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René Descartes' Foundations of Analytic Geometry and Classification of Curves

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Abstract

Descartes' *La Géométrie* of 1637 laid the foundation for analytic geometry with all its applications. This essay investigates whether the classification of curves presented in *La Géométrie*, into geometrical and mechanical curves based on their construction as well as into classes based on their equations, limited the further development of analytic geometry as a field. It also looks into why Descartes' further classification was algebraic rather than geometrical; and how it was criticized and why. In order to answer these questions, the essay touches on the historical background to Descartes' works, provides an overview and analysis of the ideas put forward by Descartes, and describes the development of analytic geometry in the 150 years following the publication of *La Géométrie*.

Content

1. Introduction

Most of our modern mathematics is possible due to the use of functions and curves that can be visualized in a coordinate plane. The concept of a function and of curves as described by a relationship between two or more variables, is a relatively recent invention in mathematical history. In 1637, René Descartes (1596-1650), a French mercenary, mathematician and philosopher, published a book called *La Géométrie* as an appendix to his great *Discours de la Methode pour bien conduire la raison, et chercher la verité dans les sciences*. In *La Géométrie*, he set out to create a new, all-encompassing field of mathematics, where the then separate fields of ”true mathematics”, geometry and algebra, were linked together and used in symbiosis[1]. It was groundbreaking in the sense that it provided the entire mathematical community with a new set of tools: a way of solving algebraic equations using geometry; and a way of describing geometrical problems in algebraic terms, thus making it possible to manipulate and solve them[2]. First presented in 1637, these tools were developed and scrutinized throughout the century until the new discoveries eventually led to the creation of the calculus in 1666 and 1684[3]. Today, calculus and coordinate geometry have an immense number of applications and are used in as separate fields as airplane coordination, astrophysics, farming and engineering. But despite its great impact, everything postulated in *La Géométrie* was far from indisputable.

One must note that independently and at roughly the same time, Pierre de Fermat presented his own version of analytic geometry, stating in a short treatise that ”[w]henever in a final equation two unknown quantities are found, we have a locus, the extremity of one of these describing a line, straight or curved”[4]. Due to less exposure (it was only present in manuscript form until 1679) and a less modern algebra, Fermat's theories did not get as much exposure as *La Géométrie*, leaving Descartes to set the main foundation of analytic geometry as a field of study. In contrast to Fermat's, Descartes' algebra was much like the one used today. He was the first to systematically use letters near the end of the alphabet to represent unknowns; to use letters from the beginning of the alphabet to represent parameters or coefficients; and to denote exponents as a^n etc. The only exception was a^2, which was written as *aa* to avoid typographical errors[5]. Descartes' symbolic algebra differed from that of, for example, François Viète, in that he saw all quantities of a polynomial expression as one-dimensional lines, thus eliminating the need to keep the homogeneity of an expression[6]. In the expression x^3+ax+c, a would traditionally be interpreted as an area and c as a volume in order to keep the whole expression in the same dimension. Descartes argued that, because of the ratio $1 : x = x : x^2 = x^2 : x^3$, expressions such as $x^2y^3 - y$ could be considered without any inherent contradictions since terms of higher powers (dimensions) can always be divided by the unit (1) to make them of a lower power (dimension)[7]. This step away from the classical idea of homogenous, and more reality-bound expressions, simplified working with terms of different powers and made it possible to use equations of higher degrees than three for geometrical problem solving. The notation in *La Géometrie* is very similar to our modern notation, and was a prerequisite for the simplicity of the concepts and methods Descartes presented[8].

During the development of his new mathematical structure, Descartes opened it up for a vast array of new curves to be used in geometrical problem solving. The need then arose to classify these new curves as geometrically acceptable or unacceptable. The traditional geometric curves had been known since Antiquity and could be constructed using straight-edge rulers and compasses; the new geometric curves (today called algebraic curves) were according to Descartes all those that could be constructed by the

[1] Sasaki, 2003, p. 3
[2] Sjöberg, 1996, p. 112f
[3] Lund, 2002, p. 48: By Sir Isaac Newton and Gottfried Wilhelm Leibniz respectively.
[4] Boyer, 2004, p. 75
[5] Suzuki, 2002, p. 347
[6] Sjöberg, 1996, p. 115
[7] op.cit, pp. 114-115
[8] Sasaki, 2003, p. 107

intersection of existing curves or could be traced using continuous motions of a known relation, thus fulfilling the criterion of geometric exactness. The third category consisted of the mechanical curves, which could not be constructed in this manner[9]. In order to make his method of problem solving mathematically legitimate, Descartes could only consider the geometric curves acceptable for it. A large part of *La Géométrie* is therefore devoted to the exact construction and further classification of these curves. Descartes' basic distinction between ”geometric” and ”mechanical” curves was based on earlier works dating as far back as 300 BC, specifically Euclid's *Elements* and Pappus' *Collection* (approximately 300 AD), as well as on works by the foremost geometer in the beginning of the 17th century, Christopher Clavius[10]. His further classification of geometric curves into classes, based on their algebraic equations, was however entirely new and has been much discussed.

This essay aims to investigate what limitations Descartes created for his new field of mathematics because of this classification of curves, how it has been criticized and why, and how the development following Descartes´ works might have changed had all curves been deemed acceptable for his new method of problem solving. This will be done through an examination of the mathematical background to Descartes' works; an overview of the three books of *La Géométrie*; a section concerning his classification of curves and the need for this; a section on what results in analytic geometry came from mathematicians following Descartes; and a discussion on whether or not those developments could have been different or made quicker had all curves been accepted for problem solving. The quotations from *La Géométrie* are translated from the 1886 edition published by Hermann for Librairie Scientifique, with some inspiration from the translation by Smith and Latham[11] and that found in Bos' *On the Representation of Curves in Descartes' Géométrie*[12].

2. Mathematical context

2i. The Mathematical Background to Descartes' Works

René Descartes was born in 1596 at La Haye, the son of a wealthy family. Of a frail constitution, the young Descartes was allowed to stay in bed until late in the morning, time that he used for contemplation and meditation and that is thought to have become “the source of the most important philosophical results that his mind produced”[13]. After receiving a law degree at the University of Poitiers in 1616, Descartes spent a year in France before travelling to Holland and enlisting at the military school in Breda[14]. During this time, in October 1618, he met with Isaac Beeckman, who would influence and inspire him to engage in the study of natural philosophy through mathematics. As Sasaki states: “in 1619 Descartes began to confess that his senior friend truly awoke his theretofore slumbering interest and stimulated him into expressing his own program for reorganizing the entire discipline of mathematics”[15]. That was the beginning of Descartes' quest to combine the mathematical branches of geometry and algebra into one “Vera Mathesis”, a “True art of Mathematics”[16]; a quest that would culminate in the publication of *La Géometrie* in 1637.

[9] Sasaki, 2003, p. 71 and *La Géométrie*, p . 16
[10] Sasaki, p. 71
[11] Smith, David Eugene & Latham, Marcia L. (1954). *The geometry of René Descartes*.
[12] Bos, Henk (1981). *On the Representation of Curves in Descartes' Géométrie*., published in Archive for History of Exact Sciences, Volume 24, Issue 4, pp. 295-338.
[13] Cottingham, 1992, p.24
[14] http://www-history.mcs.st-and.ac.uk/Biographies/Descartes.html, 2013-04-26
[15]Sasaki, 2003, p. 99
[16] Descartes stated in his *Rules for the Direction of the Mind*, Rule IV, that ”This discipline should contain the primary rudiments of human reason and extend to the discovery of truths in any field whatever”. Quoted in Rabouin, 2010, p.432

Descartes' primary education took place at the Jesuit College La Flèche, most probably between the years 1607 and 1615[17]. It was there that his basic mathematical and philosophical education was received. As he stated in his *Rules of the Direction of the Mind (Regulae ad directionem ingenii,* ca 1628*)*:

> "When I first applied my mind to the mathematical disciplines, I at once read most of the customary lore which mathematical writers pass on to us. I paid special attention to arithmetic and geometry […] But in neither subject did I come across writers who fully satisfied me."[18]

However, little focus was on mathematics as a science at La Flèche. The mathematical studies were rather thought of as a tool for further theological and philosophical reasoning[19]. Nonetheless, a certain amount of mathematics was considered essential, in large part due to the presence of the most influential mathematician in Jesuit education at the time, Christopher Clavius. A professor of mathematics at the Collegio Romano from 1563, he was sometimes called "The Euclid of the 16th century". Clavius translated Euclid's *Elements* in 1574 and published his own textbook *Algebra* in 1608[20]. *Algebra* took inspiration from Diophantus, also summarizing the rapid advances that had been made in the 16th century by for example Niccoló Tartaglia and Rafael Bombelli[21]. Descartes claimed in a conversation with mathematician John Pell that before 1616, "he had no other instructor for Algebra than ye reading of Clavy Algebra"[22]. For geometry, however, he not only read Clavius' *Elements* but also works of Apollonius, Diophantus and Archimedes[23], and it is unlikely that he did not read any other books on algebra after finishing his studies at La Flèche.

Clavius' *Algebra* was typical for the 16th century in style and despite being published ten years after Francois Viète's *In Artem Analyticam Isagoge*, it was crude in comparison. Since Descartes seemingly followed more in the footsteps of Viète than Clavius with his new notation and use of algebra, many have come to the conclusion that Descartes must have been influenced by Viète's works[24]. Descartes himself, however, stated in 1639 that he had never read Viète's works prior to the publication of La Géometrie: **"**Je n'ai aucune connaissance de ce geomètre dont vous m'écrivez [...], et je m'étonne qu'il dit, que nous avons etudié ensemble Viète á Paris; car c'est un livre dont je ne me souviens pas avoir seulement jamais vu la couverture, pendant que j'ai été en France."[25]. According to Mahoney this is supported by the development of Descartes' thoughts that can be traced in his *Rules for the Direction of the Mind*[26] but the matter is still under discussion.

Although Descartes in general did not admit to having been influenced by anyone for his new genre of mathematics, it is possible that some inspiration for finding the "vera mathesis" may have trickled down to Descartes during his time at La Flèche from the correspondence between Clavius and van Roomen, who at the time was interested in finding a "mathesis universalis"[27]. It wasn't until roughly one year after Descartes' meeting with Beeckman in 1618, however, that he really set out to find the link between algebra and geometry. While having joined the Bavarian army in battle, Descartes is said to have dreamt of how he could create a philosophy that would base all knowledge on such a solid ground so that no one could doubt that it was true[28]. The starting point for all knowledge was to Descartes the famous statement "Je pense, donc je suis" ("I think, therefore I am"). From this statement all knowledge can be

[17] Sasaki, 2003, p. 85
[18] Cottingham et al, 1985, p. 17
[19] Sasaki, 2003, pp. 31, 59
[20] http://www-history.mcs.st-and.ac.uk/Biographies/Clavius.html, 2013-04-22
[21] Sasaki, 2003, p. 74
[22] Quoted in op.cit. p. 47
[23] op.cit. pp. 45, 70
[24] See for example Katz, V., 2008, p. 436f
[25] Descartes to Mersenne, 20.II.1639, Alquié.II.126: quoted in Mahoney, 1994, p. 278f
[26] Mahoney, 1994, p. 278
[27] Sasaki, 2003, p. 83
[28] Cottingham, 1992, p. 30f.

built by reasoning, based on self-evident axioms, as it is done in mathematics since Antinquity. After having travelled Europe for nine more years[29], Descartes settled down in Holland to start writing about this new Method for finding knowledge. In 1637 he published *Discours de la Méthode pour Bien Conduire la Raison et Chercher la Verité dans les Sciences*, together with the three appendices *La Dioptrique, Les Météores* and *La Géométrie*. While *Discours de la Méthode* described the method of finding true knowledge, the three appendices were meant to show the applications of it. In Descartes' own words: "I have tried in my Dioptrique and my Météores to show that my Méthode is better than the vulgar, and in my Géométrie to have demonstrated it"[30].

La Géométrie was thus meant both as a proof of the general applicability of the new method of finding true knowledge that Descartes had devised, and as an introduction to his way of combining "[t]he logic of the schools, the geometrical analysis of the ancients, and the algebra of the moderns"[31] to solve all types of geometrical and algebraic problems. The first coherent idea for Descartes' analytical combination of geometry and algebra is thought to have been devised when he was presented in 1631[32] with the four-line problem of Pappus. Much effort was expended in the 16th century to recover and work through Ancient mathematical works. Among them was the lost book VII of Pappus' *Mathematical Collection*, which dealt mainly with geometrical analysis[33]. This specific problem from Book VII of the *Collection*, later only known as "The Pappus Problem", became the corner stone of *La Géométrie* and is solved not only for four but also for *n* lines in Books I and II. A detailed explanation of this specific problem, Descartes' solution of it, and its implications, follows in section 2ii.

Apart from during the famous feud with Pierre de Fermat from 1637 to 1638[34], Descartes spent the remainder of his life after 1637 focusing not on mathematics but on philosophy. The further development of Cartesian geometry was instead conducted by for example the Dutch mathematicians surrounding Frans van Schooten. This, and the works of mathematicians such as Barrow, Wallis and Toriccelli, eventually led to the invention of calculus by Sir Isaac Newton and Gottfried Wilhelm Leibniz in 1666 and 1684 respectively[35]. The importance of *La Géométrie* can thus not be overestimated, but hereafter follows a discussion of its mathematical contents and which possibilities and limitations it provided for the further study of curves and their equations.

2ii. Overview of *La Géométrie*

For the purpose of this essay, it is necessary to include a short summary of *La Géométrie* (hereafter LG), focusing on books I and II to explain Descartes' way of thinking. As Bos states in *The Structure of Descartes's Géométrie*, Book I explains on the "technical" level how Descartes provided "an 'analysis', that is, a universal method of finding the constructions for any problem that could occur within the tradition of geometrical problem solving"[36], using algebra. Book II deals with the "methodology" of Descartes' programme, discussing the vital question of construction and which curves could be used for construction[37]. It had been known since Antiquity that not all problems could be constructed using a ruler and a compass, but in order to validate his new findings Descartes had to define which other

[29] op.cit. p. 35
[30] http://www-history.mcs.st-and.ac.uk/Biographies/Descartes.html, 2013-04-26
[31] Sasaki, 2003, p. 63
[32] Cottingham, 1992, p. 38. Bos (2001, p.283) describes the Pappus problem as the "crucial catalyst" of Descartes work. Rabouin (2010, p.457) argues that it was when presented with this problem that "Descartes went back to his 1629 project (on the classification of geometrical problems in analogy with arithmetical ones) and merged it with that of the *Regulae* (to treat all problems as equations and to use geometrical calculus to solve them)".
[33] Mahoney, 1994, p. 74
[34] See Mahoney, 1994, *The Mathematical Career of Pierre de Fermat.*
[35] Lund, 2002, p. 48
[36] Bos, 1991, p.43
[37] Bos, 1991, pp. 43, 47

possible methods of construction were acceptable and gave exact answers. Book III, while still dealing with the "methodology", focuses more on applications of the method and finding the simplest possible curve for constructing solutions[38].

Book I, entitled "Problems that can be constructed using only circles and straight lines", gives the outline of Descartes' new method. With the opening words "All geometrical problems can easily be reduced to such terms that one need only know the lengths of a number of straight lines to construct them"[39], he moves on to describe how arithmetical operations can be constructed geometrically using a straight edge ruler and a compass. For example, as is illustrated in figure 2.1, BD * BC = BE.

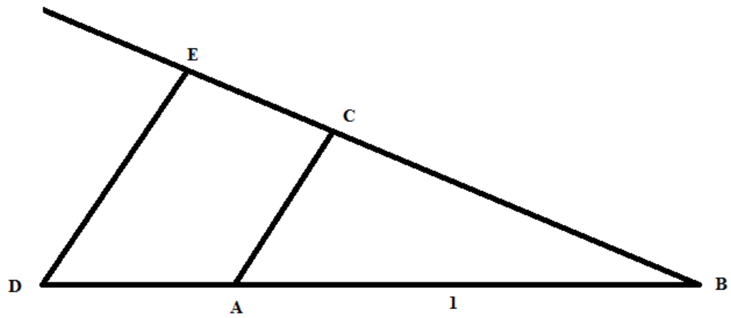

Fig 2.1: Multiplication with geometrical construction

DE is parallel to AC, which makes ΔABC similar to ΔDBE. Thus, 1 is to AC as $BD = 1 + AD$ is to DE.

If $1 : AC = (1 + AD) : DE$, then $DE = (1 + AD) \times AC$. Also, $BE : DE = BC : AC$. In that case,

$$BE = \frac{BC \times DE}{AC} = \frac{BC \times (1 + AD) \times AC}{AC} = BC \times (1 + AD) = BC \times BD.$$

Similarly, addition, subtraction, division and root taking are also demonstrated. Descartes does, however, state that "often one need not trace these lines on paper, and it is instead sufficient to name them by letters, each one a different letter"[40]. This method reoccurs when he deals with problem solving, since "When wanting to solve a problem, one must first consider it done, and name all the lines that appear necessary for its construction, also those that are unknown to the others"[41]. He skirts the problem of homogeneity in algebraic expressions, faced by his predecessors as well as by Viète and Fermat, by viewing all terms in an expression as simple lines, manipulated into squares, cubes, or likewise. All terms can thus, to Descartes, either be divided by or multiplied by a (given but arbitrary) unit a number of times to attain the dimension one would seek. Simply explained, in the expression $b^3 - a^2$, either b can be divided by the unit once to become two-dimensional (a square), or a can be multiplied by the unit once to become three-dimensional (a cube).

Descartes' second step for geometrical problem solving is to express the lines in terms of each other until there are two expressions for the same line. These can be equated to produce an equation in terms of one or two unknown. In *La Géométrie,* the expressions are found by setting one line to x and another to y, without them having to be perpendicular, and expressing all other lines in terms of this x and y. In Descartes' own words, "one can always reduce in this fashion, all the unknown quantities to a single

[38] op. cit. p. 47
[39] LG, p. 1
[40] LG, p. 2: "*souvent on n'a pas besoin de tracer ainsi les lignes sur le papier, et il suffit de les désigner par quelques lettres, chacune par une seule".*
[41] LG, p. 3: "*voulant résoudre quelque problème, on doit d'abord le considerer comme déjà fait, et donner des noms à toutes les lignes qui semblent nécessaire pour le construire, aussi bien à celles qui sont inconnues qu'aux autres".*

one, after which the problem can be constructed using circles and straight lines, or conic sections, or by some other line which is no more than one or two degrees more complex"[42]. It is by this method that Pappus' four line loci problem was solved and the foundations for Descartes' geometrical algebra were laid out.

The Pappus' problem, originally dealt with by Apollonius, is, as stated on page 9 in *La Géométrie*:

"For 3, 4 or more straight lines in given positions; First one finds a point from which one can draw as many straight lines, each intersecting a given line at a given angle, and that the rectangle made of two of these that are drawn from the same point, are of a given proportion to the square of the third, if there are but three lines; or to the rectangle of the other two, if there are four lines; or, if there are five lines, that the parallepiped made of three have the given proportion to the two who remain and a given line [...] and so on for any given number of lines."

Visualizing this problem geometrically with four lines we get:

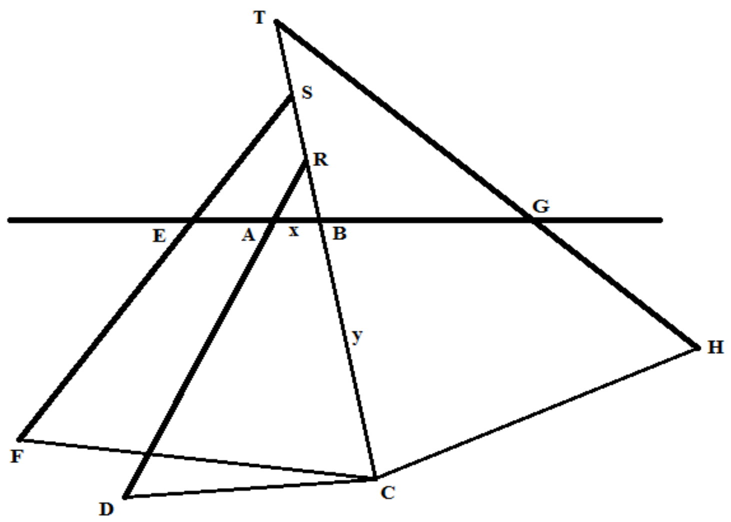

Figure 2.2: Pappus' four line problem

C is the point we are searching for, from which four straight lines can be drawn to D, F, T and H, intersecting *TH, EG, RD* and *SF* at certain given angles. By the second prerequisite, $CF \times CH$ relates to $CT \times CD$ by a given ratio. Now, "[b]ecause there are always an infinite amount of unique points that can satisfy the given conditions, one must [...] know and trace the curve in which all the points appear"[43].

Descartes sets $AB = x$, $BC = y$, $EA = k$, $AG = l$ and then expresses all the lines in terms of x and y, using the fact that all angles, in all the triangles in the figure, are given. Because of this, we would for example know the ratio between $AB = x$ and BR; we name it $z : b$. Thus, $x : BR = z : b$ and $BR = \frac{bx}{z}$.

CR then becomes $y + \frac{bx}{z}$, or $y - \frac{bx}{z}$, or $-y + \frac{bx}{z}$, depending on where C is positioned compared to B and R[44]. Since the same method applies either way, only the first expression will be used. In the same

[42] LG, p.4: "*on peut toujours réduire ainsi toutes les quantités inconnues à une seule, lorsque le problem se peut construire par des cercles et des lignes droites, ou aussi par des sections coniques, ou meme par quelque autre ligne qui ne soit que d'un ou deux degrès plus compose".*

[43] LG, p. 9

manner, Descartes sets $CR : CD = z : c$ and because $CR = y + \frac{bx}{z}$, $CD = \frac{cy}{z} + \frac{bcx}{z^2}$. Furthermore, setting the ratios $BE : BS = z : d$ and $CS : CF = z : e$ gives that $CF = \frac{ezy + dek + dex}{z^2}$.

$BG : BT = z : f$ gives that $CT = \frac{zy + fl - fx}{z}$. And $CT : CH = z : g$ gives that

$CH = \frac{gzy + fgl - fgx}{z^2}$.

All four lines are now expressed in terms of x, y and other known quantities. Setting $CF \times CT = CT \times CD$, we get an algebraic equation in which we can make x or y have infinitely many different values (magnitudes), and then find the corresponding values (magnitudes) of y or x. Thus, there are infinitely many points C for any given set of angles set in the problem[45].

The infinite amount of points form a locus, which can and must be traced, according to Descartes. When the number of lines does not exceed five, and the lines do not intersect at right angles, he argues that the locus will be described by a quadratic equation and can thus be found using a ruler and a compass. When the number of lines is between five (intersecting at right angles) and nine lines, the equation becomes either a cubic or a quartic equation and can thus be found using conic sections. For problems with nine lines intersecting at right angles, or up to thirteen lines intersecting at non-right angles, the answer is an equation of the fifth or sixth degree which can be constructed with curves more complex than the conic sections[46]. In order to maintain the validity of the answer as a geometrical solution, Book II is devoted to explaining how to construct these curves in a geometrically acceptable way.

In **Book II**, "On the nature of curved lines", Descartes expands the number of loci from a more or less known and explored set of curved lines, containing for example the parabola, the hyperbola, the circle, the ellipse, the cissoid and the conchoid, to a much larger, theretofore undiscovered, set. Setting the standard for more than a centure, Descartes defined as "geometric" all curved lines that can be traced using a continuous motion, for example by using the compass in figure 2.3, or by several successive motions where each motion is completely determined by those which precede it[47].

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[44] If B falls between C and R, CR becomes $y + \frac{bx}{z}$. If R falls between C and B, $CR = y - \frac{bx}{z}$ and if C falls between B and R, $CR = -y + \frac{bx}{z}$. Since Descartes considered negative solutions "false", the equation of $-y - \frac{bx}{z}$, which would generate only negative values, was not mentioned.

[45] LG, pp. 10-14

[46] LG, p. 14

[47] LG, p. 16: "prenant comme on fait pour géométrique ce qui est prècis et exact […] on n'en doit pas plutôt exclure les lignes les plus composes que les plus simples, pourvu qu'on les puisse imaginer être décrites par un movement continu, ou par plusieurs qui s'entre-suivent, et don't les derniers soient entièrement réglés par ceux qui les precedent; car par ce moyen on peut toujours avoir une connaissance exacte de leur mesure".

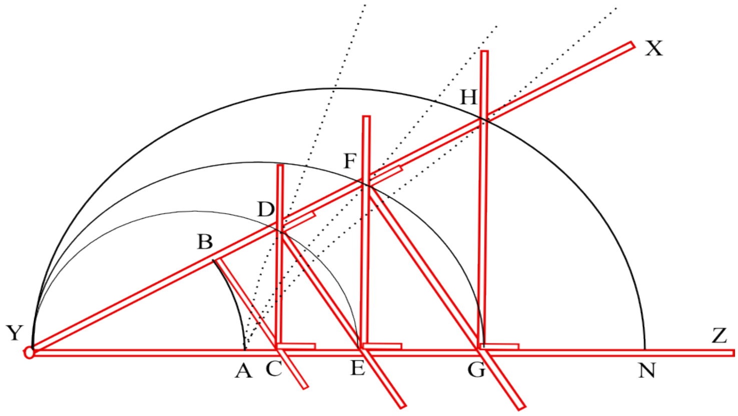


Figure 2.3: The Mesolabe compass

This new definition of a "geometrical" curve drastically increased the amount of curves available to mathematicians both for problem solving and for the general study of the properties of curved lines. Staying true to his geometrical approach to this new branch of mathematics, Descartes only found those curves acceptable which could be traced exactly by the aforesaid means, making it possible to find any given point on the curve. He also stated, rather as a by-product, that "[a]ll points of those [curves] one can call geometrical, that is that fall under some precise and exact measure, must have some relation to all the points of a straight line, which can be expressed by some equation"[48]. In contrast to Fermat, whose version of analytic geometry started from already existing equations which were solved using geometrical visualizations, Descartes always started with a curve and then derived its equation if necessary. As a result, he dealt with much more complex and, in some ways, more general curves and equations.

In order to systematize and structure this new set of curves, Descartes stepped away from the classical idea of curves being but "planar" and "solid" (geometrical) or "more complex" (mechanical) and started classifying them by the degrees of their equations instead. Due to the ever increasing complexity of the curves, he took a step away from his geometrical starting point to base this classification on the degree of their algebraic equations. Equations of the second degree, of "the square of one unknown" or the "rectangle of two unknowns" he named as "of the first and most simple class"[49]. This class contained the circle, the parabola, the hyperbola and the ellipse. Equations of the third and the fourth degree were grouped into the second class, of the fifth and the sixth degree into the third class, and so on. Descartes justified this pairing of equations of different powers with the existence of a "general rule for reducing to third degree all difficulties of the square of the square, and to the fifth degree all those of the sixth degree, in such a way that one can hardly rate them as more complex"[50]. This has been much discussed and questioned, a debate which will be further examined in Section 3.

To further explain the method presented in Book I, Descartes demonstrates it using several examples, such as for example a slightly different version of the Pappus problem. While constructing the problem geometrically, Descartes argues that the lines' positions can be manipulated to create additions, subtractions or zero lengths at will, thus making the resultant curve easier or more difficult to trace and changing the roots from real (positive) to false (negative) or vice versa. Less focused on those finer

[48] LG, p. 18: " tous les points de celles qu'on peut nommer géométriques, c'est-à-dire qui tombent sous quelque mesure précise et exact, ont nécessairement quelque rapport à tous les points d'une ligne droite, qui peut être exprimée par quelque equation, en tous par une meme"
[49] LG, p. 18: "le premier genre"
[50] LG, p. 20: "la raison est qu'il y a règle générale pour réduire au cube toutes les difficulties qui vont au carré de carré, et au sursolide toutes celles qui vont au carré de cube; de façon qu'on ne les doit point estimer plus composées."

details but a good example of the general method is his solution to the special case of Pappus five line problem, where all lines meet at right angles:

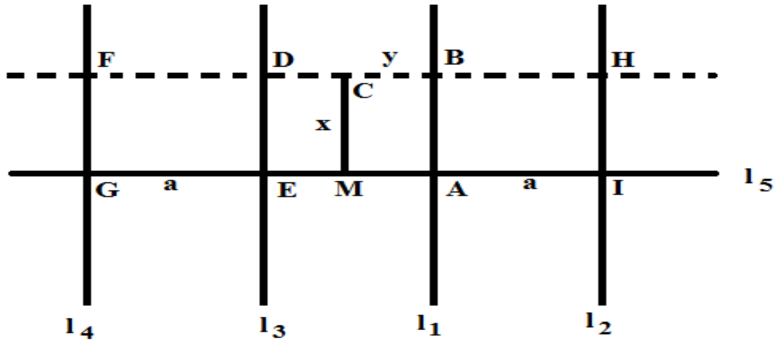


Fig 2.4: Pappus five line problem with all lines perpendicular

The problem itself is the same as in Book I, so we are looking for a point C so that for *CB, CF, CD, CH* perpendicular to l_1 and *CM* perpendicular to l_5, $CF \times CD \times CH = CB \times CM \times AI$, where *AI* is a given magnitude.

Set *CB* = *y*, *CM* = *x*, *AI* = *AE* = *EG* = *a*. Then *CF* = *2a – y*, *CD* = *a – y* and *CH* = *a + y*. Multiplied together they become

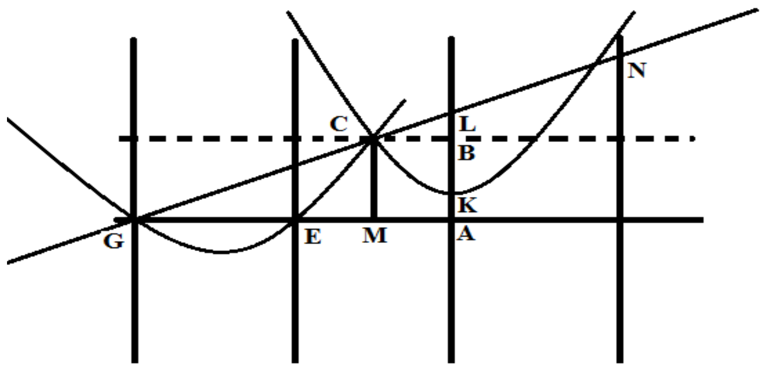
$$(2a - y)(a - y)(a + y) = (2a - y)(a^2 - y^2) = y^3 - 2ay^2 - a^2y + 2a^3$$

$CB \times CM \times AI = axy$ so since $CF \times CD \times CH = CB \times CM \times AI$,

$$y^3 - 2ay^2 - a^2y + 2a^3 = axy .$$

To find the curve which traces all possible points C, Descartes draws the curve GEC seen in figure 2.5:

Fig 2.5: The point C can be taken from anywhere on the curve GEC



GEC is described by the intersection of the parabola *CKN* which moves so that its diameter *KL* is always on the straight line *AB*, and the ruler *GL* that at the same time rotates around *G*, always passing through the point *L* on the parabola:

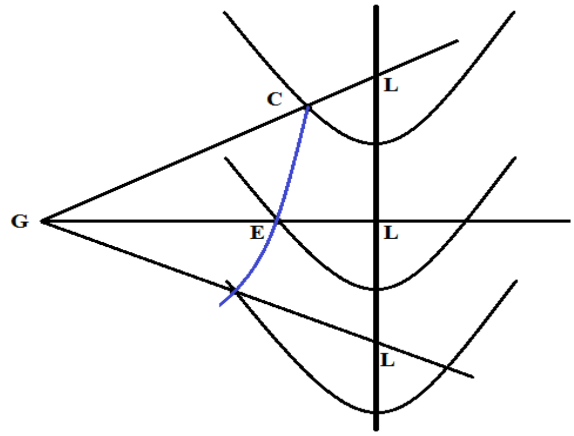


Figure 2.6: Tracing GEC through the intersection of parabola CKN, moving vertically along AB, and the straight line GC, rotating around G and always intersecting the parabola at the point L.

Set $KL = a$ and the *principal diameter* $= AB = a$[51]. As of before, $GA = 2a$, $CB = MA = y$ and $CM = AB = x$. *GMC* and *CBL* are equable because all lines cross at right angles, which makes $BL : MC = CB : GM$.

Knowing that $GM = 2a - y$, this becomes

$BL : y = x : 2a - y$

i.e. $BL = \dfrac{xy}{2a - y}$

$KL = a$ gives that

$$BK = a - BL = a - \frac{xy}{2a - y} = \frac{2a^2 - ay - xy}{2a - y}$$

Since *BK* is a segment of the diameter, it is to *BC*, to which it is linked by the parabolic properties, as *BC* is to the principal diameter, set to *a*. $BK : BC = BC : a$ gives that

$$\frac{2a^2 - ay - xy}{2a - y} : y = y : a$$

$$\frac{2a^2 - ay - xy}{2a - y} = \frac{y^2}{a}$$

$$2a^3 - a^2y - axy = 2ay^2 - y^3$$

[51] A parabola is, according to the 1911 Encyclopaedia Britannica, defined either as a section of a right circular cone by a plane parallel to a tangent plane to the cone, or as the locus of a point which moves so that its distances from a fixed point [the locus] and a fixed line [the directrix] are equal. The principal diameter is the line that passes from the directrix through the vertex of the parabola. Since the distance is set to *a*, one can assume that Descartes limited it, in this case, to the line AB and no further. http://en.wikisource.org/wiki/1911_Encyclop%C3%A6dia_Britannica/Parabola, 2013-06-19

$$y^3 - 2ay^2 - a^2 y + 2a^3 = axy$$

i.e. *C* is the desired point and can be found anywhere on the curve *CEG*[52].

Different angles set in the question will provide a different curve as the answer. Descartes does not stop to describe more examples of how to find these curves, since "having explained the way of finding an infinite amount of points through which they pass, I think I have given an adequate method of describing them"[53]. Instead he moves on to distinguish between "this method of finding several points for constructing a curved line, and that which one uses for the spiral and its likes"[54], curves which he did not consider geometrically acceptable. More shall be said later on this subject, but Descartes general reservation, quite true at the time but soon after disproved, was that "the relation between straight and curved lines is not, and, I think, cannot be known to man "[55] and any curve depending on this relation could thus not be considered exact. He did, however, accept constructions using string as long as these were kept strictly stretched throughout the process[56].

The last sections of Book II concern Descartes' famous method of finding tangents and normals to curves; finding ovals that can be used in optics; and how to apply the methods described in two dimensions also in three dimensional space. Despite the vast use and immense importance of the Cartesian "method of normals" in geometrical problem solving and the development of calculus, these last sections are more examples on applications of the method already outlined than a development of Descartes' arguments, and are thus not relevant to this essay.

The examples given in **Book III** serve to show the standard constructions for curves with equations up to the sixth degree since the reader "already knows how, when one searches for the quantities required to construct these problems, one can always reduce them to an equation with a power no higher than the sixth or the fifth"[57]. Much emphasis is put on finding the "simplest" curve to use in the construction of a geometrical problem, simplicity having been a criterion for true geometers since the days of Pappus[58]. In *La Géométrie*, simplicity entails that "a curve [is] simpler in so far as the degree of its equation is lower"[59]. This definition of simplicity contrasts with the generally geometrical approach of *La Géométrie* and will be discussed in section 3. In order to make the equations and the corresponding curves as simple as possible, Descartes then provides the reader with a number of ways to reduce the degree of equations to the lowest possible. He does not proceed to give an exhaustive instruction on how to solve problems of even higher degrees, leaving it instead to the reader to "continue in the same fashion"[60] to construct curves of ever increasing complexity. The suggested ease by which this can be done has been duly questioned, but it is outside the scope of this essay to discuss the matter.

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[52] *CEG* is one of the arms of the so called Cartesian parabola, a third-degree curve that became an important tool for the construction of curves of higher order. See [61] for a full description of the curve with both arms of the parabola.

[53] LG, p.31: "Pour les lignes qui servent aux autres cas, ja ne m'arrêterai point à les distinguer par espèces, car je n'ai pas entrepris de dire tout; et, ayant expliqué la façon de trouver une infinité de points par où elles passent, je pense avoir assez donné le moyen de les décrire".

[54] LG, p.31: "cette façon de trouver plusieurs points pour tracer une ligne courbe, et celle dont on se sert pour la spirale et ses semblables".

[55] LG, p.32: "la proportion qui est entre les droites et les courbes n'étant pas connue, et même, je crois, ne le pouvant être par les hommes, on ne pourroit rien conclure de là qui fût exact et assure".

[56] LG, p.32: "à cause qu'on ne se sert de cordes en ces constructions que pour determiner des lignes droites don't on connoit parfaitement la longueur, cela ne doit faire qu'on les rejette".

[57] LG, p.80: "Vous savez déjà comment, lorsqu'on cherche les quantités qui sont requises pour la construction de ces problems, on les peut toujours réduire à quelque équation qui ne monte que jusques au carré de cube ou au sursolide".

[58] Using a more complex curve or method than necessary was generally considered a mathematical "erreur" or "faute"

[59] Bos, 1991, p.50

[60] LG, p.87

3. On Descartes Classification of Curves

3i. The classification itself

Descartes never used a curve's equation to define it, as is done today. Instead, he defined it by the method with which it could be accurately constructed. In order to do this, he first had to define which methods of construction were "accurate" or "exact". He did state, as mentioned, that "[a]ll points on those [curves] one can call geometrical, i.e. that fall under some precise and exact measure, must have some relation to all the points of a straight line, which can be expressed by some equation"[61] and with the de-geometrization of geometrical problem solving, mathematicians eventually saw no point with actually constructing the curves which the equations described[62]. For Descartes, however, it was an absolute necessity. The relationship that Descartes described between a curve and its equation was thus the foundation on which mathematicians built the modern concept of a function, but his use of curves and equations was far in thought and idea from the algebraic geometry we know today.

By the time of *La Géométrie*'s publication, all curves used for mathematical study had names with which they could be referred to. Such was the case of for example the ellipse, the parabola, the conchoid and the Quadratrix (see figure 3.1), to name a few. Each of these curves had been thoroughly examined since Antiquity and their properties were rather well known by the start of the 17th century. Traditionally, these curves had between grouped together as either planar curves, possible to construct using ruler and compass; solid curves, the conic sections found when slicing a cone at different angles or curves made by the intersections of the said conics; or the more complex (often called "linear") curves that required other methods of construction, using curves of higher degree for its construction. Book II of *La Géométrie* begins with Descartes' ruminations as to why "out of this, [the ancients] did not distinguish between different degrees" of these linear curves, and why they called the planar and solid curves "geometrical" while other means of construction were considered less exact, resulting in "mechanical" curves[63]. "For to say that it was the need to use some sort of machine to describe them, one must for the same reason reject the circles and the straight lines, since one only writes it on paper using a compass and ruler, which one might also refer to as machines"[64]. Instead of thus ruling out many curves as geometrically inexact, Descartes set his own definition of what would be a sufficiently exact method of construction:

> "It seems very clear to me that if we take that which is precise and exact and call it geometrical, and name mechanical that which it is not […] one can no easier exclude the more complex curves than the most simple ones, given that one can imagine them described by *a continuous motion*, or by *several motions that follow on each other, the last of which are completely regulated by those which precede.* For in this way one can always have exact knowledge of their measure."[65]

This new definition formed the basis for Descartes' further reasoning. It excluded curves such as the Quadratrix and the Archimaedean spiral from the set of geometrical curves since in 1637, the relationship between curved and straight lines was not known and curves depending on the relation between a rotating and a linear motion could thus not be exactly known. As Mancosu points out, "not

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[61] LG, p.18. See [44]

[62] See Bos, 2001, *Redefining Geometrical Exactness.*

[63] LG, p.15: "Les anciens ont fort bien remarqué qu'entre les problèmes de géometrie, les uns sont plans, les autres solides et les autres linéaires […] Mais je m'étonne de ce qu'ils n'ont point outré cele distingue divers degrès entre ces lignes plus composées, et je ne saurois comprendre pourquoi ils les ont nommés mécaniques plutôt que géométriques".

[64] LG, p.15: "Car de dire que c'ait été à cause qu'il est besoin de se servir de quelque machine pour les décrire, il faudroit rejeter par même raison les cercles et les lignes droites, vu qu'on ne les décrit sur le papier qu'avec un compas et une règle, qu'on peut aussi nommer des machines".

[65] LG, p.16. See [43]. Emphasis not in the original.

any point on [these curves] can be found at pleasure". Instead, only "special points can be constructed"[66], using for example the then known ways to divide an angle.

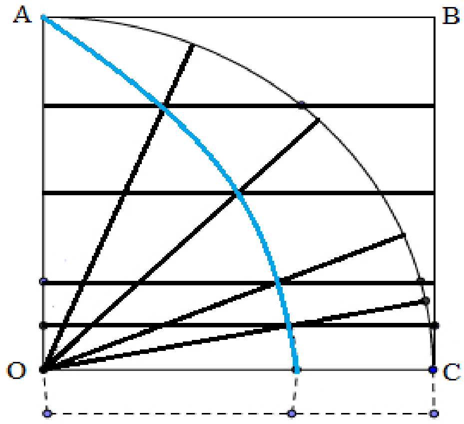


Figure 3.1: Quadratrix, constructed by the intersection of line OA, the radius of quarter circle OAC, rotating around O at a constant rate, and line AB, simultaneously moving vertically towards OC.

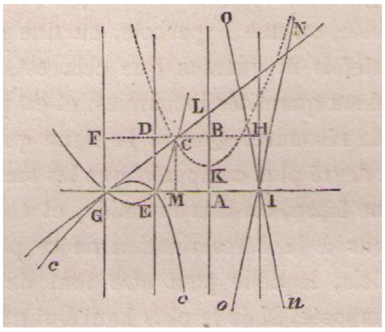
Since not all points on these curves, that we now call transcendental, could be found, Descartes referred to them as mechanical and did not use them for his problem solving method. Even without including these mechanical curves, however, Descartes' new definition of acceptable geometric curves and their methods of construction created a set of curves that could be varied in an infinite amount of ways. This raised the issue of grouping together and naming these curves, to facilitate communication and to find common properties.

Descartes kept the notion of planar, solid and "sursolide", i.e. linear, loci as a basis for his reasoning, but also divided them into "genres", or classes. Starting from the problem of Pappus, he stated that the solution to a 3, 4 or 5-line version of the problem, where in the case of five lines they are not perpendicular, would be a planar locus of degree less than or equal to 2, and could thus be constructed using a ruler and a compass. For 5 perpendicular lines or 6, 7, 8 or 9 lines, where in the nine line case they are not perpendicular, the resulting locus would be solid, of degree less than or equal to 4. For 9 perpendicular or 10, 11, 12 or 13 lines, where in the 13 line case they are not perpendicular, the resulting locus would be of degree less than or equal to 6, thus requiring a more complex curve for its construction, such as an intersection between the Cartesian parabola[67] and a circle.

The curves used to create these loci increase in complexity for each step, from straight lines and circles to the intersection of conics, etc. However, to generalize his results, Descartes also proposed a second classification. Using the algebraic equations of the curves, which he concluded on p.18 of *La Géométrie*

[66] Mancosu, 1995, p.95

[67] The Cartesian parabola, as seen on p. 30 in *La Géométrie*. It consists of the arms *GEC* (leftmost parabola that opens upwards) and the line *NIo*. According to Descartes, one could also use "son adjointe *cEGc*, qui se décrit en meme façon, excepté que le sommet de la parabole est tourné vers l'autre côté" and its related line *nIO*. The construction is shown in



Section 2.

would always exist for a geometrical curve, it is similar to the first classification but based on algebraic reasoning. He did not consider straight lines at all, instead starting from equations of the 2nd degree. These were put in the "first genre", or first class. Equations of degrees 3 and 4 were grouped into the 2nd class, of degrees 5 and 6 into the 3rd class, and so on, see Table 1.

Descartes' argument in favor of this division is quite algebraic, based on "a general rule" for reducing equations of the fourth degree to the third degree[68]. According to Bos, Descartes seems to be referring to Ferrari's rule for reducing equations of the fourth degree to the third. Since there is no corresponding rule for equations of the sixth degree, this generalization seems to have been rather rash[69], but it does connect the division based on powers to the first classification made based on the construction of the corresponding curves, as can be seen in Table 1. However faulty the reasoning[70], this connection allowed Descartes to maintain a rather traditional geometrical approach to his new field of study, while still harnessing the power of algebra.

Number of lines in the Pappus problem	Degree of the resultant locus	Curves needed for construction	Class (of equation)
3, 4 or 5 non-parallel	≤ 2	Straight line and circle	1
5 parallel or 6,7,8,9 non-parallel	≤ 4	Intersections of conics	2
9 parallel or 10, 11,12,13 non-parallel	≤ 6	Intersection of Cartesian parabola and circle	3
etc			

Table 1: Classification of curves according to construction and algebraic equation

Descartes did allow constructions using strings for for example ellipses, as long as these did not change between straightened and curved states. However, consistent with all methods of construction, the resulting curves would still be classified according to their equations. As Bos states, classification was Descartes' main use for the equations he found. They were also used in the process of finding normals and tangents, for actual calculations, but otherwise Descartes could "get through his calculations about problems without writing down the equation of the curve explicitly"[71].

3ii. The need for a classification

Why, then, if the equations of curves were so relatively unimportant compared to the construction of said curves, was Descartes' classification of them ultimately algebraic? Why, in that case, was the exactness of construction so very essential in his reasoning? And why was there a need to classify the new curves to start with? To answer these questions, one must consider the mathematical scene in the beginning of the 17th century, as well as Descartes' main goal with the publication of *La Géométrie*. It can be argued whether or not *La Géométrie* created a revolution within mathematics[72] and whatever one concludes, its significance should not be downplayed. But Descartes' wish to join the fields of geometrical analysis, logic and algebra into a "vera mathesis"[73] should be viewed in the light of current events and reigning paradigms at the time of its publication, especially regarding the emphasis placed

[68] LG, p.20
[69] Bos, 1981, p.305
[70] See p. 26 for more discussion
[71] op.cit. p.323
[72] See Mancosu (1995), Boyer (2004), etc.
[73] Sasaki, 2003, p. 91f

on exactness in geometrical constructions.

The purpose of *La Géométrie* was to show that "[a]ny problem in geometry can easily be reduced to such terms that knowledge of the lengths of certain straight lines is sufficient for its construction"[74]. It is thus ultimately a book about *geometrical* problem solving, albeit with an algebraical element. Descartes' method for solving these problems relied on the intersection of curves for the construction of solutions to geometrical problems. In 1637, however, there was no set definition of a "curve". As Bos states, "there were many ways of specifying curves. One could […] indicate how points on the curve should be constructed, one could describe a machine by which the curve could be traced, and (after analytic geometry had been introduced) one could give the equation of the curve"[75]. The problem then arose, which one of these specifications was exact enough to qualify as "geometric" and provide the "truth" sought after in Descartes' philosophy?

In a problem such as the Pappus problem, there are infinitely many solutions, or roots, for every given set of angles, depending on the magnitude of either x or y. Each solution can be plotted using the intersection of two curves, such as for example a circle and a straight line. Circles and straight lines have since Euclidean times been considered to be geometrical and possible to construct in an exact manor, since *all points on the curves can be found at will*. In order to find all solutions to a problem, all curves used for the construction of the problem had to be exact and possible to construct in such a way that all points of the curve could be found at will. This criterion ultimately discarded curves such as the Quadratrix from being classified as "geometrical", since at the time it could only be constructed pointwise, the construction depending on the division of a curved line which could only be done at certain points. Since not all points could be found with the methods then at hand[76], the Quadratrix could not be used for problem solving[77]. One criterion for geometric exactness was thus, for Descartes, that all points on a curve must be possible to find at will.

Fulfilling this criterion, Descartes' basic definition of geometrical curves is kinematic and relies on the tracing by one or several interconnected motions. Pointwise construction, instead of tracing by machines, was accepted in some cases because "curves admitting a pointwise construction in which every point of them can, in principle, be constructed, can also be traced by continuous motion"[78]. This would satisfy Descartes' basic definition and these curves could be considered geometrical. In order to construct solutions using more complex curves, Descartes also used the fact that "defining curves by continuous motion would explicitly determine intersection points"[79], meaning that the intersection of two visible curves, constructed in such a way as all point on them can be found, would be possible to find exactly. This had not at all been as evident had Descartes set an algebraic criterion for defining geometrical curves. Tracing curves from their equations was completely new territory and to Descartes it was "not at all evident that curves defined by algebraic equations had intersection points"[80]. His philosophy based all knowledge on self-evident axioms or conclusions drawn from these axioms by logical reasoning. Until it had been investigated and proved, then, that algebraic equations provided the same information of a curve as its construction, a geometrical definition was needed.

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[74] LG, p.1: "Tous les problèmes de géométrie se peuvent facilement réduire à tells termes, qu'il n'est besoin par après que de connoître la longueur de quelques lignes droites pour les construire".

[75] Bos, 1981, p.296

[76] Descartes shows no elements of infinitesimal thinking in *La Géométrie,* even though Fermat, for example, had already successfully adopted infinitesimal elements in his method for finding tangents to curves. For more on this, see for example Mahoney (1994) and Mancosu (1995).

[77] Incidentally, it could not be expressed in terms of an equation either due to the then unknown relationship between the linear and the circular motions used in its creation.

[78] Bos, 1981, p.318

[79] Katz, 2008, p.441

[80] Katz, 2008, p.441

The importance of exactness of construction of curves thus ultimately goes back to the view of geometry as "that which is precise and exact"[81]. Descartes greatly expanded the set of curves available to mathematicians and needed to justify why these, just as well as the curves already known, belonged to the realm of geometry and could be used as such. Throughout history, mathematicians have searched for truth within the science of mathematics with ever increasing demands for "exactness" or "rigor". The four different methods that existed in the beginning of the 17th century for curve construction meant that mathematicians had no set standard by which geometrical exactness of arbitrary curves could be measured. In *La Géométrie*, Descartes settled his own definition of acceptable and unacceptable curves[82]. He did not consider algebraic equations as sufficient representations, instead focusing on which methods of construction could determine whether a curve was "exact", i.e. geometrical, or "in-exact", i.e. mechanical.

The four more or less accepted ways of constructing curves by the time of *La Géométrie*'s publication were generation by the intersection of surfaces (such as the conic sections); tracing by combinations of motions (such as the Quadratrix); tracing by special instruments (such as different compasses); and pointwise construction[83]. However, the exactness of all of these methods could be and was questioned, many arguing that only curves traceable using a straight-edge ruler and a compass were truly geometrical[84]. Some, such as Mydorge, van Roomen and Clavius, advocated the virtues of pointwise construction but were contradicted by others, such as Kepler and Snellius[85] and later on by Descartes himself. Cristopher Clavius was the first mathematician to suggest that exactness in pure geometry should parallel precision in geometrical practice[86], i.e. that if the methods used to construct a curve were legitimate, then the curves themselves must be legitimate. Descartes developed this thought further in his definition of geometrical curves as being traced by continuous motions. He also put an end to the discussion of which methods of construction were acceptable by presenting a canon of construction. In it he included the tracing by continuous motion using straight lines, circles, parabolas and the like. As long as the relation between the movement of two of these curves was known, the exact points of intersection could be found, and thus the solutions to geometrical problems. He also accepted tracing by special instruments, since these also showed an exact relation between movements, and pointwise construction in the cases described above. As Bos states, Descartes' canon became a central point of reference for further discussions on geometrical exactness and was often an implicit presumption in geometrical problem solving, despite the "conceptual and technical restrictions" it presented in relation to mechanical curves[87].

Eventually, the tracing of curves became such common knowledge that more focus came on the algebra of the equations than the actual curves. For Descartes, however, the solutions to geometrical problem solving existed in the intersection of constructed curves[88]. What algebra did was to provide him with tools to structure and order the new array of curves he had made possible. As a part of his method to solve any given geometrical problem, Descartes "wished to systematize geometry on a higher level so that there should be no limitation on the degree or dimensionality of a problem"[89]. He did warn, however, that "one should take care to use the simplest possible curve that can solve the

[81] LG, p.16
[82] Bos, 2001, p.227
[83] op.cit, p. 217
[84] op.cit. p.220
[85] op.cit. p.218
[86] op.cit. p.166
[87] Bos, 2001, p. 11
[88] Geometrical problem solving has been a branch of mathematics since Antiquity and is present in for example Pappus' *Collection*. In classical Greek geometry, "geometrical figures were "known" or "given" if they could be *constructed* starting from elements that were considered given at the outset; similarly a problem was considered solved if the required configuration was geometrically *constructed*.
[89] Boyer, 2004, p.88

problem”[90] in order to not make a mathematical mistake (“erreur”). By “simplest” Descartes does not, as one might assume, refer to the curve that is easier to construct. Most of Book III, according to Boyer the “raison d'être of the work”, is devoted to solving equations of the third, fourth, and, to some extent, the fifth and the sixth, degrees, to show that the method put forward in Book II works for general cases. A very important part of this process was, in the tradition of Pappus, to find the simplest curve possible to construct the solution[91].

In *La Géométrie*, “simplicity” was linked to the curve's algebraic equation due to a succession of developments of Descartes' thoughts prior to its publication. In the very beginning of his mathematical work, Descartes tried, in a letter to Beeckman in 1619, to find the solutions to all geometrical problems using compasses that could link motions together, thus tracing new curves. Since these new compasses could be seen as an extension of the regular rulers and a compasses, the traditional tools for describing curves, and the relation between the movements was known, the curves created were considered exact enough to apply for geometrical problem solving. At this point in Descartes' reasoning, it would seem natural to assume that the classification of curves would depend on the simplicity of the compass tracing them[92]. Around 1620, however, Descartes discovered that the intersection of a parabola and a circle provided the solutions to all equations of the third and the fourth degree. Some time later, he established that the intersection of a circle and a Cartesian parabola could solve any equations of the fifth and the sixth degree. These discoveries may have shifted Descartes' focus from the simplicity of the tracing machines to the degrees of the curves' equations as criterion for geometrical simplicity[93].

Because of Descartes' strong focus on the constructibility of curves, the classification presented in *La Géométrie*, although paradoxical at first, seems to have come quite naturally as a result of his findings between 1619 and 1637. Since equations of the second degree could be solved using straight lines and circles, they were assigned to the first class. Equations of degrees three and four could be solved using the intersection of conics and were thus grouped into the second class. Equations of degrees five and six were both solvable by the intersection of a circle and the Cartesian parabola, and were thus grouped into the third class. Descartes made a mistake by saying that this pairing should continue “similarly for others”[94] and was disproved early on by for example Fermat[95]. The Cartesian definition of simplicity was, however, “a natural consequence of the hierarchy of curves, which in turn is an extension of the ancient classification of loci”[96] and was thus accepted together with the new combination of algebra and geometry in problem solving.

Descartes' distinction between geometrical and mechanical curves has in some fashion survived until today, although it was renamed by Gottfried Wilhelm Leibniz in the beginning of the 18th century to “algebraic” and “transcendental” curves[97]. The further classification of curves into classes depending on their equations has faced questioning, but was a reasonable continuation of the traditional classification of problems set down by Pappus. Where Pappus divided geometrical problems into planar, solid or linear, however, Descartes went a step further and classified not only problems but also the curves required to solve them. With the extended reach of his new method, he had to consider many problems that were beyond planar and solid. A part of defining exactness of construction of these problems was therefore an “interpretation of hierarchy with respect to simplicity”[98], which placed

[90] LG, p.54: ”il faut avoir soin de choisir toujours la plus simple par laquelle il soit possible de le résoudre”.
[91] Boyer, 2004, p.96
[92] Bos, 1981, p. 329
[93] op.cit. p.329ff
[94] LG, p. 87
[95] Boyer, 2004, p.98
[96] op.cit. p.96
[97] op.cit. p.130
[98] Bos, 2001, p.226

the new curves in relation to those already known. This translated into the classification of curves according to the power of their algebraic equations, where a lower power suggested a simpler curve. Together with algebraic methods aimed at reducing the powers of equations as far as possible and a standard method of constructing curves of the third and fourth degrees, the classification set the basis for a general method of problem solving. The proof that this method was geometrical and exact, and thus worthy of use, was the distinction between acceptable and non-acceptable curves, in symbiosis with a hierarchy of the acceptable curves[99]. With one leg in traditional mathematics, Descartes provided the proofs necessary to cement his arguments' validity while taking a step into the realm of modern mathematics with the other.

4. Progress made after 1637

4i. The development of Analytic Geometry

In the years following 1637, analytic geometry was mainly developed through commentaries *on La Géométrie*. Mathematicians such as Fermat, de Roberval and Debeaune were quick to summarize and comment on Descartes' method, and use it on new problems from different perspectives[100]. It was not until 1649, however, that Cartesian geometry really gained traction in the mathematical community thanks to the Latin translation and commentary made by Frans van Schooten. When published in Latin, the universal scientific language at the time, *La Géométrie* was spread throughout Europe, most notably to Holland, Germany and England, where mathematicians continued the development of analytic geometry into what it is today.

The elements that most notably developed over time were the terminology of analytic geometry, the use of negative and polar coordinates, and a generalization of the methods used into three dimensions. The full power of algebra was also realised when mathematicians started looking more at the properties of algebraic equations and, later on, functions, than the actual curves. This was a continuous process, proceeding with many small steps. For example, John Wallis' work served to arithmetize geometry fully. In *Tractatum de sectionibus conicis*, published in 1655, he substituted geometrical proofs in the style of Descartes with arithmetical ones, based on the same general methods[101]. He was also the first to define curves, in this case the conic sections, by their equations, and to use negative abscissas[102]. In van Schooten's 1659 edition of *La Géométrie*, an appendix by Johann Hudde contained the next step towards a generalization of equations. Hudde considered the coefficients in any algebraic equation to be either positive or negative, removing the need for expressions like $x^2 - ax - b$ and $x^2 + ax + b$ to be viewed separately. This allowed the creation of general forms and "universally applicable formulas"[103]. Sir Isaac Newton caught on to this idea and developed it further, together with the notion of polar coordinates.

By 1679, "analytic geometry had reached a point where an appropriate technical language was felt necessary"[104]. The terms that were generally adopted at the time were *origin*, *axis*, *abscissa*, *coordinate* and *ordinate*, some of which are still used today. But by the end of the 17th century, the focus of mathematicians had shifted to the new field of the calculus, leaving analytic geometry to some extent forgotten. Despite the two fields being based on the same reasoning, mainly put forward by Descartes

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[99] op.cit. p.227
[100] Boyer, 2004, p. 104ff
[101] Boyer, 2004, p.110 and Kline, 1972, p.319
[102] Boyer, 2004, p.110f: Wallis was arguably one of the first mathematicians to define the conic sections as "having nothing whatsoever to do with the cone", instead defining the ellipse as "the plane figure characterized by the property $e^2 = ld - \frac{l}{t}d^2$". See also Kline, 1972, p.319
[103] op.cit. p.113
[104] op.cit. p.121

in *La Géométrie*, the calculus and analytic geometry remained ”distinctly separate at the time”[105]. There was, however, still some progress made in the forgotten field. In 1697, Jean Bernoulli started the tradition of finding roots not where two curves cross, but where one curve crosses the axis. Also around that time, ”[e]quations had become the recognized form of representations for functional relationships” and Jean Bernoulli and Gottfried Wilhelm Leibniz started using the, then rather wide, concept of a function in their calculations[106]. Leibniz is also given credit for renaming Descartes' ”geometric” curves into today's ”algebraic” and ”mechanic” into ”transcendental”. Like many others at the time, Leibniz placed interest not only in the algebraic curves but also the transcendental.

Leibniz was the first mathematician to use coordinates in the modern sense, where both coordinates are equally strong and together describe a point in the plane or space[107]. His practice did, however, take some time to spread and was not commonly known until around 1750. About the same time as Leibniz' works in analytic geometry were published, in 1693, the first example of standardized forms for the conic sections was given by the Scotsman John Craig. Following in the steps of Wallis, he created one standard form for ellipses, one for parabolas, and two for hyperbolas[108]. Newton took up this study of the conic sections in an appendix to *Opticks* called *Enumeratio linearum tertii ordinis*, published in 1704. There, he *plotted*, not traced, curves based on their equations, using coordinates relating to two axes[109]. In G*eometria analytica,* published in 1779, Newton was also the first to present a coherent use of polar coordinates and seven other types of coordinate systems[110], allowing him to give for example the Archimaedean spiral and the Quadratrix equations.

By the beginning of the 18th century, Philippe de la Hire's, Wren's and Wallis' work on three dimensional figures was summed up and developed in Antoine Parent's *Des affections des superficies*, ”essentially the first analytic study of a curved surface”[111]. Parent here used equations to study surfaces and also expounded on methods of finding said equations. This was further developed by none other than Leonard Euler, who in 1728 ”presented for the first time a reasonably systematic analytic treatment of whole classes of surfaces”, investigating the general properties of the spehere, cylinder, cone and surfaces of revolution[112]. Not much happened in analytic geometry after this, until in 1748 two major and one crucial work was published: In *A Treatise of Algebra*, Maclaurin presented linear equations that were the starting point for the slope-intercept form of straight lines; at about the same time, Maria Gaetana Agnesi published a clear and extensive textbook on analytic geometry, based on *La Géométrie*, called *Instituzioni analytiche*, which spread knowledge of the field to a much larger audience. But most importantly, in 1748 Leonard Euler published his grand work *Introductio in analysin infinitorum*.

Introductio was groundbreaking not only for the calculus but also for analytic geometry. To quote Boyer, it is ”the work which made the function concept basic in mathematics” and that ”crystallized the distinction between algebraic and transcendental functions and between elementary and higher functions”[113]. It developed the use of polar coordinates, begun by Newton, as well as the parametric

[105] op.cit. p.126
[106] op.cit. p.130
[107] op.cit. p.133
[108] op.cit. p.131
[109] op.cit. p.138f: Contrary to the works of Leibniz, although the first line was called the axis of abscissas, the second line was not called an axis but ”the principal ordinate”, with a slightly different function than an axis. The origin was considered the origin of abscissas only, since the ordinates were not measured strictly along the principal ordinate line. Notably, negative coordinates were present and Newton showed no hesitation to use all four quadrants when plotting curves. See also Kline, 1972, p.319
[110] Boyer, 2004, p.142 and Kline, 1972, p.319
[111] Boyer, 2004, p.156
[112] op.cit. p.165. In 1729 Clairaut presented a study of space curves using three mutually perpendicular coordinate planes, quite similar to Euler's. It was in this work that the distance formula for three dimensions was first presented.
[113] op.cit. p.180. Euler defined a function as such: ”A function of a variable quantity is an analytic expression composed in any manner whatever of this variable and constants”.

representation of curves, and introduced much of the notation used today[114]. Euler also used numerical coefficients, rather than general algebraic ones, and plotted on labelled axes. What most set the *Introductio* apart, however, was the generality that we today take for granted. For the first century after the publication of *La Géométrie*, few mathematicians had really appreciated the generality that is gained by using algebra to describe many different cases in one comprehensive formulation, in Euler's case mainly functions[115]. The ease of the new general methods, with which one could handle all sorts of equations, also allowed Euler to bring in trigonometric and logarithmic functions into coordinate geometry on a rather basic level, a huge step forward in general and a leap from the geometric limitations that had ruled Descartes' thinking a century before.

The last step towards a fully mature elementary analytic geometry was the development and sustained use of general formulas. This was the result of French mathematicians', such as Lagrange, Monge, Lacroix and Hachette's, work on three dimensional analysis. Due to the difficulty in visualizing three dimensional objects, formulas were needed to find acceptable results. With the establishment of the Ècole Polytechnique in 1794, these great mathematicians were gathered in one place, producing textbooks based on their earlier works as well as their lectures, contributing new formulas and proofs relating to points, lines, planes, angles and the transformation of coordinates, solving diverse problems, and finding new properties of the conics and three dimensional surfaces[116]. Their use of notation, phraseology and methods were, around the turn of the century, ”virtually the same as those to be found in any textbook of today”, leading Boyer to claim that at this point, ”the definite form of analytic geometry finally had been achieved”[117]. This was only true for elementary coordinate geometry and the subject continued to grow rapidly in the 19th century. The mathematical field imagined by Descartes, however, had reached maturity, truly fulfilling the potential he claimed for it more than 150 years earlier.

4ii. The role of curve construction

Parallel to the study of analytic geometry in general was the study of curve construction. As stated in Section 3, Descartes' focus on the construction of curves was mainly due to the at the time vital importance of geometrical exactness, the equivalent of today's proof-based rigor. It was also necessary for him to keep many geometrical components in his problem solving to prove that his new method was a combination of both geometry and algebra. For about 115 years after the publication of *La Géométrie*, curve construction was an actual field of mathematical study, after which the questions of geometrical exactness became obsolete and ”were dissolved by later disregard and oblivion”[118]. By 1748, with the publications of Euler, Agnesi and Maclaurin, the Cartesian classification of curves had been all but abandoned in the light of new discoveries on the properties of curves and, more interestingly, their equations[119]. The fact that the actual construction of curves was studied for so long does, however, signal the importance accredited to it and its role in geometrical problem solving.

Descartes' canon for geometrical constructions and his methodical use of algebra in geometrical problem solving led to two quite separate fields of mathematical study. One was referred to as ”the construction of equations”, i.e. the exact construction of the roots of an equation, based on a problem. These equations were of one unknown. The other was based on another class of problems,

[114] This included the notation *f(x)* for functions and the naming of the constant *e*, to name the most important examples.
[115] op.cit. p. 181f
[116] op.cit. pp.218-223
[117] op.cit. p.220
[118] Bos, 2001, p.5
[119] Boyer, 2004, p.148

”in which it is required to find or construct a curve”[120]. These problems led to equations in two unknowns and the curves generated were investigated using both finite and infinitesimal analysis between 1635 and 1750. These problems dealt mainly with tangents, quadratures and centers of gravity[121]. However, the new methods created within infinitesimal analysis gradually made the actual visualization of the curves treated obsolete. The analytical expressions that were used, named *functions* in 1748, were by this time considered every bit as representative of a curve or surface as the curve or surface itself. They could be treated much more easily to find new information, such as the gradient of tangents, etc, and did not require complex and long-winded geometrical constructions. For that reason, works published by for example Lagrange in 1775[122] could be written without a single diagram. That was a long step away from the works published in the beginning of the 17^{th} century, and signalled what Bos calls ”the degeometrization of analysis”[123].

Until 1748, however, both finite and infinitesimal analysis depended in many ways on the geometrical legitimation put down in *La Géométrie*[124]. For example, when dealing with an exponential curve in the correspondence between Leibniz and Huygens in 1690-1691, Huygens understands Leibniz' arguments based not on the equation he gives to represent one of the curves ($b^t = \frac{1+v}{1-v}$), but on the geometrical construction of said curve. This was no easily done construction[125]. Despite it being pointwise constructed, however, it made much more sense to Huygens when given in terms of its construction rather than by an equation; indeed he answered right back that ”I knew the curve already for a long time”[126], but he did not see how the algebraic expression could be enough to represent the curve.

According to Descartes, the curves used for construction of problems must always be algebraic and they should be ordered into classes by the degrees of their equations to determine which the simplest possible curve to use is. The former condition was rather generally accepted, but by the middle of the 17^{th} century there was still a debate on how a curve could be determined ”simpler” than another[127]. In 1676 Newton ”abandoned the Cartesian classification [into classes] for the modern designation according to degree, thus making way for the idea of order of a curve”[128]. This provided a new, clear hierarchy of curves that replaced the disputed Cartesian classification, but it was not entirely accepted until the publication of Euler’s *Introductio* in 1748. Already by the turn of the 18^{th} century, however, it

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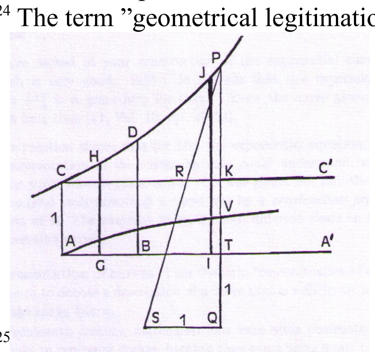
[120] Bos, 2001, p.9
[121] Op.cit. p.10
[122] *Solutions analytiques de quelques problèmes sur les pyramides triangulaires*, mentioned in Boyer, 2004, p.201
[123] Bos, 2001, p.10
[124] The term ”geometrical legitimation” comes from Bos, 2001, p.10

[125] 

Arguments and picture taken from Bos, 1986, p.1629: Let AA’ and CC’ be two parallel lines, AC=AB=1. Draw BD of length *b*. Construct the *Logarithmica* (known as the curve where for every sequence of equidistant points on the axis, the corresponding ordinates are in geometrical progression: in short, $y = b^x$) through C and D. From any point P on the *Logarithmica*, draw a line from P through K on CC’ to T on AA’, and prolong by 1 unit to Q. Draw QS = 1 and join together S and P, passing through R on CC’. Take V on PT so that TV = KR. V is on the desired curve, and more points can be found by repeating this process.
[126] op.cit. p.1631
[127] op.cit. p.1635
[128] Boyer, 2004, p.139. The work was the *Enumeratio linearum tertii ordinis*

had become generally accepted that the simplest possible curves for construction of curves of the degree n^2 were of the "degrees that are integer approximations of n"[129], which undermined the Cartesian division by pairs of degrees.

There was also a development in how curves were drawn. As could be seen in the example with Huygens and Leibniz, to them pointwise construction raised no issues and was accepted without any ado, indicating that it had by 1690 become an accepted method of construction without the extra thoughts that Descartes put behind it. A little later, in the *Introductio*, Euler used the term "plotting" instead of "constructing". Although the representation of specific graphs still was not as simplified as it is today, Euler was the first to use numerical coefficients for the equations, indicating the unit used on both axes[130]. This was a step towards the coordinate system of today, with perpendicular axes that are numbered, giving each point an ordered pair of coordinates, where curves can be easily plotted and there is no need for construction in the Cartesian manner.

As he implicitly stated himself, when it came to dealing with transcendental curves, Descartes' methods as described in *La Géométrie* did not work. Instead, new but somewhat similar methods were developed and used by for example Jakob Bernoulli. In a 1694 article about the form of elastic beams under tension[131], he was presented with a differential equation when the problem had been translated into algebra. In order to construct the solution, Bernoulli constructed it "by quadrature", i.e. instead of visualizing geometrically the solution to a differential equation he constructed the solution to its integral, in terms of the area below the curve. This was "a common way to represent transcendental curves in the seventeenth century", but "not considered the most desirable kind of representation" [132]. Interestingly, it was considered more complex than to construct "by rectification", where integrals had to be reduced to the more complicated expressions of arclengths of curves. This method was used by Jakob Bernoulli as well as Leibniz and Johann Bernoulli, presumably because "measuring length is easier than measuring area"[133].

The debate on which methods of construction were simpler and could or should be used continued until around 1750, with prominent mathematicians, such as l'Hôpital, Euler, and Cramer contributing further to it[134]. However, no real answers were actually found since "the relevant theories [of analytic geometry] (equations, differential equations) became more and more analytical, but the concepts of geometrical simplicity could not be convincingly translated and fomalized into analytical terms"[135]. When everything could be more easily described by a simple equation rather than complicated constructions of curves, involving sophisticated geometrical reasoning, there was no longer any need for the legitimation that the geometrical procedures had once provided. As Bos states, this change in mathematical thinking, starting from the radical new invention in 1637, can be thought of as a *habituation* to the new ways of thinking[136] (Bos, p.1640). Along the way, the process provided much insight into for example the properties of curves and the first techniques of solving differential equations.

[129] Bos, 1986, p.1635. Newton and l'Hôpital gave proofs of this that were later found incorrect. Euler did not attempt to prove it, but acknowledged it.
[130] Boyer, 2004, p.181
[131] Bos, 1986, p.1636: The article was published in *Acta Eruditorium*, the foremost mathematical paper at the time.
[132] Bos, 1986, p.1637
[133] op.cit. p.1639
[134] Bos, 2001, p.422
[135] Bos, 1986, p.1640
[136] op.cit p.1640

5. Conclusion

The opening questions to this paper were, what limitations did Descartes' classification of curves place on the further study of analytic geometry? How was it criticized and why? And how might the development of Descartes' work have changed had all curves been deemed acceptable for his new method? During the process of writing, the first and the third questions have seemed to become more difficult, rather than easier, to answer. Descartes was a transition figure between the traditionally geometric period[137] with its focus on ancient problems and methods of solving them, and the early modern period[138], where the connection he made between geometry and algebra laid the foundation for a veritable explosion of new mathematical tools and ways of reasoning. In order to make that connection, he had to maintain the criterion of geometrical exactness, by construction and classification, to legitimize his new method, while using algebra as a powerful tool for generalizing the problems at hand.

Descartes' kinematic classification of curves into geometric or mechanical was revolutionary in the early 17th century. It extended the realm of geometry to an infinite amount of curves, rather than a set few, which could apply to a vast array of problems. However, at the time the relation between curved lines and straight lines was not known. It was therefore natural for Descartes to disregard the curves which depended on that relation, referring to them as mechanical rather than geometric, since his philosophy of knowledge and requirement of geometrical exactness made accepting them impossible. It was only 22 years later, in 1659, that Hendrick van Heuraet, using Cartesian methods, found how to rectify a curve, i.e. find its relation to a straight line. One can therefore not claim that the division of curves into geometric or mechanical placed limitations on the development of analytic geometry, and mathematicians as contemporary to Descartes as Roberval studied both types without any second thoughts; They only used different methods[139].

By the end of the 17th century, the new techniques of the calculus brought forward by Leibniz and Newton made it possible to investigate even more carefully the properties of curves. Even though the two fields of calculus and analytic geometry were in many ways separate, the exchange between Leibniz and Huygens in 1694 shows that both men took calculations of the area under curved lines (using calculus) as a matter of fact, allowing them to construct transcendental curves by the quadrature or rectification of their equations (using analytic geometry). New methods were conjured up to deal with trigonometric, logarithmic, exponential, and other types of transcendental curves[140], based on Cartesian methods but far from them in their basic form.

Descartes published only one major work on mathematics, and it would therefore be unlikely for him to develop his method to its fullest potential. However, the division between geometric and mechanical curves still holds today and it was not until after 1748 that the general concept of a function was established, bunching the two together into a common category. What was more criticized was Descartes' classification of algebraic curves into classes of pairs, and his statement that the roots of equations of degree $2n$-1 could be found with at least one curve of degree n. Fermat disproved the latter statement in 1660[141], at the same time showing the flaws in Descartes' division of curves into classes:

On pp.19-20 of La Géométrie, Descartes claims that in the construction of a curve, the generated

[137] The term refers to the years 1550-1650, when focus was on restoring and reviving the works and methods of the Ancients.
[138] This term is *not* used as in Bos' *Redefining Geometrical* Exactness (2001). Instead it indicates the years 1650-1750.
[139] Boyer, 2004, p.105
[140] One can mention the use of polar coordinates by Newton for example.
[141] Boyer, 2004, p.98, Mancosu pp.131-140

curve will belong to a class one step higher than the generating curve[142]. Thus, curves of the second degree generate new curves of degree 3 ($2n$-1) or 4 ($2n$), while curves of the third degree generate new curves of degree 5 ($2n$-1) or 6 ($2n$), etc. Fermat easily disproved this statement by using an example from p.18, see fig 6.1. Like the example described on p.12 in this essay, the curve ECD is described by the intersection of GL, rotating around G, and the parabola KC, moving vertically. KL always remains the same. Set $GA = d$, $AB = x$, $CB = y$, $KL = b$, $NL = c$ and $BK = u$.

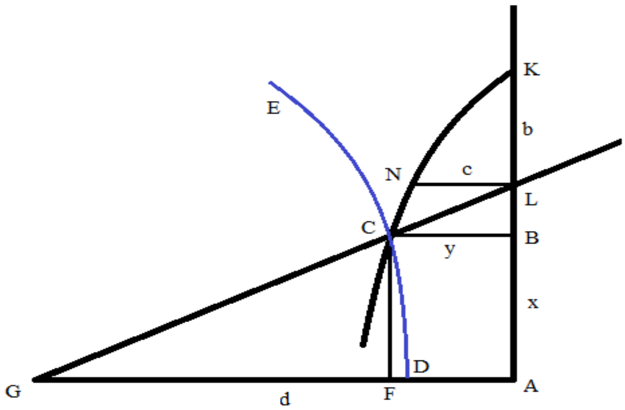


Fig 6.1: Construction of ECD by intersection of GL, rotating around G, and the parabola KC, moving vertically.

Because triangles GCF and CLB are similar, $BK = u$ can be found by $\frac{d-y}{y} = \frac{x}{BL}$, i.e. $BL = u = \frac{xy}{d-y}$.
In modern notation, if $y = f(u)$ is the parabola, the generated curve will be found where $y = f(BK) = f(b + \frac{xy}{d-y})$. Exchanging the parabola for a curve $y^3 = u$, then, would mean that the resultant curve is $y^3 = b + \frac{xy}{d-y}$, a quartic equation. This disproves Descartes' statement since quartic equations were not of a higher class than cubics, but rather of the same class[143].

Descartes' reasoning for the division of curves into classes thus held up geometrically only for some examples until the third class. Algebraically, it worked only up until the second class[144]. Despite his faulty assumptions, however, the fact that he ultimately classified his curves according to algebra signals that he understood some of the value of generalizing curves' properties by their equations. Newton discarded Descartes' classification in 1676, instead ordering all curves by their degree, as is done today. Although the reasoning behind their choices in this matter differs greatly between the two men, one can assert that Descartes' basic thought wasn't entirely wrong. Or rather, at least one cannot claim that the classification into classes, rather than by single degrees, placed limitations on the further study of analytic geometry. Rather, it provided material for mathematicians to disprove.

The program put forward by Descartes in *La Géométrie* was to combine geometry and algebra in order to solve any geometrical problem. In order to do this, he must legitimize the basic structure of his new method, the use of arbitrary curves in the construction of a solution to a problem. In the beginning of the 17th century, the way to legitimize a new method was not rigor by proof, as is emphasized today, but

[142] LG, p.19-20: "l'intersection de cette ligne et de la règle GL décrira […] une autre ligne courbe qui sera d'un second genre. […] mais si au lieu d'une de ces lignes courbes du premier genre, c'en est une du second qui termine le plan CNKL, on en décrira, par son moyen, une du troisième, ou si c'en est une du troisième, on en décrira une du quatrième, et ainsi à l'infini, comme il est fort aisé à connoître par le calcul"
[143] Argument and picture taken from Mahoney, p 135f
[144] Where Ferrari's theorem asserts that any quartic equation can be transformed into a cubic. This is not true for sixth degree equations; there is no general method of transforming them into quintic equations.

by proving the geometrical exactness of the curves used. Since no general method of constructing curves was given, geometrical exactness for Descartes meant describing an exact method of curve construction, which would place beyond doubt the utility of the curves used. It also required him to rank the curves in order to decide which ones to use, creating a hierarchy. Without this, *La Géométrie* would not have been the groundbreaking work it became, proving by traditional methods its worth while introducing new methods with immense potential. Descartes was a transition figure from the old to the new and was thus not responsible for making analytic geometry into the subject it is today. He did, however, place it on a strong foundation. No matter the criticism one can find towards some of his choices, such as the separation of curves into classes, the basic method he presented was developed and transformed into one of the most useful mathematical tools today.

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