Reciprocating Compressor

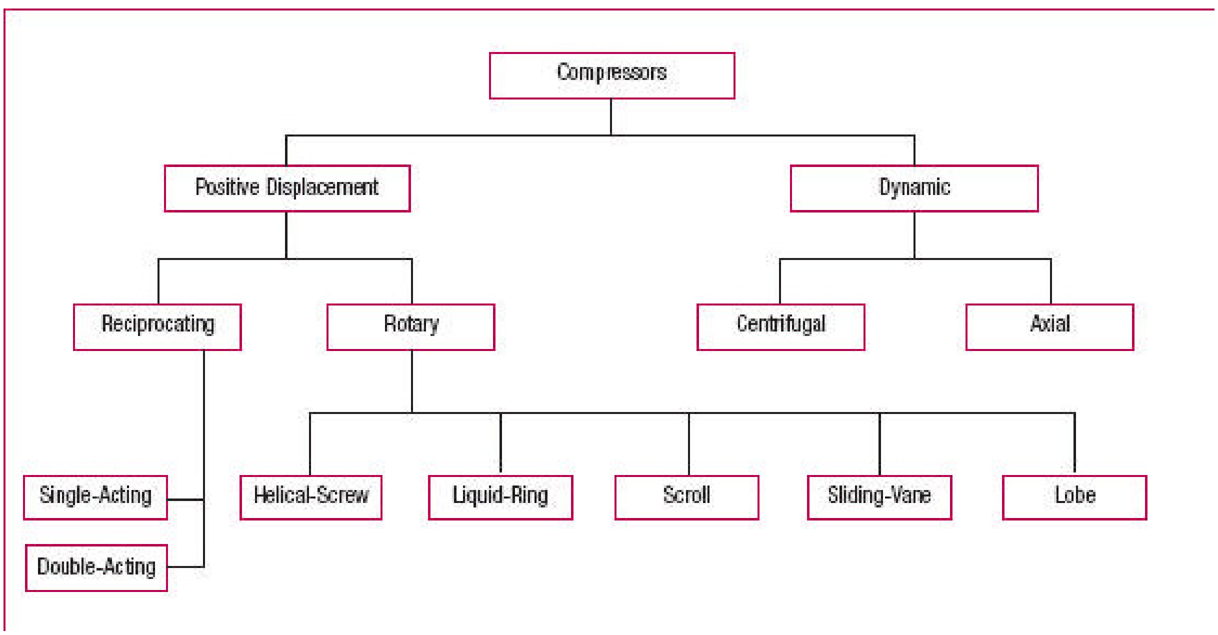
5.1 INTRODUCTION

Compressors are work absorbing devices which are used for increasing pressure of fluid at the expense or work done on fluid.

The compressors used for compressing air are called air compressors. Compressors are invariably used for all applications requiring high pressure air. Some of popular applications of compressor are, for driving pneumatic tools and air operated equipments, spray painting, compressed air engine, supercharging surface cleaning, refrigeration and air conditioning, chemical industry etc. compressors are supplied with low pressure air (or any fluid) at inlet which comes out as high pressure air (or any fluid) at outlet. Work required for increasing pressure of air is available from the prime mover driving the compressor. Generally, electric motor, internal combustion engine or steam engine, turbine etc. are used as prime movers. Compressors are similar to fans and blowers but differ in terms of pressure ratios. Fan is said to have pressure ratio up to 1.1 and blowers have pressure ratio between 1.1 to 4 while compressors have pressure ratios more than 4.

5.2 CLASSIFICATION OF COMPRESSORS

Table-5.1 Types of Compressors



Compressors can be classified in the following different ways.

(a) **Based on principle of operation:** Based on the principle of operation compressors can be classified as.

 (i) Positive displacement compressor.

 (ii) Non-positive displacement compressors.

In positive displacement compressors the compression is realized by displacement of solid boundary and preventing fluid by solid boundary from flowing back in the direction of pressure gradient. Due to solid wall displacement these are capable of providing quite large pressure ratios. Positive displacement compressors can be further classified based on the type of mechanism used for compression. These can be

(i) Reciprocating type positive displacement compressors

(ii) Rotary type positive displacement compressors.

Reciprocating compressors generally, employ piston-cylinder arrangement where displacement of piston in cylinder causes rise in pressure. Reciprocating compressors are capable of giving large pressure ratios but the mass handling capacity is limited or small. Reciprocating compressors may also be single acting compressor or double acting compressor. Single acting compressor has one delivery stroke per revolution while in double acting there are two delivery strokes per revolution of crank shaft. Rotary compressors employing positive displacement have a rotary part whose boundary causes positive displacement of fluid and thereby compression. Rotary compressors of this type are available in the names as given below;

(i) Roots blower

(ii) Vane type compressors

Rotary compressors of above type are capable of running at higher speed and can handle large mass flow rate than reciprocating compressors of positive displacement type.

Non-positive displacement compressors, also called as steady flow compressors use dynamic action of solid boundary for realizing pressure rise. Here fluid is not contained in definite volume and subsequent volume reduction does not occur as in case of positive displacement compressors. Non-positive displacement compressor may be of ‘axial flow type’ or ‘centrifugal type’ depending upon type of flow in compressor.

(b) **Based on number of stages:** Compressors may also be classfied on the basis of number of stages. Generally, the number of stages depend upon the maximum delivery pressure. Compressors can be single stage or multistage. Normally maximum compression ratio of 5 is realized in single stage compressors. For compression ratio more than 5 the multistage compressors are used.

Type values of maximum delivery pressures generally available from different type of compressor are,

(i) Single stage Compressor, for delivery pressure upto 5 bar.

(ii) Two stage Compressor, for delivery pressure between 5 to 35 bar

(iii) Three stage Compressor, for delivery pressure between 35 to 85 bar.

(iv) Four stage compressor, for delivery pressure more than 85 bar

(c) **Based on Capacity of compressors** : Compressors can also be classified depending upon the capacity of Compressor or air delivered per unit time. Typical values of capacity for different compressors are given as;

(i) Low capacity compressors, having air delivery capacity of 0.15 m^3/s or less

(ii) Medium capacity compressors, having air delivery capacity between 0.15 to 5 m^3/s.

(iii) High capacity compressors, having air delivery capacity more than 5 m^3/s

(d) Based on highest pressure developed: Depending upon the maximum pressure available from compressor they can be classified as low pressure, medium pressure, high pressure and super high pressure compressors. Typical values of maximum pressure developed for different compressors are as under:

(i) Low pressure compressor, having maximum pressure upto 1 bar

(ii) Medium pressure compressor, having maximum pressure from 1 bar to 8 bar

(iii) High pressure compressor, having maximum pressure from 8 to 10 bar

(iv) Super high pressure compressor, having maximum pressure more than 10 bar.

5.3 Reciprocating Compressors

Reciprocating Compressor has piston cylinder arrangement as shown Fig.5.1

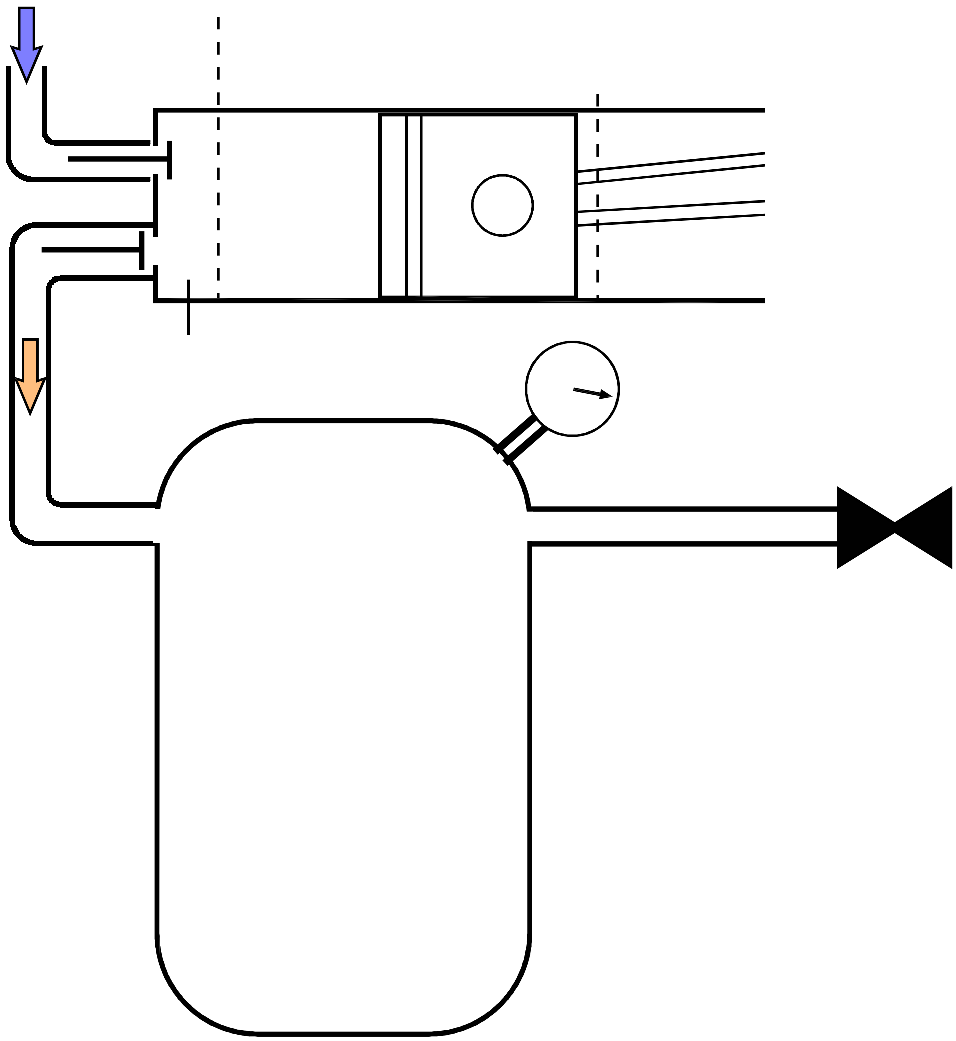


Fig.5.1 Reciprocating Compressor

Reciprocating Compressor has piston, cylinder, inlet valve, exit valve, connecting rod, crank, piston pin, crank pin and crank shaft. Inlet valve and exit valves may be of spring loaded type which get opened and closed due to pressure differential across them. Let us consider piston to be at top dead centre (TDC) and move towards bottom dead centre (BDC). Due to this piston movement from TDC to BDC suction pressure is created causing opening of inlet valve. With this opening of inlet valve and suction pressure the atmospheric air enters the cylinder.

Air gets into cylinder during this stroke and is subsequently compressed in next stroke with both inlet valve and exit valve closed. Both inlet valve and exit valves are of plate type and spring loaded so as to operate automatically as and when sufficient pressure difference is available to cause deflection in spring of valve plates to open them. After piston reaching BDC it reverses its motion and compresses the air inducted in previous stroke. Compression is continued till the pressure of air inside becomes sufficient to cause deflection in exit valve. At the moment when exit valve plate gets lifted the exhaust of compressed air takes place. This piston again reaches TDC from where downward piston movement is again accompanied by suction. This is how reciprocating compressor. Keeps on working as flow device. In order to counter for the heating of piston-cylinder arrangement during compression the provision of cooling the cylinder is there in the form of cooling jackets in the body. Reciproting compressor described above has suction, compression and discharge as three prominent processes getting completed in two strokes of piston or one revolution of crank shaft.

5.4 Thermodynamic Analysis

Compression of air in compressor may be carried out following number of thermodynamic processes such as isothermal compression, polytropic compressor or adiabatic compressor. Fig.16.3 shows the thermodynamic cycle involved in compressor. Theoretical cycle is shown neglecting clearance volume but in actual cycle clearance volume can not be negligible. Clearance volume is necessary in order to prevent collision of piston with cylinder head, accommodating valve mechanism etc., Compression process is shown by process1-2, $1\text{-}2^{1}$, $1\text{-}2^{11}$ and $1\text{-}2^{111}$ following isothermal, polytropic and adiabatic processes.

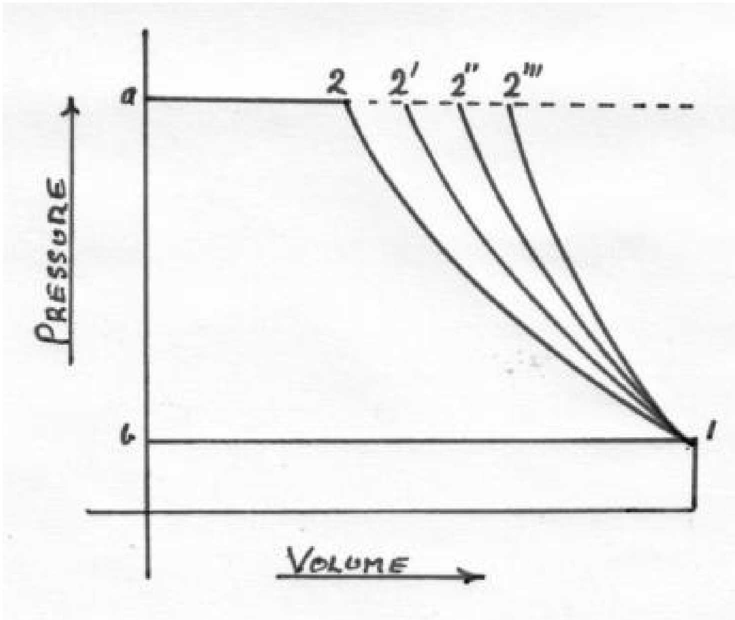


Fig.5.2 P-V diagram for Reciprocating Compressor without Clearance

On P-V diagram process 4-1 shows the suction process followed by compression during 1-2 and discharge through compressor is shown by process 2-3.

Air enters compressor at pressure p_1 and is compressed upto p_2. Compression work requirement can be estimated from the area below the each compression process. Area on p-V diagram shows that work requirement shall be minimum with isothermal process 1-2". Work requirement is maximum with process 1-2 ie., adiabatic process. As a designer one shall be interested in a compressor having minimum compression work requirement. Therefore, ideally compression should occur isothermally for minimum work input. In practice it is not possible to have isothermal compression because constancy of temperature during compression can not be realized. Generally, compressors run at substantially high speed while isothermal compression requires compressor to run at very slow speed so that heat evolved during compression is dissipated out and temperature remains constant. Actually due to high speed running of compressor the compression process may be assumed to be near adiabatic or polytropic process following law of compression as $pV^n=C$ with of 'n' varying between 1.25 to 1.35 for air. Compression process following three processes is also shown on T-s diagram in Fig.16.4. it is thus obvious that actual compression process should be compared with isothermal compression process. A mathematical parameter called isothermal efficiency is defined for quantifying the degree of deviation of actual compression process from ideal compression process. Isothermal efficiency is defined by the ratio is isothermal work and actual indicated work in reciprocating compressor.

$$\text{Isothermal efficiency} = \frac{\text{Isothermal work}}{\text{Actual indicated Work}}$$

Practically, compression process is attempted to be closed to isothermal process by air/water cooling, spraying cold water during compression process. In case of multistage compression process the compression in different stages is accompanied by intercooling in between the stages. P_2 V_2

Mathematically, for the compression work following polytropic process, $PV^n=C$. Assuming negligible clearance volume the cycle work done.

Wc = Area on p-V diagram

$$Wc = \left[p_2V_2 + \left(\frac{p_2V_2 - p_1V_1}{n-1}\right)\right] - p_1V_1 \qquad 1$$

$$= \left[\left(\frac{n}{n-1}\right)[p_2V_2 - p_1V_1]\right]$$

$$= \left(\frac{n}{n-1}\right)(p_1V_1)\left[\frac{p_2V_2}{p_1V_1} - 1\right]$$

$$= \left(\frac{n}{n-1}\right)(p_1V_1)\left[\left(\frac{p_2}{p_1}\right)^{\left(\frac{n-1}{n}\right)} - 1\right]$$

$$= \left(\frac{n}{n-1}\right)(mRT_1)\left[\left(\frac{p_2}{p_1}\right)^{\left(\frac{n-1}{n}\right)} - 1\right]$$

$$= \left(\frac{n}{n-1}\right)(mR)(T_2 - T_1)$$

In case of compressor having isothermal compression process, n = 1, ie., $p_1V_1 = p_2V_2$

$$W_{iso} = p_2V_2 + p_1V_1 \ln r - p_1V_1$$

$$W_{iso} = p_1V_1 \ln r, \qquad where, r = \frac{V_1}{V_2}$$

In case of compressor having adiabatic compression process,

$$W_{adiabatic} = \left(\frac{\gamma}{\gamma - 1}\right)(mR)(T_2 - T_1)$$

Or

$$W_{adiabatic} = (mC_p)(T_2 - T_1)$$
$$W_{adiabatic} = (m)(h_2 - h_1)$$

$$\eta_{iso} = \frac{p_1 V_1 \ln r}{\left(\frac{n}{n-1}\right)(p_1 V_1)\left[\left(\frac{p_2}{p_1}\right)^{\frac{n-1}{n}} - 1\right]}$$

The isothermal efficiency of a compressor should be close to 100% which means that actual compression should occur following a process close to isothermal process. For this the mechanism be derived to maintain constant temperature during compression process. Different arrangements which can be used are:

(i) Faster heat dissipation from inside of compressor to outside by use of fins over cylinder. Fins facilitate quick heat transfer from air being compressed to atmosphere so that temperature rise during compression can be minimized.

(ii) Water jacket may be provided around compressor cylinder so that heat can be picked by cooling water circulating through water jacket. Cooling water circulation around compressor regulates rise in temperature to great extent.

(iii) The water may also be injected at the end of compression process in order to cool the air being compressed. This water injection near the end of compression process requires special arrangement in compressor and also the air gets mixed with water and needs to be separated out before being used. Water injection also contaminates the lubricant film inner surface of cylinder and may initiate corrosion etc, The water injection is not popularly used.

(iv) In case of multistage compression in different compressors operating serially, the air leaving one compressor may be cooled upto ambient state or somewhat high temperature before being injected into subsequent compressor. This cooling of fluid being compressed between two consecutive compressors is called intercooling and is frequently used in case of multistage compressors.

Considering clearance volume: With clearance volume the cycle is represented on Fig.5.3 The work done for compression of air polytropically can be given by the are a enclosed in cycle 1-2-3-4. Clearance volume in compressors varies from 1.5% to 35% depending upon type of compressor.

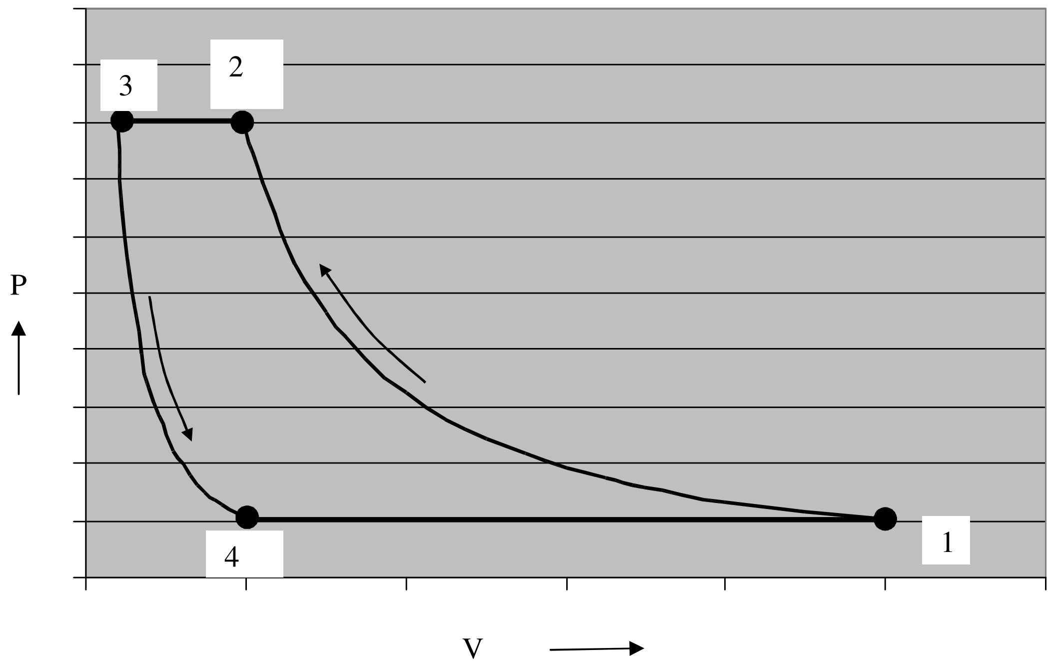


Fig.5.3 P-V diagram for Reciprocating Compressor with Clearance

$W_{c,\text{with CV}}$ = Area 1234

$$= \left(\frac{n}{n-1}\right)(p_1V_1)\left[\left(\frac{p_2}{p_1}\right)^{\left(\frac{n-1}{n}\right)} - 1\right] - \left(\frac{n}{n-1}\right)(p_4V_4)\left[\left(\frac{p_3}{p_4}\right)^{\frac{n-1}{n}} - 1\right]$$

Here $P_1 = P_4$, $P_2 = P_3$

$$= \left(\frac{n}{n-1}\right)(p_1V_1)\left[\left(\frac{p_2}{p_1}\right)^{\left(\frac{n-1}{n}\right)} - 1\right] - \left(\frac{n}{n-1}\right)(p_1V_4)\left[\left(\frac{p_2}{p_1}\right)^{\frac{n-1}{n}} - 1\right]$$

$$= \left(\frac{n}{n-1}\right)(p_1)\left[\left(\frac{p_2}{p_1}\right)^{\left(\frac{n-1}{n}\right)} - 1\right](V_1 - V_4)$$

In the cylinder of reciprocating compressor (V_1-V_4) shall be the actual volume of air delivered per cycle. $V_d = V_1 - V_4$. This ($V_1 - V_4$) is actually the volume of air in hated in the cycle and delivered subsequently.

$$W_{c,withCV}=\left(\frac{n}{n-1}\right)(p_1V_d)\left[\left(\frac{p_2}{p_1}\right)^{\left(\frac{n-1}{n}\right)}-1\right]$$

If air is considered to behave as perfect gas then pressure, temperature, volume and mass can be inter related using perfect gas equation. The mass at state 1 may be given as m_1 mass at state 2 shall be m1, but at state 3 after delivery mass reduces to m_2 and at state 4 it shall be m_2.

So, at state 1, $p_1V_1=m_1RT_1$

at state 2, $p_2V_2=m_1RT_2$

at state 3, $p_3V_3=m_2RT_3$ or $p_2V_3=m_2RT_3$

at state 4, $p_4V_4=m_2RT_4$ or $p_1V_4=m_2RT_4$

Ideally there shall be no change in temperature during suction and delivery
i.e., $T_4=T_1$ and $T_2=T_3$ from earlier equation

$$W_{c,withCV}=\left(\frac{n}{n-1}\right)(p_1)\left[\left(\frac{p_2}{p_1}\right)^{\left(\frac{n-1}{n}\right)}-1\right](V_1-V_4)$$

Temperature and pressure can be related as,

$$\left(\frac{p_2}{p_1}\right)^{\frac{(n-1)}{n}}=\frac{T_2}{T_1}\quad\text{and}\quad\left(\frac{p_4}{p_3}\right)^{\frac{(n-1)}{n}}=\frac{T_4}{T_3}\quad\Longrightarrow\quad\left(\frac{p_1}{p_2}\right)^{\frac{(n-1)}{n}}=\frac{T_4}{T_3}$$

Substiuting

$$W_{c,withCV}=\left(\frac{n}{n-1}\right)(m_1RT_1-m_2RT_4)\left[\frac{T_2}{T_1}-1\right]$$

Substituting for constancy of temperature during suction and delivery.

$$W_{c,withCV}=\left(\frac{n}{n-1}\right)(m_1RT_1-m_2RT_1)\left[\frac{T_2-T_1}{T_1}\right]$$

Or

$$W_{c,withCV}=\left(\frac{n}{n-1}\right)(m_1-m_2)R(T_2-T_1)$$

Thus $(m_1\text{-}m_2)$ denotes the mass of air sucked or delivered. For unit mass of air delivered the work done per kg of air can be given as,

$$W_{c,withCV} = \left(\frac{n}{n-1}\right)R(T_2 - T_1) \quad \text{per kg of air}$$

Thus from above expressions it is obvius that the clearance volume reduces the effective swept volume i.e., the masso f air handled but the work done per kg of air delivered remains unaffected.

From the cycle work estimated as above the theoretical power required for running compressor shall be,

For single acting compressor running with N rpm, power input required, assuming clearance volume.

$$Powerrequired = \left[\left(\frac{n}{n-1}\right)\left[\left(\frac{p_2}{p_1}\right)^{\left(\frac{n-1}{n}\right)} - 1\right] p_1(V_1 - V_4)\right](N)$$

For double acting compressor, Power

$$Powerrequired = \left[\left(\frac{n}{n-1}\right)\left[\left(\frac{p_2}{p_1}\right)^{\left(\frac{n-1}{n}\right)} - 1\right] p_1(V_1 - V_4)\right](2N)$$

Volumetric efficiency: Volumetric efficiency of compressor is the measure of the deviation from volume handling capacity of compressor. Mathematically, the volumetric efficiency is given by the ratio of actual volume of air sucked and swept volume of cylinder. Ideally the volume of air sucked should be equal to the swept volume of cylinder, but it is not so in actual case. Practically the volumetric efficiency lies between 60 to 90%.

Volumetric efficiency can be overall volumetric efficiency and absolute volumetric efficiency as given below.

$$\text{Overall volumetric efficiency} = \frac{\text{Volume of free air sucked in cylinder}}{\text{Swept volume of LP cylinder}}$$

$$(\text{Volumetric efficiency})_{\text{freeaircondition}} = \frac{\text{Volume of free air sucked in cylinder}}{(\text{Swept volume of LP cylinder})_{freeaircondition}}$$

Here free air condition refers to the standard conditions. Free air condition may be taken as 1 atm or 1.01325 bar and 15^{o}C or 288K. consideration for free air is necessary as otherwise the different compressors can not be compared using volumetric efficiency because specific volume or density of air varies with altitude. It may be seen that a compressor at datum level (sea level) shall deliver large mass than the same compressor at high altitude.

This concept is used for giving the capacity of compressor in terms of 'free air delivery' (FAD). "Free air delivery is the volume of air delivered being reduced to free air conditions". In case of air the free air delivery can be obtained using perfect gas equation as,

$$\frac{p_a V_a}{T_a} = \frac{p_1 (V_1 - V_4)}{T_1} = \frac{p_2 (V_2 - V_3)}{T_2}$$

Where subscript a or p_a, V_a, T_a denote properties at free air conditions

$$V_a = \frac{p_1 T_a}{p_a} \frac{p_1 (V_1 - V_4)}{T_1} = \text{FAD per cycle}$$

This volume V_a gives 'free air delivered' per cycle by the compressor.

Absolute volumetric efficiency can be defined, using NTP conditions in place of free air conditions.

$$\eta_{vol} = \frac{FAD}{Sweptvolume} = \frac{V_a}{(V_1 - V_2)} = \frac{p_1 T_a (V_1 - V_4)}{p_a T_1 (V_1 - V_3)}$$

$$\eta_{vol} = \left(\frac{p_1 T_a}{p_a T_1}\right) \left\{\frac{(V_s + V_c) - V_4}{V_s}\right\}$$

Here V_s is the swept volume = $V_1 - V_3$

V_c is the clearance volume = V_3

$$\eta_{vol} = \left(\frac{p_1 T_a}{p_a T_1}\right) \left\{1 + \left(\frac{V_c}{V_s}\right) - \left(\frac{V_4}{V_s}\right)\right\}$$

Here $\frac{V_4}{V_s} = \frac{V_4}{V_c} \cdot \frac{V_c}{V_s} = \left(\frac{V_4}{V_3} \cdot \frac{V_c}{V_s}\right)$

Let the ratio of clearance volume to swept volume be given by C. = $\frac{V_c}{V_s}$

$$\eta_{vol} = \left(\frac{p_1 T_a}{p_a T_1}\right) \left\{1 + C - C\left(\frac{V_4}{V_3}\right)\right\}$$

$$\eta_{vol} = \left(\frac{p_1 T_a}{p_a T_1}\right)\left\{1 + C - C\left(\frac{p_2}{p_1}\right)^{1/n}\right\}$$

Volumetric efficiency depends on ambient pressure and temperature, suction pressure and temperature, ratio of clearance to swept volume, and pressure limits. Volumetric efficiency increases with decrease in pressure ratio in compressor.

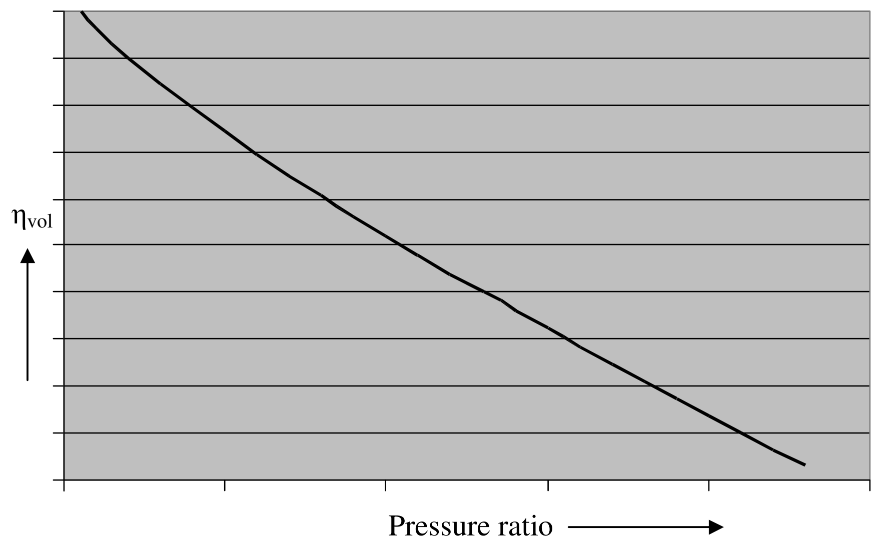


Fig.5.4 Volumetric efficiency v/s Pressure ratio

Multistage Compression

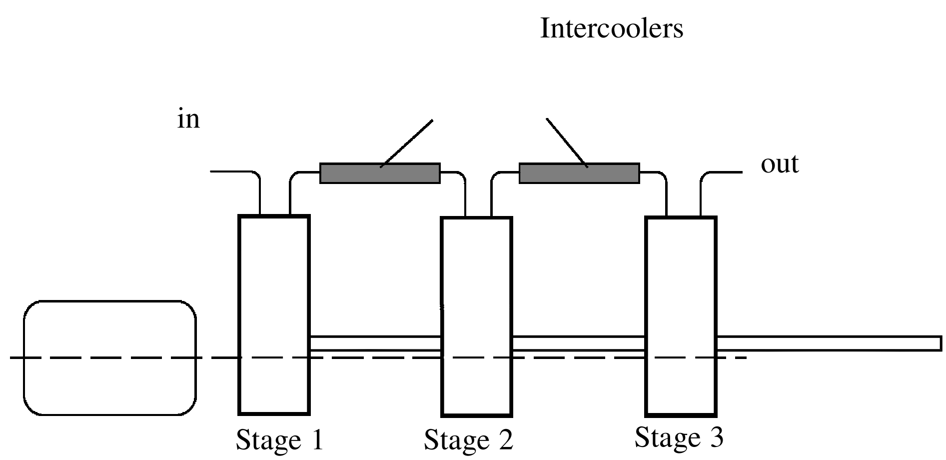


Fig.5.5 Multistage Compressor with inter coolers

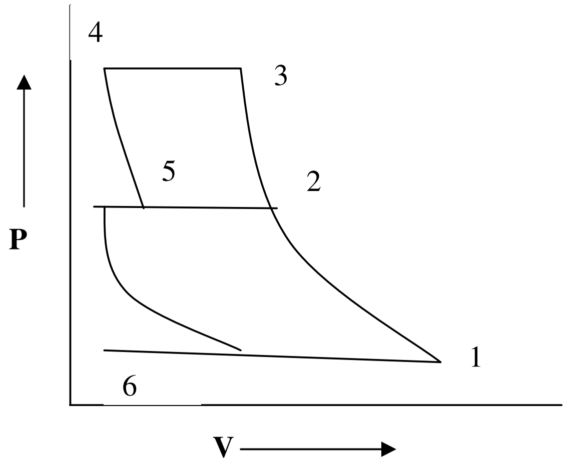


Fig.5.6 P-V diagram for Multistage Compressor

Multistage compression refers to the compression process completed in more than one stage i.e., a part of compression occurs in one cylinder and subsequently compressed air is sent to subsequent cylinders for further compression. In case it is desired to increase the compression ratio of compressor then multi-stage compression becomes inevitable. If we look at the expression for volumetric efficiency then it shows that the volumetric efficiency decreases with increase in pressure ratio. This aspect can also be explained using p-V representation shown in Fig.5.6.

Therefore, the volumetric efficiency reduces with increasing pressure ratio in compressor with single stage compression. Also for getting the same amount of free air delivery the size of cylinder is to be increased with increasing pressure ratio. The increase in pressure ratio also requires sturdy structure from mechanical strength point of view fro withstanding large pressure difference.

The solution to number of difficulties discussed above lies in using the multistage compression where compression occurs in parts in different cylinders one after the other. Fig.16.6b, shows the multistage compression occurring in two stages. Here first stage of compression occurs in cycle 12671 and after first stage compression partly compressed enters second stage of compression and occurs in cycle 2345. In case of multistage compression the compression in first stage occurs at low temperature and subsequent compression in following stages occurs at higher temperature. The compression work requirement depends largely upon the average temperature during compression. Higher average temperature during compression has larger work requirement compared to low temperature so it is always desired to keep the low average temperature during compression.

Apart from the cooling during compression the temperature of air at inlet to compressor can be reduced so as to reduce compression work. In multistage compression the partly compressed air leaving first stage is cooled upto ambient air temperature in intercooler and then sent to subsequent cylinder (stage) for compression. Thus, intercoolers when put between the stages reduce the compression work and compression is called intercooled compression. Intercooling is called perfect when temperature at inlet to subsequent stages of compression is reduced to ambient temperature. Intercooling between two stages causes temperature drop from 2 to 2' i.e discharge from first stage (at 2) is cooled upto the ambient temperature stage (at2') which lies on isothermal compression process 1-2'-3". In the absence of intercooling the discharge from first stage shall enter at 2. Final discharge from second stage occurs at 3' in case of intercooled compression compared to discharge at 3 in case of non-intercooled compression. Thus, intercooling offers reduced work requirement by the amount shown by area 22'3'3 on p-V diagram. If the intercooling is not perfect then the inlet state to second/subsequent stage shall not lie on the isothermal compression process line and this stage shall lie between actual discharge state from first stage and isothermal compression line.

Fig.5.5 shows the schematic of multi stage compressor (double stage) with inter cooler between stage T-s representation is shown in Fig.5.6. The total work requirement for running this shall be algebraic summation of work required for low pressure (LP) and high pressure (HP) stages. The size of HP cylinder is smaller than LP cylinder as HP cylinder handles high pressure air having smaller specific volume.

Mathematical analysis of multistage compressor is done with following assumptions:

(i) Compression in all the stages is done following same index of compression and there is no pressure drop in suction and delivery pressures in each stage. Suction and delivery pressure remains constant in the stages.

(ii) There is perfect intercooling between compression stages.

(iii) Mass handled in different stages is same i.e, mass of air in LP and HP stages are same.

(iv) Air behaves as perfect gas during compression.

From combined p-V diagram the compressor work requirement can be given as,

Work requirement in LP cylinder, $W_{LP} = \left(\frac{n}{n-1}\right) P_1 V_1 \left\{ \left(\frac{P_2}{P_1}\right)^{\frac{(n-1)}{n}} - 1 \right\}$

Work requirement in HP cylinder, $W_{HP} = \left(\frac{n}{n-1}\right) P_2 V_2 \left\{ \left(\frac{P_2}{P_1}\right)^{\frac{(n-1)}{n}} - 1 \right\}$

For perfect intercooling, $p_1 V_1 = p_2 V_{2'}$ and

$$W_{HP} = \left(\frac{n}{n-1}\right) P_2 V_{2'} \left\{ \left(\frac{P_2}{P_1}\right)^{\frac{(n-1)}{n}} - 1 \right\}$$

Therefore, total work requirement, $W_c = W_{LP} + W_{HP}$, for perfect inter cooling

$$W_C = \left(\frac{n}{n-1}\right) \left[P_1 V_1 \left\{ \left(\frac{P_2}{P_1}\right)^{\frac{(n-1)}{n}} - 1 \right\} + P_2 V_{2'} \left\{ \left(\frac{P_{2'}}{P_2}\right)^{\frac{n-1}{n}} - 1 \right\} \right]$$

$$= \left(\frac{n}{n-1}\right) \left[P_1 V_1 \left\{ \left(\frac{P_2}{P_1}\right)^{\frac{(n-1)}{n}} - 1 \right\} + P_1 V_1 \left\{ \left(\frac{P_{2'}}{P_2}\right)^{\frac{n-1}{n}} - 1 \right\} \right]$$

$$W_C = \left(\frac{n}{n-1}\right) P_1 V_1 \left[\left(\frac{P_2}{P_1}\right)^{\frac{n-1}{n}} + \left(\frac{P_{2'}}{P_1}\right)^{\frac{n-1}{n}} - 2 \right]$$

Power $= W_c \times N$ - Watts

If we look at compressor work then it shows that with the initial and final pressures p_1 and $P_{2'}$ remaining same the intermediate pressure p_2 may have value floating between p_1 and $P_{2'}$ and change the work requirement Wc. Thus, the compressor work can be optimized with respect to intermediate pressure P

$_2$. Mathematically, it can be differentiated with respect to P_2.

$$\frac{dW_C}{dP_2}=\frac{d}{dP_2}\left[\left(\frac{n}{n-1}\right)P_1V_1\left\{\left(\frac{P_2}{P_1}\right)^{\frac{n-1}{n}}+\left(\frac{P_{2'}}{P_2}\right)^{\frac{n-1}{n}}-2\right\}\right]$$

$$\frac{dW_C}{dP_2}=\left[\left(\frac{n}{n-1}\right)P_1V_1\frac{d}{dP_2}\left\{\left(\frac{P_2}{P_1}\right)^{\frac{n-1}{n}}+\left(\frac{P_{2'}}{P_2}\right)^{\frac{n-1}{n}}-2\right\}\right]$$

$$\frac{dW_C}{dP_2}=\left(\frac{n}{n-1}\right)P_1V\left\{\left(\frac{n-1}{n}\right)P_1^{\frac{1-n}{n}}\cdot P_2^{\frac{-1}{n}}-\left(\frac{n-1}{n}\right)\cdot P_{2'}^{\frac{1-n}{n}}\cdot P_2^{\frac{1-2n}{n}}\right\}$$

Equating to zero yields

$$P_1^{\frac{1-n}{n}}\cdot P_2^{\frac{-1}{n}}=\cdot P_{2'}^{\frac{1-n}{n}}\cdot P_2^{\frac{1-2n}{n}}$$

$$P_2^{\frac{-2+2n}{n}}=\cdot P_{2'}^{\frac{1-n}{n}}\cdot P_1^{\frac{n-1}{n}}$$

$$P_2^{\,2\left(\frac{n-1}{n}\right)}=\left(P_1\cdot P_{2'}\right)^{\left(\frac{n-1}{n}\right)}$$

$$P_2^{\,2}=\left(P_1\cdot P_{2'}\right), P_2=\sqrt{P_1\cdot P_{2'}}$$

Pressure ratio in Ist stage = Pressure ratio in IInd stage

Thus, it is established that the compressor work requirement shall be minimum when the pressure ratio in each stage is equal.

In case of multiple stages, say *i* number of stages, for the delivery and suction pressures of P_{i+1} and P_1 the optimum stage pressure ratio shall be,

Optimum stage pressure ratio = $\left(\frac{P_{i+1}}{P_1}\right)^{\frac{1}{i}}$ for pressure at stages being P_1, P_2, P_3,……P_{i-1}, P_i, P_{i+1}

Minimum work required in two stage compressor can be given by

$$W_{C,\min}=\left(\frac{n}{n-1}\right)P_1V_1\cdot 2\left\{\left(\frac{P_2}{P_1}\right)^{\frac{(n-1)}{n}}-1\right\}$$

For *i* number of stages, minimum work,

$$W_{C,\min}=i\cdot\left(\frac{n}{n-1}\right)P_1V_1\left\{\left(\frac{P_{i+1}}{P_i}\right)^{\frac{(n-1)}{n\cdot i}}-1\right\}$$

It also shows that for optimum pressure ratio the work required in different stages remains same for the assumptions made for present analysis. Due to pressure ration being equal in all stages the temperature ratios and maximum temperature in each stage remains same for perfect intercooling.

Cylinder dimensions: In case of multistage compressor the dimension of cylinders can be estimated basing upon the fact that the mass flow rate of air across the stages remains same. For perfect intercooling the temperature of air at suction of each stage shall be same.

If the actual volume sucked during suction stroke is V_1, V_2, V_3. . . . for different stages they by perfect gas law, P_1 V_1 = RT_1, P_2 V_2 = RT_2, Pc, V_3 = RT_3
For perfect intercooling

P_1 V_1 = RT_1, p_2 V_2 = RT_1, p_3, V_3 = RT_1

P_1 V_1 = P_2 V_2 = RT_2, P_3, V_3 = ...

If the volumetric efficiency of respective stages in $\eta_{V_1}, \eta_{V_2}, \eta_{V_3}, ...$

Then theoretical volume of cylinder1, $V_{1,th} = \dfrac{V_1}{\eta_{V_1}}; V_1 = \eta_{V_1} \cdot V_{1,th}$

Cylinder 2, $V_{2,th} = \dfrac{V_1}{\eta_{V_1}}; V_2 = \eta_{V_2} \cdot V_{2,th}$

Cylinder 3, $V_{3,th} = \dfrac{V_3}{\eta_{V_3}}; V_3 = \eta_{V3} \cdot V_{3,th}$

Substituting,
$P_1 \cdot \eta_{V_1} \cdot V_{1,th} = P_2 \cdot \eta_{V_2} \cdot V_{2,th} = P_3 \cdot \eta_{V_3} \cdot V_{3,th} = ...$
Theoretical volumes of cylinder can be given using geometrical dimensions of cylinder as diameters D_1, D_2, D_3 . . . and stroke lengths L_1, L_2, L_3 . . . .

Or $$V_{1,th} = \frac{\pi}{4} \cdot D_1^{\ 2} \cdot L_1$$

$$V_{2,th} = \frac{\pi}{4} \cdot D_2^{\ 2} \cdot L_2$$

$$V_{3,th} = \frac{\pi}{4} \cdot D_3^{\ 2} \cdot L_3$$

Or $$P_1 \cdot \eta_{Vi} \cdot \frac{\pi}{4} \cdot D_1^{\ 2} \cdot L_1 = P_2 \cdot \eta_{V2} \cdot \frac{\pi}{4} \cdot D_2^{\ 2} \cdot L_2$$

$$= P_3 \cdot \eta_{V3} \cdot \frac{\pi}{4} \cdot D_3^{\ 2} \cdot L_3 = ...$$

$$P_1 \cdot \eta_{V1} \cdot \frac{\pi}{4} \cdot D_1^{\ 2} \cdot L_1 = P_2 \cdot \eta_{V2} \cdot \frac{\pi}{4} \cdot D_2^{\ 2} \cdot L_2$$

$$= P_3 \cdot \eta_{V3} \cdot D_3^{\ 2} \cdot L_3 = ...$$

If the volumetric efficiency is same for all cylinders, i.e $\eta_{V1} = \eta_{V2} = \eta_{V3} = ...$ and stroke for all cylinder is same i.e. $L_1 = L_2 = L_3 = ...$

Then, $D_1^{\ 2} P_1 = D_2^{\ 2} P_2 = D_3^{\ 2} P_3 = ...$

These generic relations may be used for getting the ratio of diameters of cylinders of multistage compression.

Energy balance: Energy balance may be applied on the different components constituting multistage compression.

For LP stage the steady flow energy equation can be written as below:

$$m \cdot h_1 + W_{LP} = m \cdot h_2 + Q_{LP}$$

$$Q_{LP} = W_{LP} - m(h_2 - h_1)$$
$$Q_{LP} = W_{LP} - m \cdot C_p (T_2 - T_1)$$

For intercooling (Fig. 5.5) between LP and HP stage steady flow energy equation shall be;

$$m \cdot h_2 = m \cdot h_{2'} + Q_{Int}$$
$$Q_{Int} = m(h_2 - h_{2'})$$
$$Q_{Int} = m \cdot C_p (T_2 - T_{2'})$$

For HP stage (Fig.5.5) the steady flow energy equation yields.

$$m \cdot h_{2'} + W_{HP} = m \cdot h_{3'} + Q_{HP}$$
$$Q_{HP} = W_{HP} + m(h_{2'} - h_{3'})$$
$$Q_{HP} = W_{HP} + m \cdot C_p (T_{2'} - T_{3'}) = W_{HP} - m \cdot C_p (T_{3'} - T_{2'})$$

In case of perfect intercooling and optimum pressure ratio, $T_{2'} = T_1$ and $T_2 = T_{3'}$

Hence for these conditions,

$$Q_{LP} = W_{LP} - m \cdot C_p (T_2 - T_1)$$
$$Q_{Int} = m \cdot C_p (T_2 - T_1)$$
$$Q_{HP} = W_{HP} - m \cdot C_p (T_2 - T_1)$$

Total heat rejected during compression shall be the sum of heat rejected during compression and heat extracted in intercooler for perfect intercooling.

Heat rejected during compression for polytropic process $= \left(\frac{\gamma - n}{\gamma - 1}\right) \times Work$