Reed's Conjecture

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The Outline

ω, Δ & χ

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4. The **chromatic index** $\chi_e(G)$ of a graph G is the minimum number of colors needed to color the edges in that graph so that adjacent edges have different colors.

Some basic bounds

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Theorem (Brooks)

For graphs other than cliques and with $\Delta(G) \geq 3$ we have $\chi(G) \leq \Delta(G)$

Reed's Conjecture , 1998

The Conjecture :

$$\chi(G) \leq \lceil \frac{1+\Delta+\omega}{2} \rceil$$

The conjecture holds for $\omega \in \{2, \Delta - 1, \Delta, \Delta + 1\}$

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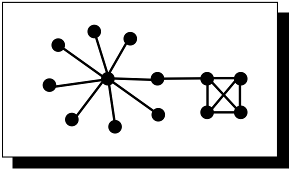
Figure: Omega and Delta



The Conjecture For Line Graphs

An upper bound for the chromatic number of line graphs
by
A.D. King, B.A. Reed, A. Vetta
2006.
`http://www.columbia.edu/~ak3074/papers/KingReedVetta-linegraphs.ps`

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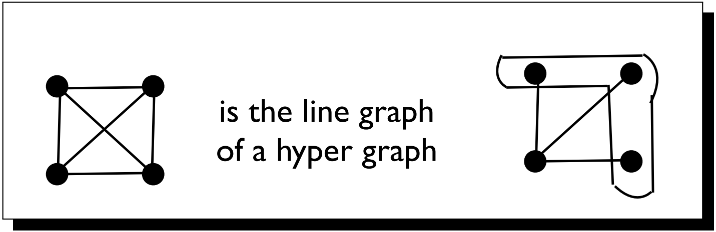
Multiplicity The multiplicity $\mu(a, b)$ of a pair of vertices a and b are the number of edges between them.

Not every graph is the line graph of a multi graph

Figure: Hyper Graphs



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4. Each color contributes fractionally. This gives rise to **fractional vertex coloring**
5. We can also set the colors available across different nodes to be different. Each node has at its disposal the same number of colors. This gives rise to **list coloring**

Definition of a Fractional Vertex Coloring

A *fractional vertex c-coloring* of a graph can be described as a set $S_1, \ldots, S_l$ of stable sets with associated non-negative real weights $w_1, \ldots, w_n$ such that for vertex ν,

$$\sum_{S_i : \nu \in S_i} w_i = 1$$

and

$$\sum_{i=1}^{l} w_i = c$$

The *fractional chromatic number* of G denoted by $\chi^f(G)$ is the smallest c for which G has a fractional vertex c coloring.

Reed's Conjecture for Fractional Coloring

The conjecture holds for fractional vertex coloring

Theorem

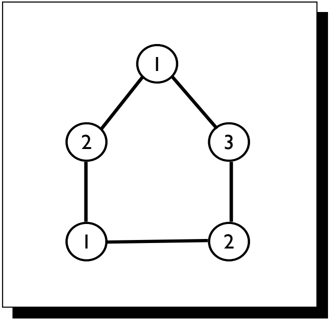
For any graph G,

$$\chi^f(G) \leq \lceil \frac{\Delta(G) + 1 + \omega(G)}{2} \rceil$$

χ and χ^f

Every integer coloring is also a fractional coloring with $w_i = 1$. So we can never have have $\chi < \chi^f$. But is $\chi^f < \chi$ possible? Yes. The cycle of 5 vertices has $\chi = 3$

Figure: Coloring of a 5 Cycle



The cycle of 5 vertices has 2.5 fractional coloring

A 2.5 Fractional Coloring

stable set	S_1	S_2	S_3	S_4	S_5
vertices	a,c	b,d	c,e	d,a	e,b
weights	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

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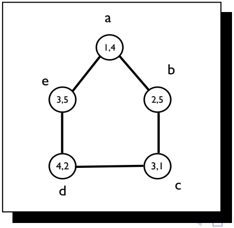

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3. The smallest list size r such that no matter how we choose the colors from each list the graph is properly colorable is the list chromatic index χ^l.
4. We have χ^l definitely not less than χ. Can χ^l be greater than χ?

List Coloring

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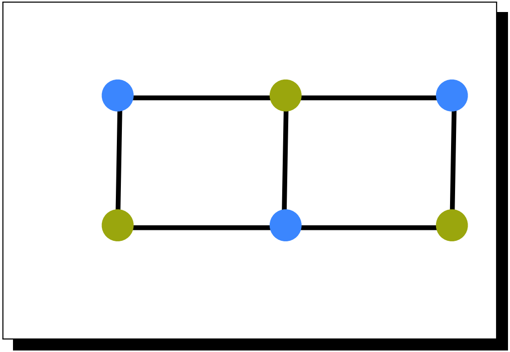

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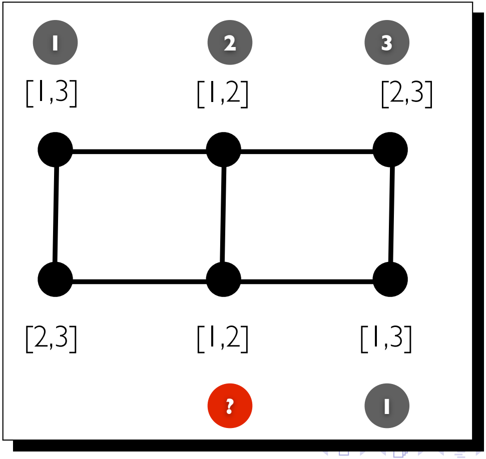

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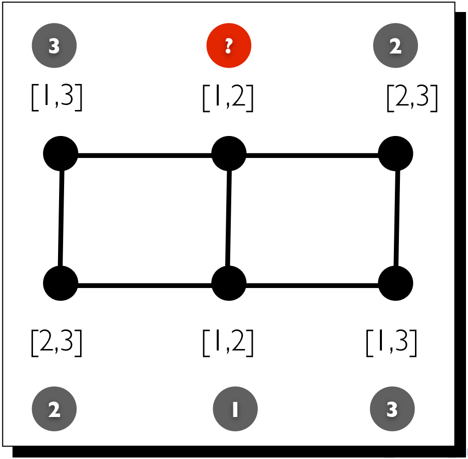

Chromatic Index $X_e(H)$

1. The chromatic index $\chi_e(H)$ of a hypergraph H is the chromatic number $\chi(G)$ of its line graph($G = L(H)$.)
2. The following theorem by Caprara and Rizzi puts an upper bound on the chromatic index of a multigraph $\chi_e(H) \leq max(\lfloor 1.1\Delta(H) + 0.7 \rfloor, \lceil \Gamma(H) \rceil)$.
3. **Goldberg-Seymour Conjecture** For a multigraph H for which $\chi_e(H) > \Delta(H) + 1$, $\chi_e(H) = \lceil \Gamma(H) \rceil$

Fractional Chromatic Index $X_e(H)$

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3. Given a nonnegative weighting w on the edges of H such that for every matching M in H we have $\sum_{e \in M} w(e) \leq 1$ then $\chi_e^f(H) \geq \sum_{e \in E(H)} w(e)$

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1. $\chi_e^f(H) = \chi^f(G) = \sum_{\nu} \sum_{\nu \in S_i} \frac{w_i}{|S_i|}$ such that $\sum_{i:\nu \in S_i} w_i = 1$

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3. Both the sums over the same set of indices.

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3. where $\Gamma(H) = max\{\frac{2|E(W)|}{|V(W)-1|} : W \subset H, |V(W)| \text{is odd}\}$
4. Edmond's theorem for matching polytypes $\chi_e^f(H) = max(\Delta(H), \Gamma(H))$

Reed's Conjecture for Line Graphs

Theorem

For any line graph G, Reed's conjecture holds.

We set $G = L(H)$ and consider two cases

1. $\Delta(G)$ is large. That is $\Delta(G) \geq \frac{3}{2}\Delta(H) - 1$
2. $\Delta(G)$ is small. That is $\Delta(G) < \frac{3}{2}\Delta(H) - 1$

Proof when $\Delta(G)$ is large

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4. Using Reed's theorem for fractional coloring $\chi(G) \leq max(\lfloor 1.1\Delta(H) + 0.7 \rfloor, \lceil \frac{\Delta(G)+1+\omega(G)}{2} \rceil)$

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5. Assumption: $\Delta(G) \geq \frac{3}{2}\Delta(H) - 1$
6. Therefore $\lceil \frac{\Delta(G)+1+\omega(G)}{2} \rceil \geq \lceil \frac{5}{4} \rceil \geq \lfloor 1.1\Delta(H) + 0.7 \rfloor$

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7. Therefore $\chi(G) \leq \frac{\Delta(G)+1\omega(G)}{2}$

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6. Therefore $\chi(G) \leq \frac{\Delta(G)+1+\omega(G)}{2}$
7. Find a maximal stable set $S \subset V(G)$ that has a vertex from every maximum clique in G.

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We need to show when $\Delta(G) < \frac{3}{2}\Delta(H) - 1$ we have a maximal stable set S which contains a vertex from every maximum clique.

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3. If $tri(H)$ is the maximum number of edges in a triangle. $\omega(G) = max\{\Delta(H), tri(H)\}$

Basic Idea Continued

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$$\therefore \mu(b,c) \geq \frac{1}{2}\Delta(H)$$

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1. T' is the set of triangles in T that contain no pair of vertices of multiplicity $> \Delta(H)/2$
2. S'_Δ are those elements of S_Δ which are not part of any pair of vertices of multiplicity $> \Delta(H)/2$
3. For a set of vertices S the union of the vertices' neighbourhoods is $N(S)$

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For any $S \subset S'_\Delta$ *we have* $|N(S)| \geq |S|$

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$$N(S)\Delta(H) \geq |S|\Delta(H)$$

$$\therefore |N(S)| \geq |S|$$

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□

Hall's Theorem

Theorem

Let G be a bipartite graph with vertex set $V = (A,B)$. There is a matching that hits every vertex in A precisely if for every $S \subset A$ we have $|N(S)| \geq |S|$

Completion of the proof

We need to show that our desired matching exists,

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We construct three matchings and combine them. Have to show the combination is still a matching and the combination is possible.

Construction of a matching

Step one: Construct matching M_1

One edge between each vertex pair with multiplicity greater than $\Delta(H)/2$. This hits $S_\Delta \setminus S'_\Delta$ and contains one edge of each triangle in $T \setminus T'$

Construction of a matching

Step two: Construct matching M_2

Using Lemma 4 (For any $S \subset S'_\Delta$ we have $|N(S)| \geq |S|$) and Hall's theorem we can construct a matching that hits S'_Δ

Lemma 7 shows that this matching cannot hit M_2 so the union on M_1 and M_2 hits S_Δ and an edge of each triangle in $T \setminus T'$

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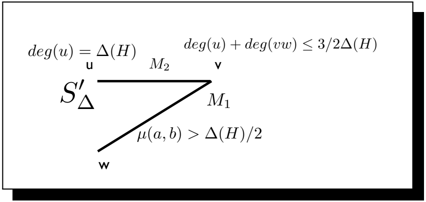
Lemma 7 shows that this matching cannot hit M_2 so the union on M_1 and M_2 hits S_Δ and an edge of each triangle in $T \setminus T'$ Every edge in the matching $M' = M_1 \cup M_2$ hits a maximum-vertex in H or has endpoints with multiplicity greater than $\Delta(H)/2$

Figure: The Final Matching



Construction of a matching

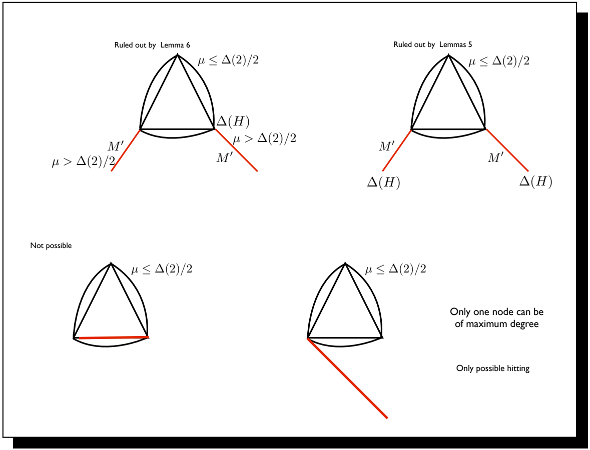
Using lemma 4 and We have to include edges from T'. We can blindly add an arbitrary edge from each triangle in T' but at least two vertices should remain uncontaminated by M'

Step three: Construct matching M

Lemmas 3 ,5 and 6 show that M' hits at most one vertex in T'. We extend M' to contain an edge of every triangle in T' and obtain M

This matching M satisfies our requirements. This matching corresponds to our maximal stable set in G that has a vertex from every maximum clique.

Figure: The Final Matching



Reed's Conjecture

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3. with $\Delta(G) \geq 3$ $\chi(G) \leq \frac{2(\Delta(G)+1)+\omega(G)}{3}$
4. There is some constant α such that for any graph $\chi(G) \leq \alpha\omega(G) + \frac{1}{2}(\Delta(G)+1)$