Renormalization Group Flows

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Abstract

In this note we study some general properties of renormalization group (RG) flow and the conformal symmetry of field theories in 2D spacetime. We first present the Zamolodchikov's c-theorem and its proof, which indicates the irreversibility of the RG flow. Then we explain the theorem with the spectral representation to reveal the physical meaning of the c-function. Finally we apply Zamolodchikov's arguments to show that global scale symmetry in 2D implies local conformal symmetry under broad conditions.

1 Introduction

2 Renormalization Group Flows of 2D Unitary QFT

2.1 Zamolodchikov's 2D c-Theorem

The basic settings of the c-theorem is a general field theory in 2D Euclidean spacetime, described by the action,

$$S = \int \mathrm{d}^2x\, \mathcal{L}(g, \Lambda, x), \tag{1}$$

where $g = (g^1, g^2, \cdots)$ is a collection of dimensionless coupling constants and Λ is the UV cut-off scale. The coupling parameters coordinatize a space of theories $\mathscr{Q}$, in the sense that each point in $\mathscr{Q}$ specifies a field theory. The corresponding coordinate basis of $\mathscr{Q}$ is given by a set of operators $\Phi_i \equiv \partial\mathcal{L}(g, \Lambda, x)/\partial g^i$. We assume that there exists a one-parameter transformation $R_t : \mathscr{Q} \to \mathscr{Q}$ generated by a vector field $\beta^i(g)$ as a section of tangent bundle $T\mathscr{Q}$. The components of the vector field are called β functions. By definition, we have

$$\mathrm{d}g^i = \beta^i(g)\mathrm{d}t. \tag{2}$$

The so-called c-theorem needs the following assumptions. 1) The presence of Galilean symmetry. In particular, the existence of translational symmetry implies the conservation of the stress tensor $T_{\mu\nu}$, which is chosen here to be symmetric. We further denote its trace by $\Theta \equiv T^{\mu}{}_{\mu}$. 2) "Renormalizability". In the present context, this means that the trace of the stress tensor Θ can be expressed linearly in terms of the basis operators Φ_i, with the coefficients being exactly the β functions $\beta^i(g)$, namely,

$$\Theta = \beta^i(g)\Phi_i. \tag{3}$$

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3) Positivity. That is, $\langle\Phi_i\Phi_j\rangle$ is a positive definite matrix, and can serve as a metric in $\mathscr{Q}$.

Then, under these assumptions, we have the following:

Theorem. 1) There exists a function $c : \mathscr{Q} \to \mathbb{R}$ with $c(g) \geq 0$, such that

$$\frac{\mathrm{d}}{\mathrm{d}t}c \equiv \beta^i(g)\frac{\partial}{\partial g^i}c(g) \leq 0, \tag{4}$$

where the equality holds if and only if $g = g_*$, with g_* a fixed point of the RG flow, namely, $\beta^i(g_*) = 0$. 2) The (critical) fixed point is the stationary point of $c(g)$, namely,

$$\beta^i(g) = 0 \qquad \Rightarrow \qquad \frac{\partial c}{\partial g^i}\bigg|_{g^i=g^i_*} = 0. \tag{5}$$

3) At the critical fixed point, the system has local conformal symmetry characterized by the Virasoro algebra with central charge $\tilde{c}(g_*)$. The central charges $\tilde{c}(g_*)$ can be different for different fixed points g_*. We have, $c(g_*) = \tilde{c}(g_*)$.

To prove this theorem, we go to the complex coordinates $(z, \bar{z})$ with $z = x^1 + \mathrm{i}x^2$. Then the stress tensor has components

$$T \equiv T_{zz} = \tfrac{1}{4}(T_{11} - T_{22} - 2\mathrm{i}T_{12}), \qquad \Theta = 4T_{z\bar{z}} = 4T_{\bar{z}z} = T_{11} + T_{22}, \qquad T_{\bar{z}\bar{z}} = T^*, \tag{6}$$

and the conservation equation $\partial^\mu T_{\mu\nu} = 0$ reads

$$0 = \bar{\partial}T + \tfrac{1}{4}\partial\Theta. \tag{7}$$

Now we consider the two point correlation functions of stress tensor $\langle T_{\mu\nu}(z)T_{\rho\sigma}(0)\rangle$. There are three independent components, which can be put into the following form:

$$\langle T(z)T(0)\rangle = \frac{F(z\bar{z}\Lambda^2)}{z^4}, \qquad \langle\Theta(z)T(0)\rangle = \frac{G(z\bar{z}\Lambda^2)}{z^3\bar{z}}, \qquad \langle\Theta(z)\Theta(0)\rangle = \frac{H(z\bar{z}\Lambda^2)}{z^2\bar{z}^2}, \tag{8}$$

where $F(\xi)$, $G(\xi)$ and $H(\xi)$ are three scalar function of the dimensionless quantity $\xi = z\bar{z}\Lambda^2$ with conformal weight zero. The form of these parameterization can be determined by the conformal weight of the correlators. For instance, without vanishing anomalous dimension, T has conformal weight[1] $(h, \bar{h}) = (2, 0)$, therefore the correlator $\langle TT\rangle$ must be proportional to z^{-4} with coefficient a conformal scalar F. Note further that Θ has conformal weight $(1, 1)$, then the correlators $\langle\Theta T\rangle$ and $\langle\Theta\Theta\rangle$ can also be parameterized in the same way.

Then, the conservation of the stress tensor (7) implies that

$$0 = \bar{\partial}\langle T(z)T(0)\rangle + \tfrac{1}{4}\partial\langle T(z)\Theta(0)\rangle, \tag{9a}$$
$$0 = \bar{\partial}\langle T(z)\Theta(0)\rangle + \tfrac{1}{4}\partial\langle\Theta(z)\Theta(0)\rangle. \tag{9b}$$

These equations can be rewritten into the following form by using (8),

$$0 = \xi F' + \tfrac{1}{4}(\xi G' - 3G), \tag{10a}$$
$$0 = (\xi G' - G) + \tfrac{1}{4}(\xi H' - 2H). \tag{10b}$$

[1]To check this statement, note that T transforms as $T \to e^{2t}T$ and $T \to e^{2\mathrm{i}\theta}T$ under the scale transformation and rotation respectively. That is it has scaling dimension $d = 2$ and spin $s = 2$. Then the conformal weight is given by $(h, \bar{h}) = \big(\tfrac{1}{2}(d+s), \tfrac{1}{2}(d-s)\big) = (2, 0)$.

Eliminating G from these two equations, we get

$$-\tfrac{3}{8}H = \xi\big(F' - \tfrac{1}{2}G' - \tfrac{3}{16}H'\big) = z\bar{z}\frac{\partial}{\partial(z\bar{z})}\big[F(z\bar{z}\Lambda^2) - \tfrac{1}{2}G(z\bar{z}\Lambda^2) - \tfrac{3}{16}H(z\bar{z}\Lambda^2)\big]. \tag{11}$$

Now we define a function C in the space of theory $\mathscr{Q}$ by

$$C = 2F - G - \tfrac{3}{8}H. \tag{12}$$

Then (11) says that

$$\frac{\partial C}{\partial \log|z|^2} = -\frac{3}{4}H. \tag{13}$$

Now, along the RG flow parameterized by t, the renormalization point $|z|^2$ scales as $e^{2t}|z|^2$. Thus we have $\mathrm{d}C/\mathrm{d}t = -\frac{3}{2}H$. On the other hand, we see from (8) that $H = z^2\bar{z}^2\langle\Theta(z)\Theta(0)\rangle = |z|^4\beta^i\beta^j\langle\Phi_i(z)\Phi_j(0)\rangle \geq 0$ by unitarity. Thus we have found a function C which does not increase along any RG flow. Now that the metrix G_{ij} in $\mathscr{Q}$ is positive definite, thus $H = 0$ if and only if $\beta^i(g) = 0$, namely, the equality holds only at fixed points. In this case the trace Θ vanishes and $C = 2F$. In addition, $C/2$ is now the coefficient of the z^{-4} in $\langle T(z)T(0)\rangle$ and thus coincides with the central charge of the CFT at this point.

2.2 Spectral Representation

In this section we will try to understand the physical meaning of the c function defined in the last section. It will be shown that this function is a measure of the density of (massless) degree of freedoms. To achieve this goal, we will use the spectral representation for two-point correlator of the stress tensor [3]. Thus we firstly recall some basic ingredients of the spectral representation, with scalar field as a simple example. Let ϕ be a scalar field interacting with some other scalar fields (which can be different from ϕ) in general n spacetime dimensions. Then the spectral representation for the two-point correlator $\langle\phi(x)\phi(0)\rangle$ is obtained by inserting a "1" made by a summation over complete basis of the Hilbert space, namely, we write

$$1 = \int \mathrm{d}\mu^2\,\mathscr{P}_\mu; \qquad \mathscr{P}_\mu \equiv \int \frac{\mathrm{d}^{n-1}\mathbf{p}}{(2\pi)^{n-1}}\frac{1}{2\sqrt{\mathbf{p}^2+\mu^2}}|\mathbf{p},\mu\rangle\langle\mathbf{p},\mu|, \tag{14}$$

where $|\mathbf{p},\mu\rangle$ is the one-particle eigenstate of the momentum operator P^μ with mass spatial momentum $\mathbf{p}$ and mass μ, and $\mathscr{P}_\mu$ is an operator projecting a state to the one-particle state with mass μ. Inserting this identity into the correlator $\langle\phi(x)\phi(0)\rangle$, we have

$$\langle\phi(x)\phi(0)\rangle = \int \mathrm{d}\mu^2 \int \frac{\mathrm{d}^{n-1}\mathbf{p}}{(2\pi)^{n-1}}\frac{1}{2\sqrt{\mathbf{p}^2+\mu^2}}\langle\phi(x)|\mathbf{p},\mu\rangle\langle\mathbf{p},\mu|\phi(0)\rangle, \tag{15}$$

where $|\phi(0)\rangle$ is the result of acting $\phi(0)$ on the interaction vacuum $|0,\mathrm{int}\rangle$. Now we use the Poincaré symmetry to recast $\langle\phi(x)|\mathbf{p},\mu\rangle = e^{\mathrm{i}p\cdot x}\langle\phi(0)|\mathbf{0},\mu\rangle$. Then we get

$$\langle\phi(x)\phi(0)\rangle = \int \mathrm{d}\mu^2 \int \frac{\mathrm{d}^n p}{(2\pi)^n}\frac{e^{\mathrm{i}p\cdot x}}{p^2+m^2}\big|\langle\mathbf{0},\mu|\phi(0)\rangle\big|^2 \equiv \int \mathrm{d}\mu^2\,\rho(\mu^2)G(x,\mu), \tag{16}$$

where $G(x,\mu)$ is the scalar's Feynman propagator with mass μ, and $\rho(\mu^2) = \big|\langle\mathbf{0},\mu|\phi(0)\rangle\big|^2$ is called the spectral density. The unitarity of the theory guarantees that $\rho(\mu^2)$ is positive definite.

Now we consider the two-point correlator of the stress tensor in 2 dimensions. Repeat the arguments above, and note that the Lorentz structure of the amplitude $\langle T_{\mu\nu}|\mathbf{p},\mu\rangle = (p^2 g_{\mu\nu} - p_\mu p_\nu)C(\mu^2)e^{\mathrm{i}p\cdot x}$ with $C(\mu^2)$ a scalar function. This structure is fixed by the conservation of the stress tensor. Then the spectral representation of the two-point correlator is given by

$$\langle T_{\mu\nu}(x)T_{\rho\sigma}(0)\rangle = \frac{1}{12\pi}\int_0^\infty \mathrm{d}\mu\, c(\mu;g,\Lambda)(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)(g_{\rho\sigma}\partial^2 - \partial_\rho\partial_\sigma)G(x,\mu^2). \tag{17}$$

Here we have normalized the spectral density function c by a factor of $1/12\pi$ and spelled out explicitly its dependence on the coupling g and mass scale Λ. Again, $c(\mu;g,\Lambda)$ is positive definite by unitarity. Furthermore, it is easy to see that the mass dimension of the spectral density is -1, and the combination $\mathrm{d}\mu\, c(\mu)$ is a dimensionless measure of the degrees of freedom. Then, in a scale invariant theory in which there is no mass scale, the form of $c(\mu)$ is completed fixed to be $c(\mu) = c_0\delta(\mu)$. In this case, consider the correlator $\langle\Theta\Theta\rangle$:

$$\langle\Theta(x)\Theta(0)\rangle = \frac{1}{12\pi}\int_0^\infty \mathrm{d}\mu\, c_0\delta(\mu)(\partial_\mu\partial^\mu)^2 G(x,\mu) = -\frac{1}{12\pi}c_0\partial_\mu\partial^\mu\delta^{(2)}(x). \tag{18}$$

As the theory flows away from its UV or IR fixed point, the spectral density develops an additional term besides a delta function at zero, namely

$$c(\mu) = c_0\delta(\mu) + c_1(\mu,\Lambda), \tag{19}$$

and the support of c_1 is away from zero. Now we go back to the complex coordinates and get

$$\langle T(z)T(0)\rangle \to \begin{cases} \dfrac{1}{8\pi^2 z^4}\displaystyle\int_0^\infty \mathrm{d}\mu c(\mu), & z\to 0 \\ \dfrac{1}{8\pi^2 z^4}\displaystyle\lim_{\epsilon\to 0}\int_0^\epsilon \mathrm{d}\mu c(\mu), & z\to\infty \end{cases} \tag{20}$$

Let's verify the case $z\to 0$. Note that the propagator $G(x,\mu^2)$ can be evaluated explicitly to be

$$G(x,\mu^2) = \frac{1}{2\pi}\mathrm{K}_0(\mu x), \tag{21}$$

by deforming the integral along a counter lying in the upper half of the p-plane. In complex coordinates, this becomes $G(|z|,\mu) = \mathrm{K}_0(\mu|z|)/2\pi$. As $z\to 0$, $\mathrm{K}_0(\mu|z|)\to -\log(\mu|z|) + \mathcal{O}(|z|^0)$. Therefore we have $\partial_z^4 G(|z|,\mu)\to 3z^{-4}/2\pi$. Then, (20) implies that when the theory approach UV ($z\to 0$) or IR ($z\to\infty$) CFTs, we get the central charges:

$$c_{\mathrm{UV}} = \int_0^\infty \mathrm{d}\mu\, c(\mu), \qquad\qquad c_{\mathrm{IR}} = \lim_{\epsilon\to 0}\int_0^\epsilon \mathrm{d}\mu\, c(\mu). \tag{22}$$

These two numbers are related to each other by

$$c_{\mathrm{UV}} = c_{\mathrm{IR}} + \int_0^\infty \mathrm{d}\mu\, c_1(\mu). \tag{23}$$

Then, from the positivity of the theory, we see that $c_{\mathrm{UV}} \geq c_{\mathrm{IR}}$.

The argument above shows that the spectral density $c(\mu)$ coincides with the UV or IR central charges as the theory approaching UV or IR CFT, a property also shared by Zamolodchikov's C function defined in (12). However, we note that Zamolodchikov's C function is defined in the parameter space, namely, is a function of couplings g^i, while the spectral density is defined for each theory along a single RG trajectory. Therefore, to find the relation between these two functions, we "smear" the spectral density over the RG flow,

$$c(g(\Lambda)) = \int \mathrm{d}\mu\, c(\mu) f(\mu) = \int \mathrm{d}\mu\, c_1(\mu, \Lambda) f(\mu) + c_{\mathrm{IR}}, \tag{24}$$

with $f(\mu)$ a smearing function being positive definite, monotonically decreasing with μ, and the boundary condition that $f(0) = 1$, $f(\mu)$ decays exponentially as $\mu \to \infty$.

2.3 Examples

In this section we consider two examples [3, 4] to illustrate the ideas outlined above.

2.3.1 RG Flow between UV and IR Minimal Models

As a first explicit example of 2D RG flow linking two CFTs at UV and IR, consider the case in which UV CFT is a unitary minimal model [5] lying in the classification of Friedan, Qiu and Shenker [6], with central charge $c(m) = 1 - 6/m(m+1)$ and m a large integer. We perturb this CFT by a slightly relevant operator $\Phi = \Phi_{1,3}$ with conformal weight[2] $h_{1,3} = 1 - 2/(m+2)$. Then its scaling dimension is $\Delta = 2h \equiv 2 - y$, with $0 < y \ll 1$. Schematically, we can write down the action of this system as

$$S = S_{\mathrm{UV}} - \lambda_0 \int \mathrm{d}^2 x\, \Phi_0(x), \tag{25}$$

where S_{UV} is the action for the UV CFT, λ_0 is a (bare) coupling with dimension $[\lambda_0] = y$ and Φ_0 the bare operator with conformal weight $h_{1,3}$ at UV, We note that since the UV CFT is not a Gaussian fixed point of the theory, thus the action S_{UV} may not correspond to the integral of an Lagrangian.

We will show in the following that the nontrivial RG flow generated by the relevant perturbation Φ_0 drives the theory to an IR CFT, which is again a minimal model, but with central charge $c(m-1)$. The basic strategy is to perform a standard calculation for the correlator $\langle \Phi\Phi \rangle$ within the perturbation theory, with regularization, renormalization and RG-improvement.

To the first order in perturbation of coupling λ_0, we have

$$\begin{aligned}\langle \Phi_0(x)\Phi_0(0) \rangle &= \frac{\left\langle \Phi_0(x)\Phi_0(0) \exp\left(\lambda_0 \int \mathrm{d}^2 x' \Phi_0(x')\right) \right\rangle_{\mathrm{UV}}}{\left\langle \exp\left(\lambda_0 \int \mathrm{d}^2 x'\, \Phi_0(x')\right) \right\rangle_{\mathrm{UV}}} \\ &= \langle \Phi_0(x)\Phi_0(0) \rangle_{\mathrm{UV}} + \lambda_0 \int \mathrm{d}^2 x'\, \langle \Phi_0(x)\Phi_0(0)\Phi_0(x') \rangle_{\mathrm{UV}} + \mathcal{O}(\lambda_0^2),\end{aligned} \tag{26}$$

where the subscript UV indicates the corresponding correlator should be evaluated with the UV CFT. The two-point function is simply given by

$$\langle \Phi_0(x)\Phi_0(0) \rangle_{\mathrm{UV}} = \frac{1}{|x|^{4h}}, \tag{27}$$

[2] Recall that the conformal weight corresponding to zeros of Kac determinant is given by

$$h_{r,s}(m) = \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)}$$

for unitary minimal models with central charge $c(m)$.

where a possible coefficient can be absorbed into the normalization of Φ_0. For three-point function, we have

$$\begin{aligned}\int \mathrm{d}^2x'\,\langle\Phi_0(x)\Phi_0(0)\Phi_0(x')\rangle_{\mathrm{UV}} &= \int \mathrm{d}^2x'\,\frac{b}{|x|^{2h}|x'|^{2h}|x-x'|^{2h}}\\ &= \left(\frac{\Gamma(1-y)\Gamma^2(1+y/2)}{\Gamma^2(1-y/2)\Gamma(1+y)}\right)\frac{4\pi b}{y}\,\frac{|x'|^y}{|x'|^{4h}},\end{aligned} \tag{28}$$

where b is the structure constant. We will use $A = A(y)$ to denote the quantity in the parenthesis in the last line. As long as y is small, it is straightforward to see that $A(y) = 1 + \mathcal{O}(y^3)$. Then we get

$$\langle\Phi_0(x)\Phi_0(0)\rangle = \frac{1}{|x|^{4h}}\left(1 + \lambda_0 A\frac{4\pi b}{y}|x|^y + \mathcal{O}(\lambda_0^2)\right). \tag{29}$$

We see that no regularization is needed since the result is finite within the first order of perturbation as long as $0 < y \ll 1$. But in order to study the marginal case with $y = 0$, renormalization is necessary because the result above is divergent as $y \to 0$. Therefore, our renormalization condition will be such that the renormalized correlator $\langle\Phi\Phi\rangle$ remains finite as $y \to 0$. We denote the renormalized coupling by $\lambda(\mu)$ with μ the renormalization scale, and define a corresponding dimensionless coupling $g(\mu)$ by $g(\mu) = \mu^{-y}\lambda(\mu)$. At the UV cut-off Λ the renormalized coupling should coincide with the bare one, namely we have $g(\Lambda) = \Lambda^{-y}\lambda_0$. The dependence of $g(\mu)$ on μ is of course dictated by the β function, $\mu\mathrm{d}g(\mu)/\mathrm{d}\mu = \beta(g)$, while the β function is in turn given by the coefficient of the trace of the stress tensor Θ, linearly expanded in terms of field operators, $\Theta = \beta(g)\Phi(x,g)$. We also define the renormalized operator $\Phi(x,g) = \Phi_0(x)/\sqrt{Z(g)}$, with the wave-function renormalization coefficient to be determined.

A number of renormalization schemes do the job, and we will (purely for convenience) pick up the following one:

$$\langle\Phi(x,g)\Phi(0,g)\rangle\big|_{|x|=\mu^{-1}} \equiv \mu^4. \tag{30}$$

Then it is straightforward to find that

$$\sqrt{Z} = \mu^{-y}\left(1 + \lambda_0 A\frac{2\pi b}{y}\mu^{-y} + \mathcal{O}(\lambda_0^2)\right). \tag{31}$$

On the other hand, for trace of the bare stress tensor, we also have a similar relation, $\Theta_0 = -y\lambda_0\Phi_0$. We also note that Θ receive no wave function renormalization since it is the trace of the conserved current corresponding to the translational symmetry, thus we should have $\Theta = \Theta_0$. Then, together with the renormalized expression $\Theta = \beta(g)\Phi(x,g)$ and $\sqrt{Z}$ in (31), we find

$$\beta(g(\lambda_0)) = -y\lambda_0\sqrt{Z(g)} = -y\lambda_0\mu^{-y} - 2\pi bA(\lambda_0\mu^{-y})^2 + \mathcal{O}(\lambda_0^3). \tag{32}$$

From this we solve the coupling g to be

$$g = \lambda_0\mu^{-y}\left(1 + \lambda_0 A\frac{\pi b}{y}\mu^{-y} + \mathcal{O}(\lambda_0^2)\right). \tag{33}$$

Conversely, λ_0 can also be solved as an expression in g order by order, $\lambda_0 = g\mu^y\big(1 - gA\pi by^{-1} + \mathcal{O}(g^2)\big)$. Inserting this expression back to the β function (32) as well as the renormalized correlator, we find

$$\beta(g) = -yg - \pi bAg^2 + \mathcal{O}(g^2), \tag{34}$$

and

$$\langle\Phi(x,g)\Phi(0,g)\rangle = \frac{\mu^4}{|\mu x|^{4h}}\bigg(1+4\pi bAg\frac{|\mu x|^y-1}{y}+\mathcal{O}(g^2)\bigg). \tag{35}$$

Now it is clear that this renormalized correlator is finite as $y\to 0$, which represents a UV CFT with $c=1$ perturbed by a marginal operator. However, we observe that (35) does not exhibit power law behavior at IR CFT, namely when g approaching its IR fixed point $g^*=-y/\pi b$. The correct power law can be got after a suitable resummation to all orders in coupling g. This can be achieved by solving the Callan-Symanzik equation. In our case, this equation reads

$$\bigg(|x|\frac{\partial}{\partial|x|}+\beta(g)\frac{\partial}{\partial g}+4h(g)\bigg)G_2(x,g)=0. \tag{36}$$

The equation can be explicitly solved by the method of characteristics, with the solution

$$G_2(x,g)=\langle\Phi(x,g)\Phi(0,g)\rangle\exp\bigg(\int^{\bar{g}(\mu)}\mathrm{d}g'\,\frac{4h(g')}{\beta(g')}\bigg). \tag{37}$$

2.3.2 Free Massive Theories

Next we consider an even more trivial example, namely a theory of free massive particle. Though trivial as it seems to be, this example provides us an explicit spectral representation, and allows us to trace the variation of the spectral density function along the RG flow.

Intuitively, the theory in the UV is a free CFT at trivial Gaussian fixed point, in which case the particle's mass can be ignored. When the theory flows to IR, nothing left. If this looks too odd to someone, we can also include any free massless field into the theory with trivial RG property, to ensure that there are something left in the IR limit. But we will never write out this massless field explicitly in the following.

The spectral density function can be evaluated explicitly in this case. For massive Majorana fermion, it is given by

$$c_1(\mu,m)=\frac{6m^2}{\mu^3}\sqrt{1-\frac{4m^2}{\mu^2}}\,\theta(\mu-2m), \tag{38}$$

and for massive boson, the spectral density is

$$c_1(\mu,m)=\frac{24m^4}{\mu^5}\bigg(1-\frac{4m^2}{\mu^2}\bigg)^{-1/2}\theta(\mu-2m). \tag{39}$$

2.4 Scale, Conformal and Weyl Symmetries in 2D

At last we apply Zamolodchikov's argument to seek for a relation among different space symmetries in 2D [7]. We will consider the rigid scale symmetry in local conformal symmetry flat 2D space, as well as scale symmetry and Weyl symmetry in curved 2D space. In particular, we will show that in flat space, rigid scale symmetry of a unitary field theory implies the local conformal symmetry.

To be definite, we firstly recall the definition of these symmetry transformations. The scale transformation in flat space is defined to be a transformation on space coordinates, $\delta x^\mu=\epsilon x^\mu$ with

ϵ a coordinate independent number. The conformal transformation, is given by $\delta x^\mu = \epsilon v^\mu(x)$ with $v^\mu(x)$ satisfying

$$\partial_\mu v_\nu(x) + \partial_\nu v_\mu(x) = \eta_{\mu\nu}\partial\cdot v(x). \tag{40}$$

In 2D this condition says that $\partial\cdot v(x)$ spans all harmonic functions of x.

Now, a scale current must be of the from

$$S^\mu(x) = x^\nu T_\nu{}^\mu + K^\mu(x), \tag{41}$$

where the first terms containing the symmetric stress tensor $T_{\mu\nu}$ comes from the transformation on space coordinate, and the second term K^μ reflect the "inner" transformation of field arise from the field's scaling dimension. Thus K^μ is a local operator with no explicit dependence on space coordinates. Then, the conservation of scale symmetry is amount to say that $\partial_\mu S^\mu = 0$, or equivalently,

$$T_\mu{}^\mu(x) = -\partial_\mu K^\mu(x). \tag{42}$$

Thus we see that the necessary and sufficient condition for existence of a conserved scale current S^μ is that the trace of any symmetric stress tensor can be written as the divergence of a local operator.

On the other hand, the most general form of the conformal current is given by

$$J^\mu(x) = v^\nu(x)T_\nu{}^\mu(x) + \partial\cdot v(x)K'^\mu(x) + \partial_\nu\partial\cdot v(x)L^{\nu\mu}(x). \tag{43}$$

Compared with the scale current, an additional term proportional to a local operator $L^{\mu\nu}$ appears, since the transformation parameter $v^\mu(x)$ is now allowed to vary with coordinates. Note that no higher order derivatives of $v^\mu(x)$ appears, because they are not independent quantities, due to the harmonic condition $\partial^2(\partial_\mu v^\mu(x)) = 0$. Then, the conformal symmetry, namely the conservation of the current j^μ, implies that

$$0 = \partial_\mu j^\mu = (\partial\cdot v)\big(T_\nu{}^\nu + \partial\cdot K'\big) + \big[\partial_\mu(\partial\cdot v)\big]\big(K'^\mu + \partial_\nu L^{\mu\nu}\big) + \big(\partial_\mu\partial_\nu(\partial\cdot v)\big)L^{\mu\nu}. \tag{44}$$

Thus we get two additional conditions, $K'^\mu = -\partial_\nu L^{\mu\nu}$ and $L^{\mu\nu} = \eta^{\mu\nu}L$ besides the condition for rigid scale symmetry. Now these three conditions combine into a single condition, $T(x) = \partial^2 L(x)$. A stress tensor satisfying this condition can always be made traceless. Thus we conclude that the system has conformal symmetry, if there exists a symmetric traceless stress tensor.

On the other hand, we see that a theory can be scale invariant but not conformal invariant only when its trace of stress tensor is the divergence of a local operator K^μ, while this operator is not a gradient of another local operator L.

3 Towards a 4D a-Theorem

3.1 Cardy's Conjecture

In our earlier presentation of Zamolodchikov's proof of 2D c-theorem, the conservation of the stress tensor play a crucial role, in that it constrains a particular combination of different components of two-point correlator $\langle T_{\mu\nu}(x)T_{\rho\sigma}(0)\rangle$ to have positive definite derivative along the RG flow. However, the same argument cannot be applied to 4D QFT directly, roughly because the increase of independent components of the correlator $\langle T_{\mu\nu}(x)T_{\rho\sigma}(0)\rangle$ is faster than that of conservation law $\partial^\mu T_{\mu\nu}=0$. More definitely, if we parameterize the correlator in any spacetime dimension n, as

$$\begin{aligned}\langle T_{\mu\nu}(x)T_{\rho\sigma}(0)\rangle = &\frac{D}{|x|^{2n+4}}x_\mu r_\nu r_\rho r_\sigma + \frac{E}{|x|^{2n+2}}(\delta_{\mu\nu}x_\rho x_\sigma+\delta_{\mu\nu}x_\rho x_\sigma)\\ &+\frac{F}{|x|^{2n+2}}(\delta_{\mu\rho}x_\nu x_\sigma+\delta_{\mu\sigma}x_\nu x_\rho+\delta_{\nu\rho}x_\mu x_\sigma+\delta_{\nu\sigma}x_\mu x_\rho)\\ &+\frac{G}{|x|^{2n}}\delta_{\mu\nu}\delta_{\rho\sigma}+\frac{H}{|x|^{2n}}(\delta_{\mu\rho}\delta_{\nu\sigma}+\delta_{\mu\sigma}\delta_{\nu\rho}),\end{aligned} \tag{45}$$

then it can be shown that

$$\frac{\mathrm{d}}{\mathrm{d}|x|}C=-4\frac{n+1}{n-1}\langle\Theta(x)\Theta(0)\rangle-2(n-2)D, \tag{46}$$

where

$$C\equiv-\frac{4}{n-1}\big[D+\tfrac{1}{2}(d^2+d+2)E+(d+3)F+\tfrac{1}{2}d(d+1)G+(d+1)H\big]. \tag{47}$$

Therefore we see that the derivative of c-function is not proportional to the positive definite trace correlator $\langle\Theta\Theta\rangle$ unless $n=2$, in which case the c-function is indeed monotonic along the RG flow, and goes to the corresponding central charges of UV and IR CFTs.

Cardy's observation is that the central charge appears not only as a coefficient before the two-point correlator of the stress tensor, but also before the Weyl anomaly. In particular, in 2-dimensional space, the Weyl anomaly due to nonzero space curvature is given by

$$\langle\Theta\rangle=-\frac{c}{12}R. \tag{48}$$

So it is possible that the Weyl anomaly coefficient can serve as a good candidate for the c-function along the RG flow, as an extension to the central charge. In its original form, Cardy proposed that a possible candidate of c-function in any even dimensional spacetime to be

$$C=(-1)^{n/2}a_n\int_{S^n}\mathrm{d}^nx\sqrt{g}\langle\Theta\rangle, \tag{49}$$

where the coefficient a_n is a normalization for C. For instance, taking $a_4=60/\pi^2$ in 4D gives $C=1$ for a massless real scalar field. The $(-1)^{n/2}$ is due to the sign of the integral flips as n increases through even integers.

To understand why the integral over the n-sphere is taken, we take $n=4$ as an example. In this case the Weyl anomaly can be generally parameterized as

$$\langle\Theta\rangle=\alpha R^2+\beta R_{\mu\nu}R^{\mu\nu}+\gamma R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}+\delta\Box R. \tag{50}$$

When taken integral over a manifold without boundary, the total derivative $\Box R$ drops off. The remaining three terms can be reorganized as a linear combination of Euler density E_4, the squared Weyl tensor, $W^2_{\mu\nu\rho\sigma}$, and the squared scalar curvature R, where

$$E_4=R^2_{\mu\nu\rho\sigma}-4R^2_{\mu\nu}+R^2, \tag{51}$$

$$W^2_{\mu\nu\rho\sigma}=R^2_{\mu\nu\rho\sigma}-2R^2_{\mu\nu}+\tfrac{1}{3}R^2. \tag{52}$$

Then we may write

$$\langle\Theta\rangle = aE_4 + bR^2 - cW^2_{\mu\nu\rho\sigma} + d\Box R. \tag{53}$$

The minus sign before c is conventional. Explicit calculation shows that

$$a = -\frac{1}{360(4\pi)^2}(N_S + 11N_F + 62N_V), \tag{54}$$

$$b = 0, \tag{55}$$

$$c = \frac{1}{120(4\pi)^2}(N_S + 6N_F + 12N_V), \tag{56}$$

$$d = \frac{2}{3}c \tag{57}$$

Here N_S, N_F and N_V are numbers of massless scalars, Dirac fermions and vectors, respectively. Then we see that after taking integral of $\langle\theta\rangle$ over the 4-sphere, only a-term remains, since the sphere is conformally flat which implies $W_{\mu\nu\rho\sigma} = 0$. Therefore the Cardy's conjecture amounts to proposing a-term coefficient (we will call a-anomaly in the following) as a candidate of monotonic function along RG flows.

The conjecture was proved for 4D's case by Komargodski and Schwimmer [10].

3.2 't Hooft Anomaly Matching Condition

3.3 A New Proof of 2D c-Theorem from Weyl Anomaly Cancellation

Now let us outline the proof of 2D c-theorem using the anomaly cancellation argument. As we shall see, this proof can be generalized directly to 4-dimensional theories, yet the calculation in 2D is much simpler than that in 4D. Thus this will serve as a warming up and we will be sketchy here.

Let us consider the case in which the UV CFT is deformed by a set of relevant operators with some characteristic mass scales M_i. When written with curved background metric, the action is not invariant under Weyl transformation. But this explicit broken Weyl symmetry can always be interpreted as if it is spontaneously broken, by introducing massless dilaton field $\tau(x)$, which behaves as $\tau(x) \to \tau(x) + \sigma(x)$ under the Weyl transformation. This can be achieved by replacing each mass parameter M_i appears in the Lagrangian with $M_i e^{-\tau(x)}$. Then the action realize the Weyl symmetry nonlinearly.

At this state the dilaton plays the role of spectator field, and couples to matter field with arbitrarily weak couplings. Then on one hand, the UV CFT contains a Weyl anomaly given by $\langle\Theta\rangle = -c_{\text{UV}}R/24\pi$; on the other hand, the IR limit of the theory may contain a nontrivial IR CFT, together with massless dilatons. The IR CFT also exhibit Weyl anomaly $\langle\Theta\rangle = -c_{\text{IR}}R/24\pi$, while the Weyl transformation property of the dilaton theory is govern by its effective action. This effective action may contain the usual term as an integral of Weyl invariant Lagrangian, which in 2D is simply the Einstein-Hilbert action. But we note that this action in 2D is a topological term and is naturally Weyl invariant. Thus to capture the Weyl anomaly contributed by matter field, the effective action of dilaton should contains a Wess-Zumino term, which in 2D reads

$$S_{\text{WZ}}[\tau, g_{\mu\nu}] = \frac{c}{24\pi}\int \mathrm{d}^2x\,\sqrt{g}\big(\tau R + (\partial\tau)^2\big). \tag{58}$$

It is easy to check that this action transforms under Weyl transformation as

$$\delta_\sigma S_{\rm WZ} = \frac{c}{24\pi}\int d^2x\,\sqrt{g}\Big[R\sigma + 2\partial_\mu\big((\partial^\mu)\sigma\big)\Big] = \frac{c}{24\pi}\int d^2x\,\sqrt{g}\sigma R. \tag{59}$$

Now, since the full theory has no explicit (operatorial) broken of Ward identity of Weyl symmetry, thus the anomaly in the UV and IR should match. That is, the anomaly of IR system, consisting of anomalies from IR CFT and that from Wess-Zumino term of dilaton action, should reproduce the anomaly of the UV CFT. It is straightforward to see that this amount to set the dilaton's effective action in the flat background limit to be

$$S = \frac{c_{\rm UV} - c_{\rm IR}}{24\pi}\int d^2x\,(\partial\tau)^2. \tag{60}$$

On the other hand, the effective action of dilaton field can be obtained by explicitly integrating out matter filed from the full theory. We would like to find the $(\partial\tau)^2$ term from this calculation. Note that dilaton couples to matter fields through $\tau\Theta$. Thus, the terms quadratic in τ with two derivatives can be extracted as follows,

$$\begin{aligned}\Big\langle \exp\Big(\int d^2x\,\tau\Theta\Big)\Big\rangle &= \frac{1}{2}\int d^2x d^2y\,\tau(x)\tau(y)\langle\Theta(x)\Theta(y)\rangle + \cdots \\ &= \frac{1}{4}\int d^2x\,\tau(x)\partial_\mu\partial_\nu\tau(x)\int d^2y\,(y-x)^\mu(y-x)^\nu\langle\Theta(x)\Theta(y)\rangle + \cdots\end{aligned} \tag{61}$$

The y-integral is x-independent due to translational invariance,

$$\int d^2y\,(y-x)^\mu(y-x)^\nu\langle\Theta(x)\Theta(y)\rangle = \frac{1}{2}\eta^{\mu\nu}\int d^2y\,y^2\langle\Theta(0)\Theta(y)\rangle. \tag{62}$$

Therefore the contribution to the dilaton effective action with two derivatives is

$$\frac{1}{8}\int d^2x\,\tau\partial^2\tau\int d^2y\,y^2\langle\Theta(y)\Theta(0)\rangle. \tag{63}$$

Comparing with (60), we see that

$$c_{\rm UV} - c_{\rm IR} = 3\pi\int d^2y\,y^2\langle\Theta(y)\Theta(0)\rangle. \tag{64}$$

Then, by unitarity of the theory, it follows immediately that $c_{\rm UV} - c_{\rm IR} \geq 0$.

3.4 A 4D a-Theorem and Its KS Proof

Now we come to the proof of the "a-theorem" in 4D. The main idea of the proof is the same with that in 2D case as sketched in the last subsection. But now we distinguish three different cases, namely, we consider the UV CFT deformed by 1) spontaneous break down of conformal symmetry in a vacuum from moduli space; 2) by a relevant deformation; 3) by a marginally relevant deformation.

The proof still consists of three steps: 1) Writing down the theory with curved background metric and interpreting explicit deformation of the conformal symmetry as spontaneous breaking by introducing dilaton field. 2)Showing that the the contribution to $\tau\tau$ scattering tin flat spacetime limit is fully governed by the a-anomaly term in the dilaton effective action. 3) Applying unitary argument to show that the dilaton scattering amplitude is positive definite.

A A Brute-Force Calculation of Wess-Zumino Term of 4D Conformal Group

We first calculate the Weyl transformation properties of dim-4 scalars made out of curvature tensor $R_{\mu\nu\rho}{}^{\sigma}$. By definition, the Weyl transformation acting on the metric $g_{\mu\nu}$ is given by

$$g'_{\mu\nu} = e^{2t} g_{\mu\nu}. \tag{65}$$

It follows directly that $g'^{\mu\nu} = e^{-2t} g^{\mu\nu}$. Then, with this we can find the curvature tensor $R_{\mu\nu\rho}{}^{\sigma}$ transforms as

$$\begin{aligned} R'_{\mu\nu\rho}{}^{\sigma} = &\, R_{\mu\nu\rho}{}^{\sigma} + 2\delta^{\sigma}_{[\mu}\nabla_{\nu]}\nabla_\rho t - 2g^{\sigma\lambda}g_{\rho[\mu}\nabla_{\nu]}\nabla_\lambda t \\ &+ 2(\nabla_{[\mu}t)\delta^{\sigma}_{\nu]}\nabla_\rho t - 2(\nabla_{[\mu}t)g_{\nu]\rho}\nabla^\sigma t - 2g_{\rho[\mu}\delta^{\sigma}_{\nu]}(\nabla_\lambda t)^2. \end{aligned} \tag{66}$$

Similarly, the Ricci tensor transforms as

$$R'_{\mu\nu} = R_{\mu\nu} - (n-2)\nabla_\mu\nabla_\nu t - g_{\mu\nu}\nabla^2 t + (n-2)\nabla_\mu t\nabla_\nu t - (n-2)g_{\mu\nu}(\nabla_\lambda t)^2, \tag{67}$$

and the scalar curvature transforms as

$$R' = e^{-2t}\big[R - 2(n-1)\nabla^2 t - (n-1)(n-2)(\nabla_\mu t)^2\big]. \tag{68}$$

With this we evaluate the Weyl transformations of $R^2_{\mu\nu\rho\sigma}$, $R^2_{\mu\nu}$ and R^2, as

$$\begin{aligned} R'^2_{\mu\nu\rho\sigma} = e^{-4t}\Big[&R^2_{\mu\nu\rho\sigma} + 8R^{\mu\nu}\big((\nabla_\mu t)(\nabla_\nu t) - \nabla_\mu\nabla_\nu t\big) - 4R(\nabla_\mu t)^2 \\ &+ 4(n-2)(\nabla_\mu\nabla_\nu t)^2 + 4(\nabla^2 t)^2 + 8(n-2)(\nabla^2 t)(\nabla_\mu t)^2 \\ &- 8(n-2)(\nabla^\mu t)(\nabla^\nu t)(\nabla_\mu\nabla_\nu t) + 2(n-1)(n-2)(\nabla_\mu t)^4\Big], \end{aligned} \tag{69}$$

$$\begin{aligned} R'^2_{\mu\nu} = e^{-4t}\Big[&R^2_{\mu\nu} + 2(n-2)R^{\mu\nu}\big((\nabla_\mu t)(\nabla_\nu t) - \nabla_\mu\nabla_\nu t\big) - 2R\nabla^2 t - 2(n-2)R(\nabla_\mu t)^2 \\ &+ (n-2)^2(\nabla_\mu\nabla_\nu t)^2 + (3n-4)(\nabla^2 t)^2 - 2(n-2)^2(\nabla^\mu t)(\nabla^\nu t)(\nabla_\mu\nabla_\nu t) \\ &+ 2(n-2)(2n-3)(\nabla^2 t)(\nabla_\mu t)^2 + (n-1)(n-2)^2(\nabla_\mu t)^4\Big], \end{aligned} \tag{70}$$

$$\begin{aligned} R'^2 = e^{-4t}\Big[&R^2 - 4(n-1)R\nabla^2 t - 2(n-1)(n-2)R(\nabla_\mu t)^2 + 4(n-1)^2(\nabla^2 t)^2 \\ &+ (n-1)^2(n-2)^2(\nabla_\mu t)^4 + 4(n-1)^2(n-2)(\nabla^2 t)(\nabla_\mu t)^2\Big]. \end{aligned} \tag{71}$$

Then it is straightforward to see that in 4D spacetime,

$$W'^2_{\mu\nu\rho\sigma} = e^{-4t} W_{\mu\nu\rho\sigma}, \tag{72}$$

$$\begin{aligned} E'_4 = e^{-4t}\Big[&E_4 - 4R\nabla^2 t - 8R^{\mu\nu}\big((\nabla_\mu t)(\nabla_\nu t) - \nabla_\mu\nabla_\nu t\big) \\ &- 8(\nabla_\mu\nabla_\nu t)^2 + 8(\nabla^2 t)^2 + 8(\nabla^2 t)(\nabla_\mu t)^2 + 8(\nabla^\mu t)(\nabla^\nu t)\nabla_\mu\nabla_\nu t\Big]. \end{aligned} \tag{73}$$

As was claimed, $\sqrt{-g}W^2_{\mu\nu\rho\sigma}$ is a Weyl invariant.

References

[1] A. B. Zamolodchikov, "*'Irreversibility' of the Flux of the Renormalization Group in a 2D Field Theory*", Pis'ma Zh. Eksp. Teor. Fiz. **43**, 565 (1986) [JETP Lett. **43**, 730 (1986)].

[2] J. L. Cardy, "*Central Charge and Universal Combinations of Amplitudes in Two-Dimensional Theories Away from Criticality*", Phys. Rev. Lett. **60**, 2709 (1988).

[3] A. Cappelli, D. Friedan, and J. I. Latorre, "*c-Theorem and Spectral Representation*", Nucl. Phys. B **352**, 616 (1991).

[4] A. Cappelli and J. I. Latorre, "*Perturbation Theoryof Higher-Spin Conserved Currents Off Criticality*", Nucl. Phys. B **340**, 659 (1990).

[5] See, for instance, P. Di Francesco, P. Mathieu, and D. Sénéchal, Conformal *Conformal Field Theory*, Springer, 1997.

[6] D. Friedan, Z. Qiu, and S. Shenker "*Conformal Invariance, Unitarity, and Critical Exponents in Two Dimensions*" Phys. Rev. Lett. **52**, 1575 (1984)

[7] J. Polchinski, "*Scale and Conformal Invariance in Quantum Field Theory*", Nucl. Phys. B **303**, 226 (1988).

[8] J. L. Cardy, "*Is There a c-Theorem in Four Dimensions?*" Phys. Lett. B **215**, 749 (1988).

[9] M. J. Duff, "*Twenty years of the Weyl anomaly*", Class. Quan. Grav. **11**, 1387 (1994) [arXiv: hep-th/9308075] and references therein.

[10] Z. Komargodski and A. Schwimmer, "On renormalization group flows in four dimensions" JHEP (2011)**12**, 1 [arXiv: 1107.3987 [hep-th]].