REE 307 - Internal Fluid Flow
Sheet 2 - Solution
Fundamentals of Fluid Mechanics

1. Is the following flows physically possible, that is, satisfy the continuity equation? Substitute the expressions for density and for the velocity field into the continuity equation to substantiate your answer:
A gas is flowing at relatively low speeds (so that its density may be assumed constant) where

$$u = -\frac{2xyz}{(x^2+y^2)^2}U_\infty L$$

$$v = \frac{(x^2-y^2)z}{(x^2+y^2)^2}U_\infty L$$

$$w = \frac{y}{x^2+y^2}U_\infty L$$

Here U∞ and L are a reference velocity and a reference length, respectively.

Let us use the continuity equation for a three-dimensional flow, i.e., equation (2.1):

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} \stackrel{?}{=} 0$$

For constant density flow, this equation becomes:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Thus,
$$\frac{\partial}{\partial x}\left\{-\frac{2xyz}{(x^2+y^2)^2}U_\infty L\right\} + \frac{\partial}{\partial y}\left\{\frac{(x^2-y^2)z}{(x^2+y^2)^2}U_\infty L\right\} + \frac{\partial}{\partial z}\left\{\frac{y}{x^2+y^2}U_\infty L\right\} = 0$$

Since U_∞ and L are constants and since they appear in every term, they can be divided out leaving:

$$-\frac{2yz}{(x^2+y^2)^2} - \frac{2xyz(-2)(2x)}{(x^2+y^2)^3} - \frac{2yz}{(x^2+y^2)^2} + \frac{(x^2-y^2)z(-2)(2y)}{(x^2+y^2)^3}$$

$$= -\frac{4yz}{(x^2+y^2)^2} - \frac{-8x^2yz + 4x^2yz - 4y^3z}{(x^2+y^2)^3}$$

$$= \frac{-4x^2yz - 4y^3z + 8x^2yz - 4x^2yz + 4y^3z}{(x^2+y^2)^3} = 0$$

Therefore, the continuity equation is satisfied.

2. Two of the three velocity components for an incompressible flow are:

$$u = x^3 + 3xz \qquad v = y^3 + 3yz$$

What is the general form of the velocity component w(x,y,z) that satisfies the continuity equation?

For incompressible flow this becomes:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Find the derivatives of the given velocity components:

$$\frac{\partial u}{\partial x} = 3(x^2 + z) \quad \frac{\partial u}{\partial y} = 3(y^2 + z)$$

Therefore:

$$\frac{\partial w}{\partial z} = -3(x^2 + y^2 + 2z)$$

Integrating yields:

$$w = -3z\,(x^2 + y^2 + z) = f(x,y,z,t)$$

Where $f(x,y,z,t)$ is an arbitrary function (x,y,z,t). Since the first two velocity components are not a function of time, it may be possible to assume the flow is steady and drop the time function from the arbitrary constant.

3. The velocity components for a two-dimensional flow are

$$u = \frac{C(y^2 - x^2)}{(x^2 + y^2)^2} \qquad v = \frac{-2Cxy}{(x^2 + y^2)^2}$$

where C is a constant. Does this flow satisfy the continuity equation?

Given: Velocity components for a 2D incompressible flow:

$$u = \frac{C\left(y^2 - x^2\right)}{\left(x^2 + y^2\right)^2} \qquad v = -\frac{2Cxy}{\left(x^2 + y^2\right)^2}$$

Assume 2D incompressible flow and that C is a constant. For 2D incompressible flow the continuity equation is:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Taking the required derivatives yields:

$$\frac{\partial u}{\partial x} = C\left(y^2 - x^2\right)(-2)\left(x^2 + y^2\right)^{-3}(2x) + C(-2x)\left(x^2 + y^2\right)^{-2}$$

$$\frac{\partial v}{\partial y} = -2Cxy(-2)\left(x^2 + y^2\right)^{-3}(2y) + C(-2x)\left(x^2 + y^2\right)^{-2}$$

$$\frac{-4Cx\left(y^2 - x^2\right)}{\left(x^2 + y^2\right)^3} - \frac{2Cx}{\left(x^2 + y^2\right)^2} + \frac{8Cxy^2}{\left(x^2 + y^2\right)^3} - \frac{2Cx}{\left(x^2 + y^2\right)^2} = 0$$

after some algebra and patience!

4. For the two-dimensional flow of incompressible air near the surface of a flat plate, the steamwise (or x) component of the velocity may be approximated by the relation

$$u = a_1 \frac{y}{\sqrt{x}} - a_2 \frac{y^3}{x^{1.5}}$$

Using the continuity equation, what is the velocity component *v* in the y direction? Evaluate the constant of integration by noting that *v* = 0 at y = 0.

Referring to the continuity equation for a two-dimensional, incompressible flow:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = \frac{1}{2}\frac{a_1 y}{x^{1.5}} - \frac{3}{2}\frac{a_2 y^3}{x^{2.5}}$$

Integrating with respect to y

$$v = +\frac{1}{4}\frac{a_1 y^2}{x^{1.5}} - \frac{3}{8}\frac{a_2 y^4}{x^{2.5}} + C$$

To evaluate the constant of integration C, we note that $v = 0$ when $y = 0$. Thus, $C = 0$ and

$$v = \frac{1}{4}\frac{a_1 y^2}{x^{1.5}} - \frac{3}{8}\frac{a_2 y^4}{x^{2.5}}$$

5. Given the velocity field

$$\vec{V} = (6 + 2xy + t^2)\hat{i} - (xy^2 + 10t)\hat{j} + 25\hat{k}$$

what is the acceleration of a particle at (3, 0, 2) at time t = 1?

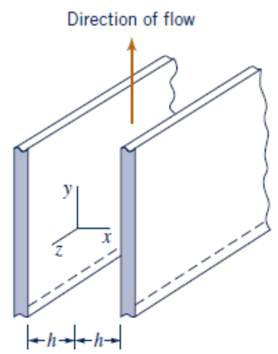
$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$$

$$\vec{a} = 2t\hat{i} - 10\hat{j} + [6 + 2xy + t^2][2y\hat{i} - y^2\hat{j}]$$
$$- [xy^2 + 10t][2x\hat{i} - 2xy\hat{j}] + 25[0]$$

when (x, y, z) is $(3, 0, 2)$ and $t = 1$

$$\vec{a} = \hat{i}[2 - 60] + \hat{j}[-10] = -58\hat{i} - 10\hat{j}$$

6. A viscous, incompressible fluid flows between the two infinite, vertical, parallel plates of Fig. Determine, by use of the Navier–Stokes equations, an expression for the pressure gradient in the direction of flow. Express your answer in terms of the mean velocity. Assume that the flow is laminar, steady, and uniform.



With the coordinate system shown $u=0$, $w=0$ and from the continuity equation $\frac{\partial v}{\partial y}=0$. Thus, from the y-component of the Navier-Stokes equations (Eq. 6.127b), with $g_y=-g$,

$$0=-\frac{\partial p}{\partial y}-\rho g+\mu\frac{d^2v}{dx^2} \quad (1)$$

Since the pressure is not a function of x, Eq. (1) can be written as

$$\frac{d^2v}{dx^2}=\frac{P}{\mu}$$

(where $\underline{P}=\frac{\partial p}{\partial y}+\rho g$) and integrated to obtain

$$\frac{dv}{dx}=\frac{P}{\mu}x+C_1 \quad (2)$$

From symmetry $\frac{dv}{dx}=0$ at $x=0$ so that $C_1=0$. Integration of Eq. (2) yields

$$v=\frac{P}{\mu}\frac{x^2}{2}+C_2$$

Since at $x=\pm h$, $v=0$ it follows that $C_2=-\frac{P}{2\mu}(h^2)$

and therefore

$$v=\frac{P}{2\mu}(x^2-h^2)$$

The flowrate per unit width in the z-direction can be expressed as

$$q=\int_{-h}^{h}v\,dx=\int_{-h}^{h}\frac{P}{2\mu}(x^2-h^2)\,dx=-\frac{2}{3}\frac{Ph^3}{\mu}$$

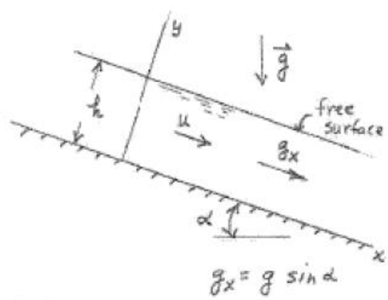
Thus, with V (mean velocity) given by the equation

$$V=\frac{q}{2h}=-\frac{1}{3}\frac{Ph^2}{\mu}$$

it follows that

$$\frac{\partial p}{\partial y}=-\frac{3\mu V}{h^2}-\rho g$$

7. A layer of viscous liquid of constant thickness (no velocity perpendicular to plate) flows steadily down an infinite, inclined plane. Determine, by means of the Navier–Stokes equations, the relationship between the thickness of the layer and the discharge per unit width. The flow is laminar, and assume air resistance is negligible so that the shearing stress at the free surface is zero.



With the coordinate system shown in the figure $v=0$, $w=0$, and from the continuity equation $\frac{\partial u}{\partial x}=0$. Thus, from the x-component of the Navier-Stokes equations (Eq. 6.127a),

$$0 = -\frac{\partial p}{\partial x} + \rho g \sin\alpha + \mu \frac{d^2u}{dy^2} \qquad (1)$$

Also, since there is a free surface, there cannot be a pressure gradient in the x-direction so that $\frac{\partial p}{\partial x}=0$ and Eq. (1) can be written as

$$\frac{d^2u}{dy^2} = -\frac{\rho g}{\mu}\sin\alpha$$

Integration yields

$$\frac{du}{dy} = -\left(\frac{\rho g}{\mu}\sin\alpha\right)y + C_1 \qquad (2)$$

Since the shearing stress

$$\tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

equals zero at the free surface ($y=h$) it follows that

$$\frac{\partial u}{\partial y} = 0 \quad \text{at} \quad y = h$$

so that the constant in Eq. (2) is

$$C_1 = \frac{\rho g}{\mu}\sin\alpha$$

Integration of Eq. (2) yields

$$u = -\left(\frac{\rho g}{\mu}\sin\alpha\right)\frac{y^2}{2} + \left(\frac{\rho g}{\mu}\sin\alpha\right)y + C_2$$

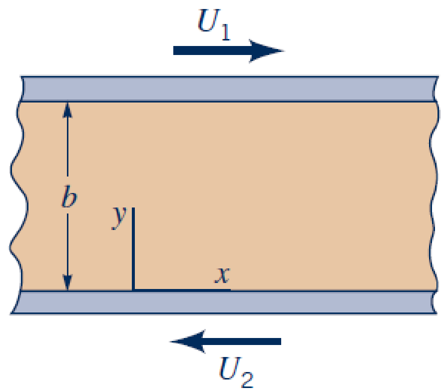
Since $u=0$ at $y=0$, it follows that $C_2=0$, and therefore

$$u = \frac{\rho g}{\mu}\sin\alpha\left(hy - \frac{y^2}{2}\right)$$

The flowrate per unit width can be expressed as $q = \int_0^h u\,dy$ so that

$$q = \int_0^h \frac{\rho g}{\mu}\sin\alpha\left(hy - \frac{y^2}{2}\right)dy = \frac{\rho g h^3 \sin\alpha}{3\mu}$$

8. An incompressible, viscous fluid is placed between horizontal, infinite, parallel plates as is shown in Fig. The two plates move in opposite directions with constant velocities, U_1 and U_2, as shown. The pressure gradient in the x direction is zero, and the only body force is due to the fluid weight. Use the Navier–Stokes equations to derive an expression for the velocity distribution between the plates. Assume laminar flow.



For the specified conditions, $v=0$, $w=0$, $\frac{\partial p}{\partial x}=0$, and $g_x=0$, so that the x-component of the Navier-Stokes equations (Eq. 6.127a) reduces to

$$\frac{d^2u}{dy^2}=0 \qquad (1)$$

Integration of Eq. (1) yields

$$u = C_1 y + C_2 \qquad (2)$$

For $y=0$, $u=-U_2$ and therefore from Eq. (2)

$$C_2 = -U_2$$

For $y=b$, $u=U_1$ so that

$$U_1 = C_1 b - U_2$$

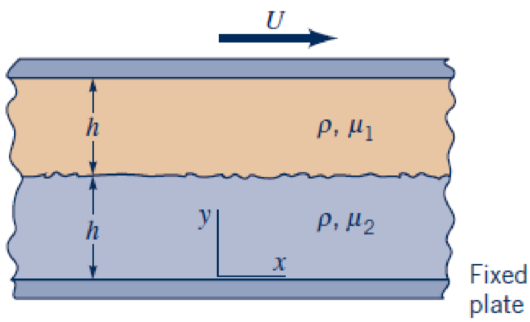
or

$$C_1 = \frac{U_1+U_2}{b}$$

Thus,

$$u = \left(\frac{U_1+U_2}{b}\right) y - U_2$$

9. Two immiscible, incompressible, viscous fluids having the same densities but different viscosities are contained between two infinite, horizontal, parallel plates (Fig.). The bottom plate is fixed, and the upper plate moves with a constant velocity U. Determine the velocity at the interface. Express your answer in terms of U, μ_1, and μ_2. The motion of the fluid is caused entirely by the movement of the upper plate; that is, there is no pressure gradient in the x direction. The fluid velocity and shearing stress are continuous across the interface between the two fluids. Assume laminar flow.



For the specified conditions, $v=0$, $w=0$, $\frac{\partial p}{\partial x}=0$, and $g_x=0$, so that the x-component of the Navier-Stokes equations (Eq. 6.127a) for either the upper or lower layer reduces to

$$\frac{d^2u}{dy^2}=0 \qquad (1)$$

Integration of Eq. (1) yields

$$u = Ay + B$$

which gives the velocity distribution in either layer.
In the upper layer at $y=2h$, $u=U$ so that

$$B_1 = U - A_1(2h)$$

where the subscript 1 refers to the upper layer.
For the lower layer at $y=0$, $u=0$ so that

$$B_2 = 0$$

where the subscript 2 refers to the lower layer. Thus,

$$u_1 = A_1(y-2h)+U$$

and

$$u_2 = A_2 y$$

At $y=h$, $u_1=u_2$ so that

$$A_1(h-2h)+U = A_2 h$$

or

$$A_2 = -A_1 + \frac{U}{h} \qquad (2)$$

Since the velocity distribution is linear in each layer the shearing stress

$$\tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \mu \frac{du}{dy}$$

is constant throughout each layer. For the upper layer

$$\tau_1 = \mu_1 A_1$$

and for the lower layer

$$\tau_2 = \mu_2 A_2$$

At the interface $\tau_1 = \tau_2$ so that

$$\mu_1 A_1 = \mu_2 A_2$$

or

$$\frac{A_1}{A_2} = \frac{\mu_2}{\mu_1} \qquad (3)$$

Substitution of Eq. (3) into Eq. (2) yields

$$A_2 = -\frac{\mu_2}{\mu_1} A_2 + \frac{U}{h}$$

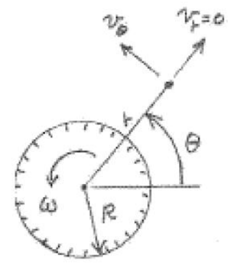
or

$$A_2 = \frac{U/h}{1 + \mu_2/\mu_1}$$

Thus, velocity at the interface is

$$u_2(y = h) = A_2 h = \frac{U}{1 + \frac{\mu_2}{\mu_1}}$$

10. An infinitely long, solid, vertical cylinder of radius R is located in an infinite mass of an incompressible fluid. Start with the Navier–Stokes equation in the u direction and derive an expression for the velocity distribution for the steady-flow case in which the cylinder is rotating about a fixed axis with a constant angular velocity ω. You need not consider body forces. Assume that the flow is axisymmetric and the fluid is at rest at infinity.



For this flow field, $v_r = 0$, $v_z = 0$, and from the continuity equation,

$$\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0 \qquad \text{(Eq. 6.35)}$$

it follows that

$$\frac{\partial v_\theta}{\partial \theta} = 0 \qquad \text{(See figure for notation.)}$$

Thus, the Navier-Stokes equation in the θ-direction (Eq. 6.128b) for steady flow reduces to

$$0 = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) - \frac{v_\theta}{r^2}\right]$$

Due to the symmetry of the flow,

$$\frac{\partial p}{\partial \theta} = 0$$

so that

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) - \frac{v_\theta}{r^2} = 0$$

or

$$\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2} = 0 \qquad (1)$$

Since v_θ is a function of only r, Eq. (1) can be expressed as an ordinary differential equation, and re-written as

$$\frac{d^2 v_\theta}{dr^2} + \frac{d}{dr}\left(\frac{v_\theta}{r}\right) = 0 \qquad (2)$$

Equation (2) can be integrated to yield

$$\frac{dv_\theta}{dr} + \frac{v_\theta}{r} = c_1$$

or

$$r\frac{dv_\theta}{dr} + v_\theta = c_1 r \qquad (3)$$

Equation (3) can be expressed as

$$\frac{d(rv_\theta)}{dr} = c_1 r$$

and a second integration yields

$$rv_\theta = \frac{c_1 r^2}{2} + c_2$$

or

$$v_\theta = \frac{c_1 r}{2} + \frac{c_2}{r}$$

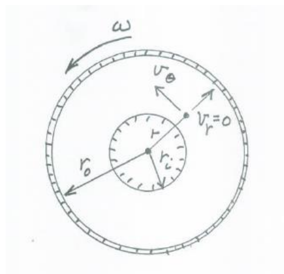
As $r \to \infty$, $v_\theta \to 0$, (since fluid is at rest at infinity) so that $c_1 = 0$. Thus,

$$v_\theta = \frac{c_2}{r}$$

and since at $r = R$, $v_\theta = R\omega$, it follows that $c_2 = R^2\omega$ and

$$\underline{v_\theta = \frac{R^2\omega}{r}}$$

11. A viscous fluid is contained between two infinitely long, vertical, concentric cylinders. The outer cylinder has a radius r_o and rotates with an angular velocity ω. The inner cylinder is fixed and has a radius r_i. Make use of the Navier–Stokes equations to obtain an exact solution for the velocity distribution in the gap. Assume that the flow in the gap is axisymmetric (neither velocity nor pressure are functions of angular position u within the gap) and that there are no velocity components other than the tangential component. The only body force is the weight.



From problem 10 solution

The velocity distribution in the annular space is given by the equation

$$v_\theta = \frac{c_1 r}{2} + \frac{c_2}{r} \qquad (1)$$

With the boundary conditions $r = r_i$, $v_\theta = 0$, and $r = r_o$, $v_\theta = r_o \omega$ (see figure for notation), it follows from Eq. (1) that:

$$0 = \frac{c_1 r_i}{2} + \frac{c_2}{r_i}$$

$$r_o \omega = \frac{c_1 r_o}{2} + \frac{c_2}{r_o}$$

Therefore,

$$c_1 = \frac{2\omega}{1 - \frac{r_i^2}{r_o^2}}$$

and

$$c_2 = \frac{-r_i^2 \omega}{1 - \frac{r_i^2}{r_o^2}}$$

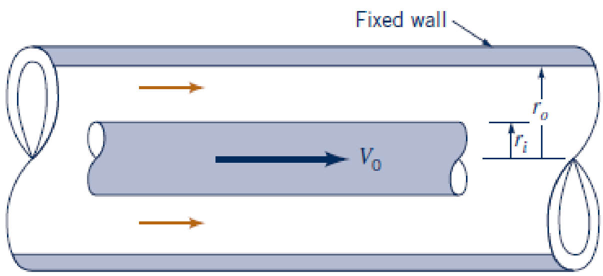
so that

$$v_\theta = \frac{r\omega}{1 - \frac{r_i^2}{r_o^2}} - \frac{r_i^2 \omega}{r\left(1 - \frac{r_i^2}{r_o^2}\right)}$$

or

$$v_\theta = \frac{r\omega}{\left(1 - \frac{r_i^2}{r_o^2}\right)}\left[1 - \frac{r_i^2}{r^2}\right]$$

12. An incompressible Newtonian fluid flows steadily between two infinitely long, concentric cylinders as shown in Fig. The outer cylinder is fixed, but the inner cylinder moves with a longitudinal velocity V_0 as shown. The pressure gradient in the axial direction is $-\Delta p/l$. For what value of V_0 will the drag on the inner cylinder be zero? Assume that the flow is laminar, axisymmetric, and fully developed.



Equation 6.147, which was developed for flow in circular tubes, applies in the annular region. Thus,

$$v_z = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right) r^2 + c_1 \ln r + c_2 \qquad (1)$$

With boundary conditions, $r = r_o$, $v_z = 0$, and $r = r_i$, $v_z = V_0$, it follows that:

$$0 = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right) r_o^2 + c_1 \ln r_o + c_2 \qquad (2)$$

$$V_0 = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right) r_i^2 + c_1 \ln r_i + c_2 \qquad (3)$$

Subtract Eq. (2) from Eq. (3) to obtain

$$V_0 = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)(r_i^2 - r_o^2) + c_1 \ln \frac{r_i}{r_o}$$

so that

$$c_1 = \frac{V_0 - \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)(r_i^2 - r_o^2)}{\ln \frac{r_i}{r_o}}$$

The drag on the inner cylinder will be zero if

$$(\tau_{rz})_{r=r_i} = 0$$

Since,

$$\tau_{rz} = \mu\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right) \qquad (Eq.\ 6.126f)$$

and with $v_r = 0$, it follows that

$$\tau_{rz} = \mu \frac{\partial v_z}{\partial r}$$

Differentiate Eq. (1) with respect to r to obtain

$$\frac{\partial v_z}{\partial r} = \frac{1}{2\mu}\left(\frac{\partial p}{\partial z}\right)r + \frac{c_1}{r}$$

so that at $r = r_i$

$$(\tau_{rz})_{r=r_i} = \mu\left[\frac{1}{2\mu}\left(\frac{\partial p}{\partial z}\right)r_i + \frac{V_0 - \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)(r_i^2 - r_0^2)}{r_i \ln \frac{r_i}{r_0}}\right]$$

Thus, in order for the drag to be zero,

$$\frac{1}{2\mu}\left(\frac{\partial p}{\partial z}\right)r_i + \frac{V_0 - \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)(r_i^2 - r_0^2)}{r_i \ln \frac{r_i}{r_0}} = 0$$

or

$$V_0 = -\frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)\left[2 r_i^2 \ln \frac{r_i}{r_0} - (r_i^2 - r_0^2)\right]$$