Relativistic Electrodynamics in Tensor Notation

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Relativistic Electrodynamics is a field of physics capable of having either an immensely complicated or encouragingly simplistic appearance, depending entirely on the particular formulation it is presented in. Under the representation which is perhaps familiar to most, systems of equations must be solved and manipulated in order to perform basic operations on, and transformations of, the familiar electric and magnetic fields of Maxwell's Equations. With adequate abstraction, however, these systems may be reduced to equations of only three to five terms and to operators with well-defined algorithms and structures. In particular, the reduction of the aforementioned fields and operators to what is commonly referred to as tensor notation, allows for a condensed presentation of some of the most fundamental laws of physics.

While the thought of representing systems of equations illustrating these laws in tensor notation may, on inspection, sound a bit daunting to many, there are many arguments to be made for this formulation. One such argument is in favor of the sheer reduction of information necessary to be presented directly. For example, under ordinary formulation, Maxwell's Equations are represented as[1]:

$$i)\ \nabla \cdot \mathbf{E} = \alpha\rho \qquad ii)\ \nabla \cdot \mathbf{B} = 0$$

$$iii)\ \nabla \times \mathbf{E} = -\gamma \frac{\partial \mathbf{B}}{\partial t} \qquad iv)\ \nabla \times \mathbf{B} = \beta \mathbf{J} + \frac{\beta}{\alpha}\frac{\partial \mathbf{E}}{\partial t}$$

This formulation requires four vector equations with intermixed fields and cumbersome coefficients which may be different depending on the unit system in which they are derived and presented, as indicated by Table 1[1].

System	α	β	γ
Gaussian	4π	$4\pi/c$	$1/c$
SI	$1/\epsilon_0$	μ_0	1
Heaviside-Lorentz	1	$1/c$	$1/c$

TABLE I: The $\alpha\beta\gamma$-system

When one then considers even the most basic of relativistic transformations*, Lorentz contraction, where the three component vectors **B** and **E** transform as (assuming relative motion in exclusively the x direction)[2]:

$$\bar{E}_x = E_x \qquad \bar{E}_y = \gamma(E_y - vB_z) \qquad \bar{E}_z = \gamma(E_z + vB_y)$$

$$\bar{B}_x = B_x \qquad \bar{B}_y = \gamma\left(B_y - \frac{v}{c^2}E_z\right) \qquad \bar{B}_z = \gamma(B_z - \frac{v}{c^2}E\_y)$$

each of which must then input to Maxwell's equations, the desire to condense information becomes evident. It should be noted that in the equations above, $\bar{E}_n$ and $\bar{B}_n$ represent the transformed n^{th} component of the respective field, v is the velocity relative to some inertial frame, and γ is the Lorentz factor, not the γ indicated in Table 1 which will henceforth be referred to as $\gamma_{U.S.}$* for clarity. To observe the extent to which Maxwell's Equations and basic relativistic operations may be condensed under tensor notation, the same system of equations become "simply"[1,2]:

$$\frac{\partial F^{\mu v}}{\partial x^v} = \beta_{U.S}J^\mu \qquad \frac{\partial G^{\mu v}}{\partial x^v} = 0$$

with the entire operation of Lorentz contraction capable of being represented as:

$$\bar{F}^{\mu v} = \Lambda^\mu{}_\lambda \Lambda_\sigma{}^v F^{\lambda\sigma}$$

What were technically 12 equations and 6 equations have respectively been reduced to 2 and 1, a remarkable degree of abstraction to be sure.

*A complete derivation of which may be found in Griffiths [2], for movement along the x direction, as is assumed here.

A reduced formulation may be desirable for such functions, but such a reduction becomes useless if the remaining information is insufficient for efficient and effective problem solving. So, what do all of these variables and subscripts mean? $F^{\mu v}$ is a second rank, antisymmetric tensor referred to as the Field Tensor. Basically, it's a 4×4 matrix of values pertaining to $\mathbf{E}$ and $\mathbf{B}$. $G^{\mu v}$, referred to as the Dual Tensor, is essentially the same matrix with substitutions of $\frac{\boldsymbol{E}}{c} \rightarrow \boldsymbol{B}$ and $\boldsymbol{B} \rightarrow -\frac{\boldsymbol{E}}{c}$. Finally, Λ_{λ}^{μ} is the Lorentz transformation matrix and all of the subscripts and superscripts are simply indicators of row, column indices in the matrices being operated on⁺, with the implication that each operator is applied across all elements of the matrices in the end. All of this is made clear with the explicit operations being provided, as they are in the following equation illustrating the calculation of $\bar{F}^{\mu v}$ where movement is assumed again to be in the x direction.

$$\bar{F}^{\mu v} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\bar{F}^{\mu v} = \begin{pmatrix} 0 & E_x/c & \gamma(E_y/c - \beta B_z) & \gamma(E_z/c + \beta B_y) \\ -E_x/c & 0 & -\gamma(B_z - \beta E_y/c) & \gamma(B_y + \beta E_z/c) \\ -\gamma(E_y/c - \beta B_z) & \gamma(B_z - \beta E_y/c) & 0 & -B_x \\ -\gamma(E_z/c + \beta B_y) & -\gamma(B_y + \beta E_z/c) & B_x & 0 \end{pmatrix}$$

If one compares the elements of $F^{\mu v}$ containing components of vectors $\mathbf{B}$ and $\mathbf{E}$ prior to the transform to the identical row, column positions post transform, the equations match identically to those obtained through ordinary algebra. Proof that tensors indeed transform identically, with proper application of operators and algorithms.

But an observant individual may notice that there are 6 additional equations with the calculation of the Lorentz contraction. Indeed, there is superfluous information with this calculation. However, the additional information is 1) correct, being only the negation of the positive values, and 2) only obtained upon elaboration of the tensors in matrix representation. Whilst in tensor notation, the observer is nonethewise to the additional information. Furthermore, the additional elements are critical when applying simple derivatives to the tensors, multiplying them by ordinary 4-vectors such as space-time location or Proper velocity, or when calculating invariants, quantities which have the same value in all inertial frames and do not change under a Lorentz transformation[3]. Such values provide extremely useful information for understanding the underlying nature of the observed fields, including profound implications such as the identical appearance of electromagnetic waves in all inertial reference frames. These invariants are obtained by multiplying a contravariant tensor by a covariant one, or equally, a covariant tensor by another covariant tensor and two applications of the metric, $\eta^{\mu v}$, where

$$\eta^{\mu v} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Covariant tensors are simply tensors with indices provided as subscripts ($F_{\mu v}$), where contravariant tensors have indices provided as superscripts ($F^{\mu v}$). The metric ($\eta^{\mu v}$) is simply a matrix used to raise and lower subscripts and superscripts on tensors and 4-vectors for proper application of matrix operations. Note, too that $\eta^{\mu v} = \eta^{\mu \rho}\eta^{\sigma v}\eta_{\rho\sigma} \equiv \eta_{\mu v}$.

While necessary attentiveness to subscripts and superscripts and the order thereof may seem a bit intensive, as with anything else, as one builds familiarity with the notation, the application of operations becomes more intuitive, and the doorway to an easier representation of truly complicated theories begins to creak open. Electromagnetic pressure (P) and tangentiality (Q), ordinarily necessitating systems to represent, are reduced to

$$P = \frac{1}{2} F_{\mu v} F^{\mu v}$$

and

$$Q = -\frac{1}{4} F_{\mu v} G^{\mu v}$$

and the Lorentz force law and rate change of a particles energy moving through an electric field are simultaneously represented by a single 4-vector, K^{μ} where

$$K^{\mu} = qU_{v}F^{\mu v}$$

Tensor notation indeed provides a spectacular generalization and streamlining of information in relativistic electrodynamics, with transformations being reduced to simple algorithmic operations, reminiscent of calculus where expansion of the formulation necessitates only knowledge of matrix algebra.

As it is often the case that working through explicit examples assists one in gaining insight on new and unfamiliar topics, an example is provided from chapter 12 David Griffiths' book[2]:

<u>Problem 12.54</u>: *Show the second of Maxwell's equations in tensor notation can be expressed in terms of the field tensor,* $F_{\mu v}$*, as follows:*

$$\frac{\partial F_{\mu v}}{\partial x^{\lambda}} + \frac{\partial F_{v\lambda}}{\partial x^{\mu}} + \frac{\partial F_{\lambda\mu}}{\partial x^{v}} = 0. \qquad (+)$$

Recall that that in tensor notation, the second of Maxwell's equations is expressed as,

$$\frac{\partial G^{\mu v}}{\partial x^{v}} = 0,$$

which simultaneously represents the equations

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{and} \qquad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}.$$

In order to solve the problem outlined, it must be shown that $(+)$ is equivalent to the two, more "simplistic" Maxwell equations. One must consider $(+)$ for each $\mu, v, \lambda \in \{0,1,2,3\}$. With 4 cases for each of the three variables, $(+)$ simultaneously represents 64 separate equations ($4^3 = 64$). An observant reader might note, however, that many of the equations will be redundant as the variables within the matrices are identical, sans a coefficient of -1. Considering now the case where 2 variables are equivalent, say μ and v, $(+)$ becomes

$$\frac{\partial F_{\mu\mu}}{\partial x^{\lambda}}+\frac{\partial F_{\mu\lambda}}{\partial x^{\mu}}+\frac{\partial F_{\lambda\mu}}{\partial x^{\mu}}=0.$$

However, $F_{\mu\mu}=0$ and $F_{\mu\lambda}=-F_{\lambda\mu}$, which leads to the trivial solution, 0 = 0. In order to get exclusively non-trivial solutions, the ones we actually care about, it must be the case that each

$$\mu\neq v\neq\lambda\neq\mu.$$

Two cases must then be considered: one where each of the indices considered describe exclusively spatial components (are equal to either 1, 2, or 3), and one where a single index is temporal in nature (equals 0), while the other describe spatial components. In the first case, considering specifically $\mu=1$, $v=2$, and $\lambda=3$, an evaluation of $(+)$ yields

$$\frac{\partial F_{12}}{\partial x^{3}}+\frac{\partial F_{23}}{\partial x^{1}}+\frac{\partial F_{31}}{\partial x^{2}}=0,$$

which implies,

$$\frac{\partial}{\partial z}(B_z)+\frac{\partial}{\partial x}(B_x)+\frac{\partial}{\partial y}(B_y)=0.$$

The above equation represents the divergence of magnetic field, or in explicit mathematical terms;

$$\frac{\partial F_{12}}{\partial x^{3}}+\frac{\partial F_{23}}{\partial x^{1}}+\frac{\partial F_{31}}{\partial x^{2}}=0\rightarrow\nabla\cdot\mathbf{B}=0. \qquad (*)$$

Any permutations maintaining the relationship $\mu, v, \lambda \in \{1,2,3\}$ and $\mu\neq v\neq\lambda\neq\mu$ yields an identical result, with the only difference being coefficients of -1 that, in the end, do not change the results.

Considering now case 2, where a single component is equal to 0, one may consider specifically the scenario of $\mu=0$, $v=1$, and $\lambda=2$, one obtains

$$\frac{\partial F_{01}}{\partial x^2} + \frac{\partial F_{12}}{\partial x^0} + \frac{\partial F_{20}}{\partial x^1} = 0,$$

where it follows that

$$\frac{\partial}{\partial y}(-\frac{E_x}{c}) + \frac{\partial}{\partial(ct)}(B_z) + \frac{\partial}{\partial x}(\frac{E_y}{c}) = 0.$$

With some manipulation, the aforementioned equation can be written as,

$$-\frac{\partial B_z}{\partial t} + \left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right) = 0,$$

which some may recognize as the z-component of the famous Maxwell equations,

$$-\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E}. \tag{$-$}$$

The x and y components of $(-)$ are obtained by altering values for v and λ, where values of $v = 1$ and $\lambda = 3$ provide $(-)$'s y-component and $v = 2$ and $\lambda = 3$ the equation's x-component. Permutations of the variables, where the solver decides to change either v or λ to remain 0, and the remaining two to vary, result again in an identical result, with any differences coming in the form of -1 coefficients that eventually may be divided out. Therefore, in the case where one of μ, v, or λ are equal to 0, one obtains

$$\frac{\partial F_{\mu\mu}}{\partial x^\lambda} + \frac{\partial F_{\mu\lambda}}{\partial x^\mu} + \frac{\partial F_{\lambda\mu}}{\partial x^v} = 0 \rightarrow \ -\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E}.$$

This demonstrates the fact that the equation provided in $(+)$ is simultaneously representing all spatial and temporal information of the magnetic field, and with the use of operators to obtain the dual tensor, electric field as well. Profound and concise indeed.

As noted previously, this condensing of information allows one to explore complicated subjects, placing emphasis on ideas and concepts rather than laborious derivations. With careful and deliberate derivations, applications of knowledge of relativistic electrodynamics allows one to explore topics such as plasmas on curved spacetimes, possibly providing indirect evidence of gravitational waves and strengthening theories of the early universe. Other topics where relativistic electrodynamics play a central role are the radiation of moving charges and the scattering and dispersion of waves and energy in lossy media, which have numerous applications to the field of engineering. Tensor notation then becomes an invaluable tool to authors attempting to explain theories and designs, without losing the reader in pages upon pages of derivations. The beauty of tensors is in the simplistic representation that they offer to immensely complex real world situations. Packing copious amounts of information into occasionally single term equations, wholly representative of a scenarios underlying physics, the provide an appealing notation to mathematicians, physicists, engineers, and interested readers alike.

<u>**Resources**</u>**:**

1. Jose A. Heras, and G. Baez. "The covariant formulation of Maxwell's equations expressed in a form independent of specific units." (2009)
2. David J. Griffiths. Introduction to Electrodynamics. 4^{th} Edition. Pearson. (2013)
3. Joel C. Corbo. "Supplemental Lecture II: Special Relativity in Tensor Notation". (2005)
4. Gerold Betschart. "General Relativistic Electrodynamics with Applications in Cosmology and Astrophysics." (2005)
5. D. Censor, "Application-Oriented Relativistic Electrodynamics," PIER 29, 107–168, (2000)