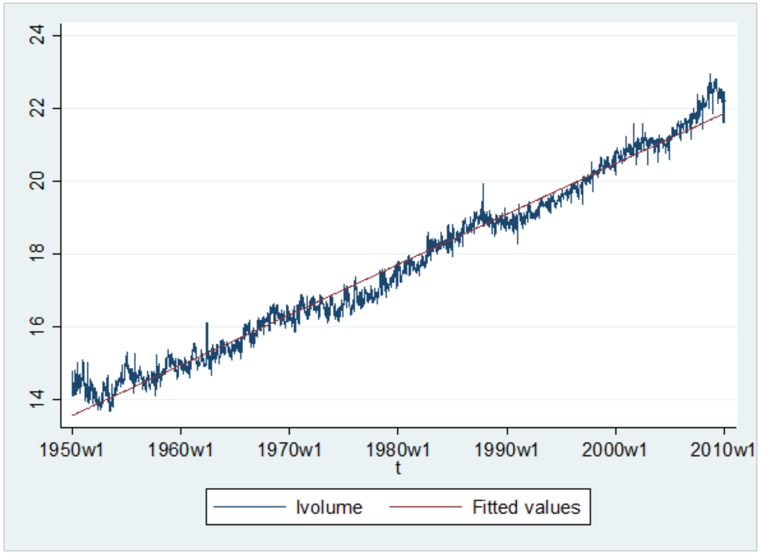
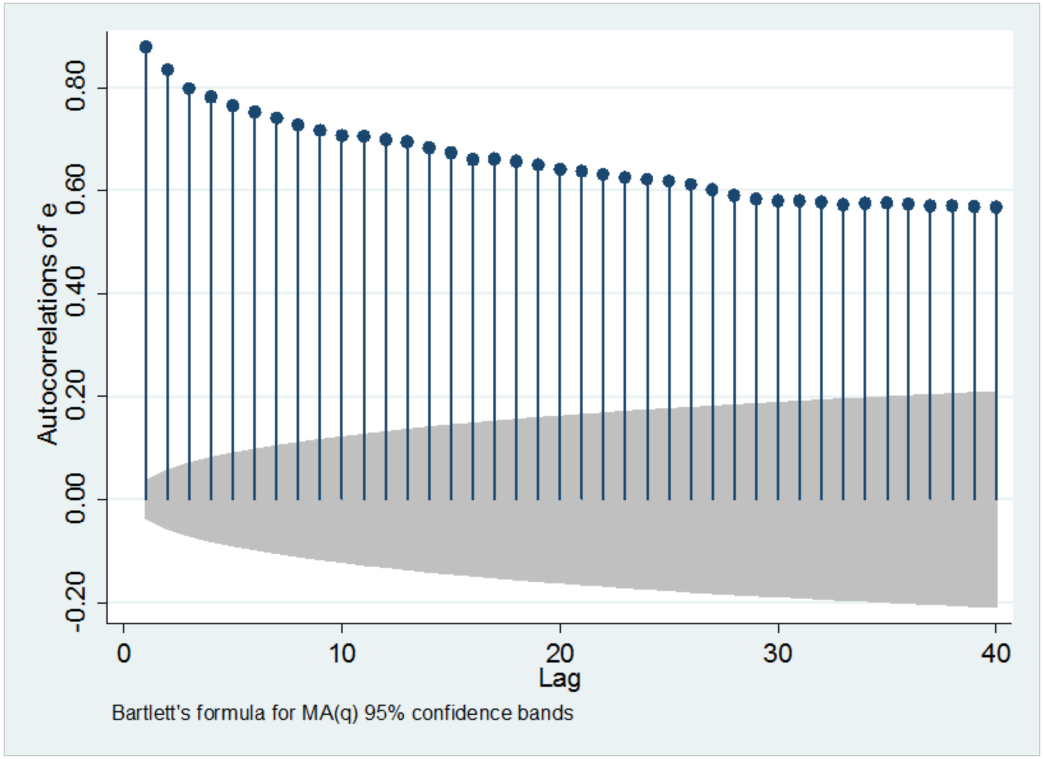
Regression with Correlated Errors

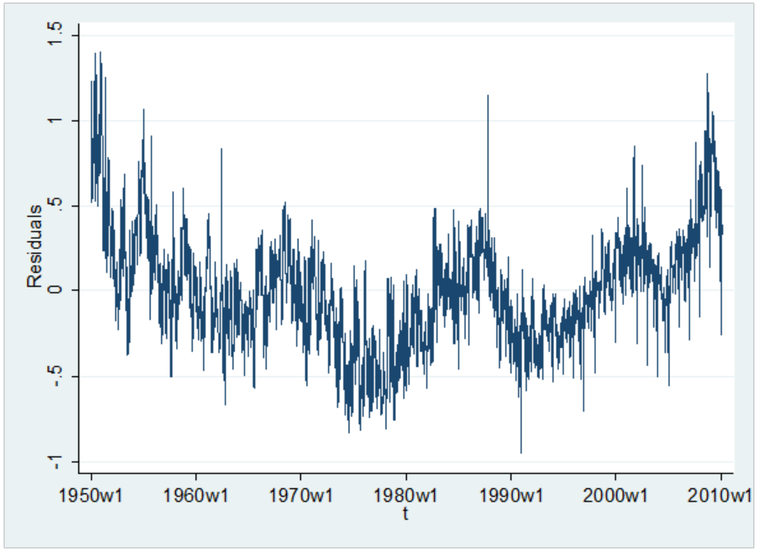
$$y_t = \alpha + \beta x_t + e_t$$

- In some regression models, the errors are correlated
  - Pure Trend Models
  - Pure Seasonality Models
- In these models the errors can be correlated
- Classical and robust standard errors are not appropriate

Example: Stock Volume







Least-Squares Variance Formula

Recall for $v_t = x_t e_t$

$$\operatorname{var}\left(\hat{\beta}\right) \overset{a}{\sim} \frac{\operatorname{var}\left(\sum_{t=1}^{T} v_t\right)}{\left[T \operatorname{var}(x_t)\right]^2}$$

When the v are uncorrelated

$$\operatorname{var}\left(\sum_{t=1}^{T} v_t\right) = \sum_{t=1}^{T} \operatorname{var}(v_t) = T \operatorname{var}(v_t)$$

$$\operatorname{var}\left(\hat{\beta}\right) \overset{a}{\sim} \frac{\operatorname{var}(v_t)}{T\left[\operatorname{var}(x_t)\right]^2}$$

General Formula

Define

$$f_T = \frac{\operatorname{var}\left(\sum_{t=1}^{T} v_t\right)}{T \operatorname{var}(v_t)}$$

When the v are uncorrelated f_T=1, otherwise not.

Then

$$\operatorname{var}\left(\hat{\beta}\right) \overset{a}{\sim} \frac{\operatorname{var}(x_t e_t)}{T\left[\operatorname{var}(x_t)\right]^2} f_T$$

Adjustment Factor

- The asymptotic variance of least-squares is the conventional, multiplied by an adjustment factor for the serial correlation

$$\operatorname{var}\left(\hat{\beta}\right) \overset{a}{\sim} \frac{\operatorname{var}\left(x_t e_t\right)}{T\left[\operatorname{var}\left(x_t\right)\right]^2} f_T$$

Autocovariance of v

- We want a useful formula for

$$f_T = \frac{\operatorname{var}\left(\sum_{t=1}^{T} v_t\right)}{T \operatorname{var}(v_t)}$$

- Since $E(v_t)=0$, then

$$E\left(v_t^2\right) = \operatorname{var}(v_t)$$

$$E\left(v_t v_j\right) = \operatorname{cov}\left(v_t v_j\right) = \gamma(t-j)$$

the autocovariance of v_t

Variance of sum of correlated v

$$\operatorname{var}\left(\sum_{t=1}^{T} v_t\right) = E\left(\sum_{t=1}^{T} v_t\right)^2$$

$$= E\left(\sum_{t=1}^{T} v_t \sum_{j=1}^{T} v_j\right)$$

$$= \sum_{t=1}^{T} \sum_{j=1}^{T} E\left(v_t v_j\right)$$

$$= \sum_{t=1}^{T} \sum_{j=1}^{T} \gamma\left(t - j\right)$$

Adjustment Factor

$$f_T = \frac{\operatorname{var}\left(\sum_{t=1}^{T} v_t\right)}{T \operatorname{var}(v_t)} = \frac{1}{T}\sum_{t=1}^{T}\sum_{j=1}^{T}\rho(t-j)$$

- Where the $\rho(t-j)$ are the autocorrelations of v_t

- This double sum is the sum of all the elements in the matrix

$$\begin{bmatrix} \rho(0) & \rho(1) & \rho(2) & \cdots & \rho(T-1) \\ \rho(1) & \rho(0) & \rho(1) & \cdots & \rho(T-2) \\ \rho(2) & \rho(1) & \rho(0) & \cdots & \rho(T-3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho(T-1) & \rho(T-2) & \rho(T-3) & \cdots & \rho(0) \end{bmatrix}$$

- There are
  - *T* of the ρ(0)
  - 2*(T-1)* of the ρ(1)
  - 2*(T-2)* of the ρ(2)
  - …

$$T + \sum_{j=1}^{T-1} 2(T-j)\rho(j)$$

Adjustment Factor

- Dividing by T

$$f_T = \frac{1}{T}\sum_{t=1}^{T}\ \sum_{j=1}^{T}\rho(t-j)$$

$$= 1 + \sum_{j=1}^{T-1} 2\left(\frac{T-j}{T}\right)\rho(j)$$

- If T is large

$$f_T \to 1 + 2\sum_{j=1}^{\infty}\rho(j) = f$$

Summary: Least-Squares Variance

- When the errors are correlated

$$\operatorname{var}\left(\hat{\beta}\right) \overset{a}{\sim} \frac{\operatorname{var}\left(x_t e_t\right)}{T\left[\operatorname{var}\left(x_t\right)\right]^2} \, f$$

$$f = 1 + 2\sum_{j=1}^{\infty} \rho\left(j\right)$$

- The conventional formula is multiplied by an adjustment for autocorrelation

HAC Estimation

- Estimation of f
  - For variances and standard errors under autocorrelation
- Called heteroskedasticity and autocorrelation consistent (HAC) variance estimation
- Multiply conventional variance estimates by estimates of f

HAC Estimation

- The adjustment is

$$f = 1 + 2\sum_{j=1}^{\infty} \rho(j)$$

  where ρ(j) are the autocorrelations of $v_t = x_t e_t$
- Estimate ρ(j) by sample autocorrelations using least-squares residuals
- But in a sample of length T we cannot estimate all autocorrelations well

Unweighted HAC Estimator

- For some **truncation parameter** m,

$$\hat{f} = 1 + 2\sum_{j=1}^{m} \hat{\rho}(j)$$

- Original proposal
  - L. Hansen, Hodrick (1978)
  - Hal White (1982)
- Deficiencies
  - This estimator is not smooth in the truncation parameter
  - The sample estimate can be negative

Lars Hansen

- Professor Lars Hansen, U Chicago
- Invented Generalized Method of Moments, the leading estimation method for applied econometrics
- Introduced unweighted HAC estimator for multi-step regression models
- Won 2013 Nobel Prize in economics

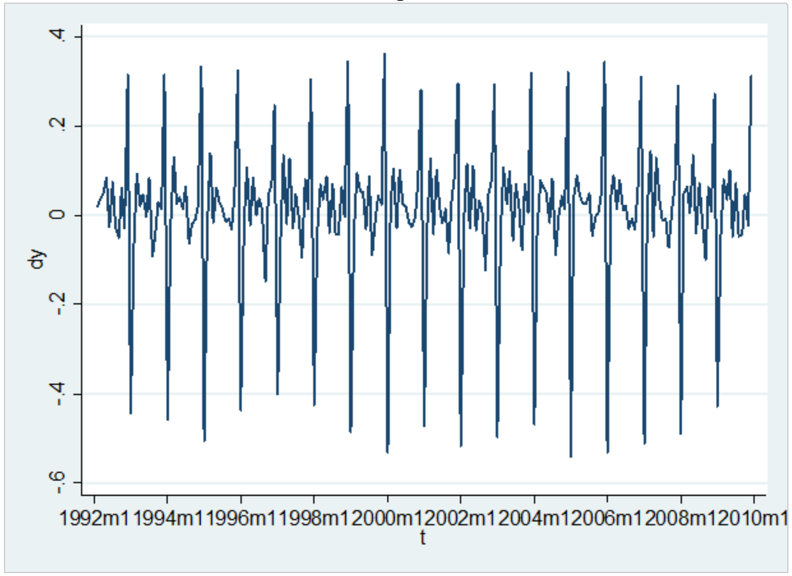
Example of Negative Estimate

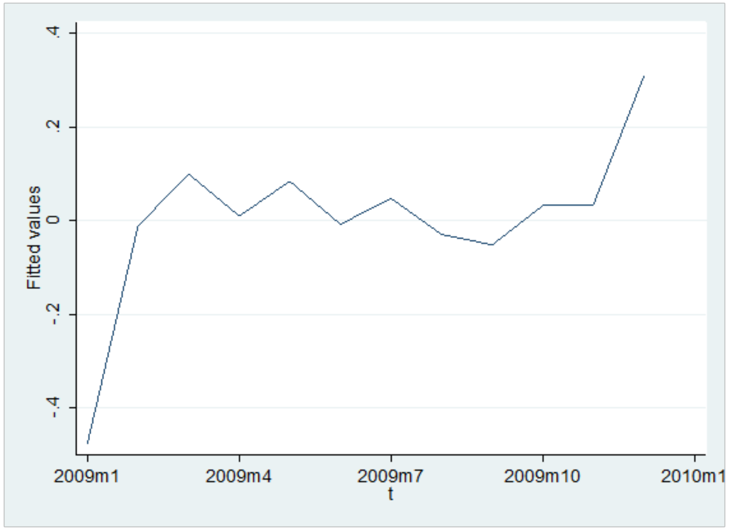
- Take m=1
- Then $\hat{f} = 1 + 2\hat{\rho}(1) < 0$

  if estimated ρ(1)<-1/2

Example: Liquor Sales

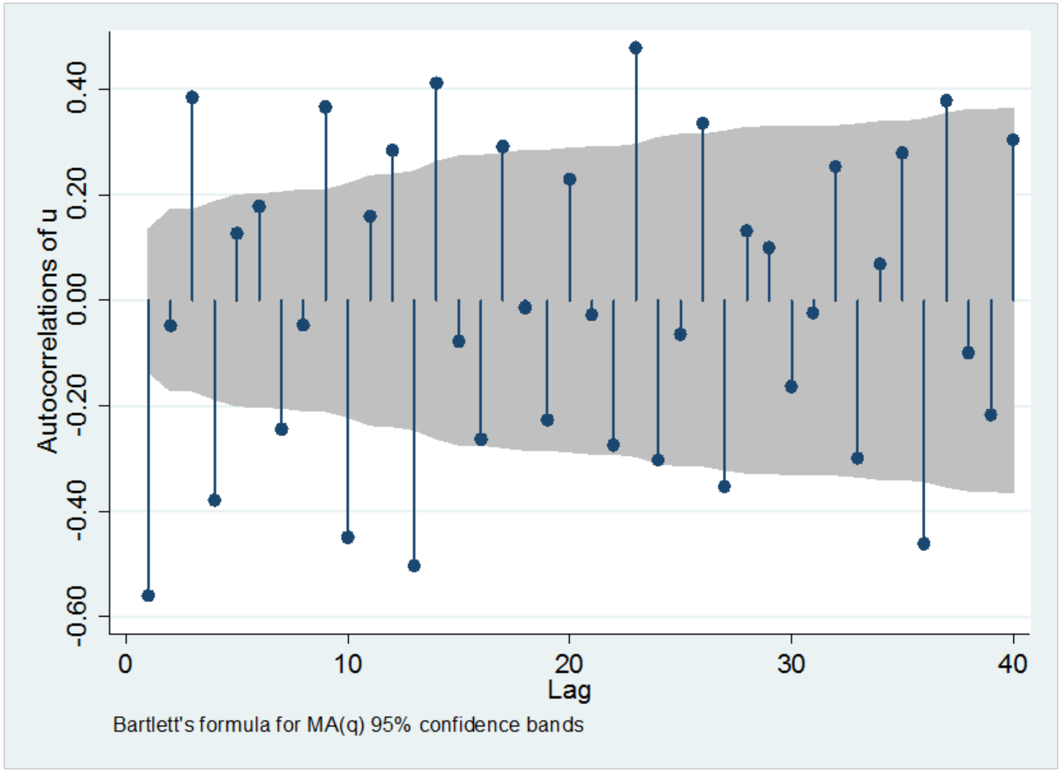
- Transform to growth rates
- Monthly change in log liquor sales
- Regress on Seasonal Dummies only to obtain seasonal pattern





Autocorrelation of Residual

- The first autocorrelation is less than -1/2



Weighted HAC Estimator

$$\hat{f} = 1 + 2\sum_{j=1}^{m}\left(\frac{m-j}{m}\right)\hat{\rho}(j)$$

- Called Newey-West variance estimator
  - Whitney Newey, Ken West (1987)
- This weighted estimator is always positive
- Smoothly changes in truncation parameter m

Whitney Newey and Ken West

- Professor Whitney Newey, MIT
  - Leading econometric theorist
- Professor Ken West, Wisconsin
  - Macroeconomist & econometrician
  - Forecast evaluation and comparison
- Joint paper in 1987
  - Weighted HAC estimator
  - One of the most referenced papers in econometrics

Computation

- In STATA, replace **regress** command with **newey** command

  .newey y x, lag(m)

- You supply the truncation parameter "m"
- Similar to regression with robust standard errors
- These are identical

  .newey y x, lag(0)

  .reg y x, r

Example: Liquor Sales

```
. reg dy b12.m,r
```

Linear regression

Number of obs	=	215
F(11, 203)	=	423.80
Prob > F	=	0.0000
R-squared	=	0.9613
Root MSE	=	.0347

dy	Coef.	Robust Std. Err.	t	P>|t|	[95% Conf.	Interval]
m						
1	-.788371	.0120765	-65.28	0.000	-.8121825	-.7645595
2	-.3218705	.0105478	-30.52	0.000	-.3426677	-.3010733
3	-.2103181	.0094619	-22.23	0.000	-.2289744	-.1916619
4	-.3002915	.010514	-28.56	0.000	-.3210222	-.2795607
5	-.2258118	.0100036	-22.57	0.000	-.245536	-.2060876
6	-.3185358	.0096047	-33.16	0.000	-.3374735	-.2995981
7	-.2618824	.0100737	-26.00	0.000	-.2817449	-.2420198
8	-.3392591	.0107775	-31.48	0.000	-.3605093	-.3180088
9	-.3624475	.0123023	-29.46	0.000	-.3867042	-.3381907
10	-.2782956	.010299	-27.02	0.000	-.2986023	-.257989
11	-.2761872	.0108553	-25.44	0.000	-.2975908	-.2547835
_cons	.3099733	.0065735	47.16	0.000	.2970122	.3229343

With Newey-West standard errors

```
. newey dy b12.m, lag(12)

Regression with Newey-West standard errors          Number of obs  =       215
maximum lag: 12                                     F( 11,    203) =    908.34
                                                    Prob > F       =    0.0000
```

dy	Coef.	Newey-West Std. Err.	t	P>|t|	[95% Conf.	Interval]
m						
1	-.788371	.0149943	-52.58	0.000	-.8179356	-.7588064
2	-.3218705	.0093479	-34.43	0.000	-.3403018	-.3034391
3	-.2103181	.0100234	-20.98	0.000	-.2300816	-.1905547
4	-.3002915	.0087418	-34.35	0.000	-.3175278	-.2830551
5	-.2258118	.0128307	-17.60	0.000	-.2511104	-.2005132
6	-.3185358	.0087245	-36.51	0.000	-.335738	-.3013336
7	-.2618824	.0090442	-28.96	0.000	-.279715	-.2440498
8	-.3392591	.0134996	-25.13	0.000	-.3658765	-.3126416
9	-.3624475	.0075171	-48.22	0.000	-.377269	-.3476259
10	-.2782956	.0116472	-23.89	0.000	-.3012606	-.2553307
11	-.2761872	.0126533	-21.83	0.000	-.3011359	-.2512384
_cons	.3099733	.0066381	46.70	0.000	.2968848	.3230618

Truncation Parameter

- *m* should be large when autocorrelation is large
- Sophistical data-dependent methods to pick m have been developed, but are not in STATA
- Stock-Watson default (explanatory x's)

$$m = 0.75T^{1/3}$$

- Trend/Seasonal default

$$m = 1.4T^{1/3}$$

Derivation of Defaults

- Due to Andrews (1991)
- The optimal m minimizes the mean-squared error of the estimate of f
- When v_t is an AR(1) with coefficient ρ, Andrews found the optimal m is

$$m = CT^{1/3}$$

$$C = \left(\frac{6\rho^2}{\left(1-\rho^2\right)^2} \right)^{1/3}$$

Donald Andrews

- Professor Donald Andrews, Yale
- Leading econometric theorist
- Contributions to time-series
  - Optimal selection of truncation parameter
  - Tests for structural change

Default Values

$$m = CT^{1/3}$$

$$C = \left(\frac{6\rho^2}{\left(1-\rho^2\right)^2} \right)^{1/3}$$

- Stock-Watson
  - If both x_t and e_t are AR(1) with coef ½, then $v_t = x_t e_t$ has AR(1) coefficient ρ=.25. Plug this in, and C=.75
- Trend-Seasonal
  - If x_t is trend and/or seasonal and e_t are AR(1) with coef ½, then $v_t = x_t e_t$ has AR(1) coefficient ρ=.5. Plug this in, and C=1.4

Liquor Sales again

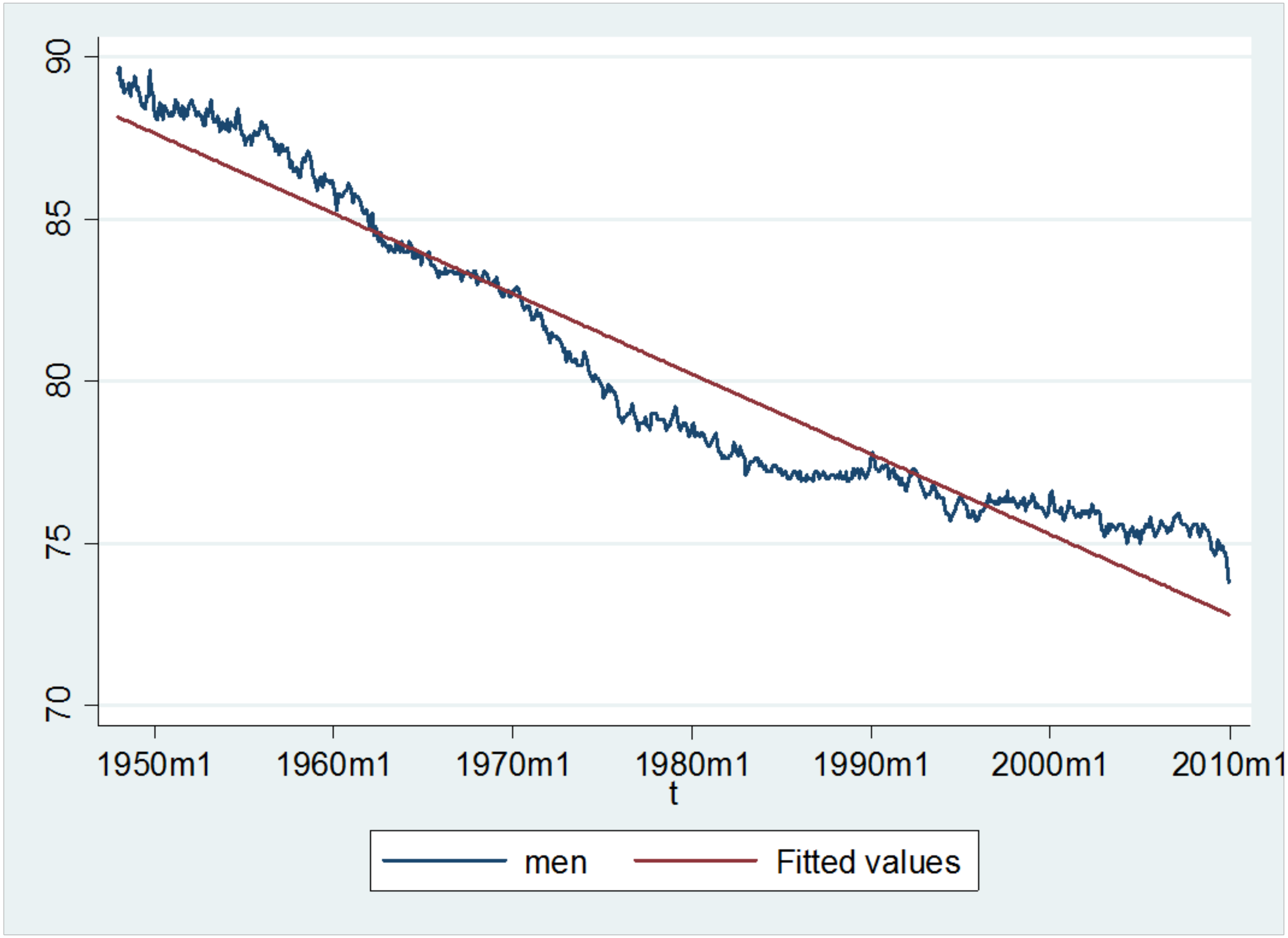
```
. dis 1.4*e(N)^(1/3)
8.387017

. newey dy b12.m, lag(8)

Regression with Newey-West standard errors          Number of obs  =       215
maximum lag: 8                                      F( 11,   203)  =    736.19
                                                    Prob > F       =    0.0000
```

dy	Coef.	Newey-West Std. Err.	t	P>|t|	[95% Conf.	Interval]
m						
1	-.788371	.0146673	-53.75	0.000	-.8172907	-.7594513
2	-.3218705	.0089781	-35.85	0.000	-.3395727	-.3041682
3	-.2103181	.0097191	-21.64	0.000	-.2294815	-.1911548
4	-.3002915	.0097151	-30.91	0.000	-.319447	-.281136
5	-.2258118	.0116748	-19.34	0.000	-.2488312	-.2027924
6	-.3185358	.0089588	-35.56	0.000	-.3362001	-.3008715
7	-.2618824	.00916	-28.59	0.000	-.2799433	-.2438214
8	-.3392591	.0126319	-26.86	0.000	-.3641655	-.3143526
9	-.3624475	.0091312	-39.69	0.000	-.3804516	-.3444434
10	-.2782956	.0106888	-26.04	0.000	-.2993709	-.2572204
11	-.2761872	.0126343	-21.86	0.000	-.3010984	-.2512759
_cons	.3099733	.0065735	47.16	0.000	.2970122	.3229343

Example: Men's Labor Force Participation Rate, Trend Model



```
. reg m t

      Source |       SS       df       MS              Number of obs =     744
-------------+------------------------------           F(  1,   742) = 9554.38
       Model |  14659.2499     1  14659.2499           Prob > F      =  0.0000
    Residual |  1138.4477    742  1.53429609           R-squared     =  0.9279
-------------+------------------------------           Adj R-squared =  0.9278
       Total |  15797.6976   743  21.2620426           Root MSE      =  1.2387

------------------------------------------------------------------------------
         men |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           t |  -.0206675   .0002114   -97.75   0.000    -.0210826   -.0202524
       _cons |   85.18169   .0661519  1287.67   0.000     85.05182    85.31156
------------------------------------------------------------------------------

. dis 1.4*e(N)^(1/3)
12.685834

. newey men t, lag(13)

Regression with Newey-West standard errors          Number of obs  =       744
maximum lag: 13                                     F(  1,   742)  =    692.69
                                                    Prob > F       =    0.0000

------------------------------------------------------------------------------
             |             Newey-West
         men |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           t |  -.0206675   .0007853   -26.32   0.000    -.0222091   -.0191259
       _cons |   85.18169   .2168636   392.79   0.000     84.75595    85.60743
```

Summary

- In one-step-ahead forecast regressions
- If the errors are serially uncorrelated
  - Use Robust standard errors
    - reg with r option
- If the errors are correlated
  - Use Newey-West standard errors
    - newey y x, lag(m)
  - In pure trend or seasonality models
    - Set $m=1.4T^{1/3}$
  - In dynamic regression
    - Set $m=.75T^{1/3}$

h-step-ahead forecasts

- In the AR(1) Model

$$y_t = \alpha + \beta y_{t-1} + e_t$$

- The optimal h-step forecasting regression takes the form

$$y_t = \alpha + \beta^h y_{t-h} + u_t$$
$$u_t = e_t + \beta e_{t-1} + \beta^2 e_{t-2} + \cdots + \beta^{h-1} e_{t-h+1}$$

- The error u_t is a correlated MA(h-1)
  - Unless β=0

h-step-ahead models

- In any h-step model

$$y_t = \alpha + \beta y_{t-h} + u_t$$

  the variable $v_t = y_{t-h} e_t$ is generally serially correlated
- Generally MA(h-1)
- Correct adjustment term

$$f = 1 + 2\sum_{j=1}^{h-1} \rho(j)$$

Newey-West Standard Errors

- Standard errors can be estimated using the Newey-West method
- Truncation parameter set to forecast horizon
  - m=h

$$\hat{f} = 1 + 2\sum_{j=1}^{h-1}\left(\frac{h-j}{h}\right)\hat{\rho}(j)$$

Example: Unemployment Rate

- 12-month-ahead forecast with 4 AR lags
  - Robust standard errors:

```
. reg ur L(12/15).ur,r

Linear regression                               Number of obs =     730
                                                F(  4,    725) =  139.36
                                                Prob > F      =  0.0000
                                                R-squared     =  0.4955
                                                Root MSE      =  1.1088
```

ur	Coef.	Robust Std. Err.	t	P>|t|	[95% Conf.	Interval]
ur						
L12.	1.686434	.2920485	5.77	0.000	1.113072	2.259795
L13.	-.0698989	.3908098	-0.18	0.858	-.837153	.6973552
L14.	-.5401552	.3461042	-1.56	0.119	-1.219641	.1393309
L15.	-.4100512	.2538791	-1.62	0.107	-.9084772	.0883747
_cons	1.94875	.1705347	11.43	0.000	1.613949	2.28355

Example: Unemployment Rate

- Newey-West standard errors:
- Standard errors on lag 13 and 14 decrease by half
- Standard error on constant more than doubles

```
. newey ur L(12/15).ur, lag(12)

Regression with Newey-West standard errors          Number of obs  =       730
maximum lag: 12                                     F(  4,    725) =     21.00
                                                    Prob > F       =    0.0000
```

ur	Coef.	Newey-West Std. Err.	t	P>|t|	[95% Conf.	Interval]
ur						
L12.	1.686434	.273372	6.17	0.000	1.149738	2.223129
L13.	-.0698989	.1564772	-0.45	0.655	-.3771014	.2373036
L14.	-.5401552	.1378278	-3.92	0.000	-.8107445	-.2695658
L15.	-.4100512	.246517	-1.66	0.097	-.8940236	.0739212
_cons	1.94875	.4550687	4.28	0.000	1.05534	2.842159

newey and forecasting

- **predict** works after **newey** command, but not with **stdf** option
- **e(rmse)** does not work, only after **regress** or **reg**
  - rmse not computed or reported
- **newey** not appropriate for iterated forecasts
- Use **newey** to assess model and examine coefficients
- Use **reg** to compute out-of-sample forecast intervals

Summary

- In one-step-ahead forecast regressions
  - If the errors are serially uncorrelated, use r option
  - If the errors are correlated
    - Use **newey** for standard errors
      - In pure trend or seasonality models set $m=1.4T^{1/3}$
      - In dynamic regression set $m=.75T^{1/3}n$
    - Use **reg** and **predict sf, stdf** for forecast intervals, or iterated forecasts with **forecast**
- In h-step-ahead forecast regressions
  - Use **newey** with m=h for standard errors
  - Use **reg** and **predict sf, stdf** for forecast intervals

Joint Tests

$$y_t = \alpha + \beta_1 y_{t-1} + \cdots + \beta_p y_{t-p} + e_t$$

- How do we assess if a subset of coefficients are jointly zero? Example: 3rd+4th lags

```
. reg gdp L(1/4).gdp,r

Linear regression                               Number of obs =     247
                                                F(  4,   242) =    8.85
                                                Prob > F      =  0.0000
                                                R-squared     =  0.1584
                                                Root MSE      =  3.8132
```

gdp	Coef.	Robust Std. Err.	t	P>|t|	[95% Conf.	Interval]
gdp						
L1.	.327656	.076895	4.26	0.000	.1761871	.479125
L2.	.1466135	.0858808	1.71	0.089	-.0225558	.3157828
L3.	-.0980287	.0728951	-1.34	0.180	-.2416186	.0455611
L4.	-.0889209	.0790354	-1.13	0.262	-.244606	.0667641
_cons	2.378427	.4731312	5.03	0.000	1.446447	3.310408

Joint Hypothesis

- This is a joint test of

$$\beta_3 = 0$$

$$\beta_4 = 0$$

- This can be done with an "F test"
- In STATA, after **regress** (**reg**) or **newey**

  .test L3.gdp L4.gdp

- List variables whose coefficients are tested for zero.

Joint Tests

- “F test” named after R.A. Fisher
  - (1890-1992)
  - A founder of modern statistical theory
- Modern form known as a “Wald test”, named after Abraham Wald (1902-1950)
  - Early contributor to econometrics

F test computation

```
. test L3.gdp L4.gdp

 ( 1)  L3.gdp = 0
 ( 2)  L4.gdp = 0

       F(  2,   242) =    1.76
            Prob > F =    0.1747
```

- You need to list each variable separately
- STATA describes the hypothesis
- The value of “F” is the F-statistic
- “Prob>F” is the p-value
  - Small p-values cause rejection of hypothesis of zero coefficients
  - Conventionally, reject hypothesis if p-value < 0.05

Example: 2-step-ahead GDP AR(4)

```
. newey gdp L(2/5).gdp, lag(2)

Regression with Newey-West standard errors          Number of obs  =       246
maximum lag: 2                                      F(  4,    241) =      3.24
                                                    Prob > F       =    0.0129
```

gdp	Coef.	Newey-West Std. Err.	t	P>|t|	[95% Conf.	Interval]
gdp						
L2.	.2410617	.0768239	3.14	0.002	.0897296	.3923938
L3.	-.0368004	.0703583	-0.52	0.601	-.1753962	.1017954
L4.	-.0910108	.0791053	-1.15	0.251	-.2468369	.0648152
L5.	-.1128763	.0687243	-1.64	0.102	-.2482533	.0225006
_cons	3.329426	.5460059	6.10	0.000	2.253873	4.404979

```
. test L3.gdp L4.gdp L5.gdp

 ( 1)  L3.gdp = 0
 ( 2)  L4.gdp = 0
 ( 3)  L5.gdp = 0

       F(  3,   241) =    1.65
            Prob > F =    0.1793
```

Testing after Estimation

- The commands **predict** and **test** are applied to the most recently estimated model
- The command test uses the standard error method specified by the estimation command
  - **reg y x** : classical F test
  - **reg r x, r**: heteroskedasticity-robust F test
  - **newey y x, lag(m)**: correlation-robust F test
    - (The robust tests are actually Wald statistics)