Relativistic Quantum Mechanics

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The aim of this chapter is to introduce a relativistic formalism which can be used to describe particles and their interactions. The emphasis is given to those elements of the formalism which can be carried on to Relativistic Quantum Fields (RQF), which underpins the theoretical framework of high energy particle physics.

We begin with a brief summary of special relativity, concentrating on 4-vectors and spinors. One-particle states and their Lorentz transformations follow, leading to the Klein–Gordon and the Dirac equations for probability amplitudes; i.e. Relativistic Quantum Mechanics (RQM). Readers who want to get to RQM quickly, without studying its foundation in special relativity can skip the first sections and start reading from the section 1.3.

Intrinsic problems of RQM are discussed and a region of applicability of RQM is defined. Free particle wave functions are constructed and particle interactions are described using their probability currents. A gauge symmetry is introduced to derive a particle interaction with a classical gauge field.

1.1 Special Relativity

Albert Einstein 1879 - 1955

Einstein's special relativity is a necessary and fundamental part of any formalism of particle physics. We begin with its brief summary. For a full account, refer to specialized books, for example (1) or (2). Theory oriented students with good mathematical background might want to consult books on groups and their representations, for example (3), followed by introductory books on RQM/RQF, for example (4). Here we are only going to present conclusions without derivations avoiding group theory language and aiming at a presentation of key concepts at a qualitative level. A chapter about spinors in (5) is recommended[1].

The basic elements of special relativity are 4-vectors (or contravariant 4-vectors) like a 4-displacement[2] $x^\mu = (t, \mathbf{x}) = (x^0, x^1, x^2, x^3) = (x^0, x^i)$ or a 4-momentum $p^\mu = (E, \mathbf{p}) = (p^0, p^1, p^2, p^3) = (p^0, p^i)$. 4-vectors have real components and form a vector space. There is a metric tensor $g_{\mu\nu} = g^{\mu\nu}$ which is used to form a dual space to the space of 4-vectors. This dual space is a vector space of linear functionals, known as 1-forms (or covariant 4-vectors), which act on 4-vectors. For every 4-vector x^μ,

[1] Please note that in that chapter, transformations are in 'active', (the coordinate system is not changing, vectors are changing) not 'passive' (the coordinate system is changing but vectors don't) sense as in this book

[2] $\mu, \nu = 0, 1, 2, 3$ and $i, j = 1, 2, 3$.

there is an associated 1-form $x_\mu = g_{\mu\nu}x^\nu$. Such a 1-form is a linear functional which acting on a 4-vector y^μ gives a real number $= g_{\mu\nu}x^\nu y^\mu$. This number is called a scalar product $x \cdot y$ of x^ν and y^μ[3]. The Lorentz transformation between two coordinate systems, $\Lambda^\mu{}_\nu$, $x'^\mu = \Lambda^\mu{}_\nu x^\nu$, leaves the scalar product unchanged which is equivalent to $g_{\rho\sigma} = g_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma$. In the standard configuration, the Lorentz transformation becomes the Lorentz boost along the first space coordinate direction and is given by

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \beta = \frac{v}{c}, \quad \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

where v is the velocity of the boost.

Two Lorentz boosts along different directions are equivalent to a single boost and a space rotation. This means that Lorentz transformations which can be seen as space-time rotations, include Lorentz boosts (rotations by a pure imaginary angle) as well as space rotations (by a pure real angle). Representing Lorentz transformations by 4-dimensional real matrices acting on 4-vectors is not well suited for combining Lorentz boosts and space rotations in a transparent way. Even a simple question like: 'What is the single space rotation which is equivalent to a combination of two arbitrary space rotations? is hard to answer. A better way is to represent Lorentz transformations by 2-dimensional complex matrices.

First we consider a 3-dimensional real space and rotations. With every rotation in that 3-dimensional real space we can associate a 2 x 2 complex spin matrix[4]

$$R = \cos(\theta/2) + i\sin(\theta/2)(\sigma_x\cos(\alpha) + \sigma_y\cos(\beta) + \sigma_z\cos(\gamma))$$

or

$$R = \cos(\theta/2) + i\sin(\theta/2)(\mathbf{n}\cdot\sigma)$$

R is unitary; $R^\dagger = R^{-1}$.

or after some algebra

$$R = \exp[i(\theta/2)(\mathbf{n}\cdot\sigma)] \tag{1.1}$$

where θ is the angle of rotation, α, β, γ are the angles[5] between the axis of rotation $\mathbf{n}$ and the coordinate axes and $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ are Pauli matrices. The vector space of spin matrices (a subspace of all 2 x 2 complex matrices) can be defined using four basis vectors, such as the unit matrix and three basis vectors formed using Pauli matrices: $i\sigma_x, i\sigma_y, i\sigma_z$. In this basis, the spin matrix R has the following coordinates: $\cos(\theta/2)$, $\sin(\theta/2)\cos(\alpha)$, $\sin(\theta/2)\cos(\beta)$ and $\sin(\theta/2)\cos(\gamma)$. Combining two rotations, one multiplies corresponding spin matrices and describes the outcome using the above basis; thus getting all the parameters of the equivalent single rotation. The next step is to associate each 3-dimensional space (real numbers) vector $\mathbf{x} = (x^1, x^2, x^3)$ with a corresponding spin matrix[6]

$$X = x^1\sigma_x + x^2\sigma_y + x^3\sigma_z. \tag{1.2}$$

[3] A similar situation occurs in the infinitely-dimensional vector space of states in quantum mechanics (complex numbers there). For every state, a vector known as a ket, for example $|x\rangle$, there is a 1-form known as a bra, $\langle x|$ which acting on a ket $|y\rangle$ gives a number $\langle x|y\rangle$ which is called a scalar product of two kets, $|x\rangle$ and $|y\rangle$.

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

[4] Also known as Hamilton's quaternion or spinor transformation or rotation operator.

[5] Only two of the angles α, β, γ are independent.

[6] No unit matrix, only three Pauli matrices as the basis.

Then, under the space rotation, $\mathbf{x}$ is transformed to $\mathbf{x}'$ and X is transformed to $X' = RXR^\dagger = x'^1\sigma_x + x'^2\sigma_y + x'^3\sigma_z$ from which we can read coordinates of $\mathbf{x}'$.

The beauty of the above approach is that it extends seamlessly to space-time rotations; i.e. to the Lorentz transformations. The spin matrix R of 1.1 becomes the Lorentz transformation

$$L = exp[(-\rho + i\theta\mathbf{n}) \cdot \sigma/2], \tag{1.3}$$

where $\rho = \rho\mathbf{n}_\rho$ is the rapidity The rapidity is related to the Lorentz β and γ parameters by:

$$\tanh(\rho) = \beta, \qquad \cosh(\rho) = \gamma, \qquad \sinh(\rho) = \beta\gamma.$$

Now, combination of two Lorentz transformations is very transparent; just addition of real and imaginary parts in the exponent. Association of a 4-vector x^μ with a Hermitian spin matrix X[7]

$$X = x^0 + x^1\sigma_x + x^2\sigma_y + x^3\sigma_z \tag{1.4}$$

allows us to get its Lorentz transformed coordinates from $X' = LXL^\dagger = x'^0 + x'^1\sigma_x + x'^2\sigma_y + x'^3\sigma_z$. Finally, the Lorentz boost alone ($\theta = 0$) along $\mathbf{n}_\rho$ is

$$L = exp[-\rho \cdot \sigma/2] = \cosh(\rho/2) - \mathbf{n}_\rho \cdot \sigma \sinh(\rho/2). \tag{1.5}$$

[7] Now, four matrices as the basis and X satisfies $X = X^\dagger$ by definition.

1.1.1 Spinors

Spin matrices act on two component complex vectors called spinors[8]. Spinors play an important role in RQM[9] and in this section we will describe them in some detail.

Under the space rotation, a spinor ξ (ξ^α to be more precise) transforms the following way

$$\xi' = R\xi.$$

For comparison, coordinates of a vector $\mathbf{x}$ transform under space rotation as

$$X' = RXR^\dagger = x'^1\sigma_x + x'^2\sigma_y + x'^z\sigma_z.$$

Thus, in a rotation of the coordinate system by $\theta = 2\pi$, $R = -1$ because the $\theta/2$ in R, gives $\xi' = -\xi$ and $\mathbf{x}' = \mathbf{x}$. Continuing the rotation by a further 2π, so all together by 4π, results in $\xi' = \xi$. Does that counter intuitive minus sign, consequent on the 2π rotation have physical significance? Yes it does, as was demonstrated in a beautiful experiment (6) using neutrons.

One of two coherent neutron beams passes through a magnetic field of variable strength. In the magnetic field the neutrons' magnetic moments precess with the Larmor frequency and the angle of the precession is easily calculated as a function of the strength of the magnetic field. After passing through the magnetic field, the beam interfers with the second beam which followed a path outside the magnetic field. As demonstrated

[8] Spinors are vectors in the mathematical sense as they form complex vector space but they are not vectors like a displacement $\mathbf{x}$ because they transform (for example under rotation) differently.

[9] Spinors like vectors or tensors are used in different parts of physics, including classical mechanics.

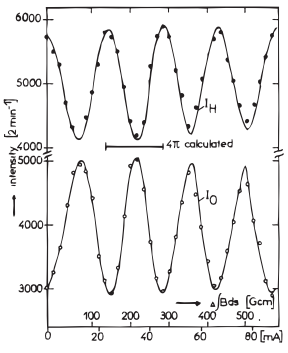

Fig. 1.1 The phase change of 4π is needed to get the same intensity from the interference of two neutron beams; taken from (6)

in fig 1.1, an angle of 4π is needed for the neutron wave function to reproduce itself. A 2π rotation gives a factor -1 in front of the original neutron wave function as predicted for a spin$-\frac{1}{2}$ spinor.

So far, one could think about spinors as being identical with Pauli spinors[10] of non-relativistic quantum mechanics. This is not quite right. In order to see that, we will look at spin matrices X of eqn 1.4 from a different angle, seeing them as tensors created by a tensor product[11] of 2-dimensional spinors, Weyl spinors are rooted in space-time, not in space only like Pauli spinors. Consider the Lorentz transformation of a spin matrix X built from spinors $\xi = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\eta = \begin{pmatrix} c \\ d \end{pmatrix}$:

$$X' = \begin{pmatrix} a' \\ b' \end{pmatrix} \begin{pmatrix} c' & d' \end{pmatrix} = L \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} c & d \end{pmatrix} L^\dagger.$$

We can see that $\xi' = L\xi$ but $\eta' = L^*\eta$ (after taking transpose, $L^{\dagger T} = L^*$). There are two different types of spinors, transforming differently. Those which transform with the complex conjugate, L^*, are called dotted Weyl spinors, distinguished from the undotted ξ^α by a dot written above the index: $\eta^{\dot\alpha}$; for example, $(\xi^\alpha)^*$ is a dotted spinor. The spin matrix X is then written as $X^{\alpha\dot\beta}$. There is a metric tensor

$$\epsilon_{\alpha\beta} = \epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(and an identical one for the dotted spinors) to create a dual space of 1-forms; $\xi_\alpha = \epsilon_{\alpha\beta}\xi^\beta$ (and $\eta_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\eta^{\dot\beta}$). The scalar product $\xi^\alpha\zeta_\alpha = \epsilon_{\alpha\beta}\xi^\alpha\zeta^\beta$ (and similarly for the dotted spinors) is invariant with respect to the Lorentz transformation. Because undotted Weyl spinors and dotted Weyl spinors are different objects, the scalar product, or in general any contraction, can only be performed on the same type of spinors; an undotted index is contracted with another undotted index and a dotted index is contracted with another dotted one; one can't contract a dotted index with an undotted one.

Hermann Weyl 1885 - 1955

In order to get more insight, we go beyond Lorentz transformations and consider a space inversion P; $P(x^0, \mathbf{x}) = (x^0, -\mathbf{x})$. The space inversion P commutes with space rotations but it does not commute with Lorentz transformations because Lorentz transformations affect the time component and P does not. One can see this, consider $P\Lambda$, the Lorentz boost followed by the space inversion in the 4-dimensional space-time: $P\Lambda = \Lambda' P$ and that $\Lambda \neq \Lambda'$. If Λ is the boost with a velocity $\mathbf{v}$, Λ' is the boost with the velocity $-\mathbf{v}$. Thus $[P, \Lambda] \neq 0$ and therefore P is not proportional to the identity operator which commutes with every operator.

Back to Weyl spinors. Because space inversion is not proportional to the identity operator, space inversion does not transform ξ^α into ξ^α times a number. It transforms ξ^α into a spinor of a different type, which transforms under the Lorentz transformation differently than ξ^α. As Pauli spinors represent spin in non-relativistic quantum mechanics, Weyl spinors are going to represent spin in RQM. If so, we know that the

[10] Eigenstates of spin operators, like the spin projection on the z axis $\frac{1}{2}\hbar\sigma_z$ for a spin$-\frac{1}{2}$ particle, in non-relativistic quantum mechanics (7).

[11] A tensor product or outer product or dyadic product; something like this: $\begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} c & d \end{pmatrix} = \begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix}$.

$\alpha = 1, 2$ and $\dot\alpha = 1, 2$.

$\xi_1 = \xi^2$ and $\xi_2 = -\xi^1$.

$\xi^\alpha\xi_\alpha = 0$

$\xi^\alpha\zeta_\alpha = -\xi_\alpha\zeta^\alpha$

space inversion leaves spin unaffected and therefore under P, ξ^α needs to be transformed to a spinor which transforms under space rotations the same way as ξ^α and represents the same spin state.[12] Out of all 3 possibilities, only the 1-form $\eta_{\dot\alpha}$ transforms the same way. So under the space inversion $\xi^\alpha \to \eta_{\dot\alpha}$ and $\eta_{\dot\alpha} \to \xi^\alpha$. In the discussion on the space inversion, $P^2 = 1$ is assumed. This is fine for all particles except Majorana particles. For a Majorana particle, $P^2 = -1$ and the transformation of spinors under P is different than given here. We will define a Majorana particle later.

[12] For a better grasp of the material presented here on spinors, the chapter on fermions in (8) is recommended.

No fundamental Majorana particle has been discovered so far but a neutrino could be one (a composite Majorana particle was discovered in condensed matter).

As spinors ξ^α and $\eta_{\dot\alpha}$ play a very important role in RQM, here is a summary of how they transform under

rotation R	$\xi^\alpha \to R\xi^\alpha$	$\eta_{\dot\alpha} \to R\eta_{\dot\alpha}$,	(1.6)
Lorentz transformation L	$\xi^\alpha \to L\xi^\alpha$	$\eta_{\dot\alpha} \to (L^\dagger)^{-1}\eta_{\dot\alpha}$,	(1.7)
Lorentz boost $L^\dagger = L$	$\xi^\alpha \to L\xi^\alpha$	$\eta_{\dot\alpha} \to L^{-1}\eta_{\dot\alpha}$,	(1.8)
space inversion P	$\xi^\alpha \to \eta_{\dot\alpha}$	$\eta_{\dot\alpha} \to \xi^\alpha$.	(1.9)

$(L^\dagger)^{-1} = \epsilon L^* \epsilon^{-1}$.

Suppose there is a spin-$\frac{1}{2}$ particle with 4-momentum p^μ described in a particular reference frame by $p^{\alpha\dot\beta}$ via equation 1.4. Note that $p^{\alpha\dot\beta}$ is identical with $p^{\dot\beta\alpha}$, it is not a transpose operation. Following our non-relativistic intuition gained from using Pauli spinors, we want to represent the spin of that particle by ξ^α. In an attempt to write a covariant[13] equation, we can try to contract the undotted index α but that would lead to something like:

$$p_{\alpha\dot\beta}\xi^\alpha = m\eta_{\dot\beta}$$

$$\begin{aligned} p^{1\dot1} &= p_{2\dot2} = p^0 + p^3 \\ p^{2\dot2} &= p_{1\dot1} = p^0 - p^3 \\ p^{1\dot2} &= -p_{2\dot1} = p^1 - ip^2 \\ p^{2\dot1} &= -p_{1\dot2} = p^1 + ip^2 \end{aligned}$$

[13] Here and in the rest of the RQM, covariant means covariant with respect to Lorentz transformations.

where $\eta_{\dot\beta}$ is a dotted spinor different from ξ^α related to the uncontracted dotted index, and m is a dimensional scalar[14] parameter appearing there because of the energy dimensionality of p^μ. So the equation is covariant only when $m = 0$ because we do not have any dotted spinor in hand to put on the right hand side. A similar outcome is obtained having only $\eta_{\dot\beta}$ instead of ξ^α. Having a column vector with two complex numbers is not enough, we need a name in addition to indicate how the two numbers transform under Lorentz transformations. Three real numbers in 3-dimensional Euclidean space are not enough; we need to know whether they represent a polar or axial vector as they transform differently under space inversion. So insisting on only one type of spinor, excludes the other one because it transforms differently.

[14] From now on, scalar means a scalar with respect to Lorentz transformations.

These attempts lead to two independent Lorentz-invariant Weyl eqns:

$$p_{\alpha\dot\beta}\xi^\alpha = 0 \qquad (p^0 - \mathbf{p}\cdot\sigma)\xi = 0 \quad (1.10)$$

and

$$p^{\alpha\dot{\beta}}\eta_{\dot{\beta}} = 0 \qquad\qquad (p^0 + \mathbf{p}\cdot\sigma)\eta = 0. \quad (1.11)$$

In quantum mechanics, a helicity operator representing a projection of particle's spin on the direction of its momentum is defined as

$$\frac{\mathbf{p}\cdot\sigma}{\mathrm{p}}.$$

For a massless particle, this is equivalent to

$$\frac{\mathbf{p}\cdot\sigma}{p^0}.$$

In the context of RQM, eqn 1.10 represents an equation of motion for a free massless spin-$\frac{1}{2}$ particle with a positive helicity and eqn 1.11, an equation of motion for a different, free massless spin-$\frac{1}{2}$ particle with a negative helicity. Each equation is not space-inversion covariant and violates parity because the space inversion, eqn 1.9, sends each spinor beyond the formalism; only one type of spinor is present in each formalism. At present there are no known particles which could be described by any of the Weyl equations. If the electron neutrino was exactly massless, it would be described by eqn 1.11 and the hypothetically massless and different electron anti-neutrino would be described by eqn 1.10.

Suppose now that we have $p^{\alpha\dot{\beta}}$ and two different spinors: ξ^{α} and $\eta_{\dot{\beta}}$ to describe a spin-$\frac{1}{2}$ particle. First contract the undotted index giving $p_{\alpha\dot{\beta}}\xi^{\alpha} = m\eta_{\dot{\beta}}$, then acting with $p^{\alpha\dot{\beta}}$ on $\eta_{\dot{\beta}}$ from that equation, gives $m\xi^{\alpha}$ under the condition that $m^2 = p^{\mu}p_{\mu}$.

$p^{\alpha\dot{\beta}}p_{\gamma\dot{\beta}} = p^{\mu}p_{\mu}\delta^{\alpha}_{\gamma}$

The result is a covariant set of equations

$$\begin{aligned} p^{\alpha\dot{\beta}}\eta_{\dot{\beta}} &= m\xi^{\alpha} & (p^0 + \mathbf{p}\cdot\sigma)\eta &= m\xi \\ p_{\alpha\dot{\beta}}\xi^{\alpha} &= m\eta_{\dot{\beta}} & (p^0 - \mathbf{p}\cdot\sigma)\xi &= m\eta \end{aligned} \qquad (1.12)$$

Requiring that under the space inversion ξ^{α} and $\eta_{\dot{\beta}}$ are transformed into each other as in eqn 1.9, makes the set of eqns 1.12 space inversion invariant as, simultaneously, $p^{\alpha\dot{\beta}}$ and $p_{\alpha\dot{\beta}}$ are also transformed into each other. Spinors ξ^{α} and $\eta_{\dot{\beta}}$ are combined into a single four-component bispinor, called the Dirac spinor and the two equations become one equation 1.12 called the Dirac equation. In the context of RQM, the Dirac equation describes a spin-$\frac{1}{2}$ particle like the electron.[15]

[15] P.A.M. Dirac 1902 - 1984

In order to have more insight into the origin of the Dirac equation, consider the Lorentz boost, eqns 1.5 and 1.8, from the rest frame, momentum $\mathbf{p} = 0$, to the frame in which the particle has the energy E and the momentum $\mathbf{p}$. The relevant spinors transform as[16]

[16]

$$\cosh(\rho/2) = \frac{E+m}{(2m(E+m))^{1/2}}$$

$$\sinh(\rho/2) = \frac{\mathrm{p}}{(2m(E+m))^{1/2}}$$

$$\xi(\mathbf{p}) = (\cosh(\rho/2) - \mathbf{n}_{\rho}\cdot\sigma\sinh(\rho/2))\xi(0) \qquad (1.13)$$

$$\eta(\mathbf{p}) = (\cosh(\rho/2) + \mathbf{n}_{\rho}\cdot\sigma\sinh(\rho/2))\eta(0) \qquad (1.14)$$

which can be written as

$$\xi(\mathbf{p}) = \frac{E+m+\mathbf{p}\cdot\sigma}{(2m(E+m))^{1/2}}\xi(0) \qquad (1.15)$$

$$\eta(\mathbf{p}) = \frac{E+m-\mathbf{p}\cdot\sigma}{(2m(E+m))^{1/2}}\eta(0). \qquad (1.16)$$

In the particle's rest frame and in all frames moving with respect to it slowly enough, such that the Lorentz boost can be approximated by the Galilean transformation (not affecting time) when transforming between those frames, any differences in how spinors with dotted or undotted

indexes transform disappear. In that case, spinors effectively live in 3 real dimensions. Inspecting eqn 1.5, one can see that at the limit $\beta \to 0$ $L \to$ the unit matrix and therefore under the Galilean transformation, spinors don't change. Thus, at rest, both Weyl spinors, ξ^α and $\eta_{\dot\beta}$, become effectively identical with the same Pauli spinor and we can write $\xi^\alpha(0) = \eta_{\dot\alpha}(0)$. This allows us, after some algebra, to remove $\mathbf{p} = 0$ spinors from the eqns 1.15 and 1.16 and to obtain the Dirac equation 1.12.

So the Dirac equation 1.12 is equivalent to the Lorentz boost. This should be expected. Once an object, like a bispinor, is found to represent a particle in its rest frame, the only thing left to do is to boost it to another frame as needed.

1.2 One-particle states

The fact that quantum states of free relativistic particles are fully defined by the Lorentz transformation supplemented by the space-time translation was discovered by Wigner. Here we will follow his idea in a qualitative way just to get the main concept across.

Eugene Wigner 1902 - 1995

First, we note that Lorentz transformations are not able to transform a given arbitrary 4-momentum p^μ to every possible p'^μ. Instead, the vector space of 4-momenta is divided into sub-spaces of 4-momenta which can be Lorentz transformed into each other. Three of those sub-spaces represent experimentally known states. The simplest, at this stage, is the vacuum state given by the conditions $p^\mu = 0$ and $p^\mu p_\mu = 0$. There is no Lorentz transformation which would transform a 4-momentum not satisfying these conditions to the one which does and vice-versa. We will not study vacuum states in this book and therefore we move directly to consider two other possibilities.

A 4-momentum sub-space related to massive particles, like the electron, is given by the condition $p^\mu p_\mu > 0$. In addition to a 4-momentum p^μ, what are other degrees of freedom present and which geometrical object represent them? To answer this question, we can consider Lorentz transformations which leave p^μ invariant[17]. To see what they are, we can transform p^μ to the particle rest frame where $p'^\mu = (mass, 0, 0, 0)$, find the largest subset of the Lorentz transformations leaving p'^μ invariant, and then transform back to the same p^μ. It turns out, as intuitively expected, that the desired transformations are space rotations acting on $2s + 1$ spinors representing $2s + 1$ spin projections of spin s particle. Thus, the electron, $s = 1/2$, is represented by two Dirac spinors. In fact, by two Dirac spinors multiplied be a dimensionless scalar. To get the scalar, we add space-time translations. Looking for a theory which is space-time translation invariant,[18] we are looking for the free particle energy and momentum eigenstates which in the position representation, lead to the scalar $exp(-ip^\mu x_\mu)$.

[17] The group of such transformations is known as the little group.

[18] Implying energy and momentum conservation.

The third 4-momentum sub-space is defined by the conditions $p^\mu \neq 0$ and $p^\mu p_\mu = 0$. Photons belong to this class. The question is again to

find the largest sub-set of Lorentz transformations leaving p^μ invariant. There is no rest frame in this case and therefore and instead, we transform an arbitrary p^μ to the frame where $p'^\mu = (\omega, 0, 0, \omega)$. We can see that the largest[19] sub-set of the Lorentz transformations leaving p'^μ invariant are the rotations in the x^1x^2 plane. As a result, a spin s massless particle is represented by only one state, a helicity eigenstate, and not by $2s+1$ states as in the massive case. This is an important difference. In order to get parity conserving electromagnetism with photons having either helicity + or helicity – states, one puts those two, in principle different, helicity states into one theory.

[19] For a discussion of some subtle issues see (4).

1.2.1 Fields and probability amplitudes

We have now everything needed to develop RQM and to describe fundamental particles and their interactions. But before we move on, we pause to look at a larger picture of which RQM is only a part. The Dirac equation, for example, can be studied in the context of classical fields (CF) or RQF or RQM. The algebra would often be identical but the basic objects and the interpretation different.

The most natural way to proceed from here would be to study a CF theory. The paradigm for this is classical electromagnetism described in terms of the tensor field $F^{\mu\nu}$ or 4-vector potential field A^μ;

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

Then such a field, for example a classical electron field, i.e. a classical Dirac spinor field Ψ, would be quantized, promoting Ψ of CF to a relativistic quantum field operator Ψ of RQF. Probability amplitudes would be obtained by taking matrix elements of Ψ of RQF sandwiched between particle states living in a suitably constructed space. In RQM, Ψ of RQM represents a particle state and it is not an operator. Taking the vacuum to one-particle matrix element of the field operator Ψ of RQF one gets Ψ of RQM which in the position representation is called the wave-function. In RQM one can also have states describing many particles but only a fixed number of them. At high energies, much larger than the masses of the particles involved, particles can be created and the number of particles can not be fixed; RQM is not adequate for this and RQF has to be used instead.

It is not appropriate in this text to go into sufficient detail to enable a proper understanding of CF and RQF. Fortunately, considering the most important aspects of physics we require, RQF gives the same results as those which we will obtain in RQM. Differences will be in details beyond leading effects. The one important exception is that we will be missing the idea of a vacuum state. In RQF, a vacuum is not a nothingness, although particles are absent. For example, the QCD vacuum is a very complicated state.

1.3 The Klein–Gordon equation

Relativistic quantum mechanics of spin-0 particles was considered by Schroedinger first, before he published his famous equation for the non-relativistic case. He abandoned RQM because of formal difficulties which were only understood many years later. Here, we will see what they are and then define an area of RQM applicability.

As argued earlier, a spin-0 particle with

$$p^\mu p_\mu = m^2 > 0, \tag{1.17}$$

in the position representation, is expected to be described by a scalar wave function $\sim exp(-ip^\mu x_\mu)$. Replacing the energy by $i\frac{\partial}{\partial t}$ and the momentum by $-i\nabla$ in eqn 1.17, one gets the Klein–Gordon (K–G) equation[20] of RQM in the position representation:

$$(\Box + m^2)\Psi(t, \mathbf{x}) = 0, \tag{1.18}$$

where $\Box = \partial^\mu \partial_\mu = \frac{\partial^2}{\partial t^2} - \nabla^2$. For a particle at rest, $-i\nabla\Psi(t, \mathbf{x}) = 0$, only the time (proper time τ) derivative would be present in eqn 1.18 and there would be two independent solutions: $\Psi^\pm(\tau, \mathbf{x}) = \exp(\mp im\tau)\Psi^\pm(0, 0)$. Therefore in a frame in which the particle has a momentum p and the energy $E_p = +\sqrt{p^2 + m^2} > 0$ (the subscript p for the plus sign; indicating that E_p is positive), we get, as expected,[21]

$$\Psi^+(t, \mathbf{x}) = N \exp(-ip \cdot x) = N \exp(-iE_p t + i\mathbf{p} \cdot \mathbf{x}) \tag{1.19}$$

Instead of boosting the other solution, i.e. taking the $(-E_p, \mathbf{p})$ eigenstate, we take the complex conjugate of $\Psi^+(t, \mathbf{x})$, corresponding to the $(-E_p, -\mathbf{p})$ eigenstate and to get[22]

$$\Psi^-(t, \mathbf{x}) = N \exp(+ip \cdot x) = N \exp(+iE_p t - i\mathbf{p} \cdot \mathbf{x}) \tag{1.20}$$

where N is a normalization constant which will be defined shortly. By a direct substitution, one can check that a general solution of eqn 1.18 is indeed a linear combination of $\Psi^+(t, \mathbf{x})$ and $\Psi^-(t, \mathbf{x})$.

We have obtained, as expected, $\Psi^+(t, \mathbf{x})$ but in addition, we also have $\Psi^-(t, \mathbf{x})$. This is the first puzzle of RQM, the nature of which will become clearer when we progress a little further. Both solutions of the K–G equation are eigenfunctions of the energy operator $i\frac{\partial}{\partial t}$; $\Psi^+(t, \mathbf{x})$ with an eigenvalue of E_p and $\Psi^-(t, \mathbf{x})$ with $-E_p$, a negative energy for a free particle!

In exactly the same way as for the non-relativistic Schroedinger eqn, one can derive the continuity equation for a probability density ρ and a probability current $\mathbf{j}$:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0 \tag{1.21}$$

where

$$\rho = i(\Psi^* \frac{\partial \Psi}{\partial t} - \Psi \frac{\partial \Psi^*}{\partial t}) \tag{1.22}$$

[20] Sometimes known as the Klein–Gordon–Fock equation.

$$p^\mu = i\partial^\mu$$
$$p^0 = i\frac{\partial}{\partial x^0} = i\frac{\partial}{\partial t}$$
$$p^i = i\partial^i = -i\partial_i = -i\frac{\partial}{\partial x^i}$$

[21] $m\tau = p^\mu x_\mu$

[22] Why one is doing that should become clear after reading section 1.3.1.

and

$$\mathbf{j} = -i(\Psi^* \nabla\Psi - \Psi\nabla\Psi^*). \tag{1.23}$$

The probability current comes out to be given by the same expression as the non-relativistic one but the probability density is different, although nicely symmetric to the current and we can define a 4-vector current $j^\mu \equiv (\rho, \mathbf{j}) = i(\Psi^*\partial^\mu\Psi - \Psi\partial^\mu\Psi^*)$. The continuity eqn **??** can be then written as $\partial_\mu j^\mu = 0$. The corresponding conserved quantity is the total probability which one gets integrating $j^0 = \rho$ over the 3D space. The underlying symmetry is invariance with respect to multiplication by a global phase factor: physics described by Ψ is identical to physics described by $\exp(i\theta)\Psi$ for any fixed real parameter θ.[23]

[23] One can prove this from the Klein–Gordon Lagrangian and Noether's theorem.

Substituting $\Psi^+(t, \mathbf{x})$ from eqn 1.19 into 1.22 and 1.23 we obtain

$$\rho^+ = 2\,|N|^2\, E_p \qquad \text{and} \qquad \mathbf{j}^+ = 2\,|N|^2\,\mathbf{p}. \tag{1.24}$$

Now we can fix the normalization N. In non-relativistic quantum mechanics, the volume integral of the probability density is a constant with value 1 for one particle in the whole space. This doesn't work in RQM because there is the Lorentz contraction which modifies the volume, contracting one side of a cube, parallel to the Lorentz boost, by the Lorentz factor γ. To keep the integral independent of the Lorentz transformation, the probability density, should grow by the same γ[24]. So putting $N = 1$ would do the job, as would any other constant. The choice of $N = 1$ is called the covariant normalization and corresponds to $2E_p$ particles in a unit volume. Another popular choice is $N = 1/\sqrt{2m}$ which in the non-relativistic limit, $E_p \to m$ makes $\rho \to \Psi^*\Psi$ and $\mathbf{j} \to$ **velocity** approach the expressions from non-relativistic quantum mechanics.

[24] No problem, as ρ is the time-like component of a 4-vector

For $\Psi^-(t, \mathbf{x})$, eqn 1.24 becomes

$$\rho^- = -2\,|N|^2\, E_p \quad \text{and} \quad \mathbf{j}^- = -2\,|N|^2\,\mathbf{p}. \tag{1.25}$$

Summarizing, $\Psi^+(t, \mathbf{x})$ and related observables, the energy, probability density and the probability current come out as expected and behave nicely in the non-relativistic limit. In contrast to $\Psi^+(t, \mathbf{x})$ an unexpected additional wave function $\Psi^-(t, \mathbf{x})$, describes a free particle with negative energy, negative probability density and with the probability current flowing in the opposite direction to the particle's momentum; all properties unexpected and difficult to accept.

1.3.1 The Feynman–Stueckelberg interpretation of negative energy states

In this section we outline the Feynman–Stueckelberg interpretation of negative energy states following the approach described by Feynman in his Dirac Lecture (21) which is recommended as further reading.

Suppose there is a particle in a state ϕ_0 as indicated in fig 1.2a. At a time t_1 a potential U_1 is turned on for a moment, acting on the particle and changing its state to an intermediate state. At a time t_2, a second

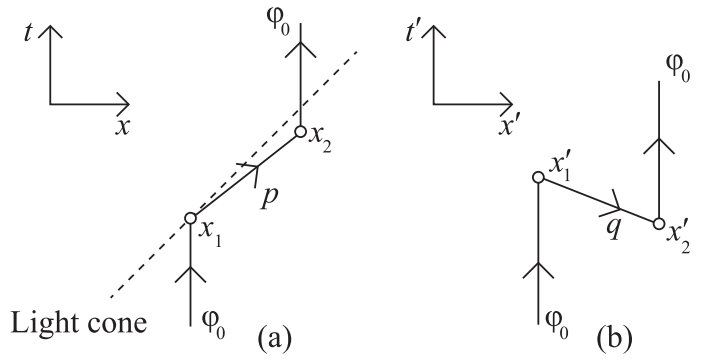

Fig. 1.2 A contribution to the transition amplitude viewed in two different reference frames; adapted from (21).

perturbation U_2 changes that intermediate state to the final one which could be the same as the original state ϕ_0. An amplitude for the particle to go from the initial state ϕ_0 to the same state ϕ_0 after time t_2 has a contribution from an amplitude with an intermediate state, existing for a period of time from t_1 to t_2, of energy $E_p > 0$. All possible intermediate states of different energies $E_p > 0$ are contributing. Among them, there are amplitudes for particles travelling faster than the speed of light. This is the result of insisting that all energies E_p are positive. If one starts a series of waves from a point keeping all energies positive, those waves cannot be confined to be inside the light cone. A sketch in fig 1.2a corresponds to an amplitude where the particle in the intermediate state travels faster than the speed of light. An observer in the reference frame (a) with coordinates (t, x) observes one particle in quantum state ϕ_0 which moves from x_1 to x_2 ending in the same quantum state ϕ_0.

As indicated in fig 1.2b, there is another reference frame (b) with coordinates (t', x') in which the sequence of events is different; t'_2 happens first, before t'_1. An observer in this reference frame has a different story to tell. A particle at x'_1, is in a quantum state ϕ_0. Nothing happens until a time t'_2 when suddenly two particles emerge from a point x'_2. One of those particles travels to x'_1 and at a time t'_1 collides with the original particle. Both particles annihilate each other, disappearing from the scene, leaving the third particle at x'_2 in state ϕ_0. In frame (b) three particles were present between t'_2 and t'_1. The second observer can argue that the particle which travelled from x'_2 to x'_1 is the antiparticle of the original particle and therefore they were able to annihilate each other. So antiparticles must exist and their properties are defined from particles such that the annihilation works. The first observer might argue that the antiparticle of the second observer is her/his particle travelling backwards in time and that is the interpretation of the negative energy states. The negative energy states correspond to particles travelling backwards in time and therefore the phase of the wave function in eqn 1.20 has a $-iE_p(-t) \equiv +iE_pt$ contribution instead of $-iE_pt$ as in

eqn 1.19. To make the picture complete we must also take into account that a particle travelling backwards in time has its momentum reversed. Mathematically, all this is equivalent to taking the complex conjugate of the positive energy solution Ψ^+ (eqn 1.19) to obtain the negative energy solution Ψ^- (eqn 1.20).

In summary, negative energy solutions of the Klein–Gordon equation represent antiparticles. The probability density represent the charge density and can be either negative or positive. Similarly for the probability current, representing the charged current; the number of charges passing through the unit area per unit time.

Inclusion of interactions via a potential

We introduce interactions using a potential and see what happens. Following the way a potential V was introduced into the non-relativistic Schroedinger eqn, we modify the energy operator:[25] $i\frac{\partial}{\partial t} \rightarrow i\frac{\partial}{\partial t} - V$ transforming eqn 1.18 into

$$(i\frac{\partial}{\partial t} - V)^2\Psi = (-\nabla^2 + m^2)\Psi$$

which for a time independent, time-like potential V in 1-dimension and for energy eigenstates with energy E_p, becomes[26]

$$(E_p - V(s))^2\psi(s) = (-\frac{\partial^2}{\partial s^2} + m^2)\psi(s). \qquad (1.26)$$

Consider a time-like potential barrier of fixed height $V > 0$ for $s \geq 0$, as shown in fig. 1.3 The wave function $\psi(s)$ consists of incident, $I\exp(ips)$, reflected, $R\exp(-ips)$, and transmitted, $T\exp(iks)$ waves:

$$\psi_L(s) = I\exp(ips) + R\exp(-ips)$$

$$\psi_R(s) = T\exp(iks),$$

where $\psi(s) = \psi_L(s)$ for $s < 0$ and $\psi(s) = \psi_R(s)$ for $s \geq 0$.

Substituting ψ_L and ψ_R into eqn 1.26, one gets $E^2 = p^2 + m^2$ and $(E - V)^2 = k^2 + m^2$ leading to $p = \pm\sqrt{E^2 - m^2}$ and $k = \pm\sqrt{(E-V)^2 - m^2}$. In both cases one chooses a + sign in front of the $\sqrt{()}$ to match expected propagation directions as in fig. 1.3.

From the continuity condition at $s = 0$ for $\psi(s)$ and $\frac{d\psi(s)}{ds}$ one gets

$$I + R = T \qquad\qquad pI - pR = kT$$

and $$T = \frac{2p}{p+k}I \qquad\qquad R = \frac{p-k}{p+k}I. \qquad (1.27)$$

Probability currents along s for $s < 0$ and $s \geq 0$ are

$$j_L = \frac{p}{m}(|I|^2 - |R|^2) \qquad\qquad j_R = \frac{k}{m}|T|^2 \qquad (1.28)$$

[25] A proper discussion of interactions will be given in section 1.5.

[26]
$$i\frac{\partial}{\partial t}\Psi = E_p\Psi$$
$$\Psi(t,s) = \psi(s)\exp(-iE_pt)$$
$$s = 0$$

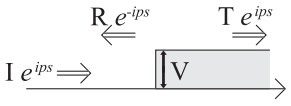

Fig. 1.3 A time-like potential barrier of hight V. Incoming, reflected and transmitted waves are also indicated.

Keeping the energy E fixed, we consider three different regions of the potential strength.

The first region is of a weak potential, $E > V + m$ where k is real and $k < p$. Probability densities in two regions are

$$\rho_L = \frac{E}{m} |\psi_L|^2 > 0 \qquad \rho_R = \frac{E-V}{m} |\psi_R|^2 > 0.$$

This case looks like the non-relativistic one; nothing special, a small fraction of the incoming wave is reflected and the rest is transmitted.

In the second region, the potential is of a moderate strength, $V - m < E < V + m$ and $k = i\sqrt{m^2 - (E-V)^2} = i\kappa$ is pure imaginary (κ real) and

$$R = \frac{p - i\kappa}{p + i\kappa} \qquad \Rightarrow \qquad |R| = |I| \qquad \text{and} \qquad j_L = 0.$$

The incoming wave is totally reflected and the probability density in the barrier shows expected exponential decay

$$\rho_R = \frac{E-V}{m} |\psi_R|^2 = \frac{E-V}{m} \exp(-2\kappa s)$$

as in the non-relativistic case. The situation is however not identical because the increasing potential V changes the sign of the probability density from positive (ρ_R was > 0 in the first case; weak potential) to negative:

$$E > V \qquad \Rightarrow \qquad \rho_R > 0$$

but if

$$E < V \qquad \Rightarrow \qquad \rho_R < 0. \quad (1.29)$$

We will come back to this after discussing the case of the strong potential, $E < V - m$, when k becomes pure real and $k^2 > p^2$. The probability, which became negative when the potential became strong enough in the previous case, stays negative and the probability current becomes real inside the barrier:

$$\rho_R = \frac{E-V}{m} |T|^2 < 0 \qquad \text{and} \qquad j_R = \frac{k}{m} |T|^2 .$$

Consider next, the previously unphysical case of $k < 0$, because, in a counter-intuitive way, this case now corresponds to a particle moving to the right. To see that we will calculate the group velocity

$$\mathcal{V}_g = \frac{\partial E}{\partial k} = \frac{k}{E-V} > 0.$$

A consequence is that T and R given by eqn 1.27 can be arbitrarily large, possibly making the reflected current bigger than the incoming one.[27] The reason for this is that the very strong potential provides enough energy to produce particle-antiparticle pairs. The probability current and the probability density within the barrier are negative because created antiparticles (see eqn 1.25) are attracted to the barrier; moving to

[27] Known as the Klein paradox, another puzzle of RQM - after O. Klein 1894 - 1977

the right ($\mathcal{V}_g > 0$). The created particles are repelled by the barrier, moving to the left, increasing the reflected probability current. Such a situation can be created by focusing light from a high power laser, making a very strong electric field which in turn produces electron-positron pairs from the vacuum. Another example is the Hawking radiation in the neighbourhood of a black hole. The fact that the probability density became negative already in the case of a moderate strength potential, corresponds to vacuum polarization by the creation of virtual particle-antiparticle pairs. They do not affect the probability currents because there is not enough energy in the system to promote them to become real particles. An analogy is the Lamb shift in atomic physics where the vacuum polarizations affects energy levels.

The problem of RQM is now clear: the formalism describes one particle (or a fixed number of particles) but physics needs many particles; the number of which can not be fixed; particles can be created and particles can be annihilated. One needs RQF to describe such physics. RQM can be used, as long as the number of particles is fixed. Using the uncertainty relation $\triangle p \triangle s \sim \hbar$, we see that pair creation which starts at $\triangle p \sim mc$ sets the limit on $\triangle s \sim \hbar/(mc)$. So as long as we are studying physics at a scale bigger than $\hbar/(mc)$, known as the Compton wavelength, RQM can be applied. Atomic physics is an example when this condition is usually fulfilled[28]. But RQM can also be applied for many processes in high energy particle physics. A representative example is electron-positron annihilation producing hadrons; many hadrons. The fundamental process in that case is $\mathrm{e}^+ + \mathrm{e}^- \to \mathrm{q} + \bar{\mathrm{q}}$. The number of particles is 2 and is fixed and the change from 2 leptons to 2 quarks can be handled by RQM. Fragmentation of quarks to hadrons takes place on a different, much slower time scale and therefore can be separated from the fundamental process of the $\mathrm{e}^+ + \mathrm{e}^-$ annihilation.

[28]The Compton wavelength for the electron $\simeq 3.86 \times 10^{-13}$ m.

Is there any limit on $\triangle p$? How well can one measure momentum? In the non-relativistic quantum mechanics, momentum can be measured with any precision but due to the limit speed $< c$,

$$\triangle p \triangle t \sim \frac{\hbar}{c},$$

see the Introduction chapter in (8). Infinite precision $\triangle p \to 0$ requires infinite measurement time $t \to \infty$.

1.4 The Dirac equation

This is the most important section of the chapter on RQM. The Dirac equation provides a relativistically consistent equation describing a massive point-like spin-$\frac{1}{2}$ particle such as the electron and it led to the prediction of the positron - the first antiparticle.

First, we consider different representations of the Dirac equation, the probability current and bilinear covariants. Then we will find the free particle states, examine their properties and introduce chirality and helicity operators. The formalism is then applied to describe simple SM

processes such as $e^+ + e^- \to \mu^+ + \mu^-$ at energies well above the muon rest mass, but well below the Z^0 mass.[29]

[29]The domain of the e^+e^- colliders PETRA (DESY) and PEP (SLAC).

The properties of Dirac particles under the discrete sysmetrries (P, C, T) are then discussed. Electromagnetic interactions are introduced via the so called minimal coupling and the non-relativistic limit is obtained leading to the prediction of $g = 2$ for the magnetic dipole moment of the electron. Finally, there is a brief discussion of the Aharonov-Bohm effect and the pre-eminence of the electromagnetic 4-vector potential in relativistic quantum mechanics.

Before we move on, few words of introduction for readers who skipped section 1.1: Unlike non-relativistic case where electron spin is described by a column of two complex numbers, called Pauli spinor[30], in RQM one needs a pair of two component spinors: ξ^α and $\eta_{\dot\beta}$; called undotted and dotted spinor[31], distinguished by a dot above the index; just names, could be apples and pears as they are different as apples and pears. The reason for different names is that corresponding spinors transform differently under Lorentz transformations (see 1.7 and 1.8). Combining ξ^α and $\eta_{\dot\beta}$ into one 4 component column one gets Dirac spinor Ψ, $\Psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$. Now this Dirac spinor can be transformed using a unitary operator mixing ξ and η and we must follow this different ξ and η spinors all the way through to know how to Lorentz transform resulting Dirac spinor; having just a Dirac spinor, i.e. 4 complex numbers is not enough to know how to apply the Lorentz transformation to them.

[30]Following experimental observations suggesting that the electron has a property called spin, Pauli extended the Schroedinger eqn describing the electron interaction with electromagnetic field by inserting into it a two component spinor and corresponding magnetic dipole moment.

[31]$\alpha = 1, 2$ and $\dot\beta = 1, 2$.

The Dirac eqn 1.12 can be written as

$$\begin{pmatrix} 0 & p_0 + \mathbf{p}\cdot\sigma \\ p_0 - \mathbf{p}\cdot\sigma & 0 \end{pmatrix} \Psi = m\Psi. \tag{1.30}$$

Instead of Ψ, we could use $\Psi' = U\Psi$, where U is a unitary operator. In the new basis, eqn 1.30 would look different. In general, the Dirac equation can be written as

$$(\gamma p - m)\Psi = 0 \tag{1.31}$$

where

$$\gamma p \equiv \gamma^\mu p_\mu = p_0\gamma^0 - \mathbf{p}\cdot\boldsymbol{\gamma} = i\gamma^0\frac{\partial}{\partial t} + i\boldsymbol{\gamma}\cdot\nabla. \tag{1.32}$$

Comparing equations 1.31 and 1.30 one can see that in the representation which was used to get eqn 1.30,

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \boldsymbol{\gamma} = \begin{pmatrix} 0 & -\sigma \\ \sigma & 0 \end{pmatrix}. \tag{1.33}$$

This representation in known as the Weyl or symmetric or chiral representation.[32] Multiplying eqn 1.31 by γp from the left, one gets representation independent constraints on the γ matrices:

[32] In some books, ξ^α and $\eta_{\dot\beta}$ swap places leading to different space-like γ matrices; multiplied by -1.

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}. \tag{1.34}$$

The matrix γ^0 is Hermitian and the matrices γ^i are anti-Hermitian (in

$$(\gamma^0)^2 = 1$$
$$(\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -1$$
$$\gamma' = U\gamma U^\dagger = U\gamma U^{-1}$$

any representation):

$$\gamma^{0\dagger} = \gamma^0, \qquad (\gamma^i)^\dagger = -\gamma^i \tag{1.35}$$

Applying the Hermitian conjugation to eqn 1.31, using properties of the γ matrices given by eqn 1.35 and after some algebra[33] one gets the adjoint Dirac equation

$$\bar{\Psi}(\gamma p + m) = 0 \tag{1.36}$$

where the adjoint spinor is

$$\bar{\Psi} \equiv \Psi^\dagger \gamma^0 \tag{1.37}$$

and p acts on the left.

Multiplying eqn 1.31 by $\bar{\Psi}$ from the left and eqn 1.36 by Ψ from the right and adding the resulting equations, one gets

$$\bar{\Psi}\gamma^\mu \partial_\mu \Psi + (\partial_\mu \bar{\Psi})\gamma^\mu \Psi = \partial_\mu(\bar{\Psi}\gamma^\mu \Psi) = 0,$$

which is the continuity equation, $\partial_\mu j^\mu = 0$, for the probability current 4-vector

$$j^\mu = \bar{\Psi}\gamma^\mu \Psi. \tag{1.38}$$

The probability density

$$\rho \equiv j^0 = \bar{\Psi}\gamma^0 \Psi = \sum_{i=1}^{4} |\Psi_i|^2 \tag{1.39}$$

is a time-like component of the probability current, it is positive definite and it has a similar form to the non-relativistic expression.

Ψ and $\bar{\Psi}$ may be used to form quantities with well-defined space-time transformation properties – known as bilinear covariants. The simplest is the Lorentz scalar $\bar{\Psi}\Psi$.[34] The next simplest is $\bar{\Psi}\gamma^\mu\Psi$ which transforms as a Lorentz 4-vector.

The Hamiltonian H of the Dirac equation is obtained by multiplying eqn 1.31 by γ^0 from the left and separating the time derivative:

$$H\Psi = i\frac{\partial \Psi}{\partial t}, \tag{1.40}$$

where

$$H = \boldsymbol{\alpha} \cdot \mathbf{p} + \beta m \tag{1.41}$$

and

$$\boldsymbol{\alpha} = \gamma^0 \boldsymbol{\gamma}, \qquad \beta = \gamma^0. \tag{1.42}$$

Matrices $\boldsymbol{\alpha}$ and β are Hermitian and in the Weyl representation are given by

$$\boldsymbol{\alpha} = \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1.43}$$

The Weyl representation is very well suited to the ultra-relativistic limit, when the mass can be neglected because the Dirac bispinor is effectively reduced to a single Weyl spinor. In the non-relativistic limit,

[33] Eqn 1.31 is

$$i\gamma^0 \frac{\partial \Psi}{\partial t} + i\gamma^k \frac{\partial \Psi}{\partial x^k} - m\Psi = 0.$$

Applying † one gets

$$i\frac{\partial \Psi^\dagger}{\partial t}\gamma^0 + i\frac{\partial \Psi^\dagger}{\partial x^k}(-\gamma^k) + m\Psi^\dagger = 0$$

and multiplying by γ^0 from the right, using eqn 1.34, gives

$$i\frac{\partial \bar{\Psi}}{\partial t}\gamma^0 + i\frac{\partial \bar{\Psi}}{\partial x^k}\gamma^k + m\bar{\Psi} = 0.$$

[34] The matrix γ^0 which is inside $\bar{\Psi}$ swaps spinors in the bi-spinor such that the upper (undotted) spinor meets the upper one and the lower (undotted) spinor meets the lower one.

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij}$$
$$\beta\alpha + \alpha\beta = 0$$
$$\beta^2 = 1$$

however, both Weyl spinor components of the Dirac bispinor contribute equally so another representation, called the standard or the Dirac representation, is more suitable.[35]

[35] This representation is the most common in textbooks.

The transformation from the Weyl representation to the Dirac representation is effected by the unitary transformation

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

which gives

$$\Psi(Dirac) = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = U\Psi(Weyl) = U \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi + \eta \\ \xi - \eta \end{pmatrix}. \tag{1.44}$$

The transformation of γ matrices; $\gamma(Dirac) = U\gamma(Weyl)U^{-1}$ gives

$$\gamma^0 = \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}, \qquad \boldsymbol{\alpha} = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}.$$

The Dirac equation in the (standard) Dirac representation is then given by:

$$\begin{aligned} E\varphi - \mathbf{p} \cdot \sigma\chi &= m\varphi \\ -E\chi + \mathbf{p} \cdot \sigma\varphi &= m\chi. \end{aligned} \tag{1.45}$$

In the non-relativistic limit, $\chi \to 0$ and the Dirac spinor becomes effectively a two component Pauli spinor.

The fact that in the relativistic theory one needs two Weyl spinors to describe the electron and in the non-relativistic world one Pauli spinor is enough to do the same, is always difficult to accept. To give some insight why, consider yet another representation, that of Foldy and Wouthuysen (FW). We start with the Dirac representation and apply a momentum dependent unitary transformation U_{FW} given by

$$U_{FW} = \exp(\frac{1}{2} \frac{\beta\boldsymbol{\alpha} \cdot \mathbf{p}}{\mid \mathbf{p} \mid} \arctan \frac{\mid \mathbf{p} \mid}{m}).$$

The wave function in the FW representation is then

$$\Psi(FW) = \begin{pmatrix} u \\ w \end{pmatrix} = U_{FW}\Psi(Dirac) = U_{FW} \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$

and after the transformation of the Hamiltonian, eqn 1.41 splits into components becoming

$$\sqrt{\mathbf{p}^2 + m^2}\ u = i\frac{\partial u}{\partial t}, \tag{1.46}$$

$$-\sqrt{\mathbf{p}^2 + m^2}\ w = i\frac{\partial w}{\partial t}. \tag{1.47}$$

Now we have two decoupled eqns for positive and negative energy solutions respectively.

But if we want to drop one of them and consider, say, only the one for the positive energies, then there is a problem because we would not know how to transform the positive energy spinor without any knowledge of the other one.[36] In the non-relativistic limit, however, $\sqrt{\mathbf{p}^2+m^2} \simeq m + \frac{\mathbf{p}^2}{2m}$ which leads to the Schroedinger Hamiltonian and the Lorentz transformation becomes a Galilean one which doesn't affect spin. So, in the non-relativistic limit, we can just take one equation, for example, the one for the positive energy, use it to describe a non-relativistic electron with its spinor u being effectively the Pauli spinor. We can 'forget' about the negative energy solution.

[36] We only know how to transform ξ^α and $\eta_{\dot\beta}$ and they are buried inside the u and w spinors. We would need to get ξ^α and $\eta_{\dot\beta}$ out of u and w, transforming u and w back to ξ^α and $\eta_{\dot\beta}$, do the Lorentz transformation and transform back to get Lorentz transformed u and w.

Majorana particles

This is a short, rather technical detour, from the main track to introduce the concept of the Majorana particle.[37] It is not essential for what follows. A neutrino might be the first fundamental particle of that type. From the discussion in section 1.1.1 of spinors, one could get an impression that a massless spin-$\frac{1}{2}$ particle is described by the Weyl spinor and a massive one by the Dirac spinor which has two independent Weyl spinors as components. Although all particles we know either fit, or could fit, this scenario, this is not the only scenario. One can start with the Weyl spinor, say, an undotted spinor ξ^α, get a dotted one by the complex conjugation and then lower the index with the metric tensor ϵ getting the spinor which transforms like $\eta_{\dot\beta}$. Insert these ξ and η-like objects into the Dirac spinor getting the Dirac equation in the Weyl representation for a massive particle, the Majorana particle, effectively defined by one Weyl spinor; not two independent Weyl spinors as for the electron.

[37] Named after Ettore Majorana 1906 - 1938

1.4.1 Free particle solutions

The Dirac eqn 1.45 in the Dirac (standard) representation can be written as

$$\begin{pmatrix} m & \mathbf{p}\cdot\sigma \\ \mathbf{p}\cdot\sigma & -m \end{pmatrix} \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = E \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$

which for the electron at rest simplifies to

$$\begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix} \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = i\frac{\partial}{\partial t} \begin{pmatrix} \varphi \\ \chi \end{pmatrix}.$$

There are 4 independent, not normalized solutions for energy eigenstates:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \exp(-im\tau), \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \exp(-im\tau) \quad \text{for} \quad E = m$$

and

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \exp(+im\tau), \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \exp(+im\tau) \quad \text{for} \quad E = -m$$

By boosting those solutions to the frame where $\mathbf{p} \neq 0$ one gets

$$u^{(s)} \exp(-i(E_p t - \mathbf{p}\cdot\mathbf{x}) = u^{(s)} \exp(-ip\cdot x) \quad \text{for} \quad E > 0$$

and, for $-E_p$ energy eigenstate and $\mathbf{p}$ momentum eigenstate,

$$u^{(s+2)} \exp(+iE_p t) \exp(i\mathbf{p}\cdot\mathbf{x}) = u^{(s+2)} \exp(+i(+E_p t + \mathbf{p}\cdot\mathbf{x}) \quad \text{for} \quad E < 0$$

where

$$u^{(s)} = N \begin{pmatrix} \vartheta^{(s)} \\ \frac{\sigma\cdot\mathbf{p}}{E_p+m}\vartheta^{(s)} \end{pmatrix}, u^{(s+2)} = N \begin{pmatrix} \frac{-\sigma\cdot\mathbf{p}}{E_p+m}\vartheta^{(s)} \\ \vartheta^{(s)} \end{pmatrix}$$

and N is the normalization constant,

$$E_p = +\sqrt{\mathbf{p}^2 + m^2}, \quad \vartheta^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \vartheta^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ s = 1, 2.$$

One can go back to the Weyl representation, boost ξ, eqn 1.15, and η, eqn 1.16, and return to the Dirac representation.

As in the case of K–G eqn, instead of the above two solutions for $E < 0$ energy eigenstates,[38] we will follow the Feynman–Stueckelberg interpretation of negative energy states as antiparticles which are equivalent to particles travelling backwards in time. This requires us to replace the momentum $\mathbf{p}$ by $-\mathbf{p}$ and to decide which spin state to choose in going from a particle to an anti-particle.

[38] In principle we could carry on with them, as done in a number of textbooks.

Taken together these steps give the four independent solutions of the Dirac equation:

$$\Psi^+(x) = u^{(s)} \exp(-ip\cdot x) \quad \text{and} \quad \Psi^-(x) = v^{(s)} \exp(+ip\cdot x) \tag{1.48}$$

where $u^{(s)}$ is as above and

$$v^{(1)}(\mathbf{p}) = u^{(4)}(-\mathbf{p}) \quad \text{and} \quad v^{(2)}(\mathbf{p}) = u^{(3)}(-\mathbf{p}). \tag{1.49}$$

Both Ψ^+ describing a free electron, and Ψ^- describing a free positron, are two fold degenerate. We need a quantum number (label) to distinguish the pairs of states with the same energy. A suitable operator, commuting with the free particle Hamiltonian, is the helicity operator

$$h(\mathbf{p}) = \begin{pmatrix} \frac{\sigma\cdot\mathbf{p}}{|\mathbf{p}|} & 0 \\ 0 & \frac{\sigma\cdot\mathbf{p}}{|\mathbf{p}|} \end{pmatrix}. \tag{1.50}$$

In a convenient reference frame where $\mathbf{p} = (0, 0, p)$,

$$h(\mathbf{p}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \text{ and, for example, } u^{(1)} = N \begin{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \frac{p}{E_p+m}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}$$

thus $u^{(s)}$ (as well as $v^{(s)}$) are helicity eigenstates,

$$h(\mathbf{p})u^{(1)} = +u^{(1)} \quad \text{and} \quad h(\mathbf{p})u^{(2)} = -u^{(2)}.$$

Helicity eigenvalues, correspond to the spin component along the direction of motion. Helicity + means that, in its rest frame, the electron has $+\frac{1}{2}$ spin projection on the axis parallel to $\mathbf{p}$ (likewise for the helicity – and $-\frac{1}{2}$ spin projection); $\vartheta^{(1)}$ and $\vartheta^{(2)}$ are also defined with respect to this axis, which has to be the z-axis, given the chosen representation of the Pauli matrices.

We can see now, that changing $\mathbf{p}$ to $-\mathbf{p}$ changes the direction of the quantization axis in the rest frame of the electron and the spin projection changes sign as in eqn 1.49 (see the s label).[39]

[39] See (10) for further reading.

As for the K–G eqn, we use covariant normalization requiring that the integral of the probability density over the unit volume gives $2E_p$ particles. This gives the normalization constant as $N = \sqrt{E_p + m}$.

1.4.2 Chiriality ≠ helicity

Chiriality, also called handedness, is defined by a pair of projection operators

$$P_L = \frac{1-\gamma^5}{2} \quad \text{and} \quad P_R = \frac{1+\gamma^5}{2} \quad \text{where} \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{1.51}$$

$P_L + P_R = 1, \qquad P_L P_R = P_R P_L = 0,$
$P_L P_L = P_L,\ P_R P_R = P_R.$

They divide the space of wave functions into right-handed and left-handed half-spaces. P_R projects a wave function onto the right-handed half-space, giving the right-handed component of the wave function. P_L does the same with respect to the left-handed half-space and left-handed component of the wave function[40]. Describing the Dirac spinor in the Dirac representation (ignoring the normalization) in terms of the Weyl spinors ξ and η of the Weyl representation, see eqn 1.44, one gets

[40] $\Psi = P_R\Psi + P_L\Psi = \Psi_R + \Psi_L$.

$$\frac{1-\gamma^5}{2}\begin{pmatrix} \xi+\eta \\ \xi-\eta \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \xi+\eta \\ \xi-\eta \end{pmatrix} = \begin{pmatrix} \eta \\ -\eta \end{pmatrix}$$

and

$$\frac{1+\gamma^5}{2}\begin{pmatrix} \xi+\eta \\ \xi-\eta \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \xi+\eta \\ \xi-\eta \end{pmatrix} = \begin{pmatrix} \xi \\ \xi \end{pmatrix}.$$

Thus, we can define left-handed and right handed spinors as:

$$\eta_L \equiv \eta \quad \text{and} \quad \xi_R \equiv \xi. \tag{1.52}$$

The projection operators P_L and P_R are important in particle physics because weak interactions are sensitive to these components.[41] The W boson couples only to the left-handed part of a particle wave function and Z boson couples to both parts but with different strengths.

[41] P_L and P_R pull the dotted and undotted components out of the Dirac spinor, respectively.

In order to get a better insight, we will consider chiral components of the Dirac spinor u:

$$\frac{1-\gamma^5}{2}u = \frac{1-\gamma^5}{2}\begin{pmatrix} \vartheta \\ \frac{\sigma\cdot\mathbf{p}}{E_p+m}\vartheta \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (1-\frac{\sigma\cdot\mathbf{p}}{E_p+m})\vartheta \\ -(1-\frac{\sigma\cdot\mathbf{p}}{E_p+m})\vartheta \end{pmatrix}$$

where $\vartheta = \vartheta^+ + \vartheta^-$ is a superposition of helicity + (ϑ^+) and – (ϑ^-) eigenstates (not normalized). Because

$$\frac{\sigma \cdot \mathbf{p}}{\mid \mathbf{p} \mid}\vartheta^+ = \vartheta^+ \quad \text{and} \quad \frac{\sigma \cdot \mathbf{p}}{\mid \mathbf{p} \mid}\vartheta^- = -\vartheta^-$$

it follows that

$$\frac{\sigma \cdot \mathbf{p}}{\mid \mathbf{p} \mid}(1 - \frac{\sigma \cdot \mathbf{p}}{\mid \mathbf{p} \mid})\vartheta^+ = 0, \qquad \frac{\sigma \cdot \mathbf{p}}{\mid \mathbf{p} \mid}(1 - \frac{\sigma \cdot \mathbf{p}}{\mid \mathbf{p} \mid})\vartheta^- = -2\vartheta^-,$$
$$\frac{\sigma \cdot \mathbf{p}}{\mid \mathbf{p} \mid}(1 + \frac{\sigma \cdot \mathbf{p}}{\mid \mathbf{p} \mid})\vartheta^+ = 2\vartheta^+, \qquad \frac{\sigma \cdot \mathbf{p}}{\mid \mathbf{p} \mid}(1 + \frac{\sigma \cdot \mathbf{p}}{\mid \mathbf{p} \mid})\vartheta^- = 0.$$

Finally:

$$(1 - \frac{\sigma \cdot \mathbf{p}}{E_p + m})\vartheta = \frac{a}{2}(1 + \frac{\sigma \cdot \mathbf{p}}{\mid \mathbf{p} \mid})\vartheta + \frac{b}{2}(1 - \frac{\sigma \cdot \mathbf{p}}{\mid \mathbf{p} \mid})\vartheta$$

where

$$a = 1 - \frac{\mid \mathbf{p} \mid}{E_p + m}, \qquad b = 1 + \frac{\mid \mathbf{p} \mid}{E_p + m},$$

gives a decomposition of the final state into helicity + and helicity – components with $a/2$ and $b/2$ weights accordingly. In summary,

$$\frac{1 - \gamma^5}{2}u = \frac{a}{2}(\text{helicity}+) + \frac{b}{2}(\text{helicity}-) = u_L; \quad \text{left chiral state}$$

and

$$\frac{1 + \gamma^5}{2}u = \frac{b}{2}(\text{helicity}+) + \frac{a}{2}(\text{helicity}-) = u_R; \quad \text{right chiral state.}$$

It should be clear now that chirality and helicity are two different things. However, in the limit

$$\text{speed} \to c \Longrightarrow a \to 0 \quad \text{and} \quad b \to 2,$$

chiriality and helicity become identical, which is the subject of the next section.

Before that, we will explore consequences, relevant at energies when masses cannot be neglected, of chiriality and helicity being different; for example affecting weak decay rates of particles. As an example, we will consider $\pi^- \to \mu^- + \bar{\nu}_\mu$ decay.

In the rest frame of the π^-, the momenta of μ^- and $\bar{\nu}_\mu$ are back to back and the helicities are as indicated in fig 1.4. The $\bar{\nu}_\mu$ is in the helicity + state[42] with its spin along its momentum. The μ^- has to be in the helicity + state to conserve the total angular momentum as the pion has spin zero. But the W^- couples to the left-handed component of the μ^- wave function and therefore the μ^- needs, simultaneously, to have the helicity + (to conserve the angular momentum) and left-handed chirality (for the W^- to couple). The probability for this is:

$$\frac{\mid a \mid^2}{\mid a \mid^2 + \mid b \mid^2} = \frac{1}{2}(1 - \frac{\text{speed}}{c}).$$

We can see that as the speed $\to c$, the decay rate tends to 0 so the π^- cannot decay to a massless μ^-. This explains why, although favoured by the energy phase-space factor (not considered here), the decay rate for $\pi^- \to e^- + \bar{\nu}_e$ is much smaller than that for $\mu^- + \bar{\nu}_\mu$.[43]

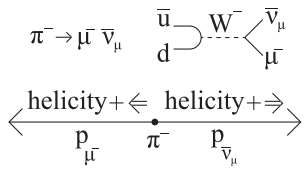

Fig. 1.4 $\pi^- \to \mu^- + \bar{\nu}_\mu$ decay.

[42] We are not going to consider the helicity − state as such a state has never been observed. A neutrino with non-zero mass is either described by a Dirac spinor with two Weyl spinors contributing but with one sterile helicity state or neutrinos are massive Majorana particles having only one helicity state.

[43] See Qxx at the end of the chapter.

1.4.3 Helicity conservation and interactions via currents

It is worth repeating the conclusion of the previous section that chirality (handedness) is different from helicity and this has consequences at low energies where masses cannot be neglected. At high energies, where masses can be neglected, helicity and chirality can be treated as identical and in many textbooks this is the working assumption right from the start.

It can be shown (see for example (9)) that in the probability current

$$j^\mu = \bar{u}\gamma^\mu u = (\bar{u}_L + \bar{u}_R)\gamma^\mu(u_L + u_R) = \bar{u}_L\gamma^\mu u_L + \bar{u}_R\gamma^\mu u_R$$

no cross terms like $\bar{u}_L\gamma^\mu u_R$ are present. In the high energy limit where masses can be neglected

$$\frac{1}{2}(1-\gamma^5)u = u_L \simeq u_L^- \text{ is the helicity } - \text{ eigenstate,}$$

$$\frac{1}{2}(1+\gamma^5)u = u_R \simeq u_R^+ \text{ is the helicity } + \text{ eigenstate}$$

and therefore, in the probability current, there are no helicity mixing terms like $\bar{u}_L^-\gamma^\mu u_R^+$. The significance of that will be discussed in this section.

In classical electromagnetism, two parallel wires with electric currents, interact with each other with a force proportional to the product of the currents. One can try a similar idea to describe scattering of particles. Multiplying the probability current of a free electron by its electric charge will give us a 4-vector with the time-like component representing the charge density and the space-like component representing the number of electric charges crossing unit area per unit time, i.e. the electric current corresponding to moving electrons. If we now take that current and another one, for say a free muon, then we can expect that the dot product of these currents will have something to do with the electromagnetic interaction of these particles. Multiplying that by a propagator[44] (which contains the $g_{\mu\nu}$ tensor for the above dot product) allows for momentum transfer between the interacting particles and indeed gives the matrix element of the lowest order approximation[45] to the scattering amplitude. That matrix element can be visualized by a Feynman diagram, see fig 1.5.

It is the property of the Standard Model (SM) that interactions between any two SM fermions can be described, in a leading approximation, by the current–current interaction as outlined above, although in case of the weak interactions, treating the left-handed and the right-handed parts of it separately because weak interaction bosons couple to them differently. Since in the probability current, there are no helicity mixing terms like $\bar{u}_L^-\gamma^\mu u_R^+$, there are only two fundamental SM vertices as illustrated in fig **??**a, where f stands for a fermion and the wiggly line of the exchanged particle represents either of the SM vector bosons, the photon, W^+, W^-, Z^0 or any of the eight gluons. The helicity is the same

[44] For an introduction to propagators, see Chap xx and for example (9) for greater detail.

[45] "lowest order" refers to a perturbative expansion in powers of the coupling constant - here $\alpha^2/4\pi$.

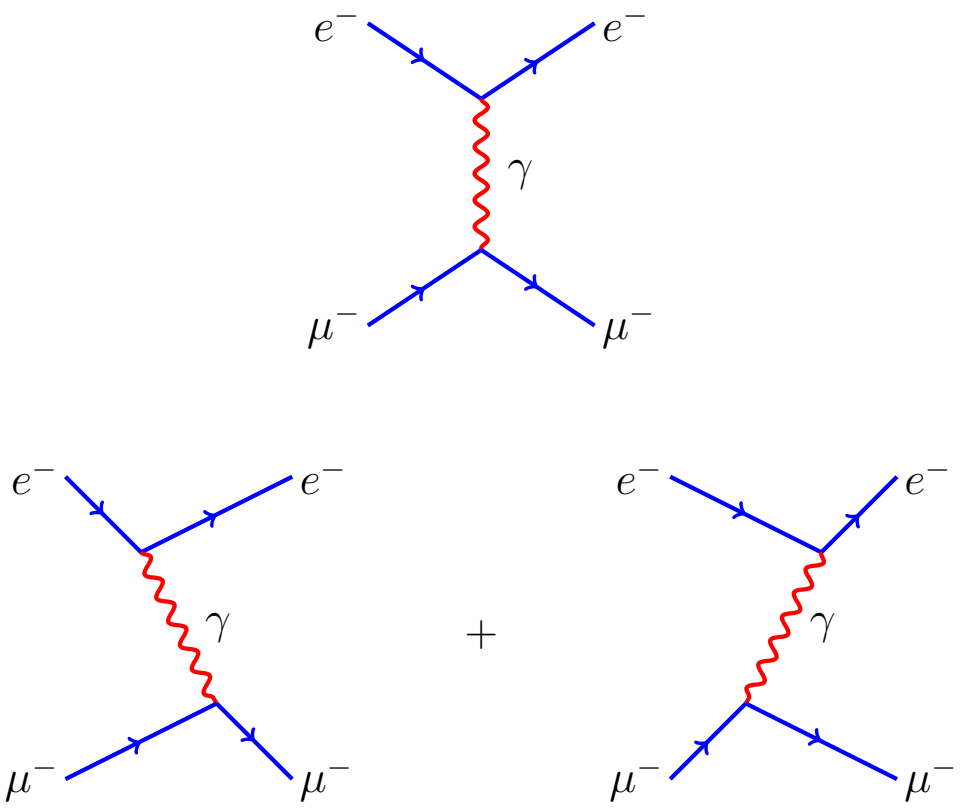

Fig. 1.5 $e^- + \mu^- \to e^- + \mu^-$ Feynman diagram (the top diagram). The top diagram is in fact a sum of two diagrams. The vertical wiggly line representing the exchanged particle propagator is the sum of two scenarios depending on which particle was the emitter and which one was the absorber of the exchanged particle (for example a photon or Z^0) as sketched below the main, top, diagram; time goes from left to right

before and after the scattering (coupling to the exchanged particle). One says that the helicity is conserved, not changed by the interaction.

The annihilation or pair creation vertices are obtained from the scattering ones by *crossing symmetry*, see for example (9) or (10). Keeping in mind that in fig **??** time is going from left to right, we "cross" the incoming particle to the other side of the reaction equation by inverting its 4-momentum and swapping the helicity state, so it travels backwards in time, representing the outgoing anti-particle with the opposite helicity travelling forward in time[46] , fig **??**b:

$$\Psi^+(x) = u^{(1)}(\mathbf{p})\exp(-ip{\cdot}x) \to u^{(4)}(-\mathbf{p})\exp(+ip{\cdot}x) = v^{(1)}(\mathbf{p})\exp(+ip{\cdot}x).$$

Crossing symmetry allows us to get the annihilation amplitude, for example $e^+ + e^- \to \mu^+ + \mu^-$, from the scattering one, here $e^- + \mu^- \to e^- + \mu^-$ by "crossing" the relevant particles to the other side of the reaction equation by changing in the scattering amplitude the corre-

[46] The arrow on the fermion line indicates in which direction with respect to the time arrow the fermion specified by the label at the end of the line is travelling.

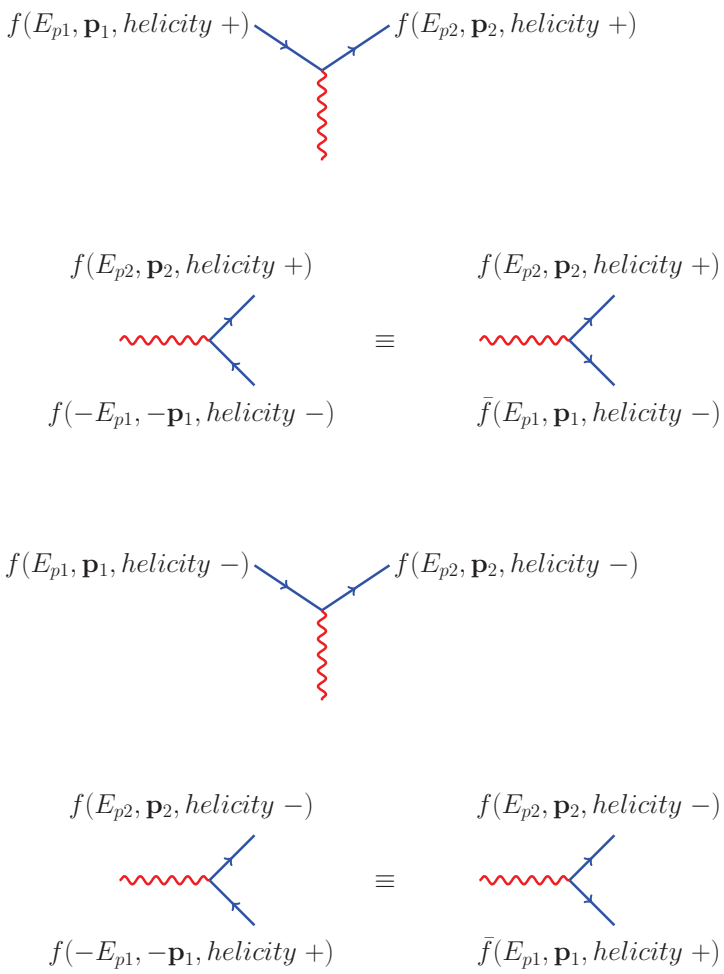

Fig. 1.6 Fundamental vertices of the SM at the high energy limit where masses can be neglected.

sponding 4-momenta $p \to -p$, the helicities and spinors $u \to v$ plus, what is beyond the formalism we are using, putting in "by hand" the minus sign in front of the whole amplitude (QFT is needed for that). This transformation also affects the description of the propagator. In the scattering amplitude, we have, in the example considered , $p1$ and $p2$ 4-momenta for the incoming and outgoing electrons respectively and therefore the 4-momentum of the exchanged boson is the difference $p1 - p2$ which dotted with itself gives $t \equiv (p1 - p2) \cdot (p1 - p2)$; for that reason the scattering is called the t-channel process. By changing $p2 \to -p2$, $t \to s \equiv (p1 + p2) \cdot (p1 + p2)$ which is the square of the centre-of-mass energy; the resulting annihilation reaction is therefore called the s-channel process[47]

[47] There is also a u-channel process, see for example (9) - *also check Ch2 RD* .

1.4.4 P, T and a comment on C

By construction, the Dirac equation is covariant with respect to Lorentz transformations and space inversion. It is also covariant with respect to the time inversion. Space inversion $\mathrm{P}(\mathbf{r} \to -\mathbf{r})$ and time inversion $\mathrm{T}(t \to -t)$ are discussed in this section. We make a comment about charge conjugation C(particle $\to$ antiparticle) at the end of this section.

Consider two observers with reference frames $\mathcal{O}$ and $\mathcal{O}'$. They are describing the same system or physics process, using their coordinates and wave functions, $\Psi(x)$ and $\Psi'(x')$, respectively. The coordinates are related by a linear coordinate transformation $(x^\nu)' = a^\nu_\mu x^\mu$, where a^ν_μ could be the Lorentz transformation or space or time inversions. Correspondingly, gamma matrices in their Dirac eqns are related because of that transformation but, after a lot of algebra, one can show[48] that both observers can neglect differences between their gamma matrices. The covariance of the Dirac eqn requires that the wave functions transform as:[49]

$$\Psi'(x') = \Psi'(ax) = S(a)\Psi(x) = S(a)\Psi(a^{-1}x')$$

where $S(a)$ is a matrix, $S^{-1}(a)$ exists and $S^{-1}(a) = S(a^{-1})$. One says that the coordinate transformation a induces a transformation $S(a)$ in the space of the wave functions; $\Psi'(x') = S(a)\Psi(x)$.

[48] For proofs of all theorems in this section, please refer to (11) for example.

[49] One can also show that if the wave functions transform this way, the Dirac equation is covariant with respect to this coordinate transformation.

$\mathbf{P}(\mathbf{r} \to -\mathbf{r})$

The space inversion transformation is

$$a = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

We want to find $S(a)$ satisfying $\Psi'(x') = S(a)\Psi(x)$. Considering, for example, $\Psi^+(x) = u^{(1)} \exp(-ip\cdot x)$ and following classical physics expectations that $\mathbf{r} \to -\mathbf{r}$ makes $\mathbf{r}' = -\mathbf{r}$, $\mathbf{p}' = -\mathbf{p}$ and $\sigma' = \sigma$, (the angular momentum axial vector), we obtain

$$(\Psi^+)'(x') = \begin{pmatrix} \vartheta^{(1)} \\ \frac{\sigma'\cdot\mathbf{p}'}{E_p+m}\vartheta^{(1)} \end{pmatrix} \exp(-i(E_p t - \mathbf{p}'\cdot\mathbf{x}') =$$

$$= \begin{pmatrix} \vartheta^{(1)} \\ \frac{-\sigma\cdot\mathbf{p}}{E_p+m}\vartheta^{(1)} \end{pmatrix} \exp(-i(E_p t - \mathbf{p}\cdot\mathbf{x}) = \gamma^0\Psi^+(x)$$

giving $S = \gamma^0$, up to a fixed phase, which is set to 1 by a convention. We could get that result immediately by looking at the transformation properties of Weyl spinors, see eqn 1.9, and the transformation from the Weyl to the Dirac representation, eqn 1.44.

Eigenstates of the parity operator $S = \gamma^0$ are states of defined intrinsic parity. Positive energy states in the particle rest frame,

$$u^{(1)} \exp(-im\tau) \quad \text{and} \quad u^{(2)} \exp(-im\tau)$$

are eigenstates of γ^0 with the eigenvalue +1, i.e. they have intrinsic parity +1 but negative energy states

$$v^{(1)}\exp(+im\tau) \quad \text{and} \quad v^{(2)}\exp(+im\tau)$$

are eigenstates with the eigenvalue -1; thus having the intrinsic parity -1. We can state then that the intrinsic parity of spin-$\frac{1}{2}$ particle (defined in its rest frame) is the negative of the intrinsic parity of its antiparticle. This applies to higher spin fermions as well. Free particle states with $\mathbf{p} \neq 0$ are not eigenstates of the parity operator γ^0.

$\mathbf{T}(t \to -t)$

For the time inversion

$$a = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The derivation of $S(a)$ is more complicated than that for the space inversion. We consider only one aspect of it and then give the result. From classical physics we expect that $t \to -t$ makes $\mathbf{r}' = \mathbf{r}$, $\mathbf{p}' = -\mathbf{p}$, $t' = -t$ and therefore

$$\exp(-i(E_p t' - \mathbf{p}' \cdot \mathbf{x}') = \exp(-i(-E_p t + \mathbf{p} \cdot \mathbf{x}) = (\exp(-i(E_p t - \mathbf{p} \cdot \mathbf{x}))^*,$$

so complex conjugation is involved. The exact derivation gives $\Psi'(t') = S(a)\Psi(t) = i\gamma^1\gamma^3\Psi(t)^*$, again up to an arbitrary fixed phase which, by a convention, is taken to be 1.

It is useful to consider how $S(\mathrm{T}) \equiv S(a)$ acts on a free particle state, for example

$$S(\mathrm{T}) \begin{pmatrix} \vartheta^{(1)} \\ \frac{\sigma\cdot\mathbf{p}}{E_p+m}\vartheta^{(1)} \end{pmatrix} \exp(-i(E_p t - \mathbf{p} \cdot \mathbf{x}) = -i \begin{pmatrix} \vartheta^{(2)} \\ \frac{\sigma\cdot\mathbf{p}'}{E_p+m}\vartheta^{(2)} \end{pmatrix} \exp(-i(E_p t' - \mathbf{p}' \cdot \mathbf{x}').$$

Note that $\vartheta^{(1)} \to \vartheta^{(2)}$, but the helicity does not flip as well because the direction of the momentum changes too, so a positive helicity state remains a positive helicity state in the time reversed system.

$S(\mathrm{T})$ is antiunitary, $S^2(\mathrm{T}) = -1$ and this has interesting consequences. If we have interactions which are invariant with respect to the time inversion then $S(\mathrm{T})$ commutes with the Hamiltonian. If Ψ is an eigenstate of the Hamiltonian then $S(\mathrm{T})\Psi$ is also an eigenstate of the Hamiltonian with the same energy. But if

$$S(\mathrm{T})\Psi = \zeta\Psi \quad \text{where } \zeta \text{ is a phase then}$$

$$S(\mathrm{T})S(\mathrm{T})\Psi = S(\mathrm{T})\zeta\Psi = \zeta^*\zeta\Psi = \Psi$$

in contradiction to $S^2(\mathrm{T}) = -1$. For that reason $S(\mathrm{T})\Psi$ is a different state than Ψ; there is at least a two-fold degeneracy, known as the

Kramers degeneracy. A spin-$\frac{1}{2}$ particle has a natural two-fold degeneracy due to the two spin projections ($2j+1$ states). A static magnetic field can lift that degeneracy by coupling to the particle's magnetic dipole moment but a static magnetic field is not invariant with respect to time inversion, so nothing is wrong with having non degenerate states in the magnetic field. The situation is different if the particle is put into a static electric field which is invariant with respect to time inversion. If the particle has an electric dipole moment, the electric field will couple to it and would lift the $2j+1$ degeneracy shifting the energy levels up or down depending on the electric dipole orientation. The Kramers degeneracy forbids this to happen because shifted energy states require to be at least two-fold degenerate. So unless there is an extra degree of freedom to guarantee that, electric dipole moments are forbidden by the Kramers degeneracy. A molecule of water has a large electric dipole moment but in molecular or atomic systems, there are extra energy degenerate (or nearly degenerate) states which allow for that, see for example (12).

In a series of beautiful experiments, pioneered by N.F. Ramsey (13), using a beam of ultra-cold neutrons coming from a nuclear reactor at the ILL in Grenoble, the absolute value of the neutron electron dipole moment d was measured to be smaller than 2.9 10^{-26} e cm at 90% confidence level.[50]

[50] The SM does predict a very tiny violation of time inversion symmetry leading to the prediction of a very tiny d, much smaller than the above limit. A number of extensions of the SM were rejected because they predicted d to be larger than the current limit.

A comment on C

For every particle, there is a corresponding antiparticle (and vice versa). This symmetry of nature is called charge conjugation symmetry. It has nothing to do with Lorentz invariance and is the same in all inertial frames. Charge conjugation leads to a unitary operator C in RQF (all states have positive energies in RQF). In RQM one can construct an antiunitary operator transforming a positive energy state into a negative energy state (u to v spinor). Because it is antiunitary, i.e. different from a common unitary charge conjugation operator of RQF, we will not spend time deriving it to avoid possible confusion.

We will however briefly outline basic properties of the charge conjugation, see for example (14). C changes the sign of the electric charge, magnetic moment, baryon number and lepton number. Dynamical quantities like the energy, momentum and helicity are left unchanged. Except for weak interactions, all the other interactions obey the charge conjugation symmetry. Eigenstates of charge conjugation are neutral particles like the photon, π^0, η, ρ^0 with eigenvalues +1 or -1. For the photon it is -1 and therefore it is +1 for $\pi^0 \to \gamma\gamma$. Charge conjugation symmetry forbids π^0 to decay to three photons. For particles built out of two fermions, $f\bar{f}$, the eigenvalue is $(-1)^{l+S}$, where S is the total spin of the $f\bar{f}$ state and l is the relative orbital angular momentum of the fermions. Thee^+e^- bound state, positronium, decays to two photons if it is in the singlet state ($S=0$, $l=0$) and decays to three photons if it is in the triplet state ($S=1$, $l=0$).

In conclusion, we note that the weak interaction violates the symmetry of each of the three discrete transformations P, T, C, as well as any superposition of any two of them. For all other interactions, each of the three discrete transformations is a symmetry of the interaction. But all known interactions, including the weak interaction, obey CPT symmetry.[51] In RQF it is impossible to construct Lorentz-invariant interactions which would violate the CPT symmetry.

[51] It should be noted that, in RQF, CPT is an anti-unitary operator.

1.4.5 Electromagnetic interactions and the non-relativistic limit

So far we have considered only the free particle Dirac equation. The electron (or indeed any electrically charged spin-$\frac{1}{2}$ structureless fermion) interaction with a classical electromagnetic field is introduced through the so called "minimal coupling", following a prescription of how the electromagnetic interactions are introduced in classical mechanics. The physics of this prescription will be discussed in section 1.5 below.

The classical electromagnetic field is given by a 4-vector potential $A^\mu = (\phi, \mathbf{A})$ and its interaction is introduced into the Dirac equation by modifications of the derivatives: $i\frac{\partial}{\partial t} \to i\frac{\partial}{\partial t} - q\phi$ and $-i\nabla \to -i\nabla - q\mathbf{A}$, where $q < 0$ is the electron's charge. The Dirac equation then becomes

$$(i\frac{\partial}{\partial t} - q\phi)\Psi(\mathbf{r},t) = (\boldsymbol{\alpha}\cdot(-i\nabla - q\mathbf{A}) + \beta m)\Psi(\mathbf{r},t).$$

Solutions of this equation exist for some cases of the electromagnetic field, see for example (15). We will find the non-relativistic limit of the above equation by applying an iterative method of approximations.

At low energies, the mass m of a particle is the main part of the energy, so we will factor that part out of the wave function:

$$\Psi(\mathbf{r},t) = \exp(-imt)\begin{pmatrix}\psi_U(\mathbf{r},t)\\ \psi_L(\mathbf{r},t)\end{pmatrix};$$

$\psi_U(\mathbf{r},t)$ and $\psi_U(\mathbf{r},t)$ are known as the upper and lower components of the Dirac spinor. They contain all "non-rest" energy information relevant for the non-relativistic limit. Inserting this Ψ into the Dirac equation we find:

$$i\frac{\partial\psi_U}{\partial t} = \sigma\cdot(\mathbf{p} - q\mathbf{A})\psi_L + q\phi\psi_U \tag{1.53}$$

and

$$i\frac{\partial\psi_L}{\partial t} = \sigma\cdot(\mathbf{p} - q\mathbf{A})\psi_U + q\phi\psi_L - 2m\psi_L. \tag{1.54}$$

Equation 1.54 may be re-arranged to give:

$$\psi_L = \frac{\sigma\cdot(\mathbf{p} - q\mathbf{A})}{2m}\psi_U - \frac{i\frac{\partial}{\partial t} - q\phi}{2m}\psi_L. \tag{1.55}$$

Because m is relatively large, ψ_L is then small relative to ψ_U. In the 1^{st} order approximation, we neglect the last term in eqn 1.55 and take

$$\psi_L \simeq \psi_{L1} = \frac{\sigma\cdot(\mathbf{p} - q\mathbf{A})}{2m}\psi_U.$$

Inserting this into eqn 1.53 to get the 1^{st} order approximation ψ_{U1} for ψ_U:

$$i\frac{\partial\psi_{U1}}{\partial t} = q\phi\psi_{U1} + \frac{(\sigma\cdot(\mathbf{p}-q\mathbf{A}))(\sigma\cdot(\mathbf{p}-q\mathbf{A}))}{2m}\psi_{U1}.$$

Using vector identities[52] this equation can be written as

$$i\frac{\partial\psi_{U1}}{\partial t} = q\phi\psi_{U1} + \frac{(\mathbf{p}-q\mathbf{A})^2}{2m}\psi_{U1} - \frac{q}{2m}\sigma\cdot(\nabla\times\mathbf{A}+\mathbf{A}\times\nabla)\psi_{U1}$$

and finally as

$$i\frac{\partial\psi_{U1}}{\partial t} = H_P\psi_{U1}$$

where

$$H_P \equiv q\phi + \frac{(\mathbf{p}-q\mathbf{A})^2}{2m} - \frac{q}{2m}\sigma\cdot\mathbf{B}$$

is the Pauli Hamiltonian for the Schroedinger QM and $\mathbf{B}$ is the magnetic field. Since the Hamiltonian of the interaction of a magnetic dipole moment $\boldsymbol{\mu}$ with an external magnetic field $\mathbf{B}$ is $-\boldsymbol{\mu}\cdot\mathbf{B}$, we can identify $\frac{q}{2m}\sigma$ with the electron magnetic dipole moment $\boldsymbol{\mu}$. On the other hand, the electron magnetic moment is related to the electron spin as $\boldsymbol{\mu} = -g\frac{1}{2}\sigma\mu_B$, where g is the proportionality factor and $\mu_B = \frac{\lceil q \rfloor}{2m}$ is the Bohr magneton (q is the electron charge). From that we get that $g = 2$ which was a triumph for Dirac and his equation.

The value of g was known from atomic physics measurements but until the Dirac eqn, there was no explanation why the experimental value was as it was. In fact the value for g, for the electron, as well as for the muon, is not exactly 2. The small difference is explained by RQF. The value of $(g-2)/2$ is measured with fantastic precision: 2.8 10^{-13} for the electron (16), and 6 10^{-10} for the muon (17). As for the precision measurements of the neutron electric dipole moment, in these cases also the precision of the measurements and theoretical calculations give constraints on extensions of the SM.

One should note that the magnetic dipole moment of the proton is quite different from that predicted by the Dirac eqn. The g value is about 5.6 instead of about 2 (using the proper magneton for the proton). For the neutron it is even more surprising, instead of 0 as the electric charge is 0, g is about -3.8. These significant deviations from the predictions of the Dirac eqn, applicable for point-like particles, were the first indications that protons and neutrons are not point-like particles. As we know now, they have a complicated substructures of guarks and gluons.

We can carry on the iterative procedure and get relativistic corrections beyond the Pauli Hamiltonian. The next one is obtained by substituting ψ_{L1} for ψ_L in the last term of eqn 1.55 thus getting the 2^{nd} order approximation:

$$\psi_L \simeq \psi_{L2} = \frac{\sigma\cdot(\mathbf{p}-q\mathbf{A})}{2m}\psi_U - \frac{(i\frac{\partial}{\partial t}-q\phi)(\sigma\cdot(\mathbf{p}-q\mathbf{A}))}{2m}\psi_U.$$

[52]

$$(\sigma\cdot\mathbf{A})(\sigma\cdot\mathbf{A}) = \mathbf{A}\cdot\mathbf{A} + i\sigma\cdot(\mathbf{A}\times\mathbf{A})$$

and

$$(\nabla\times\mathbf{A}+\mathbf{A}\times\nabla)f = f(\nabla\times\mathbf{A}) = f\mathbf{B}$$

Unfortunately, after inserting this into eqn 1.53 one gets a Hamiltonian which is not Hermitian and the electron acquires an imaginary electric dipole moment. Formal problems of this type took many years to solve after Dirac published his equationn. Eventually, a consistent second order Hermitian Hamiltonian was found:

$$\begin{aligned} H &= \frac{(\mathbf{p}-q\mathbf{A})^2}{2m} - \frac{\mathbf{p}^4}{8m^3} \\ &\quad +q\phi - \frac{q}{2m}\sigma\cdot\mathbf{B} \\ &\quad -\frac{iq}{8m^2}\mathbf{p}\cdot\mathbf{E} \\ &\quad -(\frac{iq}{8m^2}\sigma\cdot(\nabla\times\mathbf{E}) + \frac{q}{4m^2}\sigma\cdot(\mathbf{E}\times\mathbf{p})). \end{aligned}$$

The first line in the Hamiltonian expression represents the kinetic term with its first relativistic correction, the third,the Darwin term and the fourth, the spin-orbit interaction where $\mathbf{E}$ is the electric field. For $\mathbf{E} = -\nabla\phi$ and a spherically symmetric potential ϕ, $\sigma\cdot(\nabla\times\mathbf{E}) = 0$, the spin-orbit term takes the familiar: form

$$\begin{aligned} H_{\rm SO} &= -\frac{q}{4m^2}\sigma\cdot(\mathbf{E}\times\mathbf{p}) = \frac{q}{4m^2}\sigma\cdot\frac{\partial\phi}{\partial r}\frac{\mathbf{r}}{r}\times\mathbf{p} \\ &= \frac{q}{4m^2}\frac{1}{r}\frac{\partial\phi}{\partial r}\sigma\cdot\mathbf{l}, \end{aligned}$$

where $\mathbf{l}$ is the orbital angular momentum of the electron. One should note that the Thomas precession is automatically included, as it should be in the relativistic formalism.

Historically, the first formally successful derivation of the non-relativistic limit of the Dirac eqn, which had interactions with the (classical) electromagnetic field included, was obtained via the F–W transformation. It is interesting that one has to introduce the interaction first into the Dirac equation in the Weyl or Dirac representation (or any other representation related to them by a transformation not depending on the momentum) and only then make the F–W transformation. One might think that doing the F–W transformation for the free particle Dirac equation first, thus decoupling the lower and upper components of the Dirac spinor, and then introducing the interactions via the "minimal coupling" would work, but it doesn't.

1.5 Gauge symmetry

By considering transformations of space-time one gets a relativistic description of free particles. In order to describe their interactions, one needs to consider transformations, gauge transformations, in another space, an internal space. Symmetries of the gauge transformations in that internal space are at the heart of the SM. Before discussing gauge symmetries, we need to revise the three essential ingredients of the formalism. Here they are:

1.5.1 Covariant derivative

Consider two coordinates systems as sketched in fig 1.7. The spherical basis vectors are related to the Cartesian ones in the following way:

$$\mathbf{e_r} = \frac{\partial x}{\partial r}\mathbf{e_x} + \frac{\partial y}{\partial r}\mathbf{e_y} = \cos\varphi\ \mathbf{e_x} + \sin\varphi\ \mathbf{e_y},\quad |\ \mathbf{e_r}\ | = 1,$$

$$\mathbf{e}_\varphi = \frac{\partial x}{\partial \varphi}\mathbf{e_x} + \frac{\partial y}{\partial \varphi}\mathbf{e_y} = -r\sin\varphi\ \mathbf{e_x} + r\cos\varphi\ \mathbf{e_y},\quad |\ \mathbf{e}_\varphi\ | = r.$$

Next introduce a constant vector field $\mathbf{a}$ of unit length, as sketched in fig 1.8, using both coordinate systems. In the Cartesian basis

$$\mathbf{a} = 1\ \mathbf{e_x} + 0\ \mathbf{e_y} = a^x\ \mathbf{e_x} + a^y\ \mathbf{e_y}$$

and in the spherical one

$$\mathbf{a} = \cos\varphi\ \mathbf{e_r} - 1/r\sin\varphi\ \mathbf{e}_\varphi = a^r\ \mathbf{e_r} + a^\varphi\ \mathbf{e}_\varphi.$$

Although

$$\frac{\partial a^x}{\partial x} = \frac{\partial a^x}{\partial y} = \frac{\partial a^y}{\partial x} = \frac{\partial a^y}{\partial y} = 0,$$

in the spherical basis

$$\frac{\partial a^r}{\partial \varphi} = -\sin\varphi \neq 0 \quad \text{and} \quad \frac{\partial a^\varphi}{\partial \varphi} = -\frac{1}{r}\cos\varphi \neq 0$$

which looks wrong because the field $\mathbf{a}$ is constant, nothing is changing from a point to point. It is wrong because the differentiation has not taken into account that the spherical basis vectors are changing from one space point to another. If we take this into account, differentiating not only coordinates but also basis vectors, everything is fine. For example

$$\begin{aligned}
\frac{\partial \mathbf{a}}{\partial \varphi} &= \frac{\partial}{\partial \varphi}(\cos\varphi\ \mathbf{e_r} - \frac{1}{r}\sin\varphi\ \mathbf{e}_\varphi) \\
&= \frac{\partial}{\partial \varphi}(\cos\varphi)\ \mathbf{e_r} + \cos\varphi\frac{\partial}{\partial \varphi}(\mathbf{e_r}) + \frac{\partial}{\partial \varphi}(-\frac{1}{r}\sin\varphi)\ \mathbf{e}_\varphi - \frac{1}{r}\sin\varphi\frac{\partial}{\partial \varphi}(\mathbf{e}_\varphi) \\
&= -\sin\varphi\ \mathbf{e_r} + \cos\varphi(\frac{1}{r}\ \mathbf{e}_\varphi) - \frac{1}{r}\cos\varphi\ \mathbf{e}_\varphi - \frac{1}{r}\sin\varphi(-r\ \mathbf{e_r}) = 0.
\end{aligned}$$

So in general, for a vector $\mathbf{V}$,

$$\frac{\partial \mathbf{V}}{\partial x^\beta} = \frac{\partial V^\alpha}{\partial x^\beta}\ \mathbf{e}_\alpha + V^\alpha\frac{\partial \mathbf{e}_\alpha}{\partial x^\beta}.$$

The last derivative, $\frac{\partial \mathbf{e}_\alpha}{\partial x^\beta}$, is a vector and can be described as a linear combination of the basis vectors $\mathbf{e}_\alpha$:

$$\frac{\partial \mathbf{e}_\alpha}{\partial x^\beta} = \Gamma^\mu_{\alpha\beta}\mathbf{e}_\mu.$$

The geometrical object $\Gamma^\mu_{\alpha\beta}$ is called a connection. Changing names of the indices in the above equation, $\mu \to \alpha,\ \alpha \to \mu$, we can write

$$\frac{\partial \mathbf{V}}{\partial x^\beta} = (\frac{\partial V^\alpha}{\partial x^\beta} + V^\mu\Gamma^\alpha_{\mu\beta})\ \mathbf{e}_\alpha.$$

This is the covariant derivative which takes care of the changing coordinates as well as the basis vectors.

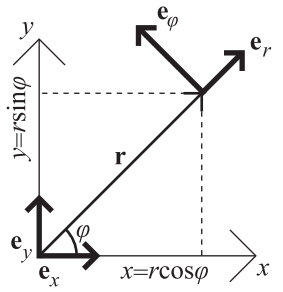

Fig. 1.7 Cartesian and spherical coordinates systems.

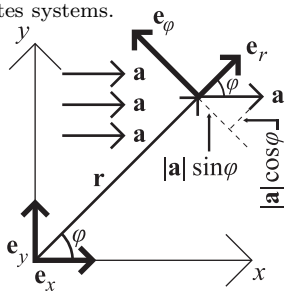

Fig. 1.8 A constant vector field $\mathbf{a}$.

1.5.2 Gauge invariance in electromagnetism

In classical electromagnetism, because $\nabla \cdot \mathbf{B} = 0$, we can introduce a vector potential $\mathbf{A}$, such that the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$. Then

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0 \quad \text{can be written as} \quad \nabla \times (\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}) = 0,$$

allowing the introduction of a scalar potential ϕ, such that

$$\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = -\nabla\phi,$$

rearranging this gives the electric field as

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}.$$

If we now take an arbitrary but differentiable scalar function $\lambda(\mathbf{r}, t)$ and make the transformation

$$\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \nabla\lambda \tag{1.56}$$

then $\mathbf{B}$ remains unchanged. If simultaneously with this transformation 1.56 we change

$$\phi \to \phi' = \phi - \frac{1}{c}\frac{\partial \lambda}{\partial t}, \tag{1.57}$$

then $\mathbf{E}$ will also stay unchanged. Transformations 1.56 and 1.57 are called gauge transformations[53] and the fact that electric and magnetic fields stay the same is called gauge invariance.

[53] In manifestly Lorentz covariant form (and $c = 1$): $A^\mu \to A'^\mu = A^\mu - \partial^\mu \lambda$.

1.5.3 The Aharonov–Bohm effect

Following Feynman et al (18), we will examine two-slit electron diffraction. To study the Aharonov–Bohm effect we use the set-up sketched in fig 1.9. With no current in the solenoid, electrons are diffracted by the slits and form an interference pattern on the screen. As soon as current is flowing through the solenoid, the diffraction pattern changes to another one. The probability amplitude ψ_1 for an electron to follow path 1 and the amplitude ψ_2 for the path 2 are modified in the following way:

$$\psi_1 = \psi_{01}\exp(-iqS_1/\hbar) \quad \text{and} \quad \psi_2 = \psi_{02}\exp(-iqS_2/\hbar),$$

where ψ_{01} and ψ_{02} are the amplitudes without the current, q the electron's charge and S_1 and S_2 are extra phases due to the presence of the current in the solenoid, which are given by

$$S_1 = \frac{q}{\hbar}\int_{\text{path1}} \mathbf{A}\cdot d\mathbf{r} \quad \text{and} \quad S_2 = \frac{q}{\hbar}\int_{\text{path2}} \mathbf{A}\cdot d\mathbf{r}$$

The modification of the phase on the screen is then

$$\frac{q}{\hbar}(S_1 - S_2) = \frac{q}{\hbar}(\int_{\text{path1}} \mathbf{A}\cdot d\mathbf{r} - \frac{q}{\hbar}\int_{\text{path2}} \mathbf{A}\cdot d\mathbf{r}) = \frac{q}{\hbar}\oint_{\text{closed path}} \mathbf{A}\cdot d\mathbf{r},$$

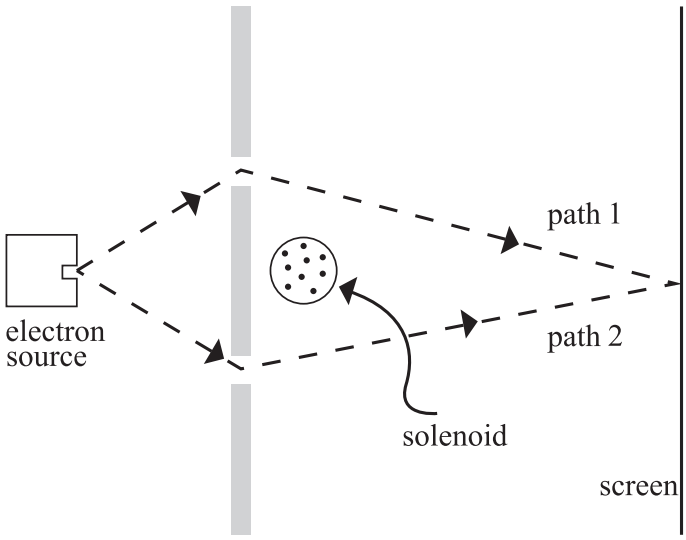

Fig. 1.9 A diagram of a two-slit electron diffraction experiment to demonstrate the Aharonov–Bohm effect; adapted from (18).

which by Stokes' theorem is proportional to the magnetic flux in the solenoid.

For an infinitely long thin (therefore not obstructing the slits) solenoid there is no magnetic field **B** outside the solenoid. The experimentally observed modification of the interference pattern (19) is therefore due to the vector potential **A**, present outside the solenoid, with a magnitude inversely proportional to the distance from the solenoid axis. Classically, the magnetic field **B** and the vector potential **A** are equivalent, in the sense that one can use eitherr. This is not true at the quantum level where **A** is apparently more fundamental. One should note, that the same applies to the scalar potential ϕ and the electric field **E**, where instead of space dimensions one considers time.

The final conclusion, we take from the Aharonov–Bohm effect, is the observation that the vector potential **A** is related to the space dependent phase of the probability amplitude and the scalar potential ϕ is related to the time dependent phase of the probability amplitude. So from the phase of the probability amplitude we can get the 4-vector potential, and the other way around.

1.5.4 Interactions from gauge symmetry

The electromagnetic interactions were introduced into the Dirac equation following the classical "minimal coupling" procedure which required modification of the derivatives:

$$\frac{\partial}{\partial t} \to D^0 \equiv \frac{\partial}{\partial t} + iq\phi \quad \text{and} \quad \nabla \to \mathbf{D} \equiv \nabla - iq\mathbf{A}.$$

which can be written in manifestly Lorentz covariant form as

$$D^\mu \equiv \partial^\mu + iqA^\mu \tag{1.58}$$

The derivative D^μ is the covariant derivative and one can start thinking that the potential ϕ and the vector potential $\mathbf{A}$ are connections in a space to be defined. The outcome is the Dirac equation for the electron interacting with a classical electromagnetic field represented by the 4-vector potential $(\phi, \mathbf{A})$:

$$(i\frac{\partial}{\partial t} - q\phi)\Psi(\mathbf{r},t) = (\boldsymbol{\alpha}\cdot(-i\nabla - q\mathbf{A}) + \beta m)\Psi(\mathbf{r},t). \tag{1.59}$$

We know that the potentials are not unique and we can perform gauge transformations eqn 1.56 and 1.57 without affecting Maxwell's equations and their solutions. Would the same apply to the Dirac eqn 1.59? The answer is negative. A solution of 1.59 will differ from that obtained solving 1.59 after the transformations 1.56 and 1.57. But if, simultaneously with 1.56 and 1.57, we also transform

$$\Psi(\mathbf{r},t) \rightarrow \Psi'(\mathbf{r},t) = \exp(iq\lambda(\mathbf{r},t))\Psi(\mathbf{r},t), \tag{1.60}$$

then the Dirac eqn 1.59 will be covariant with respect to the three combined gauge transformations, 1.56, 1.57, 1.60, and the solution will describe the same physics as that of eqn 1.59 before the transformations. We see that changes to the 4-vector potential affect the space-time dependent phase of the wave function; it needs to be changed accordingly as well. As in the Aharonov–Bohm effect, they are connected.

Now, the crucial step. We reverse the flow of arguments. Demand first that we want interactions which are invariant with respect to the transformation 1.60. For that to happen, we need to modify the derivatives of the free particle Dirac equation in such a way that the symmetry is obeyed. This requires a move from space-time derivatives to covariant derivatives and the introduction of the 4-vector potential $(\phi, \mathbf{A})$. So demanding the gauge symmetry with respect to the gauge transformation 1.60 we get the classical electromagnetic field with which our particle is interacting. If in addition, in all formulae, for example the one for the probability current, we replace derivatives by the covariant derivatives, they will also be form invariant.

The last step is to introduce the internal space with its basis vectors on which the connection (the 4-vector potential) operates and extend the formalism to other interactions beyond electromagnetism[54].

One can imagine that a particle, moving in space-time, is carrying its internal space with it as sketched in fig 1.10. In mathematical language, the internal space is called a fibre and the whole structure which locally is a product of the fibre and space-time, is called a fibre bundle. In the case of electromagnetism, the fibre is a circle, as sketched in fig 1.11 . Each point on the circle, parametrised by a real function $\lambda(\mathbf{r},t)$, corresponds to a complex number with modulus 1, i.e. a one-dimensional unitary matrix $\exp(iq\lambda(\mathbf{r},t))$. All such matrices form a group called U(1), 1 for

[54]The formalism will be only outlined here. For more details see (20).

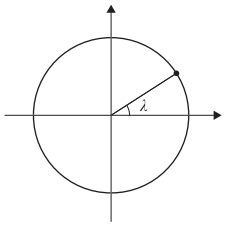

Fig. 1.11 The internal space of electromagnetism; adapted from (20).

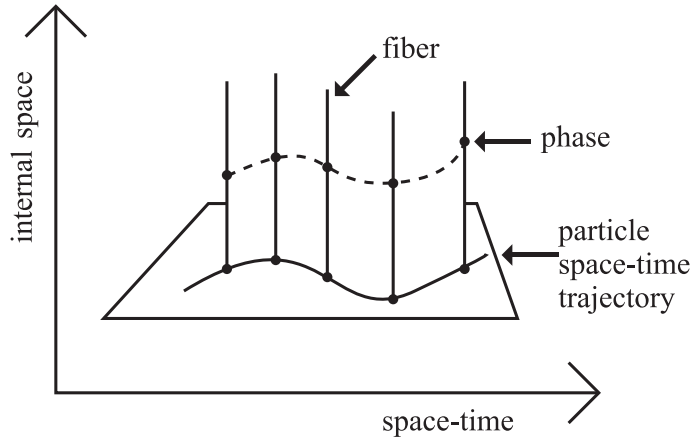

Fig. 1.10 A particle travelling in space-time and its internal space; adapted from (20).

one-dimensionality. When the particle moves from one space-time point to another, the phase of its wave function is modified (eqn 1.60) and the 4-vector potential $(\phi, \mathbf{A})$ is modified as well (eqns 1.57 and 1.56). A basis in the space U(1), consists of one particular matrix, for example the unit matrix (in this case the real number 1) which corresponds to $\lambda = 0$.

When the particle moves in space-time, $\lambda(\mathbf{r}, t)$ changes corresponding to the changing basis.[55] The change of the basis is represented by the connection which in turn gives us the change of the 4-vector potential $(\phi, \mathbf{A})$, eqns 1.57 and 1.56. So our "differential" equation is: the connection equals the 4-vector potential.

[55] With respect to four coordinates requiring four derivatives: one time and three space directions.

We are ready now to extend the formalism to include other interactions. What will happen if in eqn 1.60, instead of the real function $\lambda(\mathbf{r}, t)$ we insert a matrix $\mathcal{M}(\mathbf{r}, t)$? Suppose $\mathcal{M}$ belongs to SU(2), the group of complex unitary matrices 2×2 with the unit determinant (the letter S $\equiv$ special, stands for that). SU(2) has 3 basis vectors, which can be the Pauli matrices, σ_x, σ_y, σ_z. Each element of the group can then be defined by 3 real numbers, λ^1, λ^2, λ^3, and can be represented as a point within a sphere of radius 2π in 3 dimensions. So our particle carries with itself such a sphere, its internal space, and the interaction we get this way, would be the weak interaction. Generalizing eqn 1.60, we require gauge symmetry with respect to the following transformation:

$\exp \mathcal{M} \equiv 1 + \mathcal{M} + \frac{1}{2}\mathcal{M}^2 + ...$

$$\Psi(\mathbf{r}, t) \rightarrow \Psi'(\mathbf{r}, t) = \exp(iq \sum_{k=1}^{3} \lambda^k(\mathbf{r}, t)\sigma_k)\Psi(\mathbf{r}, t). \tag{1.61}$$

The operator acting on the wave function Ψ is now a series of 2×2 matrices and therefore our wave function Ψ gets one extra dimension and is represented by two components (for historical reasons called projections of the isotopic spin) related to the SU(2) internal space. In order

to get the 4-vector potential of the weak interaction, one considers the infinitesimal change of the wave function when the space-time point is changed from $(\mathbf{r}, t)$ to $(\mathbf{r} + d\mathbf{r}, t + dt)$. All components of the wave function are affected but to get the 4-vector potential one selects only the connection part of the covariant derivative, i.e. the change of the basis vectors in the internal space. That change is represented by the change of λ^1, λ^2, λ^3, giving finally:

$$A^\mu = \sum_{k=1}^{3} (\partial^\mu \lambda^k(\mathbf{r}, t))\sigma_k. \tag{1.62}$$

So, if we want to introduce the weak interactions into the free particle Dirac eqn, we change space-time derivatives to covariant derivatives, including the connection, i.e. the 4-vector potential given by eqn 1.62.[56]

[56]The 4-vector potential is a 2×2 matrix in the isotopic spin space generated by the Pauli matrices.

The strong interaction is introduced in a similar way. The internal space is now SU(3), a group of unitary complex matrices with unit determinant. There are 8 basis vectors in that space and therefore 8 real numbers identify each matrix belonging to the group. The Pauli matrices generating the weak interactions are replaced by those 8 basis matrices of the SU(3) group and the wave function gains extra degrees of freedom, becoming a three dimensional column vector in the colour space of the strong interaction. The corresponding 4-vector potential becomes a 3×3 matrix. Changing derivatives to covariant derivatives causes the particle to interact with the classical colour field.

After quantization of the classical weak and strong fields we have three spin-1 bosons for the weak interaction and 8 spin-1 gluons for the strong interaction. All of them are massless at this stage. To get massive physical Z^0, W^+ and W^- bosons, one needs an extra mechanism, like the Higgs mechanism which will be discussed later in this book. Elements of U(1) commute with each other but those of SU(2) or SU(3) do not commute with each other. The physical consequences are that photons do not interact with each other but weak interactions bosons and strong interactions gluons do interact with each other. The SM vertices representing their interactions are sketched in fig 1.12.

For further reading, (10) is particularly recommended.

Chapter summary

- Special Relativity, Lorentz transformation, invariance and covariance
- Klein-Gordan and Dirac equations, covariant derivative
- Introduction of electromagnetic interactions, spin-oribit interaction

(1) Using the definitions given in Section 1.3, derive the continuity equation (1.3) and show that it may also be written in the covariant form $\partial_\mu j^\mu = 0$.

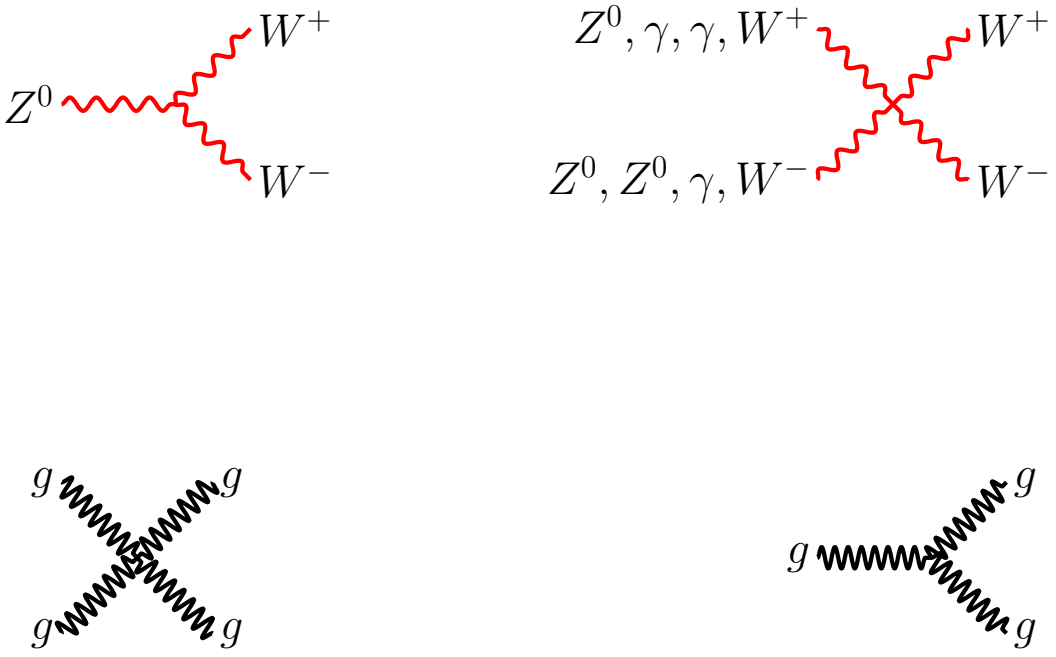

Fig. 1.12 SM vertices of gauge bosons interactions with each other

(2) Follow Dirac's derivation of 'a relativistic equation for the electron'. Start from $(p^2 - m2)\psi = 0$, where $p^2 = p_0^2 - \mathbf{p}^2$ and $c = 1$. Factorise the 4-momentum operator as $(p_0 + \sqrt{\mathbf{p}^2 + m^2})(p_0 - \sqrt{\mathbf{p}^2 + m^2})$ and apply only the second bracket to give $(p_0 - \sqrt{\mathbf{p}^2 + m^2})\psi = 0$). The difficulty is that quantum mechanics gives operator expressions for p_0 and p_i, which in the Schroedinger representation are first order derivaties in time and space respectively. The spatial derivates are now under a square root. Dirac solved the problem by writing:

$$\sqrt{\mathbf{p}^2 + m^2} = \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \beta \quad \text{where}$$
$$p_r = -i\hbar\partial/\partial x_r, \quad r = 1, 2, 3 \quad \text{and} \quad p_0 = i\hbar\partial/\partial t.$$

Show that to satisfy $p^2 = m^2$ requires:

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij} \quad i, j = 1, 2, 3 \quad \text{and}$$
$$\alpha_i\beta + \beta\alpha_i = 0 \quad i = 1, 3 \quad \text{with} \quad \beta^2 = m^2.$$

Dirac showed that the simplest realisation of these constraints are given by 4-dimensional matrices.

(3) The covariant Dirac matrices are defined by $\gamma^0 = \beta$, $\gamma^i = \beta\alpha_i$, so that $\gamma \cdot p$ is a 4-vector scalar product - often written as $\not{p}$ and the Dirac equation as $(\not{p} - m)\psi = 0$. Using the definitions of the Dirac

matrices given in Section 1.4 follow the text to derive the defiining relation:

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}. \tag{1.63}$$

(4) Using the information given at the end of Section 1.4.2 on helicity conservation in $\pi^\pm$ decays, estimate the relative decay rates for $e\nu_e$ and $\mu\nu_\mu$ modes. (The two-body phase factor is discussed in Chapter 2).

(5) Work through the derivation of the interaction of a point-like spin-$\frac{1}{2}$ charged particle with an electromagnetic field (given in Sec. 1.5.4).

(6) Given $\Psi(x) = u(p)\exp(-\imath p\cdot x)$ is a solution of the Dirac equation, where $\bar{u}(p)u(p) = 2m$, derive expressions for $u(p)$ with $p = (E, 0, 0, p_z)$. Here, p is the 4-momentum, $p\cdot p = m^2$, and x is the 4-displacement.
What are the $u(p)$ spinors in the ultra-relativistic limit? What are their helicities? Why is the Dirac representation more suitable in the low energy limit?

(7) A free particle solution of the Dirac equation is $\Psi(x) = C(p)\exp(-\imath p\cdot x)$, where $p = (E, 0, 0, p_3)$. Find the normalized $E > 0$ spinors $C_+(p)$ and $C_-(p)$ for positive and negative helicity states in the standard Dirac representation and using covariant normalization.
The operator for spin projection on the x-axis is

$$\Sigma_1 = \frac{1}{2}\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}.$$

Show that neither $C_+(p)$ nor $C_-(p)$ is an eigenstate of Σ_1.
The expectation value of Σ_1 is

$$\langle\Sigma_1\rangle = \frac{1}{2E}C^\dagger(p)\Sigma_1 C(p),$$

where $C = C_+(p)\cos\alpha + C_-(p)\sin\alpha$, with α a real constant, and normalization is to unit volume. Calculate $\langle\Sigma_1\rangle$ and interpret the result in the non-relativistic and the ultra-relativistic limits for $\alpha = \pm\pi/4$.

(8) Give the expression for the total energy operator H, including the electromagnetic potential $(A^0, \mathbf{A})$ for a particle of mass m.
Introducing the upper and lower bi-spinor components (φ, χ) of ψ and writing $\mathrm{H} = \mathrm{m}+\mathrm{H_n}$, derive an expression for $\mathrm{H_n}\psi$. Using $\mathrm{H_n}\psi$, show that under certain conditions to be specified, $\chi \sim \frac{\text{velocity}}{\text{c}} \times \varphi$, for a stationary state $\mathrm{H}\psi = E\psi$.
Under these conditions, show how the Dirac equation reduces to the non-relativistic Schrödinger-Pauli equation for a spin-$\frac{1}{2}$ particle with energy $E_n = E - m$. Identify the term giving the interaction of the particle's spin with the magnetic field $\mathbf{B} = \nabla\times\mathbf{A}$ and comment on its magnitude.
[The identity $(\sigma\cdot\mathbf{a})(\sigma\cdot\mathbf{b}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{i}\sigma\cdot\mathbf{a}\times\mathbf{b}$ may be assumed.]

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g-2 electron

g-2 muon

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