) stable under the six operations, admitting a perverse t-structure, satisfying affine vanishing, etc.

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However, NO interesting properties of these sheaves are obvious!

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Until further notice: K a nonarchimedean field of **characteristic zero** and residue characteristic $p \geq 0$, $\Lambda = \mathbf{Z}/n\mathbf{Z}$.

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Main theorem (Bhatt-H.)

On rigid spaces X/K, $D^{(b)}_{zc}(X,\Lambda)$ is stable under the operations f^*, Rf_* for proper f, $Rf_!$ and Rf_* on lisse sheaves for Zariski-compactifiable f, $Rf^!$ if $p \nmid n$ or f is finite, $\otimes$ and $R\mathscr{H}\mathrm{om}$ (under a finite tor-dimension assumption), and Verdier duality.

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Proof requires many auxiliary ingredients, possibly of independent interest.

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Algebraization theorem (Bhatt-H.)

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Upshot: In the proof of the main theorem, all claims can be checked locally in the analytic topology.

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1) Let $f : \mathfrak{X} \to \mathfrak{Y}$ be any finite type map of locally finite type $\mathrm{Spec} A$-schemes. Then for any $F \in D_c^+(\mathfrak{X}, \Lambda)$, the natural map $(Rf_* F)^{\mathrm{an}} \to Rf_*^{\mathrm{an}} F^{\mathrm{an}}$ is an isomorphism.

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Essential case of X smooth and $X - U$ an snc divisor treated by Lütkebohmert. General case can be deduced by resolution of singularities. $\rightsquigarrow$ If $\mathfrak{X}$ is a scheme of finite type over $\mathrm{Spec} A$ - e.g. an open subscheme of $\mathrm{Spec} A$ - finite étale covers of $\mathfrak{X}^{\mathrm{an}}$ are uniquely algebraizable.

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Essential case of X smooth and $X - U$ an snc divisor treated by Lütkebohmert. General case can be deduced by resolution of singularities. $\rightsquigarrow$ If $\mathfrak{X}$ is a scheme of finite type over $\mathrm{Spec}A$ - e.g. an open subscheme of $\mathrm{Spec}A$ - finite étale covers of $\mathfrak{X}^{\mathrm{an}}$ are uniquely algebraizable. $\rightsquigarrow$ Same statement for lcc sheaves of Λ-modules. Algebraization theorem follows. Also get the following very useful result:

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If X is a quasicompact rigid space, $D^b_{zc}(X, \Lambda)$ is the thick triangulated subcategory of $D(X, \Lambda)$ generated by $f_* M$ for all finite maps $f : X' \to X$ and finite Λ-modules M.

We now use the second ingredient:

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Upshot: In the proof of the main theorem, we can (usually) reduce to checking claims in the special case of constant sheaves.

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All claims now follow by judiciously combining everything.
Sample argument i.: stability under Rf_* for proper f follows from the locality theorem + generation lemma + 2) + 3).
Sample argument ii.: compatibility of six operations with extension of base field L/K follows from locality theorem + algebraization theorem + regular base change and consequences thereof + regularity of $A \to A\widehat{\otimes}_K L$.

Open problems

One key consequence of all this: now have a good theory of IC sheaves on (char. 0) rigid spaces. In particular, for any proper rigid space X/K, get intersection cohomology groups $IH^*(X_{\overline{K}}, \mathbf{Q}_\ell)$. Finite-dimensional; Poincaré duality holds for $\ell \neq p$.

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Reminder on perverse sheaves

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Let X be an algebraic variety over a field K, ℓ a prime invertible in K. Recall: $A \in D_c^b(X, \mathbf{Q}_\ell)$ is **perverse** if $\dim \mathrm{supp} \mathcal{H}^n(A) \leq -n$ for all $n \in \mathbf{Z}$, and likewise with $\mathbf{D}A$ in place of A.

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In 2018, I began (publicly) asking: is there a "relative" / "in families" version of perverse sheaves?

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There are **hints** towards a positive answer in the geometric Langlands literature. For instance, one can extract from various geometric Langlands papers the following theorem:

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Let $f : X \to S$ be any morphism of varieties with S smooth (of pure dimension d). If $A \in D_c^b(X, \mathbf{Q}_\ell)$ is perverse and f-ULA, then $(A|X_s)[-d]$ is perverse for all points $s \to S$. More generally, for any $g : T \to S$ with T smooth, $f^*A[\dim T - \dim S]$ is perverse and f_T-ULA.

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This is good enough for constructing the fusion product in geometric Satake. Suggests that (for a smooth base S) one should consider the category $\mathrm{Perv}^{\mathrm{ULA}}(X/S)$ of objects $A \in D_c^b(X, \mathbf{Q}_\ell)$ which are f-ULA and with $A[\dim S]$ perverse.

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Let $f : X \to S$ be any morphism of varieties with S smooth. If $A \in D_c^b(X, \mathbf{Q}_\ell)$ is f-ULA, then all perverse cohomologies ${}^p\mathcal{H}^n(A)$ are f-ULA, and moreover any perverse subquotient of any ${}^p\mathcal{H}^n(A)$ is f-ULA.

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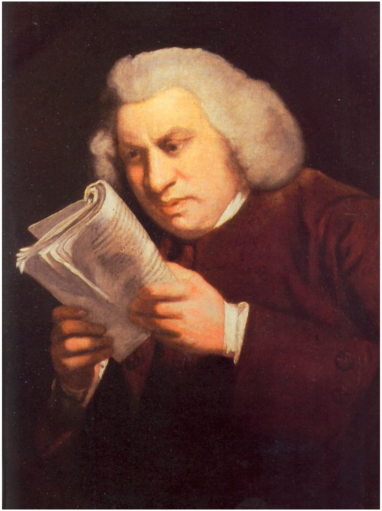

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What is going on??

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The heart of this t-structure is exactly the objects in $D^b_c(X, \mathbf{Q}_\ell)$ which restrict to a perverse sheaf on each geometric fiber of f. In particular, objects of this type naturally form an abelian category $\mathrm{Perv}(X/S)$. No idea how to see this directly!

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We also show that for regular S, perverse and relative perverse t-structures agree up to (explicit) shift on ULA objects. $\rightsquigarrow$ New proof of Gaitsgory's theorem.

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Sketch of argument over rank one aic valuation rings

Let $S = \mathrm{Spec}\, V$ be the spectrum of a rank one aic valuation ring, with generic point η and special point s. For any finite type S-scheme X, get $j : X_\eta \to X$ and $i : X_s \to X$ as usual.

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Key point: Look at the triangle

$$Ri^!A \to i^*A \to i^*Rj_*j^*A \to,$$

and use the fact that $i^*Rj_* : D(X_\eta, \Lambda) \to D(X_s, \Lambda)$ is perverse t-exact (Gabber). This + condition on j^*A implies that $Ri^!A$ and i^*A have same perverse cohomology in negative degrees. Done.

From the case where S is the spectrum of a rank one aic valuation ring, some small arguments extend the result first to the case where S is the spectrum of any aic valuation ring, and then to the case where S is qcqs and **all connected components of S are spectra of aic valuation rings**.

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Since the t-structure we are seeking is supposed to behave well with respect to any base change on S, we're now in a position to define it in the general case by descent from this funny case.

Descent

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Back to a general $X \to S$ as before. Can pick a v-hypercover $S_\bullet \to S$ as in 1. Then 2. gives $\mathcal{D}^+(X, \mathbf{Z}/n) \simeq \lim_m \mathcal{D}^+(X \times_S S_m, \mathbf{Z}/n)$, and we can now descend the t-structure as desired since all pullbacks

$$\mathcal{D}^+(X \times_S S_m, \mathbf{Z}/n) \to \mathcal{D}^+(X \times_S S_{m'}, \mathbf{Z}/n)$$

are t-exact.

Thank you for listening!

Featured art:

- *A Young Man Writing at a Cloth Covered Table* by Christian van Donck (circa 1653)
- *Portrait of Samuel Johnson* by Joshua Reynolds (1775)