Recent Progress in Gyrokinetic Theory

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Gyrokinetic Particle Simulation:
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The Paper that Launched my Gyrokinetics Career

Nonlinear gyrokinetic theory for finite-beta plasmas

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A self-consistent and energy-conserving set of nonlinear gyrokinetic equations, consisting of the averaged Vlasov and Maxwell's equations for finite-beta plasmas, is derived. The method utilized in the present investigation is based on the Hamiltonian formalism and Lie transformation. The resulting formulation is valid for arbitrary values of $k_\perp \rho_i$ and, therefore, is most suitable for studying linear and nonlinear evolution of microinstabilities in tokamak plasmas as well as other areas of plasma physics where the finite Larmor radius effects are important. Because the underlying Hamiltonian structure is preserved in the present formalism, these equations are directly applicable to numerical studies based on the existing gyrokinetic particle simulation techniques.

Recent Progress in Guiding-center & Gyrokinetic Theories

Guiding-center Theory

- Polarization effects in higher-order guiding-center theory
  (Brizard & Tronko, 2015 & 2016; Brizard, 2013)
- Variational formulations of guiding-center Vlasov-Maxwell theory
  (Brizard & Tronci, 2016)
- Lifting of the Vlasov-Maxwell bracket by Lie-transform method
  (Brizard, Morrison, Burby, *et al.*, 2016)
- Monte Carlo implementation of 5D guiding-center collisions
  (Hirvijoki, Brizard, *et al.*, 2015)

Gyrokinetic Theory

- Energetically-consistent collisional gyrokinetics
  (Burby, Brizard & Qin, 2015)
- Hamiltonian formulation of gyrokinetic theory
  (Burby, Brizard, Morrison & Qin, 2015)
- Higher-order gyrokinetic theory (nonlinear polarization)
  (Mishchenko & Brizard, 2011)

Outline

I. Higher-order Guiding-center Theory

- Guiding-center polarization

II. Guiding-center (pre-Gyrokinetic) Vlasov-Maxwell Theory

- Guiding-center angular-momentum conservation law

III. Lecture Notes on Gyrokinetic Theory

- Graduate-level textbook to be completed by Fall 2017

IV. Ongoing Work Related to Gyrokinetic Theory

I. Higher-order Guiding-center Theory

Guiding-center Phase-space Lagrangian $(e \to \epsilon^{-1} e)$

$$\Gamma_{\rm gc} = (\epsilon^{-1} e/c)\,\mathbf{A}^{*}\cdot{\rm d}\mathbf{X} + \;\epsilon\,J\,{\rm d}\zeta \;-\; \left(J\,\Omega + p_{\|}^{2}/2m\right)\;{\rm d}t$$

- Guiding-center polarization correction: $(\widehat{\rm b}\cdot\nabla\times\widehat{\rm b})\,\widehat{\rm b} \to \nabla\times\widehat{\rm b}$

$$\mathbf{A}^{*} \equiv \mathbf{A} + \frac{c}{e}\left[\epsilon\,p_{\|}\,\widehat{\rm b} - \epsilon^{2}\,J\left(\mathbf{R} + \frac{1}{2}\nabla\times\widehat{\rm b}\right)\right]$$

Guiding-center displacement: $\boldsymbol{\rho}_{\rm gc} \equiv {\sf T}_{\rm gc}^{-1}\mathbf{x} - \mathbf{X} \;\;\to\;\; \langle\boldsymbol{\rho}_{\rm gc}\rangle \neq 0$

- Guiding-center polarization (Pfirsch, 1984 & Kaufman, 1986)

$$\boldsymbol{\pi}_{\rm gc} = e\,\langle\boldsymbol{\rho}_{\rm gc}\rangle \;-\; \nabla\cdot\left(\frac{e}{2}\langle\boldsymbol{\rho}_{\rm gc}\boldsymbol{\rho}_{\rm gc}\rangle\right) + \cdots = \frac{e\,\widehat{\rm b}}{\Omega}\times\frac{d_{\rm gc}\mathbf{X}}{dt}$$

- Guiding-center magnetization (intrinsic/moving electric-dipole)

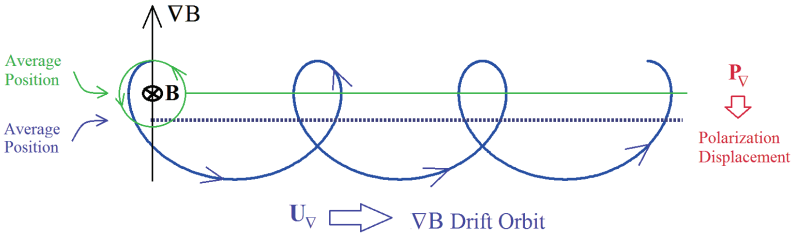
$$\boldsymbol{\mu}_{\rm gc} \;=\; \frac{e}{2c}\left\langle\boldsymbol{\rho}_{\rm gc}\times\frac{d_{\rm gc}\boldsymbol{\rho}_{\rm gc}}{dt}\right\rangle + \frac{e}{c}\langle\boldsymbol{\rho}_{\rm gc}\rangle\times\frac{d_{\rm gc}\mathbf{X}}{dt} = -\,\mu\widehat{\rm b} + \boldsymbol{\pi}_{\rm gc}\times\frac{p_{\|}\,\widehat{\rm b}}{mc}$$

Example: Guiding-center grad-B Polarization

- Ratio of polarization displacement $e^{-1}|\langle \boldsymbol{\pi}_{\rm gc}\rangle|$ to $|\boldsymbol{\rho}_0| = \rho_0$

$$\frac{\mu}{m\Omega^2\,\rho_0}\,|\nabla_\perp B| \;\sim\; \rho_0\,|\nabla_\perp \ln B|$$

- Standard guiding-center ordering $\rho_0\,|\nabla_\perp \ln B| \ll 1$



Guiding-center Lagrangian Constraint

- Lagrangian constraint on guiding-center transformation

$$\mathbf{p} \;=\; m\,\frac{d\mathbf{x}}{dt} \quad\to\quad \mathsf{T}_{\rm gc}^{-1}\mathbf{p} \;=\; m\left(\frac{d_{\rm gc}\mathbf{X}}{dt} \;+\; \frac{d_{\rm gc}\boldsymbol{\rho}_{\rm gc}}{dt}\right)$$

Guiding-center Hamiltonian (exact to all orders)

$$H_{\rm gc} \;=\; \frac{m}{2}\left|\frac{d_{\rm gc}\mathbf{X}}{dt} \;+\; \frac{d_{\rm gc}\boldsymbol{\rho}_{\rm gc}}{dt}\right|^2 \;\equiv\; \frac{m}{2}\left\langle\left|\frac{d_{\rm gc}\mathbf{X}}{dt} \;+\; \frac{d_{\rm gc}\boldsymbol{\rho}_{\rm gc}}{dt}\right|^2\right\rangle$$

$$=\; \frac{p_{\|}^2}{2m} \;+\; \left(J\,\Omega \;+\; \epsilon^2\,\Psi_2 \;+\; \cdots\right)$$

- Hamiltonian ($\Psi = J\Omega + \cdots$) or Symplectic ($\mathbf{\Pi} = p_{\|}\,\widehat{\mathsf{b}} + \cdots$) representations

$$\Gamma_{\rm gc} \equiv \left(\frac{e\mathbf{A}}{\epsilon\,c} \;+\; \mathbf{\Pi}\right)\cdot \mathsf{d}\mathbf{X} + \epsilon\,J\,\left(\mathsf{d}\zeta \;-\; \mathbf{R}\cdot\mathsf{d}\mathbf{X}\right) \;-\; \left(\frac{p_{\|}^2}{2m} \;+\; \Psi\right)\,\mathsf{d}t$$

Guiding-center Toroidal Canonical Momentum

$$p_{\mathrm{gc}\varphi} \;\equiv\; \left[\frac{e}{\epsilon\, c}\,\mathbf{A} \;+\; p_{\|}\,\widehat{\mathsf{b}} \;-\; \epsilon\, J \left(\mathbf{R} \;+\; \frac{1}{2}\,\nabla\times\widehat{\mathsf{b}} \right) \right] \cdot \frac{\partial \mathbf{X}}{\partial \varphi}$$

$$= \;-\,\frac{e}{c} \left[\epsilon^{-1}\psi \;+\; \nabla\cdot\left(\frac{\epsilon\, J}{2\, m\Omega}\,\nabla\psi \right) \right] \;+\; p_{\|}\, b_{\varphi} \;-\; 2\epsilon\, J\, b_{z}$$

Guiding-center transformation: $p_{\varphi\mathrm{gc}} \;\equiv\; \mathsf{T}_{\mathrm{gc}}^{-1} p_{\varphi} = \left\langle \mathsf{T}_{\mathrm{gc}}^{-1} p_{\varphi} \right\rangle$

$$p_{\varphi\mathrm{gc}} \;\equiv\; -\,\frac{e}{\epsilon c}\,\mathsf{T}_{\mathrm{gc}}^{-1}\psi + m \left(\frac{d_{\mathrm{gc}}\mathbf{X}}{dt} \;+\; \frac{d_{\mathrm{gc}}\boldsymbol{\rho}_{\mathrm{gc}}}{dt} \right) \cdot \left(\frac{\partial_{\mathrm{gc}}\mathbf{X}}{\partial\varphi} \;+\; \frac{\partial_{\mathrm{gc}}\boldsymbol{\rho}_{\mathrm{gc}}}{\partial\varphi} \right)$$

- Exact (and faithful) guiding-center conservation law

$$p_{\varphi\mathrm{gc}} \;\equiv\; p_{\mathrm{gc}\varphi} \;\;\rightarrow\;\; \frac{d_{\mathrm{gc}}\, p_{\mathrm{gc}\varphi}}{dt} \;\equiv\; \mathsf{T}_{\mathrm{gc}}^{-1}\left(\frac{dp_{\varphi}}{dt} \right) \;=\; 0$$

II. Guiding-center (pre-Gyrokinetic) Vlasov-Maxwell Theory: No background-fluctuation separation

Guiding-center Lagrangian (lowest order; μ = orbit label)

$$L_{\rm gc} = \left(\frac{e}{c}\mathbf{A} + p_\parallel\,\widehat{\sf b}\right)\cdot\dot{\mathbf{X}} - \left(\frac{p_\parallel^2}{2m} + \mu\,B + e\,\Phi\right)$$

$$\equiv \frac{e}{c}\mathbf{A}^*\cdot\dot{\mathbf{X}} - \left(\frac{p_\parallel^2}{2m} + e\,\Phi^*\right)$$

- Guiding-center Euler-Lagrange equations ($\mathbf{E}^* \equiv -\nabla\Phi^* - c^{-1}\partial_t\mathbf{A}^*,\ \mathbf{B}^* \equiv \nabla\times\mathbf{A}^*$)

$$\dot{\mathbf{X}} = \frac{p_\parallel}{m}\frac{\mathbf{B}^*}{B_\parallel^*} + \mathbf{E}^*\times\frac{c\widehat{\sf b}}{B_\parallel^*} \quad\text{and}\quad \dot{p}_\parallel = e\,\mathbf{E}^*\cdot\frac{\mathbf{B}^*}{B_\parallel^*}$$

- Guiding-center magnetization

$$\frac{\partial L_{\rm gc}}{\partial\mathbf{B}} = p_\parallel\,\frac{\partial\widehat{\sf b}}{\partial\mathbf{B}}\cdot\dot{\mathbf{X}} - \mu\,\frac{\partial B}{\partial\mathbf{B}} \equiv \boldsymbol{\mu}_{\rm gc}$$

Guiding-center Vlasov-Maxwell Equations

Guiding-center (Low) Lagrange Variational Principle

$$\mathcal{A}_{\rm gc}^{L} \;=\; \int L_{\rm gc}\; F_0\, d^4 z_0\, d\mu\, dt \;+\; \int \frac{d^3x\, dt}{8\pi}\;\left(|\mathbf{E}|^2 \;-\; |\mathbf{B}|^2 \right)$$

- Guiding-center Vlasov equation ($F_0\, d^4 z_0 d\mu \equiv F_\mu\, d^4 z\, d\mu$)

$$F_\mu \;\equiv\; \mathcal{J}_{\rm gc}\, f_\mu \;\;\rightarrow\;\; \frac{\partial F_\mu}{\partial t} \;+\; \frac{\partial}{\partial z^a}\left(\dot{z}^a\, F_\mu \right) \;=\; 0$$

- Guiding-center Maxwell equations

$$\begin{aligned}
\nabla\cdot\mathbf{E} &= 4\pi\;\varrho_{\rm gc} \;\equiv\; -\,4\pi \int \frac{\partial L_{\rm gc}}{\partial\Phi} F_\mu\; dp_\| \, d\mu \\
\nabla\times\mathbf{B} - \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} &= \frac{4\pi}{c}\left(\mathbf{J}_{\rm gc} + c\,\nabla\times\mathbf{M}_{\rm gc}\right) \\
&\equiv 4\pi \int \left[\frac{\partial L_{\rm gc}}{\partial\mathbf{A}}\, F_\mu + \nabla\times\left(\frac{\partial L_{\rm gc}}{\partial\mathbf{B}}\, F_\mu \right)\right] dp_\|\, d\mu
\end{aligned}$$

Guiding-center Euler Variational Principle

Guiding-center Euler variational principle in extended guiding-center phase space $Z^{\alpha} = (\mathbf{X}, p_{\|}, w, t)$

$$\mathcal{A}_{\rm gc}^{E} \;=\; -\;\int \mathcal{F}_{\mu}\,\mathcal{H}\;d^{6}Z\;d\mu \;+\; \int \frac{d^{4}x}{8\pi}\;\left(|\mathbf{E}|^{2}\;-\;|\mathbf{B}|^{2}\right)$$

∘ Extended Vlasov phase-space density

$$\mathcal{F}_{\mu} \;\equiv\; F_{\mu}\,\delta(w - H_{\rm gc}) \;\;\text{and}\;\; \mathcal{H} \;\equiv\; H_{\rm gc} \;-\; w$$

∘ Eulerian Hamiltonian variation → intrinsic magnetization

$$\delta\mathcal{H} \;\equiv\; e\,\delta\Phi^{*} \;=\; e\,\delta\Phi \;+\; \mu\,\widehat{\mathsf{b}}\cdot\delta\mathbf{B}$$

Guiding-center Eulerian Variation Constraint

Guiding-center Eulerian Vlasov variation

$$\delta\mathcal{F}_\mu \;\equiv\; \frac{e}{c}\,\delta\mathbf{A}^*\cdot\left(B_\|^*\left\{\mathbf{X},\;\mathcal{F}_\mu/B_\|^*\right\}_{\rm gc}\right) \;+\; \delta B_\|^*\;\mathcal{F}_\mu/B_\|^* \;+\; B_\|^*\;\left\{\delta\mathcal{S},\;\mathcal{F}_\mu/B_\|^*\right\}_{\rm gc}$$

- Eulerian magnetic variations $\rightarrow$ moving electric-dipole

$$\frac{e}{c}\,\delta\mathbf{A}^* = \frac{e}{c}\,\delta\mathbf{A} + p_\|\,\delta\mathbf{B}\cdot\frac{\partial\widehat{b}}{\partial\mathbf{B}} \quad\text{and}\quad \delta B_\|^* = \delta\mathbf{B}^*\cdot\widehat{b} + \left(\delta\mathbf{B}\cdot\frac{\partial\widehat{b}}{\partial\mathbf{B}}\right)\cdot\mathbf{B}^*$$

Guiding-center Vlasov constraint

$$\int\delta\mathcal{F}_\mu\,d^6Z \;=\; 0 \quad\rightarrow\quad \delta\mathcal{F}_\mu \;\equiv\; \frac{\partial}{\partial Z^\alpha}\left(\mathcal{F}_\mu\,\delta Z^\alpha\right)$$

- Guiding-center phase-space virtual displacement

$$\delta Z^\alpha \;\equiv\; \{\delta\mathcal{S}, Z^\alpha\}_{\rm gc} + (e/c)\,\delta\mathbf{A}^*\cdot\{\mathbf{X}, Z^\alpha\}_{\rm gc}$$

Guiding-center Noether Equation

Guiding-center Noether equation

$$\delta\mathcal{L}_{\rm gc} \;\equiv\; -\left(\delta t\,\frac{\partial}{\partial t} + \delta\mathbf{x}\cdot\nabla\right)\mathcal{L}_{\rm M} \;=\; \frac{\partial\delta\mathcal{J}}{\partial t} \;+\; \nabla\cdot\delta\mathbf{\Gamma}$$

∘ Noether components ($\mathbf{H} \equiv \mathbf{B} - 4\pi\,\mathbf{M}_{\rm gc}$)

$$\delta\mathcal{J} \;\equiv\; \int \delta\mathcal{S}\;\mathcal{F}_{\mu}\;dp_{\|}\,d\mu\,dw \;-\; \frac{\mathbf{E}\cdot\delta\mathbf{A}}{4\pi\,c}$$

$$\delta\mathbf{\Gamma} \;\equiv\; \int \delta\mathcal{S}\;\mathcal{F}_{\mu}\;\dot{\mathbf{X}}\;dp_{\|}\,d\mu\,dw - \frac{1}{4\pi}\left(\delta\Phi\,\mathbf{E} \;+\; \delta\mathbf{A}\times\mathbf{H}\right)$$

Energy-momentum conservation law ($\delta\chi \equiv \mathbf{A}\cdot\delta\mathbf{x} - \Phi\;c\,\delta t$)

$$\delta\mathcal{S} \;=\; \frac{e}{c}\,\mathbf{A}^{*}\cdot\delta\mathbf{x} \;-\; w\,\delta t \;\equiv\; \mathbf{P}\cdot\delta\mathbf{x} \;-\; w\,\delta t$$

$$\delta\Phi \;\equiv\; \delta\mathbf{x}\cdot\mathbf{E} \;+\; c^{-1}\partial\delta\chi/\partial t$$

$$\delta\mathbf{A} \;\equiv\; c\,\delta t\,\mathbf{E} \;+\; \delta\mathbf{x}\times\mathbf{B} \;-\; \nabla\delta\chi$$

Guiding-center energy conservation law

Guiding-center energy conservation law

$$\frac{\partial \mathcal{E}_{\rm gc}}{\partial t} + \nabla\cdot\mathbf{S}_{\rm gc} = 0$$

- Guiding-center energy density ($K_{\rm gc} = \mu\, B + p_{\|}^{2}/2m$)

$$\mathcal{E}_{\rm gc} \equiv \int F_{\mu}\, K_{\rm gc}\, d{\rm p}_{\|}\, d\mu + \frac{1}{8\pi}\left(|\mathbf{E}|^{2} + |\mathbf{B}|^{2}\right)$$

- Guiding-center energy-density flux

$$\mathbf{S}_{\rm gc} \equiv \int F_{\mu}\, K_{\rm gc}\, \dot{\mathbf{X}}\, d{\rm p}_{\|}\, d\mu + \frac{c}{4\pi}\,\mathbf{E}\times\mathbf{H}$$

Guiding-center momentum conservation law

Guiding-center momentum conservation law

$$\frac{\partial \mathbf{P}_{\rm gc}}{\partial t} \;+\; \nabla\cdot\mathsf{T}_{\rm gc} \;=\; 0$$

- Guiding-center momentum density

$$\mathbf{P}_{\rm gc} \;\equiv\; \int \mathsf{p}_{\|}\,\widehat{\mathsf{b}}\,F_{\mu}\,d\mathsf{p}_{\|}\,d\mu \;+\; \frac{\mathbf{E}\times\mathbf{B}}{4\pi\,c}$$

- Symmetric guiding-center stress tensor $\mathsf{T}_{\rm gc} \equiv \mathsf{T}_{\rm M} + \mathsf{T}_{\rm gcV}$

$$\mathsf{T}_{\rm M} \;\equiv\; \left(|\mathbf{E}|^2 \;+\; |\mathbf{B}|^2\right)\frac{\mathbf{I}}{8\pi} - \frac{1}{4\pi}\left(\mathbf{EE} \;+\; \mathbf{BB}\right)$$

$$\mathsf{T}_{\rm gcV} \;\equiv\; \mathsf{P}_{\rm CGL} + \int\left(\dot{\mathbf{X}}_{\perp}\;\;\mathsf{p}_{\|}\,\widehat{\mathsf{b}} + \mathsf{p}_{\|}\,\widehat{\mathsf{b}}\;\;\dot{\mathbf{X}}_{\perp}\right)\;F_{\mu}\,d\mathsf{p}_{\|}\,d\mu$$

- CGL pressure tensor

$$\mathsf{P}_{\rm CGL} \equiv \int\left[\frac{\mathsf{p}_{\|}^2}{m}\,\widehat{\mathsf{b}}\,\widehat{\mathsf{b}} + \;\mu\,B\;\left(\mathbf{I} - \widehat{\mathsf{b}\mathsf{b}}\right)\right]\;F_{\mu}\,d\mathsf{p}_{\|}\,d\mu$$

Guiding-center toroidal angular momentum conservation law

- Toroidal covariant component $P_{\mathrm{gc}\varphi} \;\equiv\; \mathbf{P}_{\mathrm{gc}} \cdot \partial\mathbf{x}/\partial\varphi$

$$\frac{\partial P_{\mathrm{gc}\varphi}}{\partial t} \;+\; \nabla\cdot\left(\mathsf{T}_{\mathrm{gc}}\cdot\frac{\partial\mathbf{x}}{\partial\varphi}\right) \;=\; \nabla\left(\frac{\partial\mathbf{x}}{\partial\varphi}\right) : \mathsf{T}_{\mathrm{gc}}^{\top} \;\equiv\; 0$$

Symmetric guiding-center stress tensor

$$\mathsf{T}_{\mathrm{gc}}^{\top} \;\equiv\; \mathsf{T}_{\mathrm{gc}}$$

- Guiding-center stress tensor T_{gc} was previously only assumed to be symmetric (e.g., Similon 1985).

- Guiding-center polarization is crucial in establishing symmetry

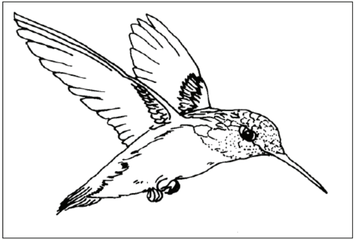
$$\mathsf{T}_{\mathrm{gcV}} \;\equiv\; \mathsf{P}_{\mathrm{CGL}} + \int\left(\dot{\mathbf{X}}_{\perp}\;\; p_{\|}\,\widehat{\mathsf{b}} + p_{\|}\,\widehat{\mathsf{b}}\;\;\dot{\mathbf{X}}_{\perp}\right)\; F_{\mu}\, dp_{\|}\, d\mu$$

III. Lecture Notes on Gyrokinetic Theory

Two-week course on gyrokinetic theory (LANL 2013)

- Over 450 slides with references (pdf copy available upon request)

Lectures on Gyrokinetic Theory



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Tentative Table of Contents (expected length 500 pages)

IV. Ongoing Work Related to Gyrokinetic Theory

Linear and Nonlinear Hybrid kinetic-MHD Variational Principles for Energetic Ions & Electrons (APS-DPP 2016)

- Physical nature of current-coupling and pressure-coupling formulations: Particle versus Reduced Vlasov-Maxwell equations

- Resonant three-wave interactions in kinetic-MHD models described by Manley-Rowe relations (involving wave action)

Geometric Methods in Gyrokinetic Theory (APS-DPP 2016)

- Gyrokinetic Hamiltonian field theory (with Morrison & Burby)

- Functional Lie-transform field perturbation theory