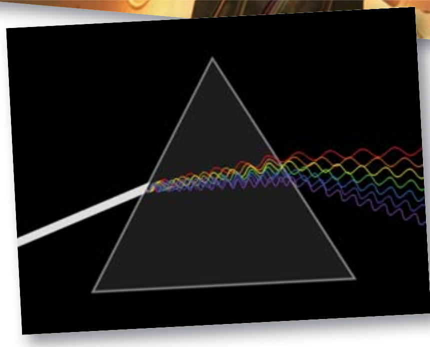
6.1

Reciprocal, Quotient, and Pythagorean Identities



Engineer using an electronic spin resonance spectroscope

Focus on…

- verifying a trigonometric identity numerically and graphically using technology
- exploring reciprocal, quotient, and Pythagorean identities
- determining non-permissible values of trigonometric identities
- explaining the difference between a trigonometric identity and a trigonometric equation

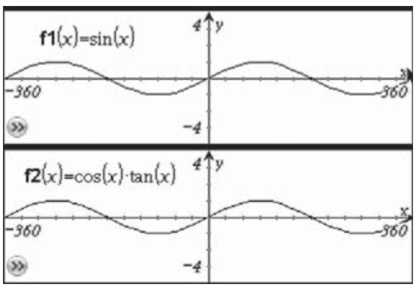
Digital music players store large sound files by using trigonometry to compress (store) and then decompress (play) the file when needed. A large sound file can be stored in a much smaller space using this technique. Electronics engineers have learned how to use the periodic nature of music to compress the audio file into a smaller space.

Investigate Comparing Two Trigonometric Expressions

Materials

- graphing technology

1. Graph the curves $y = \sin x$ and $y = \cos x \tan x$ over the domain $-360° \leq x < 360°$. Graph the curves on separate grids using the same range and scale. What do you notice?



2. Make and analyse a table of values for these functions in multiples of 30° over the domain $-360° \leq x < 360°$. Describe your findings.

3. Use your knowledge of $\tan x$ to simplify the expression $\cos x \tan x$.

Reflect and Respond

4. a) Are the curves $y = \sin x$ and $y = \cos x \tan x$ identical? Explain your reasoning.

b) Why was it important to look at the graphs *and* at the table of values?

5. What are the non-permissible values of x in the equation $\sin x = \cos x \tan x$? Explain.

6. Are there any permissible values for x outside the domain in step 2 for which the expressions $\sin x$ and $\cos x \tan x$ are not equal? Share your response with a classmate.

Link the Ideas

The equation $\sin x = \cos x \tan x$ that you explored in the investigation is an example of a **trigonometric identity**. Both sides of the equation have the same value for all permissible values of x. In other words, when the expressions on either side of the equal sign are evaluated for any permissible value, the resulting values are equal. Trigonometric identities can be verified both numerically and graphically.

trigonometric identity

- a trigonometric equation that is true for all permissible values of the variable in the expressions on both sides of the equation

You are familiar with two groups of identities from your earlier work with trigonometry: the reciprocal identities and the quotient identity.

Reciprocal Identities

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{1}{\tan x}$$

Quotient Identities

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

Example 1

Verify a Potential Identity Numerically and Graphically

a) Determine the non-permissible values, in degrees, for the equation $\sec \theta = \frac{\tan \theta}{\sin \theta}$.

b) Numerically verify that $\theta = 60°$ and $\theta = \frac{\pi}{4}$ are solutions of the equation.

c) Use technology to graphically decide whether the equation could be an identity over the domain $-360° < \theta \leq 360°$.

Solution

a) To determine the non-permissible values, assess each trigonometric function in the equation individually and examine expressions that may have non-permissible values. Visualize the graphs of $y = \sin x$, $y = \cos x$ and $y = \tan x$ to help you determine the non-permissible values.

First consider the left side, sec θ:

$\sec \theta = \frac{1}{\cos \theta}$, and $\cos \theta = 0$ when $\theta = 90°, 270°, \ldots$.

So, the non-permissible values for sec θ are $\theta \neq 90° + 180°n$, where $n \in I$.

Now consider the right side, $\frac{\tan \theta}{\sin \theta}$:

tan θ is not defined when $\theta = 90°, 270°, \ldots$.

Why must these values be excluded?

So, the non-permissible values for tan θ are $\theta \neq 90° + 180°n$, where $n \in I$.

How do these non-permissible values compare to the ones found for the left side?

Also, the expression $\frac{\tan \theta}{\sin \theta}$ is undefined when $\sin \theta = 0$.

$\sin \theta = 0$ when $\theta = 0°, 180°, \ldots$.

So, further non-permissible values for $\frac{\tan \theta}{\sin \theta}$ are $\theta \neq 180°n$, where $n \in I$.

Are these non-permissible values included in the ones already found?

The three sets of non-permissible values for the equation $\sec \theta = \frac{\tan \theta}{\sin \theta}$ can be expressed as a single restriction, $\theta \neq 90°n$, where $n \in I$.

b) Substitute $\theta = 60°$.

$$\begin{aligned}\text{Left Side} &= \sec \theta \\ &= \sec 60° \\ &= \frac{1}{\cos 60°} \\ &= \frac{1}{0.5} \\ &= 2\end{aligned}$$

$$\begin{aligned}\text{Right Side} &= \frac{\tan \theta}{\sin \theta} \\ &= \frac{\tan 60°}{\sin 60°} \\ &= \frac{\sqrt{3}}{\frac{\sqrt{3}}{2}} \\ &= 2\end{aligned}$$

Why does substituting 60° in both sides of the equation not prove that the identity is true?

Left Side = Right Side

The equation $\sec \theta = \frac{\tan \theta}{\sin \theta}$ is true for $\theta = 60°$.

Substitute $\theta = \frac{\pi}{4}$.

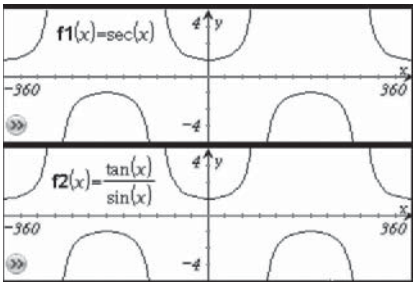
$$\begin{aligned}\text{Left Side} &= \sec \theta \\ &= \sec \frac{\pi}{4} \\ &= \frac{1}{\cos \frac{\pi}{4}} \\ &= \frac{1}{\frac{1}{\sqrt{2}}} \\ &= \sqrt{2}\end{aligned}$$

$$\begin{aligned}\text{Right Side} &= \frac{\tan \theta}{\sin \theta} \\ &= \frac{\tan \frac{\pi}{4}}{\sin \frac{\pi}{4}} \\ &= \frac{1}{\frac{1}{\sqrt{2}}} \\ &= \sqrt{2}\end{aligned}$$

Left Side = Right Side

The equation $\sec \theta = \frac{\tan \theta}{\sin \theta}$ is true for $\theta = \frac{\pi}{4}$.

c) Use technology, with domain $-360° < x \leq 360°$, to graph $y = \sec \theta$ and $y = \frac{\tan \theta}{\sin \theta}$. The graphs look identical, so $\sec \theta = \frac{\tan \theta}{\sin \theta}$ could be an identity.



How do these graphs show that there are non-permissible values for this identity?

Does graphing the related functions on each side of the equation prove that the identity is true? Explain.

Your Turn

a) Determine the non-permissible values, in degrees, for the equation $\cot x = \frac{\cos x}{\sin x}$.

b) Verify that $x = 45°$ and $x = \frac{\pi}{6}$ are solutions to the equation.

c) Use technology to graphically decide whether the equation could be an identity over the domain $-360° < x \leq 360°$.

Example 2

Use Identities to Simplify Expressions

a) Determine the non-permissible values, in radians, of the variable in the expression $\frac{\cot x}{\csc x \cos x}$.

b) Simplify the expression.

Solution

a) The trigonometric functions cot x and csc x both have non-permissible values in their domains.

For $\cot x$, $x \neq \pi n$, where $n \in \mathrm{I}$.
For $\csc x$, $x \neq \pi n$, where $n \in \mathrm{I}$.

Why are these the non-permissible values for both reciprocal functions?

Also, the denominator of $\frac{\cot x}{\csc x \cos x}$ cannot equal zero. In other words, $\csc x \cos x \neq 0$.

There are no values of x that result in $\csc x = 0$.

However, for $\cos x$, $x \neq \frac{\pi}{2} + \pi n$, where $n \in \mathrm{I}$.

Combined, the non-permissible values for $\frac{\cot x}{\csc x \cos x}$ are $x \neq \frac{\pi}{2}n$, where $n \in \mathrm{I}$.

Why can you write this single general restriction?

b) To simplify the expression, use reciprocal and quotient identities to write trigonometric functions in terms of cosine and sine.

$$\frac{\cot x}{\csc x \cos x} = \frac{\frac{\cos x}{\sin x}}{\frac{1}{\sin x}\cos x}$$

$$= \frac{\frac{\cos x}{\sin x}}{\frac{\cos x}{\sin x}}$$ Simplify the fraction.

$$= 1$$

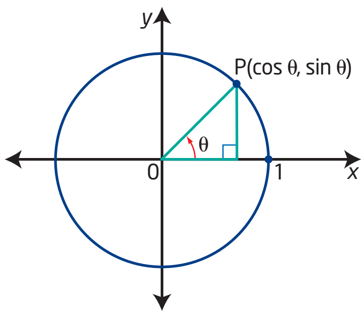
Your Turn

a) Determine the non-permissible values, in radians, of the variable in the expression $\frac{\sec x}{\tan x}$.

b) Simplify the expression.

Pythagorean Identity

Recall that point P on the terminal arm of an angle θ in standard position has coordinates (cos θ, sin θ). Consider a right triangle with a hypotenuse of 1 and legs of cos θ and sin θ.



The hypotenuse is 1 because it is the radius of the unit circle. Apply the Pythagorean theorem in the right triangle to establish the Pythagorean identity:

$$x^2 + y^2 = 1^2$$
$$\cos^2 \theta + \sin^2 \theta = 1$$

Example 3

Use the Pythagorean Identity

a) Verify that the equation $\cot^2 x + 1 = \csc^2 x$ is true when $x = \frac{\pi}{6}$.

b) Use quotient identities to express the Pythagorean identity $\cos^2 x + \sin^2 x = 1$ as the equivalent identity $\cot^2 x + 1 = \csc^2 x$.

Solution

a) Substitute $x = \frac{\pi}{6}$.

$$\begin{aligned}\text{Left Side} &= \cot^2 x + 1 \\ &= \cot^2 \frac{\pi}{6} + 1 \\ &= \frac{1}{\tan^2 \frac{\pi}{6}} + 1 \\ &= \frac{1}{\frac{1}{(\sqrt{3})^2}} + 1 \\ &= (\sqrt{3})^2 + 1 \\ &= 4\end{aligned}$$

$$\begin{aligned}\text{Right Side} &= \csc^2 x \\ &= \csc^2 \frac{\pi}{6} \\ &= \frac{1}{\sin^2 \frac{\pi}{6}} \\ &= \frac{1}{\left(\frac{1}{2}\right)^2} \\ &= 2^2 \\ &= 4\end{aligned}$$

Left Side = Right Side

The equation $\cot^2 x + 1 = \csc^2 x$ is true when $x = \frac{\pi}{6}$.

b) $\cos^2 x + \sin^2 x = 1$

Since this identity is true for all permissible values of x, you can multiply both sides by $\frac{1}{\sin^2 x}$, $x \neq \pi n$, where $n \in \mathrm{I}$.

$$\begin{aligned}\left(\frac{1}{\sin^2 x}\right)\cos^2 x + \left(\frac{1}{\sin^2 x}\right)\sin^2 x &= \left(\frac{1}{\sin^2 x}\right)1 \\ \frac{\cos^2 x}{\sin^2 x} + 1 &= \frac{1}{\sin^2 x} \\ \cot^2 x + 1 &= \csc^2 x\end{aligned}$$

Why multiply both sides by $\frac{1}{\sin^2 x}$? How else could you simplify this equation?

Your Turn

a) Verify the equation $1 + \tan^2 x = \sec^2 x$ numerically for $x = \frac{3\pi}{4}$.

b) Express the Pythagorean identity $\cos^2 x + \sin^2 x = 1$ as the equivalent identity $1 + \tan^2 x = \sec^2 x$.

The three forms of the Pythagorean identity are

$\cos^2 \theta + \sin^2 \theta = 1$ $\quad$ $\cot^2 \theta + 1 = \csc^2 \theta$ $\quad$ $1 + \tan^2 \theta = \sec^2 \theta$

Key Ideas

- A trigonometric identity is an equation involving trigonometric functions that is true for all permissible values of the variable.
- You can verify trigonometric identities
  - numerically by substituting specific values for the variable
  - graphically, using technology
- Verifying that two sides of an equation are equal for given values, or that they appear equal when graphed, is not sufficient to conclude that the equation is an identity.
- You can use trigonometric identities to simplify more complicated trigonometric expressions.
- The reciprocal identities are

  $\csc x = \frac{1}{\sin x}$ $\quad \sec x = \frac{1}{\cos x}$ $\quad \cot x = \frac{1}{\tan x}$
- The quotient identities are

  $\tan x = \frac{\sin x}{\cos x}$ $\quad \cot x = \frac{\cos x}{\sin x}$
- The Pythagorean identities are

  $\cos^2 x + \sin^2 x = 1$ $\quad 1 + \tan^2 x = \sec^2 x$ $\quad \cot^2 x + 1 = \csc^2 x$

Check Your Understanding

Practise

1. Determine the non-permissible values of x, in radians, for each expression.

a) $\frac{\cos x}{\sin x}$ **b)** $\frac{\sin x}{\tan x}$

c) $\frac{\cot x}{1 - \sin x}$ **d)** $\frac{\tan x}{\cos x + 1}$

2. Why do some identities have non-permissible values?

3. Simplify each expression to one of the three primary trigonometric functions, $\sin x$, $\cos x$ or $\tan x$. For part a), verify graphically, using technology, that the given expression is equivalent to its simplified form.

a) $\sec x \sin x$

b) $\sec x \cot x \sin^2 x$

c) $\frac{\cos x}{\cot x}$

4. Simplify, and then rewrite each expression as one of the three reciprocal trigonometric functions, $\csc x$, $\sec x$, or $\cot x$.

a) $\left(\frac{\cos x}{\tan x}\right)\left(\frac{\tan x}{\sin x}\right)$

b) $\csc x \cot x \sec x \sin x$

c) $\frac{\cos x}{1 - \sin^2 x}$

5. a) Verify that the equation $\frac{\sec x}{\tan x + \cot x} = \sin x$ is true for $x = 30°$ and for $x = \frac{\pi}{4}$.

b) What are the non-permissible values of the equation in the domain $0° \leq x < 360°$?

6. Consider the equation
$\frac{\sin x \cos x}{1 + \cos x} = \frac{1 - \cos x}{\tan x}$.

a) What are the non-permissible values, in radians, for this equation?

b) Graph the two sides of the equation using technology, over the domain $0 \leq x < 2\pi$. Could it be an identity?

c) Verify that the equation is true when $x = \frac{\pi}{4}$. Use exact values for each expression in the equation.

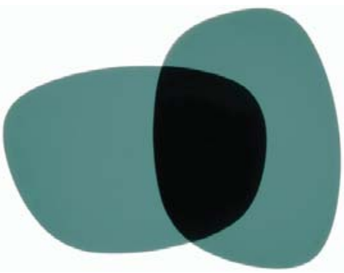
Apply

7. When a polarizing lens is rotated through an angle θ over a second lens, the amount of light passing through both lenses decreases by $1 - \sin^2 \theta$.

a) Determine an equivalent expression for this decrease using only cosine.

b) What fraction of light is lost when $\theta = \frac{\pi}{6}$?

c) What percent of light is lost when $\theta = 60°$?



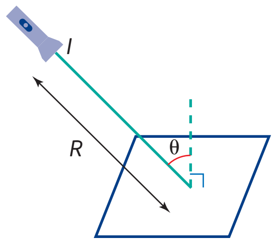
8. Compare $y = \sin x$ and $y = \sqrt{1 - \cos^2 x}$ by completing the following.

a) Verify that $\sin x = \sqrt{1 - \cos^2 x}$ for $x = \frac{\pi}{3}$, $x = \frac{5\pi}{6}$, and $x = \pi$.

b) Graph $y = \sin x$ and $y = \sqrt{1 - \cos^2 x}$ in the same window.

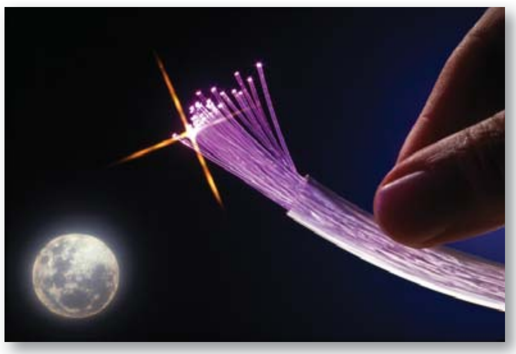
c) Determine whether $\sin x = \sqrt{1 - \cos^2 x}$ is an identity. Explain your answer.

9. Illuminance (E) is a measure of the amount of light coming from a light source and falling onto a surface. If the light is projected onto the surface at an angle θ, measured from the perpendicular, then a formula relating these values is $\sec \theta = \frac{I}{ER^2}$, where I is a measure of the luminous intensity and R is the distance between the light source and the surface.



a) Rewrite the formula so that E is isolated and written in terms of $\cos \theta$.

b) Show that $E = \frac{I \cot \theta}{R^2 \csc \theta}$ is equivalent to your equation from part a).



Fibre optic cable

10. Simplify $\frac{\csc x}{\tan x + \cot x}$ to one of the three primary trigonometric ratios. What are the non-permissible values of the original expression in the domain $0 \leq x < 2\pi$?

11. a) Determine graphically, using technology, whether the expression $\frac{\csc^2 x - \cot^2 x}{\cos x}$ appears to be equivalent to csc x or sec x.

b) What are the non-permissible values, in radians, for the identity from part a)?

c) Express $\frac{\csc^2 x - \cot^2 x}{\cos x}$ as the single reciprocal trigonometric ratio that you identified in part a).

12. a) Substitute $x = \frac{\pi}{4}$ into the equation $\frac{\cot x}{\sec x} + \sin x = \csc x$ to determine whether it could be an identity. Use exact values.

b) Algebraically confirm that the expression on the left side simplifies to csc x.

13. Stan, Lina, and Giselle are working together to try to determine whether the equation $\sin x + \cos x = \tan x + 1$ is an identity.

a) Stan substitutes $x = 0$ into each side of the equation. What is the result?

b) Lina substitutes $x = \frac{\pi}{2}$ into each side of the equation. What does she observe?

c) Stan points out that Lina's choice is not permissible for this equation. Explain why.

d) Giselle substitutes $x = \frac{\pi}{4}$ into each side of the equation. What does she find?

e) Do the three students have enough information to conclude whether or not the given equation is an identity? Explain.

14. Simplify $(\sin x + \cos x)^2 + (\sin x - \cos x)^2$.

Extend

15. Given $\csc^2 x + \sin^2 x = 7.89$, find the value of $\frac{1}{\csc^2 x} + \frac{1}{\sin^2 x}$.

16. Show algebraically that

$$\frac{1}{1 + \sin \theta} + \frac{1}{1 - \sin \theta} = 2 \sec^2 \theta$$

is an identity.

17. Determine an expression for m that makes $\frac{2 - \cos^2 x}{\sin x} = m + \sin x$ an identity.

Create Connections

C1 Explain how a student who does not know the $\cot^2 x + 1 = \csc^2 x$ form of the Pythagorean identity could simplify an expression that contained the expression $\cot^2 x + 1$ using the fact that $1 = \frac{\sin^2 x}{\sin^2 x}$.

C2 For some trigonometric expressions, multiplying by a conjugate helps to simplify the expression. Simplify $\frac{\sin \theta}{1 + \cos \theta}$ by multiplying the numerator and the denominator by the conjugate of the denominator, $1 - \cos \theta$. Describe how this process helps to simplify the expression.

C3 MINI LAB Explore the effect of different domains on apparent identities.

Materials
- graphing calculator

Step 1 Graph the two functions $y = \tan x$ and $y = \left|\frac{\sin x}{\cos x}\right|$ on the same grid, using a domain of $0 \le x < \frac{\pi}{2}$. Is there graphical evidence that $\tan x = \left|\frac{\sin x}{\cos x}\right|$ is an identity? Explain.

Step 2 Graph the two functions $y = \tan x$ and $y = \left|\frac{\sin x}{\cos x}\right|$ again, using the expanded domain $-2\pi < x \le 2\pi$. Is the equation $\tan x = \left|\frac{\sin x}{\cos x}\right|$ an identity? Explain.

Step 3 Find and record a different trigonometric equation that is true over a restricted domain but is not an identity when all permissible values are checked. Compare your answer with that of a classmate.

Step 4 How does this activity show the weakness of using graphical and numerical methods for verifying potential identities?

6.2

Sum, Difference, and Double-Angle Identities

Focus on...

- applying sum, difference, and double-angle identities to verify the equivalence of trigonometric expressions
- verifying a trigonometric identity numerically and graphically using technology


Paris gold box

In addition to holograms and security threads, paper money often includes special Guilloché patterns in the design to prevent counterfeiting. The sum and product of nested sinusoidal functions are used to form the blueprint of some of these patterns. Guilloché patterns have been created since the sixteenth century, but their origin is uncertain. They can be found carved in wooden door frames and etched on the metallic surfaces of objects such as vases.

Web Link

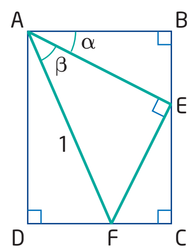
To learn more about Guilloché patterns, go to www.mcgrawhill.ca/school/learningcentres and follow the links.

Investigate Expressions for sin (α + β) and cos (α + β)

Materials

- ruler
- protractor

1. a) Draw a large rectangle and label its vertices A, B, C, and D, where BC < 2AB. Mark a point E on BC. Join AE and use a protractor to draw EF perpendicular to AE. Label all right angles on your diagram. Label ∠BAE as α and ∠EAF as β.



b) Measure the angles α and β. Use the angle sum of a triangle to determine the measures of all the remaining acute angles in your diagram. Record their measures on the diagram.

2. a) Explain how you know that ∠CEF = α.

b) Determine an expression for each of the other acute angles in the diagram in terms of α and β. Label each angle on your diagram.

3. Suppose the hypotenuse AF of the inscribed right triangle has a length of 1 unit. Explain why the length of AE can be represented as $\cos \beta$. Label AE as $\cos \beta$.

4. Determine expressions for line segments AB, BE, EF, CE, CF, AD, and DF in terms of $\sin \alpha$, $\cos \alpha$, $\sin \beta$, and $\cos \beta$. Label each side length on your diagram using these sines and cosines. Note that AD equals the sum of segments BE and EC, and DF equals AB minus CF.

5. Which angle in the diagram is equivalent to $\alpha + \beta$? Determine possible identities for $\sin (\alpha + \beta)$ and $\cos (\alpha + \beta)$ from $\triangle$ADF using the sum or difference of lengths. Compare your results with those of a classmate.

Reflect and Respond

6. a) Verify your possible identities numerically using the measures of α and β from step 1. Compare your results with those of a classmate.
   b) Does each identity apply to angles that are obtuse? Are there any restrictions on the domain? Describe your findings.

7. Consider the special case where $\alpha = \beta$. Write simplified equivalent expressions for $\sin 2\alpha$ and $\cos 2\alpha$.

Link the Ideas

In the investigation, you discovered the angle sum identities for sine and cosine. These identities can be used to determine the angle sum identity for tangent.

The sum identities are

$$\sin (A + B) = \sin A \cos B + \cos A \sin B$$
$$\cos (A + B) = \cos A \cos B - \sin A \sin B$$
$$\tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

The angle sum identities for sine, cosine, and tangent can be used to determine angle difference identities for sine, cosine, and tangent.

For sine,

$$\begin{aligned} \sin (A - B) &= \sin (A + (-B)) \\ &= \sin A \cos (-B) + \cos A \sin (-B) \\ &= \sin A \cos B + \cos A (-\sin B) \\ &= \sin A \cos B - \cos A \sin B \end{aligned}$$

Why is $\cos (-B) = \cos B$?
Why is $\sin (-B) = -\sin B$?

Web Link

To see a derivation of the difference $\cos (A - B)$, go to www.mcgrawhill.ca/school/learningcentres and follow the links.

The three angle difference identities are

$$\sin (A - B) = \sin A \cos B - \cos A \sin B$$
$$\cos (A - B) = \cos A \cos B + \sin A \sin B$$
$$\tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

A special case occurs in the angle sum identities when A = B. Substituting B = A results in the double-angle identities.

For example,

$$\begin{aligned}\sin 2A &= \sin (A + A)\\ &= \sin A \cos A + \cos A \sin A\\ &= 2 \sin A \cos A\end{aligned}$$

Similarly, it can be shown that

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

The double-angle identities are

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Example 1

Simplify Expressions Using Sum, Difference, and Double-Angle Identities

Write each expression as a single trigonometric function.

a) $\sin 48° \cos 17° - \cos 48° \sin 17°$

b) $\cos^2 \frac{\pi}{3} - \sin^2 \frac{\pi}{3}$

Solution

a) The expression $\sin 48° \cos 17° - \cos 48° \sin 17°$ has the same form as the right side of the difference identity for sine, $\sin (A - B) = \sin A \cos B - \cos A \sin B$.

Thus,

$$\begin{aligned}\sin 48° \cos 17° - \cos 48° \sin 17° &= \sin (48° - 17°)\\ &= \sin 31°\end{aligned}$$

b) The expression $\cos^2 \frac{\pi}{3} - \sin^2 \frac{\pi}{3}$ has the same form as the right side of the double-angle identity for cosine, $\cos 2A = \cos^2 A - \sin^2 A$.

Therefore,

$$\begin{aligned}\cos^2 \frac{\pi}{3} - \sin^2 \frac{\pi}{3} &= \cos \left(2\left(\frac{\pi}{3}\right)\right)\\ &= \cos \frac{2\pi}{3}\end{aligned}$$

How could you use technology to verify these solutions?

Your Turn

Write each expression as a single trigonometric function.

a) $\cos 88° \cos 35° + \sin 88° \sin 35°$

b) $2 \sin \frac{\pi}{12} \cos \frac{\pi}{12}$

Example 2

Determine Alternative Forms of the Double-Angle Identity for Cosine

Determine an identity for cos 2A that contains only the cosine ratio.

Solution

An identity for cos 2A is $\cos 2A = \cos^2 A - \sin^2 A$.

Write an equivalent expression for the term containing sin A.

Use the Pythagorean identity, $\cos^2 A + \sin^2 A = 1$.

Substitute $\sin^2 A = 1 - \cos^2 A$ to obtain another form of the double-angle identity for cosine.

$$\begin{aligned} \cos 2A &= \cos^2 A - \sin^2 A \\ &= \cos^2 A - (1 - \cos^2 A) \\ &= \cos^2 A - 1 + \cos^2 A \\ &= 2\cos^2 A - 1 \end{aligned}$$

Your Turn

Determine an identity for cos 2A that contains only the sine ratio.

Example 3

Simplify Expressions Using Identities

Consider the expression $\dfrac{1 - \cos 2x}{\sin 2x}$.

a) What are the permissible values for the expression?

b) Simplify the expression to one of the three primary trigonometric functions.

c) Verify your answer from part b), in the interval $[0, 2\pi)$, using technology.

Solution

a) Identify any non-permissible values. The expression is undefined when $\sin 2x = 0$.

Method 1: Simplify the Double Angle

Use the double-angle identity for sine to simplify $\sin 2x$ first.

$\sin 2x = 2 \sin x \cos x$

$2 \sin x \cos x \neq 0$

So, $\sin x \neq 0$ and $\cos x \neq 0$.

$\sin x = 0$ when $x = \pi n$, where $n \in I$.

$\cos x = 0$ when $x = \dfrac{\pi}{2} + \pi n$, where $n \in I$.

When these two sets of non-permissible values are combined, the permissible values for the expression are all real numbers except $x \neq \dfrac{\pi n}{2}$, where $n \in I$.

Method 2: Horizontal Transformation of sin *x*

First determine when $\sin x = 0$. Then, stretch the domain horizontally by a factor of $\frac{1}{2}$.

$\sin x = 0$ when $x = \pi n$, where $n \in \mathrm{I}$.

Therefore, $\sin 2x = 0$ when $x = \frac{\pi n}{2}$, where $n \in \mathrm{I}$.

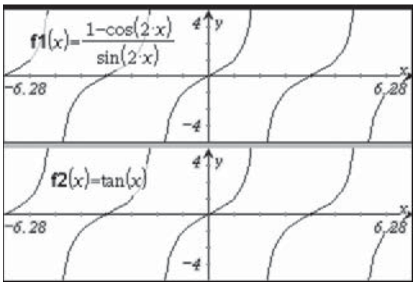
The permissible values of the expression $\frac{1 - \cos 2x}{\sin 2x}$ are all real numbers except $x \neq \frac{\pi n}{2}$, where $n \in \mathrm{I}$.

b)
$$\begin{aligned}\frac{1 - \cos 2x}{\sin 2x} &= \frac{1 - (1 - 2\sin^2 x)}{2 \sin x \cos x} \\ &= \frac{2\sin^2 x}{2 \sin x \cos x} \\ &= \frac{\sin x}{\cos x} \\ &= \tan x\end{aligned}$$

Replace sin 2*x* in the denominator. Replace cos 2*x* with the form of the identity from Example 2 that will simplify most fully.

The expression $\frac{1 - \cos 2x}{\sin 2x}$ is equivalent to $\tan x$.

c) Use technology, with domain $0 \leq x < 2\pi$, to graph $y = \frac{1 - \cos 2x}{\sin 2x}$ and $y = \tan x$. The graphs look identical, which verifies, but does not prove, the answer in part b).



Your Turn

Consider the expression $\frac{\sin 2x}{\cos 2x + 1}$.

a) What are the permissible values for the expression?

b) Simplify the expression to one of the three primary trigonometric functions.

c) Verify your answer from part b), in the interval $[0, 2\pi)$, using technology.

Example 4

Determine Exact Trigonometric Values for Angles

Determine the exact value for each expression.

a) $\sin \frac{\pi}{12}$

b) $\tan 105°$

Solution

a) Use the difference identity for sine with two special angles.

For example, because $\frac{\pi}{12} = \frac{3\pi}{12} - \frac{2\pi}{12}$, use $\frac{\pi}{4} - \frac{\pi}{6}$.

$$\begin{aligned}\sin \frac{\pi}{12} &= \sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right)\\ &= \sin \frac{\pi}{4} \cos \frac{\pi}{6} - \cos \frac{\pi}{4} \sin \frac{\pi}{6}\\ &= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)\\ &= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}\\ &= \frac{\sqrt{6} - \sqrt{2}}{4}\end{aligned}$$

The special angles $\frac{\pi}{3}$ and $\frac{\pi}{4}$ could also be used.

Use $\sin (A - B) = \sin A \cos B - \cos A \sin B$.

How could you verify this answer with a calculator?

b) Method 1: Use the Difference Identity for Tangent

Rewrite tan 105° as a difference of special angles.

$\tan 105° = \tan (135° - 30°)$

Are there other ways of writing 105° as the sum or difference of two special angles?

Use the tangent difference identity, $\tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$.

$$\begin{aligned}\tan (135° - 30°) &= \frac{\tan 135° - \tan 30°}{1 + \tan 135° \tan 30°}\\ &= \frac{-1 - \frac{1}{\sqrt{3}}}{1 + (-1)\left(\frac{1}{\sqrt{3}}\right)}\\ &= \frac{-1 - \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}}\\ &= \left(\frac{-1 - \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}}\right)\left(\frac{-\sqrt{3}}{-\sqrt{3}}\right)\\ &= \frac{\sqrt{3} + 1}{1 - \sqrt{3}}\end{aligned}$$

Simplify.

Multiply numerator and denominator by $-\sqrt{3}$.

How could you rationalize the denominator?

Method 2: Use a Quotient Identity with Sine and Cosine

$$\tan 105° = \frac{\sin 105°}{\cos 105°}$$

$$= \frac{\sin (60° + 45°)}{\cos (60° + 45°)}$$

$$= \frac{\sin 60° \cos 45° + \cos 60° \sin 45°}{\cos 60° \cos 45° - \sin 60° \sin 45°}$$

$$= \frac{\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)}{\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)}$$

$$= \frac{\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}}{\frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}}$$

$$= \left(\frac{\sqrt{6} + \sqrt{2}}{4}\right)\left(\frac{4}{\sqrt{2} - \sqrt{6}}\right)$$

$$= \frac{\sqrt{6} + \sqrt{2}}{\sqrt{2} - \sqrt{6}}$$

Use sum identities with special angles. Could you use a difference of angles identity here?

How could you verify that this is the same answer as in Method 1?

Your Turn

Use a sum or difference identity to find the exact values of

a) $\cos 165°$ **b)** $\tan \frac{11\pi}{12}$

Key Ideas

- You can use the sum and difference identities to simplify expressions and to determine exact trigonometric values for some angles.

Sum Identities

$$\sin (A + B) = \sin A \cos B + \cos A \sin B$$

$$\cos (A + B) = \cos A \cos B - \sin A \sin B$$

$$\tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Difference Identities

$$\sin (A - B) = \sin A \cos B - \cos A \sin B$$

$$\cos (A - B) = \cos A \cos B + \sin A \sin B$$

$$\tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

- The double-angle identities are special cases of the sum identities when the two angles are equal. The double-angle identity for cosine can be expressed in three forms using the Pythagorean identity, $\cos^2 A + \sin^2 A = 1$.

Double-Angle Identities

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$\cos 2A = 2 \cos^2 A - 1$$

$$\cos 2A = 1 - 2 \sin^2 A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Check Your Understanding

Practise

1. Write each expression as a single trigonometric function.

a) $\cos 43° \cos 27° - \sin 43° \sin 27°$

b) $\sin 15° \cos 20° + \cos 15° \sin 20°$

c) $\cos^2 19° - \sin^2 19°$

d) $\sin \frac{3\pi}{2} \cos \frac{5\pi}{4} - \cos \frac{3\pi}{2} \sin \frac{5\pi}{4}$

e) $8 \sin \frac{\pi}{3} \cos \frac{\pi}{3}$

2. Simplify and then give an exact value for each expression.

a) $\cos 40° \cos 20° - \sin 40° \sin 20°$

b) $\sin 20° \cos 25° + \cos 20° \sin 25°$

c) $\cos^2 \frac{\pi}{6} - \sin^2 \frac{\pi}{6}$

d) $\cos \frac{\pi}{2} \cos \frac{\pi}{3} - \sin \frac{\pi}{2} \sin \frac{\pi}{3}$

3. Using only one substitution, which form of the double-angle identity for cosine will simplify the expression $1 - \cos 2x$ to one term? Show how this happens.

4. Write each expression as a single trigonometric function.

a) $2 \sin \frac{\pi}{4} \cos \frac{\pi}{4}$

b) $(6 \cos^2 24° - 6 \sin^2 24°) \tan 48°$

c) $\frac{2 \tan 76°}{1 - \tan^2 76°}$

d) $2 \cos^2 \frac{\pi}{6} - 1$

e) $1 - 2 \cos^2 \frac{\pi}{12}$

5. Simplify each expression to a single primary trigonometric function.

a) $\frac{\sin 2\theta}{2 \cos \theta}$

b) $\cos 2x \cos x + \sin 2x \sin x$

c) $\frac{\cos 2\theta + 1}{2 \cos \theta}$

d) $\frac{\cos^3 x}{\cos 2x + \sin^2 x}$

6. Show using a counterexample that the following is not an identity: $\sin (x - y) = \sin x - \sin y$.

7. Simplify $\cos (90° - x)$ using a difference identity.

8. Determine the exact value of each trigonometric expression.

a) $\cos 75°$

b) $\tan 165°$

c) $\sin \frac{7\pi}{12}$

d) $\cos 195°$

e) $\csc \frac{\pi}{12}$

f) $\sin \left(-\frac{\pi}{12}\right)$

Apply

Yukon River at Whitehorse

9. On the winter solstice, December 21 or 22, the power, P, in watts, received from the sun on each square metre of Earth can be determined using the equation $P = 1000 (\sin x \cos 113.5° + \cos x \sin 113.5°)$, where x is the latitude of the location in the northern hemisphere.

a) Use an identity to write the equation in a more useful form.

b) Determine the amount of power received at each location.

i) Whitehorse, Yukon, at 60.7° N

ii) Victoria, British Columbia, at 48.4° N

iii) Igloolik, Nunavut, at 69.4° N

c) Explain the answer for part iii) above. At what latitude is the power received from the sun zero?

10. Simplify $\cos(\pi + x) + \cos(\pi - x)$.

11. Angle θ is in quadrant II and $\sin\theta = \frac{5}{13}$. Determine an exact value for each of the following.

a) $\cos 2\theta$

b) $\sin 2\theta$

c) $\sin\left(\theta + \frac{\pi}{2}\right)$

12. The double-angle identity for tangent in terms of the tangent function is $\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$.

a) Verify numerically that this equation is true for $x = \frac{\pi}{6}$.

b) The expression $\tan 2x$ can also be written using the quotient identity for tangent: $\tan 2x = \frac{\sin 2x}{\cos 2x}$. Verify this equation numerically when $x = \frac{\pi}{6}$.

c) The expression $\frac{\sin 2x}{\cos 2x}$ from part b) can be expressed as $\frac{2\sin x\cos x}{\cos^2 x - \sin^2 x}$ using double-angle identities. Show how the expression for $\tan 2x$ used in part a) can also be rewritten in the form $\frac{2\sin x\cos x}{\cos^2 x - \sin^2 x}$.

13. The horizontal distance, d, in metres, travelled by a ball that is kicked at an angle, θ, with the ground is modelled by the formula $d = \frac{2(v_0)^2\sin\theta\cos\theta}{g}$, where v_0 is the initial velocity of the ball, in metres per second, and g is the force of gravity ($9.8\ \text{m/s}^2$).

a) Rewrite the formula using a double-angle identity.

b) Determine the angle $\theta \in (0°, 90°)$ that would result in a maximum distance for an initial velocity v_0.

c) Explain why it might be easier to answer part b) with the double-angle version of the formula that you determined in part a).

14. If $(\sin x + \cos x)^2 = k$, then what is the value of $\sin 2x$ in terms of k?

15. Show that each expression can be simplified to $\cos 2x$.

a) $\cos^4 x - \sin^4 x$

b) $\frac{\csc^2 x - 2}{\csc^2 x}$

16. Simplify each expression to the equivalent expression shown.

a) $\frac{1 - \cos 2x}{2}$ $\qquad$ $\sin^2 x$

b) $\frac{4 - 8\sin^2 x}{2\sin x\cos x}$ $\qquad$ $\frac{4}{\tan 2x}$

17. If the point $(2, 5)$ lies on the terminal arm of angle x in standard position, what is the value of $\cos(\pi + x)$?

18. What value of k makes the equation $\sin 5x\cos x + \cos 5x\sin x = 2\sin kx\cos kx$ true?

19. **a)** If $\cos\theta = \frac{3}{5}$ and $0 < \theta < 2\pi$, determine the value(s) of $\sin\left(\theta + \frac{\pi}{6}\right)$.

b) If $\sin\theta = -\frac{2}{3}$ and $\frac{3\pi}{2} < \theta < 2\pi$, determine the value(s) of $\cos\left(\theta + \frac{\pi}{3}\right)$.

20. If $\angle A$ and $\angle B$ are both in quadrant I, and $\sin A = \frac{4}{5}$ and $\cos B = \frac{12}{13}$, evaluate each of the following,

a) $\cos(A - B)$

b) $\sin(A + B)$

c) $\cos 2A$

d) $\sin 2A$

Extend

21. Determine the missing primary trigonometric ratio that is required for the expression $\frac{\blacksquare\ \sin 2x}{2 - 2\cos^2 x}$ to simplify to

a) $\cos x$

b) 1

22. Use a double-angle identity for cosine to determine the half-angle formula for cosine, $\cos\frac{x}{2} = \pm\sqrt{\frac{1 + \cos x}{2}}$.

23. a) Graph the curve $y = 4 \sin x - 3 \cos x$. Notice that it resembles a sine function.

b) What are the approximate values of a and c for the curve in the form $y = a \sin (x - c)$, where $0 < c < 90°$?

c) Use the difference identity for sine to rewrite the curve for $y = 4 \sin x - 3 \cos x$ in the form $y = a \sin (x - c)$.

24. Write the following equation in the form $y = A \sin Bx + D$, where A, B, and D are constants:
$y = 6 \sin x \cos^3 x + 6 \sin^3 x \cos x - 3$

Create Connections

C1 a) Determine the value of $\sin 2x$ if $\cos x = -\frac{5}{13}$ and $\pi < x < \frac{3\pi}{2}$ using

i) transformations

ii) a double-angle identity

b) Which method do you prefer? Explain.

C2 a) Graph the function $f(x) = 6 \sin x \cos x$ over the interval $0° \leq x \leq 360°$.

b) The function can be written as a sine function in the form $f(x) = a \sin bx$. Compare how to determine this sine function from the graph versus using the double-angle identity for sine.

C3 a) Over the domain $0° \leq x \leq 360°$, sketch the graphs of $y_1 = \sin^2 x$ and $y_2 = \cos^2 x$. How do these graphs compare?

b) Predict what the graph of $y_1 + y_2$ looks like. Explain your prediction. Graph to test your prediction.

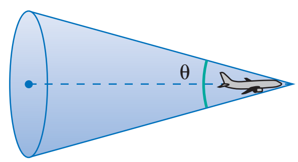
c) Graph the difference of the two functions: $y_1 - y_2$. Describe how the two functions interact with each other in the new function.

d) The new function from part c) is sinusoidal. Determine the function in the form $f(x) = a \cos bx$. Explain how you determined the expression.

Project Corner — Mach Numbers

- In aeronautics, the Mach number, M, of an aircraft is the ratio of its speed as it moves through air to the speed of sound in air. An aircraft breaks the sound barrier when its speed is greater than the speed of sound in dry air at 20 °C.

- When an aircraft exceeds Mach 1, $M > 1$, a shock wave forms a cone that spreads backward and outward from the aircraft. The angle at the vertex of a cross-section of the cone is related to the Mach number by $\frac{1}{M} = \sin \frac{\theta}{2}$.



- How could you use the half-angle identity, $\sin \frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos \theta}{2}}$, to express the Mach number, M, as a function of θ?

- If plane A is travelling twice as fast as plane B, how are the angles of the cones formed by the planes related?

6.3

Proving Identities

Focus on...

- proving trigonometric identities algebraically
- understanding the difference between verifying and proving an identity
- showing that verifying that the two sides of a potential identity are equal for a given value is insufficient to prove the identity

Many formulas in science contain trigonometric functions. In physics, torque (τ), work (W), and magnetic forces (F_B) can be calculated using the following formulas:

$$\tau = rF \sin \theta \qquad W = F\delta r \cos \theta \qquad F_B = qvB \sin \theta$$

In dynamics, which is the branch of mechanics that deals with motion, trigonometric functions may be required to calculate horizontal and vertical components. Skills with identities reduce the time it takes to work with formulas involving trigonometric functions.

Investigate the Equivalence of Two Trigonometric Expressions

Materials

- graphing calculator

Two physics students are investigating the horizontal distance, d, travelled by a model rocket. The rocket is launched with an angle of elevation θ. Katie has found a formula to model this situation: $d = \frac{(v_0)^2 \sin 2\theta}{g}$, where g represents the force of gravity and v_0 represents the initial velocity. Sergey has found a different formula: $d = \frac{2(v_0)^2}{g}(\tan \theta - \tan \theta \sin^2 \theta)$.

1. Are the two expressions, $\frac{(v_0)^2 \sin 2\theta}{g}$ and $\frac{2(v_0)^2}{g}(\tan \theta - \tan \theta \sin^2 \theta)$, equivalent? Use graphical and numerical methods to explain your answer. The initial velocity, v_0, of the rocket is 14 m/s and g is 9.8 m/s^2, so first substitute these values and simplify each expression.
2. Which parts are common to both formulas?
3. Write an identity with the parts of the formulas that are not common. Use your knowledge of identities to rewrite each side and show that they are equivalent.
4. Compare your reasoning with that of a classmate.

Reflect and Respond

5. How does this algebraic method for verifying an identity compare to verifying an identity graphically or numerically? Why do numerical and graphical verification fail to prove that an identity is true?

Link the Ideas

To prove that an identity is true for all permissible values, it is necessary to express both sides of the identity in equivalent forms. One or both sides of the identity must be algebraically manipulated into an equivalent form to match the other side.

You cannot perform operations across the equal sign when proving a potential identity. Simplify the expressions on each side of the identity independently.

Example 1

Verify Versus Prove That an Equation Is an Identity

a) Verify that $1 - \sin^2 x = \sin x \cos x \cot x$ for some values of x. Determine the non-permissible values for x. Work in degrees.

b) Prove that $1 - \sin^2 x = \sin x \cos x \cot x$ for all permissible values of x.

Solution

a) First, determine the non-permissible values.

The only function in the equation that has non-permissible values in its domain is $\cot x$.

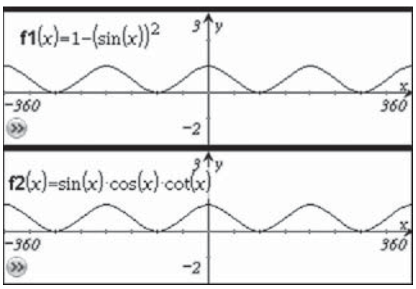
Recall that $\cot x$ is undefined when $\sin x = 0$.

Therefore, $x \neq 180°n$, where $n \in I$.

Verify the identity graphically and numerically.

Method 1: Verify Graphically

Use technology to graph $y = 1 - \sin^2 x$ and $y = \sin x \cos x \cot x$ over the domain $-360° \leq x \leq 360°$. The graphs appear to be the same. So, graphically, it seems that $1 - \sin^2 x = \sin x \cos x \cot x$ is an identity.



Why are the non-permissible values not apparent from these graphs?

Method 2: Verify Numerically

Use $x = 30°$. Why is 30° a good choice?

$$\begin{aligned}\text{Left Side} &= 1 - \sin^2 x \\ &= 1 - \sin^2 30° \\ &= 1 - \left(\frac{1}{2}\right)^2 \\ &= 1 - \frac{1}{4} \\ &= \frac{3}{4}\end{aligned}$$

$$\begin{aligned}\text{Right Side} &= \sin x \cos x \cot x \\ &= \sin 30° \cos 30° \cot 30° \\ &= \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{1}\right) \\ &= \frac{3}{4}\end{aligned}$$

Left Side = Right Side

The equation $1 - \sin^2 x = \sin x \cos x \cot x$ is verified for $x = 30°$.

b) To prove the identity algebraically, examine both sides of the equation and simplify each side to a common expression.

$$\begin{aligned}\text{Left Side} &= 1 - \sin^2 x \\ &= \cos^2 x\end{aligned}$$

Why is this true?

$$\begin{aligned}\text{Right Side} &= \sin x \cos x \cot x \\ &= \sin x \cos x \left(\frac{\cos x}{\sin x}\right) \\ &= \cos^2 x\end{aligned}$$

Left Side = Right Side

Therefore, $1 - \sin^2 x = \sin x \cos x \cot x$ is an identity for $x \neq 180°n$, where $n \in \mathrm{I}$.

Your Turn

a) Determine the non-permissible values for the equation $\dfrac{\tan x \cos x}{\csc x} = 1 - \cos^2 x$.

b) Verify that the equation may be an identity, either graphically using technology or by choosing one value for x.

c) Prove that the identity is true for all permissible values of x.

Example 2

Prove an Identity Using Double-Angle Identities

Prove that $\tan x = \dfrac{1 - \cos 2x}{\sin 2x}$ is an identity for all permissible values of x.

Solution

$$\text{Left Side} = \tan x$$

$$\begin{aligned}\text{Right Side} &= \frac{1 - \cos 2x}{\sin 2x} \\ &= \frac{1 - (1 - 2\sin^2 x)}{2 \sin x \cos x} \\ &= \frac{2\sin^2 x}{2 \sin x \cos x} \\ &= \frac{\sin x}{\cos x} \\ &= \tan x\end{aligned}$$

Recall the double-angle identities.

Remove common factors.

Left Side = Right Side

Therefore, $\tan x = \dfrac{1 - \cos 2x}{\sin 2x}$ is an identity for all permissible values of x.

Your Turn

Prove that $\dfrac{\sin 2x}{\cos 2x + 1} = \tan x$ is an identity for all permissible values of x.

In the previous example, you did not need to simplify the left side of the identity. However, tan x could have been expressed as $\frac{\sin x}{\cos x}$ using the quotient identity for tangent. In this case, the right side of the proof would have ended one step earlier, at $\frac{\sin x}{\cos x}$. Sometimes it is advisable to convert all trigonometric functions to expressions of sine or cosine.

Example 3

Prove More Complicated Identities

Prove that $\frac{1 - \cos x}{\sin x} = \frac{\sin x}{1 + \cos x}$ is an identity for all permissible values of x.

Solution

$$\text{Left Side} = \frac{1 - \cos x}{\sin x}$$

$$\begin{aligned}\text{Right Side} &= \frac{\sin x}{1 + \cos x}\\ &= \frac{\sin x}{1 + \cos x} \times \frac{1 - \cos x}{1 - \cos x}\\ &= \frac{\sin x\,(1 - \cos x)}{1 - \cos^2 x}\\ &= \frac{\sin x\,(1 - \cos x)}{\sin^2 x}\\ &= \frac{1 - \cos x}{\sin x}\end{aligned}$$

How does multiplying by $1 - \cos x$, which is the conjugate of $1 + \cos x$, let you express the denominator in terms of $\sin x$?

Left Side = Right Side

Therefore, $\frac{1 - \cos x}{\sin x} = \frac{\sin x}{1 + \cos x}$ is an identity for all permissible values of x.

Your Turn

Prove that $\frac{1}{1 + \sin x} = \frac{\sec x - \sin x \sec x}{\cos x}$ is an identity for all permissible values of x.

Example 4

Prove an Identity That Requires Factoring

Prove the identity $\cot x - \csc x = \frac{\cos 2x - \cos x}{\sin 2x + \sin x}$ for all permissible values of x.

Solution

Left Side $= \cot x - \csc x$

$$= \frac{\cos x}{\sin x} - \frac{1}{\sin x}$$

$$= \frac{\cos x - 1}{\sin x}$$

Right Side $= \frac{\cos 2x - \cos x}{\sin 2x + \sin x}$

$$= \frac{(2\cos^2 x - 1) - \cos x}{2\sin x \cos x + \sin x}$$

Why is $2\cos^2 x - 1$ substituted for $\cos 2x$?

$$= \frac{2\cos^2 x - \cos x - 1}{\sin x\,(2\cos x + 1)}$$

$$= \frac{(2\cos x + 1)(\cos x - 1)}{\sin x\,(2\cos x + 1)}$$

$$= \frac{\cos x - 1}{\sin x}$$

Left Side = Right Side

Therefore, $\cot x - \csc x = \frac{\cos 2x - \cos x}{\sin 2x + \sin x}$ is an identity for all permissible values of x.

Your Turn

Prove the identity $\frac{\sin 2x - \cos x}{4\sin^2 x - 1} = \frac{\sin^2 x \cos x + \cos^3 x}{2\sin x + 1}$ for all permissible values of x.

Key Ideas

- Verifying an identity using a specific value validates that it is true for that value only. Proving an identity is done algebraically and validates the identity for all permissible values of the variable.
- To prove a trigonometric identity algebraically, separately simplify both sides of the identity into identical expressions.
- It is usually easier to make a complicated expression simpler than it is to make a simple expression more complicated.
- Some strategies that may help you prove identities include:
  - Use known identities to make substitutions.
  - If quadratics are present, the Pythagorean identity or one of its alternate forms can often be used.
  - Rewrite the expression using sine and cosine only.
  - Multiply the numerator and the denominator by the conjugate of an expression.
  - Factor to simplify expressions.

Check Your Understanding

Practise

1. Factor and simplify each rational trigonometric expression.
   a) $\dfrac{\sin x - \sin x \cos^2 x}{\sin^2 x}$
   b) $\dfrac{\cos^2 x - \cos x - 2}{6 \cos x - 12}$
   c) $\dfrac{\sin x \cos x - \sin x}{\cos^2 x - 1}$
   d) $\dfrac{\tan^2 x - 3 \tan x - 4}{\sin x \tan x + \sin x}$

2. Use factoring to help to prove each identity for all permissible values of x.
   a) $\cos x + \cos x \tan^2 x = \sec x$
   b) $\dfrac{\sin^2 x - \cos^2 x}{\sin x + \cos x} = \sin x - \cos x$
   c) $\dfrac{\sin x \cos x - \sin x}{\cos^2 x - 1} = \dfrac{1 - \cos x}{\sin x}$
   d) $\dfrac{1 - \sin^2 x}{1 + 2 \sin x - 3 \sin^2 x} = \dfrac{1 + \sin x}{1 + 3 \sin x}$

3. Use a common denominator to express the rational expressions as a single term.
   a) $\dfrac{\sin x}{\cos x} + \sec x$
   b) $\dfrac{1}{\sin x - 1} + \dfrac{1}{\sin x + 1}$
   c) $\dfrac{\sin x}{1 + \cos x} + \dfrac{\cos x}{\sin x}$
   d) $\dfrac{\cos x}{\sec x - 1} + \dfrac{\cos x}{\sec x + 1}$

4. a) Rewrite the expression $\dfrac{\sec x - \cos x}{\tan x}$ in terms of sine and cosine functions only.
   b) Simplify the expression to one of the primary trigonometric functions.

5. Verify graphically that $\cos x = \dfrac{\sin 2x}{2 \sin x}$ could be an identity. Then, prove the identity. Determine any non-permissible values.

6. Expand and simplify the expression $(\sec x - \tan x)(\sin x + 1)$ to a primary trigonometric function.

7. Prove each identity.
   a) $\dfrac{\csc x}{2 \cos x} = \csc 2x$
   b) $\sin x + \cos x \cot x = \csc x$

Apply

8. As the first step of proving the identity $\dfrac{\cos 2x - 1}{\sin 2x} = -\tan x$, Hanna chose to substitute $\cos 2x = 1 - 2 \sin^2 x$, while Chloe chose $\cos 2x = 2 \cos^2 x - 1$. Which choice leads to a shorter proof? Explain. Prove the identity.

9. The distance, d, in metres, that a golf ball travels when struck by a golf club is given by the formula $d = \dfrac{(v_0)^2 \sin 2\theta}{g}$, where v_0 is the initial velocity of the ball, θ is the angle between the ground and the initial path of the ball, and g is the acceleration due to gravity (9.8 m/s^2).

   a) What distance, in metres, does the ball travel if its initial velocity is 21 m/s and the angle θ is 55°?
   b) Prove the identity $\dfrac{(v_0)^2 \sin 2\theta}{g} = \dfrac{2(v_0)^2(1 - \cos^2 \theta)}{g \tan \theta}$.

10. Verify each potential identity by graphing, and then prove the identity.
   a) $\dfrac{\csc x}{2 \cos x} = \csc 2x$
   b) $\dfrac{\sin x \cos x}{1 + \cos x} = \dfrac{1 - \cos x}{\tan x}$
   c) $\dfrac{\sin x + \tan x}{1 + \cos x} = \dfrac{\sin 2x}{2 \cos^2 x}$

11. Prove each identity.
   a) $\dfrac{\sin 2x}{\cos x} + \dfrac{\cos 2x}{\sin x} = \csc x$
   b) $\csc^2 x + \sec^2 x = \csc^2 x \sec^2 x$
   c) $\dfrac{\cot x - 1}{1 - \tan x} = \dfrac{\csc x}{\sec x}$

12. Prove each identity.

a) $\sin(90° + \theta) = \sin(90° - \theta)$

b) $\sin(2\pi - \theta) = -\sin\theta$

13. Prove that
$2\cos x\cos y = \cos(x + y) + \cos(x - y)$.

14. Consider the equation
$\cos 2x = 2\sin x\cos x$.

a) Graph each side of the equation. Could the equation be an identity?

b) Either prove that the equation is an identity or find a counterexample to show that it is not an identity.

15. Consider the equation $\dfrac{\sin 2x}{1 - \cos 2x} = \cot x$.

a) Determine the non-permissible values for x.

b) Prove that the equation is an identity for all permissible values of x.

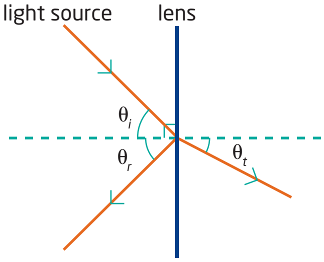
Extend

16. Use double-angle identities to prove the identity $\tan x = \dfrac{\sin 4x - \sin 2x}{\cos 4x + \cos 2x}$.

17. Verify graphically and then prove the identity $\dfrac{\sin 2x}{1 - \cos 2x} = 2\csc 2x - \tan x$.

18. Prove the identity
$\dfrac{1 - \sin^2 x - 2\cos x}{\cos^2 x - \cos x - 2} = \dfrac{1}{1 + \sec x}$.

19. When a ray of light hits a lens at angle of incidence θ_i, some of the light is refracted (bent) as it passes through the lens, and some is reflected by the lens. In the diagram, θ_r is the angle of reflection and θ_t is the angle of refraction. Fresnel equations describe the behaviour of light in this situation.



a) Snells's law states that $n_1 \sin\theta_i = n_2 \sin\theta_t$, where n_1 and n_2 are the refractive indices of the mediums. Isolate $\sin\theta_t$ in this equation.

b) Under certain conditions, a Fresnel equation to find the fraction, R, of light reflected is
$$R = \left(\frac{n_1\cos\theta_i - n_2\cos\theta_t}{n_1\cos\theta_i + n_2\cos\theta_t}\right)^2.$$
Use identities to prove that this can be written as
$$R = \left(\frac{n_1\cos\theta_i - n_2\sqrt{1 - \sin^2\theta_t}}{n_1\cos\theta_i + n_2\sqrt{1 - \sin^2\theta_t}}\right)^2.$$

c) Use your work from part a) to prove that
$$\left(\frac{n_1\cos\theta_i - n_2\sqrt{1 - \sin^2\theta_t}}{n_1\cos\theta_i + n_2\sqrt{1 - \sin^2\theta_t}}\right)^2 = \left(\frac{n_1\cos\theta_i - n_2\sqrt{1 - \left(\frac{n_1}{n_2}\right)^2\sin^2\theta_i}}{n_1\cos\theta_i + n_2\sqrt{1 - \left(\frac{n_1}{n_2}\right)^2\sin^2\theta_i}}\right)^2$$

Did You Know?

Fresnel equations were developed by French physicist Augustin-Jean Fresnel (1788–1827). A Fresnel lens is also named for him, and is a common lens in lights used for movies, TV, and live theatre. A new use for Fresnel lenses is to focus light in a solar array to allow for much more efficient collection of solar energy.

Create Connections

C1 Why is verifying, either numerically or graphically, that both sides of an equation seem to be equal not sufficient to prove that the equation is an identity?

C2 Use the difference identity for cosine to prove the identity $\cos\left(\dfrac{\pi}{2} - x\right) = \sin x$.

C3 Consider the equation
$\cos x = \sqrt{1 - \sin^2 x}$.

a) What are the non-permissible values for x in this equation?

b) What is a value for x that makes this equation true?

c) What is a value for x that does not work in this equation and provides evidence that this equation is not an identity?

d) Explain the difference between an identity and an equation.

6.4

Solving Trigonometric Equations Using Identities

Focus on…

- solving trigonometric equations algebraically using known identities
- determining exact solutions for trigonometric equations where possible
- determining the general solution for trigonometric equations
- identifying and correcting errors in a solution for a trigonometric equation

Sound from a musical instrument is composed of sine waves. Technicians often fade the sound near the end of a song. To create this effect, the sound equipment is programmed to use mathematical damping techniques. The technicians have three choices: a linear fade, a logarithmic fade, or an inverse logarithmic fade. You will explore logarithmic functions in Chapter 8.

Knowledge of trigonometric identities can help to simplify the expressions involved in the trigonometric equations of sound waves in music.

Did You Know?

The musical instrument with the purest sound wave is the flute. The most complex musical sound wave can be created with a cymbal.

Investigate Solving Trigonometric Equations

Materials

- graphing technology

1. Graph the function $y = \sin 2x - \sin x$ over the domain $-720° < x \leq 720°$. Make a sketch of the graph and describe it in words.
2. From the graph, determine an expression for the zeros of the function $y = \sin 2x - \sin x$ over the domain of all real numbers.
3. Algebraically solve the equation $\sin 2x - \sin x = 0$ over the domain of all real numbers. Compare your answer and method with those of a classmate.

Reflect and Respond

4. Which method, graphic or algebraic, do you prefer to solve the equation $\sin 2x - \sin x = 0$? Explain.

Link the Ideas

To solve some trigonometric equations, you need to make substitutions using the trigonometric identities that you have studied in this chapter. This often involves ensuring that the equation is expressed in terms of one trigonometric function.

Example 1

Solve by Substituting Trigonometric Identities and Factoring

Solve each equation algebraically over the domain $0 \leq x < 2\pi$.

a) $\cos 2x + 1 - \cos x = 0$

b) $1 - \cos^2 x = 3 \sin x - 2$

Solution

a)

$$\cos 2x + 1 - \cos x = 0$$

$$(2 \cos^2 x - 1) + 1 - \cos x = 0$$ Why is this version of the identity for $\cos 2x$ chosen?

$$2 \cos^2 x - \cos x = 0$$ Simplify.

$$\cos x (2 \cos x - 1) = 0$$ Factor.

$\cos x = 0$ or $2 \cos x - 1 = 0$ Use the zero product property.

$x = \frac{\pi}{2}$ or $x = \frac{3\pi}{2}$ or $\cos x = \frac{1}{2}$

$x = \frac{\pi}{3}$ or $x = \frac{5\pi}{3}$

There are no non-permissible values for the original equation, so the solutions over the domain $0 \leq x < 2\pi$ are $x = \frac{\pi}{3}$, $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$, and $x = \frac{5\pi}{3}$.

b)

$$1 - \cos^2 x = 3 \sin x - 2$$

$$\sin^2 x = 3 \sin x - 2$$ Use the Pythagorean identity.

$$\sin^2 x - 3 \sin x + 2 = 0$$

$$(\sin x - 1)(\sin x - 2) = 0$$ Use the zero product property.

$\sin x - 1 = 0$ or $\sin x - 2 = 0$

$\sin x = 1$ $\quad$ $\sin x = 2$

$x = \frac{\pi}{2}$ $\quad$ $\sin x = 2$ has no solution.

Why is there no solution for $\sin x = 2$?

There are no non-permissible values for the original equation, so the solution over the domain $0 \leq x < 2\pi$ is $x = \frac{\pi}{2}$.

Your Turn

Solve each equation algebraically over the domain $0 \leq x < 2\pi$.

a) $\sin 2x - \cos x = 0$

b) $2 \cos x + 1 - \sin^2 x = 3$

Example 2

Solve an Equation With a Quotient Identity Substitution

a) Solve the equation $\cos^2 x = \cot x \sin x$ algebraically in the domain $0° \leq x < 360°$.

b) Verify your answer graphically.

Solution

a)

$$\cos^2 x = \cot x \sin x$$

What is the quotient identity for cot *x*?

$$\cos^2 x = \left(\frac{\cos x}{\sin x}\right) \sin x$$

$$\cos^2 x = \cos x$$

Why is it incorrect to divide by cos *x* here?

$$\cos^2 x - \cos x = 0$$

$$\cos x (\cos x - 1) = 0$$

Factor.

$$\cos x = 0 \text{ or } \cos x = 1$$

Apply the zero product property.

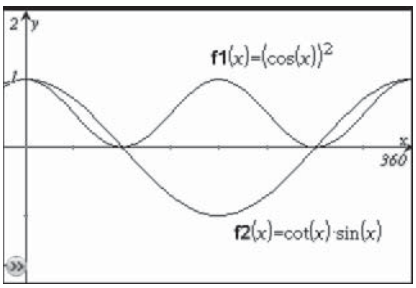
For $\cos x = 0$, $x = 90°$ and $x = 270°$.
For $\cos x = 1$, $x = 0°$.

Check whether there are any non-permissible values for the initial equation.

For cot *x*, the domain has the restriction $\sin x \neq 0$, which gives the non-permissible values $x \neq 0°$ and $x \neq 180°$.

Therefore, the solution for $\cos^2 x = \cot x \sin x$ is limited to $x = 90°$ and $x = 270°$.

b) Graph $y = \cos^2 x$ and $y = \cot x \sin x$ over the domain $0° \leq x < 360°$. Determine the points of intersection of the two functions.



It appears from the graph that a solution is $x = 0$. Note that $y = \cot x \sin x$ is not defined at $x = 0$ because it is a non-permissible value for cot *x*.

What solutions are confirmed by the graph?

Your Turn

a) Solve the equation $\sin^2 x = \frac{1}{2} \tan x \cos x$ algebraically over the domain $0° \leq x < 360°$.

b) Verify your answer graphically.

Example 3

Determine the General Solution for a Trigonometric Equation

Solve the equation $\sin 2x = \sqrt{2} \cos x$ algebraically. Give the general solution expressed in radians.

Solution

$$\sin 2x = \sqrt{2} \cos x$$

$$2 \sin x \cos x = \sqrt{2} \cos x$$ Use the double-angle identity for sin 2*x*.

$$2 \sin x \cos x - \sqrt{2} \cos x = 0$$

$$\cos x (2 \sin x - \sqrt{2}) = 0$$ Why is it incorrect to divide by cos *x* here?

Then, $\cos x = 0$ or $2 \sin x - \sqrt{2} = 0$

$$\sin x = \frac{\sqrt{2}}{2}$$

For $\cos x = 0$, $x = \frac{\pi}{2} + \pi n$, where $n \in I$.

For $\sin x = \frac{\sqrt{2}}{2}$, $x = \frac{\pi}{4} + 2\pi n$ and $x = \frac{3\pi}{4} + 2\pi n$, where $n \in I$.

Since there are no non-permissible values for the original equation, the solution is $x = \frac{\pi}{2} + \pi n$, $x = \frac{\pi}{4} + 2\pi n$, and $x = \frac{3\pi}{4} + 2\pi n$, where $n \in I$.

Your Turn

Algebraically solve $\cos 2x = \cos x$. Give general solutions expressed in radians.

Example 4

Determine the General Solution Using Reciprocal Identities

Algebraically solve $2 \sin x = 7 - 3 \csc x$. Give general solutions expressed in radians.

Solution

$$2 \sin x = 7 - 3 \csc x$$

$$2 \sin x = 7 - \frac{3}{\sin x}$$ Use the reciprocal identity for cosecant.

$$\sin x (2 \sin x) = \sin x \left(7 - \frac{3}{\sin x}\right)$$ Why multiply both sides by sin *x*?

$$2 \sin^2 x = 7 \sin x - 3$$

$$2 \sin^2 x - 7 \sin x + 3 = 0$$

$$(2 \sin x - 1)(\sin x - 3) = 0$$ Factor.

For $2 \sin x - 1 = 0$, Use the zero product property.

$$\sin x = \frac{1}{2}$$

$$x = \frac{\pi}{6} + 2\pi n \text{ and } x = \frac{5\pi}{6} + 2\pi n$$

For $\sin x - 3 = 0$,

$$\sin x = 3$$ Why is there no solution for sin *x* = 3?

There is no solution for $\sin x = 3$.

The restriction on the original equation is $\sin x \neq 0$ because of the presence of $\csc x$.

Since $\sin x = 0$ does not occur in the solution, all determined solutions are permissible.

The solution is $x = \frac{\pi}{6} + 2\pi n$ and $x = \frac{5\pi}{6} + 2\pi n$, where $n \in I$.

Your Turn

Algebraically solve $3 \cos x + 2 = 5 \sec x$. Give general solutions expressed in radians.

Key Ideas

- Reciprocal, quotient, Pythagorean, and double-angle identities can be used to help solve a trigonometric equation algebraically.
- The algebraic solution for a trigonometric equation can be verified graphically.
- Check that solutions for an equation do not include non-permissible values from the original equation.
- Unless the domain is restricted, give general solutions. For example, for $2 \cos x = 1$, the general solution is $x = \frac{\pi}{3} + 2\pi n$ and $x = \frac{5\pi}{3} + 2\pi n$, where $n \in I$. If the domain is specified as $0° \leq x < 360°$, then the solutions are $60°$ and $300°$.

Check Your Understanding

Practise

1. Solve each equation algebraically over the domain $0 \leq x < 2\pi$.

a) $\tan^2 x - \tan x = 0$

b) $\sin 2x - \sin x = 0$

c) $\sin^2 x - 4 \sin x = 5$

d) $\cos 2x = \sin x$

2. Solve each equation algebraically over the domain $0° \leq x < 360°$. Verify your solution graphically.

a) $\cos x - \cos 2x = 0$

b) $\sin^2 x - 3 \sin x = 4$

c) $\tan x \cos x \sin x - 1 = 0$

d) $\tan^2 x + \sqrt{3} \tan x = 0$

3. Rewrite each equation in terms of sine only. Then, solve algebraically for $0 \leq x < 2\pi$.

a) $\cos 2x - 3 \sin x = 2$

b) $2 \cos^2 x - 3 \sin x - 3 = 0$

c) $3 \csc x - \sin x = 2$

d) $\tan^2 x + 2 = 0$

4. Solve $4 \sin^2 x = 1$ algebraically over the domain $-180° \leq x < 180°$.

5. Solve $2 \tan^2 x + 3 \tan x - 2 = 0$ algebraically over the domain $0 \leq x < 2\pi$.

Apply

6. Determine the mistake that Sanesh made in the following work. Then, complete a correct solution.

Solve $2\cos^2 x = \sqrt{3}\cos x$. Express your answer(s) in degrees.

Solution:

$$\frac{1}{\cos x}(2\cos^2 x) = (\sqrt{3}\cos x)\frac{1}{\cos x}$$

$$2\cos x = \sqrt{3}$$

$$\cos x = \frac{\sqrt{3}}{2}$$

$$x = 30^\circ + 360^\circ n \text{ and } x = 330^\circ + 360^\circ n$$

7. a) Solve algebraically $\sin 2x = 0.5$, $0 \le x < 2\pi$.

b) Solve the equation from part a) using a different method.

8. Solve $\sin^2 x = \cos^2 x + 1$ algebraically for all values of x. Give your answer(s) in radians.

9. Solve $\cos x \sin 2x - 2\sin x = -2$ algebraically over the domain of real numbers. Give your answer(s) in radians.

10. How many solutions does the equation $(7\sin x + 2)(3\cos x + 3)(\tan^2 x - 2) = 0$ have over the interval $0° < x \le 360°$? Explain your reasoning.

11. Solve $\sqrt{3}\cos x \csc x = -2\cos x$ for x over the domain $0 \le x < 2\pi$.

12. If $\cos x = \frac{2}{3}$ and $\cos x = -\frac{1}{3}$ are the solutions for a trigonometric equation, what are the values of B and C if the equation is of the form $9\cos^2 x + B\cos x + C = 0$?

13. Create a trigonometric equation that includes $\sin 2x$ and that can be solved by factoring. Then, solve it.

14. Solve $\sin 2x = 2\cos x \cos 2x$ algebraically. Give the general solution expressed in radians.

15. Algebraically determine the number of solutions for the equation $\cos 2x \cos x - \sin 2x \sin x = 0$ over the domain $-360° < x \le 360°$.

16. Solve $\sec x + \tan^2 x - 3\cos x = 2$ algebraically. Give the general solution expressed in radians.

Extend

17. Solve $4\sin^2 x = 3\tan^2 x - 1$ algebraically. Give the general solution expressed in radians.

18. Solve $\dfrac{1 - \sin^2 x - 2\cos x}{\cos^2 x - \cos x - 2} = -\dfrac{1}{3}$ algebraically over the domain $-\pi \le x \le \pi$.

19. Find the general solution for the equation $4(16^{\cos^2 x}) = 2^{6\cos x}$. Give your answer in radians.

20. For some angles α and β, $\sin^2\alpha + \cos^2\beta = m^2$ and $\cos^2\alpha + \sin^2\beta = m$. Find the possible value(s) for m.

Create Connections

C1 Refer to the equation $\sin x - \cos 2x = 0$ to answer the following.

a) Which identity would you use to express the equation in terms of one trigonometric function?

b) How can you solve the resulting equation by factoring?

c) What is the solution for the domain $0° \le x < 360°$?

d) Verify your solution by graphing.

C2 Refer to the equation $3\cos^2 x + \cos x - 1 = 0$ to answer the following.

a) Why is not possible to factor the left side of the equation?

b) Solve the equation using the quadratic formula.

c) What is the solution over the domain $0° \le x < 720°$?

C3 Use the double-angle identity for sine to create an equation that is not an identity. Solve the equation and explain why it is not an identity.