Report for MA502 Presentation: Lebesgue's Criterion for Riemann Integrability

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March 26, 2015

1 Introduction

We covered Riemann integrals in the first three weeks in MA502 this semester (Chapter 11 in [1]). This report explores a necessary and sufficient condition for determining Riemann integrability of $f(x)$ solely from its properties. This condition is known as Lebesgue's criterion and elucidating the proof of this condition is the aim of this report.

This report revisits the concepts learnt in MA501/502 in Section 2. Section 2 also covers an example showing that functions discontinuous everywhere need not be Riemann integrable. Towards the end, this section reminds the readers a few results from measure theory useful in proving Lebesgue's criterion for Riemann integrability. Section 3 states the Lebesgue's criterion and provides examples of functions with countably infinite and uncountably infinite discontinuities which are Riemann integrable to motivate the usefulness of this criterion. Finally, Section 4 provides the proof for the Lebesgue's criterion for Riemann integrability. The references used in this report are attached in the end.

2 Revisiting what we learnt in MA501/502

2.1 Known conditions on f for Riemann integrability

We define a partition using Definition 11.1.10 in [1] as follows:

Definition 2.1. *Let I be a bounded interval. A partition of I is a finite set $\boldsymbol{P}$ of bounded intervals contained in I, such that every x in I lies in exactly one of the bounded intervals J in $\boldsymbol{P}$.*

Partitions are allowed to contain empty sets. Using Definition 11.3.2, Lemma 11.3.3, Definition 11.3.4 and Proposition 11.3.12 in [1], we define Riemann integrability of a function $f(x)$ as follows

Definition 2.2. *Let $f : I \to \mathbb{R}$ be a bounded function on a bounded interval I. Then, f is Riemann integrable if and only if given $\epsilon > 0$, $\exists$ $\boldsymbol{P}$ such that $0 \leq U(f, \boldsymbol{P}) - L(f, \boldsymbol{P}) < \epsilon$ where $U(f, \boldsymbol{P}) = \sum\limits_{J \in \boldsymbol{P}: J \neq \phi} (\sup\limits_{x \in J} f(x))|J|$ and $L(f, \boldsymbol{P}) = \sum\limits_{J \in \boldsymbol{P}: J \neq \phi} (\inf\limits_{x \in J} f(x))|J|$.*

Note that while Definition 2.2 is a necessary and sufficient condition for Riemann integrability, it requires some sort of computation on $f(x)$ in order to determine if f is Riemann integrable.

Sections $11.5 - 6$ in [1] list a few theorems for determining the Riemann integrability of a function without any computation and just its properties. From Theorem 11.5.1 in [1], we have

Theorem 2.1. *Let I be a bounded interval, and let f be a function which is uniformly continuous on I. Then, f is Riemann integrable.*

Proof. Given as proof of Theorem 11.5.1 in pages $284 - 286$ of [1]. It should be noted from Proposition 9.9.15, a uniformly continuous function in a bounded interval is bounded. □

From Proposition 11.5.3 in [1], we have

Theorem 2.2. *Let I be a bounded interval, and let $f : I \to \mathbb{R}$ be both continuous and bounded. Then, f is Riemann integrable.*

Proof. Given as proof of Proposition 11.5.3 in pages $286 - 287$ of [1]. $\square$

We use the definition of piecewise continuous functions from Definition 11.5.4 in [1] to give Definition 2.3 and use Proposition 11.5.6 in [1] to give Theorem 2.3. $f|_J$ is defined as the restriction of a function f to a subset J of its domain.

Definition 2.3. *A function f is piecewise continuous on I iff $\exists$ $\boldsymbol{P}$ of I such that $f|_J$ is continuous on J $\forall$ $J \in \boldsymbol{P}$.*

Theorem 2.3. *Let I be a bounded interval, and let $f : I \to \mathbb{R}$ be both piecewise continuous and bounded. Then, f is Riemann integrable.*

Proof. Consider the function $f : [a, b] \to \mathbb{R}$ defined in Figure 1 with two discontinuities at points $x = c$ and $x = d$. We will illustrate with this example how to proceed with the proof of Riemann integrability of a bounded piecewise continuous function with 2 discontinuities. The proof of Riemann integrability of bounded piecewise continuous functions having N discontinuities follows from simple modifications to the arguments made in this proof.

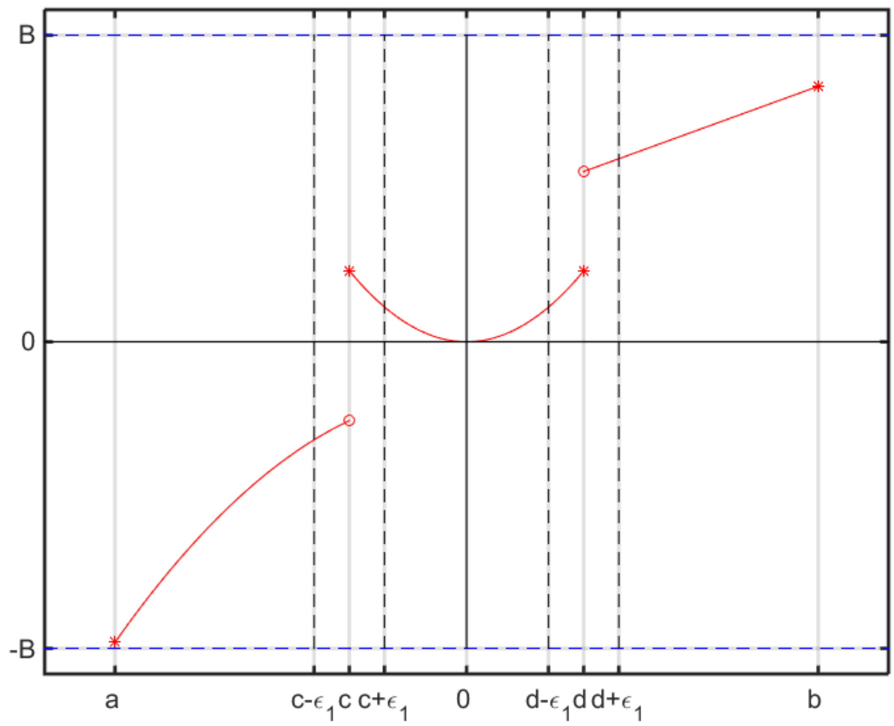

Figure 1: A piecewise function with two discontinuities

From Definition 2.3, if we consider a partition $\boldsymbol{P}_{cts} = \{[a, c), [c, d], (d, b]\}$, we know that the function f is continuous in every interval of $\boldsymbol{P}_{cts}$. Let us choose an $\epsilon_1 > 0$ small enough to define a partition $\boldsymbol{P} = \{[a, c - \epsilon_1], (c - \epsilon_1, c + \epsilon_1), [c + \epsilon_1, d - \epsilon_1], (d - \epsilon_1, d + \epsilon_1), [d + \epsilon_1, b]\}$. Given $\epsilon > 0$. From Theorem 2.2, we know that $\exists$ $\boldsymbol{P}_{[a,c-\epsilon_1]}$ such that

$$U(f|_{[a,c-\epsilon_1]}, \boldsymbol{P}_{[a,c-\epsilon_1]}) - L(f|_{[a,c-\epsilon_1]}, \boldsymbol{P}_{[a,c-\epsilon_1]}) < \frac{\epsilon}{5}.$$

Similarly, $\exists\ \boldsymbol{P}_{[c+\epsilon_1, d-\epsilon_1]}$ and $\boldsymbol{P}_{[d+\epsilon_1, b]}$ satisfying the same requirements for intervals $[c+\epsilon_1, d-\epsilon_1]$ and $[d+\epsilon_1, b]$ respectively. We define

$$\boldsymbol{Q} = \boldsymbol{P}_{[a,c-\epsilon_1]} \cup \{(c-\epsilon_1, c+\epsilon_1)\} \cup \boldsymbol{P}_{[c+\epsilon_1,d-\epsilon_1]} \cup \{(d-\epsilon_1, d+\epsilon_1)\} \cup \boldsymbol{P}_{[d+\epsilon_1,b]}$$

such that

$$\begin{aligned} U(f,\boldsymbol{Q}) - L(f,\boldsymbol{Q}) &\leq \frac{\epsilon}{5} + 2B\epsilon_1 + \frac{\epsilon}{5} + 2B\epsilon_1 + \frac{\epsilon}{5} \qquad (1) \\ &\leq \left[\text{Choosing } \epsilon_1 \text{ such that } 0 \leq \epsilon_1 \leq \tfrac{\epsilon}{5\times 2B}\right] \\ &\leq \epsilon \end{aligned}$$

From Definition 2.2, we conclude that f is a Riemann integrable. The trick used here can be extended to finitely many discontinuities. Specifically, if there are N discontinuities, replace $\frac{\epsilon}{5}$ by $\frac{\epsilon}{2N+1}$ in (1). □

Remark 2.1. *Theorem 2.3 comments only about functions with finite number of discontinuities. This is because Definition 2.3 uses the notion of partitions which are finite sets.*

There are two important observations to note from Theorems 2.1-2.3:

1. All of these theorems have two common conditions:
   (a) the candidate function is bounded, and
   (b) the interval over which the Riemann integration is to be performed is bounded.
2. All of these theorems give *necessary* conditions expected from the candidate function to be Riemann integrable.
3. There seems to be a trend of decreasing strength expected in the continuity of the candidate function.

The last observation prompts us to ask the question covered in the Subsection 2.3: *Can a function discontinuous everywhere be Riemann integrable?* Before answering that question, we revisit some properties of $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ in Subsection 2.2.

2.2 Some properties of $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$

Lemma 2.1. *The set of rationals, $\mathbb{Q}$, is countable.*

Proof. If $x \in \mathbb{Q}$, then $x = \frac{p}{q}$ where $p \in \mathbb{Z}$ and $q \in \mathbb{N} \setminus \{0\}$. Hence, we can enumerate the rational numbers by listing down all possible combinations of p, q and then sweeping them across diagonals as shown in Figure 2. This process of sweeping creates a bijection between rationals and natural numbers, making the set of rationals $\mathbb{Q}$ countable. □

Lemma 2.2. *The set of rationals, $\mathbb{Q}$, is dense. In other words, given $a, b \in \mathbb{R},\ a < b,\ \exists\ x \in \mathbb{Q}$ such that $a < x < b$.*

Proof. Assume for contradiction, there is no x. By Archimedian principle, we can find $N \in \mathbb{N}$ such that $\frac{1}{b-a} < N \Rightarrow \frac{1}{N} < (b-a)$. Let $A = \{\frac{m}{N} : m \in \mathbb{N}\}$. Then, for the assumption to hold, if m_1 is the greatest integer such that $\frac{m_1}{N} < a$, then $\frac{m_1+1}{N} > b$. This means, $\frac{m_1+1}{N} - \frac{m_1}{N} > b - a$ from an extension[1] of Proposition 5.4.7(d) in [1]. Hence, $\frac{1}{N} > b - a$, which is a contradiction. □

Lemma 2.3. *The set of irrationals, $\mathbb{R} \setminus \mathbb{Q}$, is dense. In other words, given $a, b \in \mathbb{R},\ a < b,\ \exists\ x \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < x < b$.*

[1]If $a, b, c, d \in \mathbb{R}$ and $a < b, c < d$, then by Proposition 5.4.7(d) in [1], we have $a + c < b + c$ and $b + c < b + d$. Hence, $a + c < b + d$.

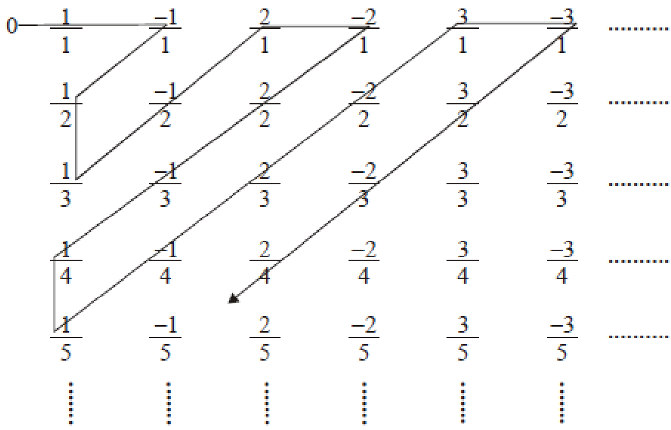

Figure 2: Enumeration process used in proving $\mathbb{Q}$ is a countable set. (Source: [2])

Proof. Given $a < b$, consider the reals $\frac{a}{\sqrt{2}}$ and $\frac{b}{\sqrt{2}}$. We know $\frac{a}{\sqrt{2}} < \frac{b}{\sqrt{2}}$. From Lemma 2.2, we know there is a rational number q such that $\frac{a}{\sqrt{2}} < q < \frac{b}{\sqrt{2}}$. In order to get a non-zero q, we apply Lemma 2.2 to $\frac{a}{\sqrt{2}}$ and 0 when $a < 0 < b$.

By Proposition 5.4.7(e) in [1], we have $a < \sqrt{2}q < b$. If $\sqrt{2}q = q' \in \mathbb{Q}$, then $\frac{q'}{q}$ is rational[2] but $\frac{q'}{q} = \sqrt{2}$ which is irrational. Hence, $\sqrt{2}q = q' \in \mathbb{R} \setminus \mathbb{Q}$. □

Proofs for the density of $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ were inspired from [3].

2.3 Can a function discontinuous everywhere be Riemann integrable?

We demonstrate this by giving a counter-example, the Dirichlet function which is a real-valued function which is discontinuous everywhere in $\mathbb{R}$. The Dirichlet function is defined as $g(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$. Dirichlet function is not Riemann integrable in any subset of $\mathbb{R}$.

Consider $\int_0^1 g(x)dx$**.** Owing to Lemma 2.2 and Lemma 2.3, we know that, for any partition $\boldsymbol{P}$, we have $\sup\limits_{x\in[0,1]} g(x) = 1$ and $\inf\limits_{x\in[0,1]} g(x) = 0$. Therefore, $L(g, \boldsymbol{P}) = 0$ and $U(g, \boldsymbol{P}) = 1$ for any partition $\boldsymbol{P}$. Hence, $g(x)$ is not Riemann integrable from Definition 2.2. Hence, functions which are discontinuous everywhere are **not necessarily** Riemann integrable.

2.4 Set of measure zero

Denoting the length of an interval I as $|I|$, we have the definition of a set of measure zero from [4], as follows:

Definition 2.4. *A set S is said to have* measure zero *if given $\epsilon > 0$, $\exists$ a countable collection of open intervals $\{I_k\}_{k=1}^{\infty}$ such that*

$$S \subseteq \bigcup_{k=1}^{\infty} I_k \text{ and } \sum_{k=1}^{\infty} |I_k| < \epsilon.$$

Lemma 2.4. *Any countable set has its measure equal to zero. In particular, the empty set has measure zero.*

Lemma 2.5. *Cantor set has uncountably infinite points and has measure zero.*

The reader is requested to use *Trevor, Angel and Michael's report* on set of measure zero for the proofs of Lemma 2.4 and Lemma 2.5.

[2]Division of rationals give rationals.

2.5 Some results from measure theory

We borrow a few results from measure theory which can intuitively understood by appealing to the notion of measure as "length".

Remark 2.2. *Riemann integral of a function, when it exists, equals the Lebesgue integral of the function. In other words, Riemann integrable functions are Lebesgue integrable. The integral of a characteristic function of an interval X, $1_X(x)$, is its length given by $\int 1_X(x)dx = m(X)$ where $m(A)$ denotes the Lebesgue measure of the set A.*

Remark 2.3. *If $\{A_i\}_{i=0}^{\infty}$ is a sequence of disjoint sets which are Lebesgue-measurable and $A = \bigcup\limits_{i=0}^{\infty} A_i$, then $m(A) = \sum\limits_{i=0}^{\infty} m(A_i)$. This result can be used to conclude that for some $N \in \mathbb{N}$ and disjoint sets A_i, $m(\bigcup\limits_{i=0}^{N} A_i) = \sum\limits_{i=0}^{N} m(A_i)$.*

Lemma 2.6. *Set of irrationals, $\mathbb{R} \setminus \mathbb{Q}$, form a measurable set of non-zero measure.*

Proof. From Lemma 2.1 and Lemma 2.4, we know that the set of rationals in $\mathbb{R}$, $\mathbb{Q}$, is of measure zero. Since $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})$ and $\mathbb{Q} \cap (\mathbb{R} \setminus \mathbb{Q}) = \phi$, we have, from Remark 2.3, $m(\mathbb{R}) = m(\mathbb{Q}) + m(\mathbb{R} \setminus \mathbb{Q}) \Rightarrow m(\mathbb{R} \setminus \mathbb{Q}) = m(\mathbb{R}) > 0$. $\square$

Remark 2.4. *If $\{A_i\}_{i=0}^{\infty}$ is a sequence of measurable sets, then $0 \leq m(\bigcup\limits_{i=0}^{\infty} A_i) \leq \sum\limits_{i=0}^{\infty} m(A_i)$. Hence, if $m(A_i) = 0 \; \forall \; i$, then $m(\bigcup\limits_{i=0}^{\infty} A_i) = 0$ and if $m(\bigcup\limits_{i=0}^{\infty} A_i) > 0$, then $\exists \; i$ such that $m(A_i) > 0$.*

Remark 2.5. *If $A \subseteq B$, then $m(A) \leq m(B)$.*

3 Lebesgue's criterion for Riemann integrability and some motivating examples

From [4], we state Lebesgue's criterion for Riemann integrability, which will be proven in Section 4, as follows:

Theorem 3.1. *If $f(x)$ is a bounded function defined on $[a, b]$, then f is Riemann integrable iff the set of points on which f is discontinuous, say S, is a set of measure zero.*

From Subsection 2.3, we remarked that $g(x)$ defined as the Dirichlet function is discontinuous everywhere. Specifically, $g(x)$ is discontinuous on $\mathbb{R} \setminus \mathbb{Q}$. Therefore, from Lemma 2.6, $g(x)$ is discontinuous on a set of measure greater than zero. Hence, $g(x)$ is not Riemann integrable according to Lebesgue's criterion of Riemann integrability. This matches with the observations made in Subsection 2.3.

We now look at two famous examples of Riemann integrable functions having infinite discontinuities. By Definition 2.3, they are not piecewise continuous and hence Theorem 2.3 can not be used to determine their Riemann integrability. From Theorem 3.1, we can ensure that these examples are Riemann integrable if their corresponding sets of points of discontinuities are of measure zero.

3.1 Countably infinite discontinuities example: Thomae's function

Thomae's function is defined as follows:

$$t(x) = \begin{cases} 1 & x = 0 \\ \frac{1}{q} & x \in \mathbb{Q} \Rightarrow x = \frac{p}{q}, \ \mathrm{g.c.d}(p, q) = 1 \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} .$$

We list some important properties of Thomae's function as lemmas. The proof for these lemmas were inspired from [5].

Lemma 3.1. *$t(x)$ is discontinuous in $\mathbb{Q}$.*

Proof. Let $r = \frac{p}{q} \in \mathbb{Q}$ such that $g.c.d(p,q) = 1$ implying $f(r) = \frac{1}{q} > 0$. Consider a sequence $x_k = r + \frac{1}{k\sqrt{2}}$. Clearly, x_k is a sequence in $\mathbb{R} \setminus \mathbb{Q}$ and hence, $f(x_k) = 0$. Even though, $x_k \to r$ as $k \to \infty$, we observe that $f(x_k) \not\to f(r) \neq 0$ as $k \to \infty$. Hence, the function is not continuous in $\mathbb{Q}$ by Proposition 9.4.7 in [1]. $\square$

Lemma 3.2. *$t(x)$ is continuous in $\mathbb{R} \setminus \mathbb{Q}$.*

Proof. Let x_0 be an irrational number and hence, $f(x_0) = 0$. By definition, the function is periodic with a time period 1. Hence, without loss of generality, choose $x_0 \in [0,1] \setminus \mathbb{Q}$. Given $\epsilon > 0$, we need to find $\delta > 0$ such that $\sup\limits_{x \in B(x_0,\delta)} f(x) \leq \epsilon$.

Extending interspersing of reals with integers (Exercise 5.4.3 in [1]), if we choose $m \in \mathbb{N}$, then $\exists\ k \in \mathbb{Z}$, $x_0 \in \left(\frac{k}{m}, \frac{k+1}{m}\right)$. We define $\delta_m = \min\{|\frac{k}{m} - x_0|, |\frac{k+1}{m} - x_0|\} > 0$ since $x_0 \notin \mathbb{Q}$. Similarly, $\exists\ N \in \mathbb{N}$ such that $\frac{1}{N} \leq \epsilon < \frac{1}{N-1}$. If $x \in [0,1]$ is such that $f(x) > \epsilon$, then $x \in Q$ since irrationals are mapped to 0 by f. Also, we can say that $x = \frac{p}{q}$ is such that $\frac{1}{q} > \epsilon \geq \frac{1}{N} \Rightarrow q < N$. Hence, the number of possible values x can take is only finitely many and is bounded by N^2 ($x = \frac{p}{q} \leq 1$). So, we can choose $\delta = \min\{\delta_1, \delta_2, \ldots, \delta_m\} > 0$ since the set of δ_m is finite, which completes the proof.

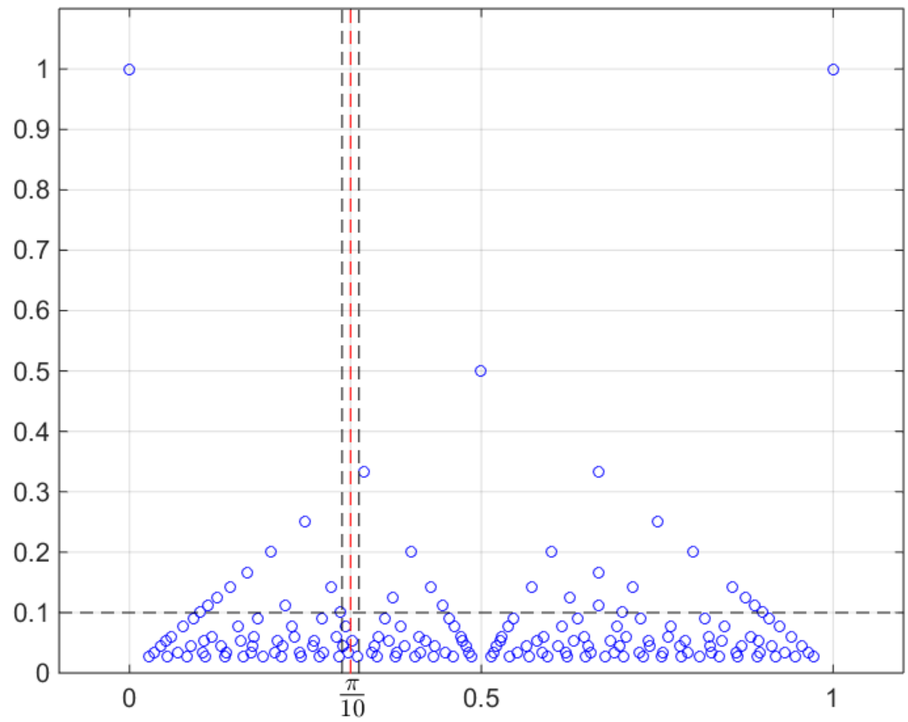

Figure 3: Thomae's function (only plotted for prime numbers $q \leq 37$) with $x_0 = \frac{\pi}{10}$ and $N = 10$

Figure 3 demonstrates the choice of δ when $x_0 = \frac{\pi}{10} \in [0,1] \setminus \mathbb{Q}$ and $\epsilon = 0.1 \Rightarrow N = 10$. We find that $\delta = 0.012$ is sufficient to ensure that $\forall\ x \in B(x_0, \delta),\ 0 \leq f(x) < \epsilon$. $\square$

From Lemma 3.1 and Lemma 3.2, $t(x)$ is known to be discontinuous only in $\mathbb{Q}$. From Lemma 2.1, Lemma 2.4 and the fact that $t(x)$ is bounded in $[0,1]$, we can conclude that $t(x)$ is Riemann integrable in $[0,1]$ using Theorem 3.1. For any partition $\boldsymbol{P}$ of $[0,1]$, we have $L(t, \boldsymbol{P}) = 0$ since $\inf\limits_{x \in [0,1]} t(x) = 0$. Hence, from Definition 2.2, $\int_0^1 t(x)dx = 0$.

3.2 Uncountably infinite discontinuities example: $1_C(x)$

From Lemma 2.5, we know that $1_C(x)$ has uncountably infinite discontinuities and the set of discontinuities is the Cantor set. Since $1_C(x)$ is a bounded function in $[0,1]$, $1_C(x)$ is Riemann integrable in $[0,1]$ according to Lebesgue's criterion (Theorem 3.1). From Remark 2.2 and Lemma 2.5, $\int_0^1 1_C(x)dx = m(C) = 0$.

4 Proof of Lebesgue's Criterion

We restate Theorem 3.1: *If $f(x)$ is a bounded function defined on $[a,b]$, then f is Riemann integrable iff the set of points on which f is discontinuous, say S, is a set of measure zero.*

We first define D_α as a set containing all discontinuity points who have a jump greater than α as:

$$D_\alpha = \left\{ x \in [a,b] : \left(\sup_{B(x,r)} f(x) - \inf_{B(x,r)} f(x) \right) > \alpha, \ \forall\ r > 0 \right\}$$

For the example function discussed in the proof of Theorem 2.3, let us say Figure 4 describes the jumps of f at $c, d \in [a,b]$, then $c \in D_\alpha$ but $d \notin D_\alpha$.

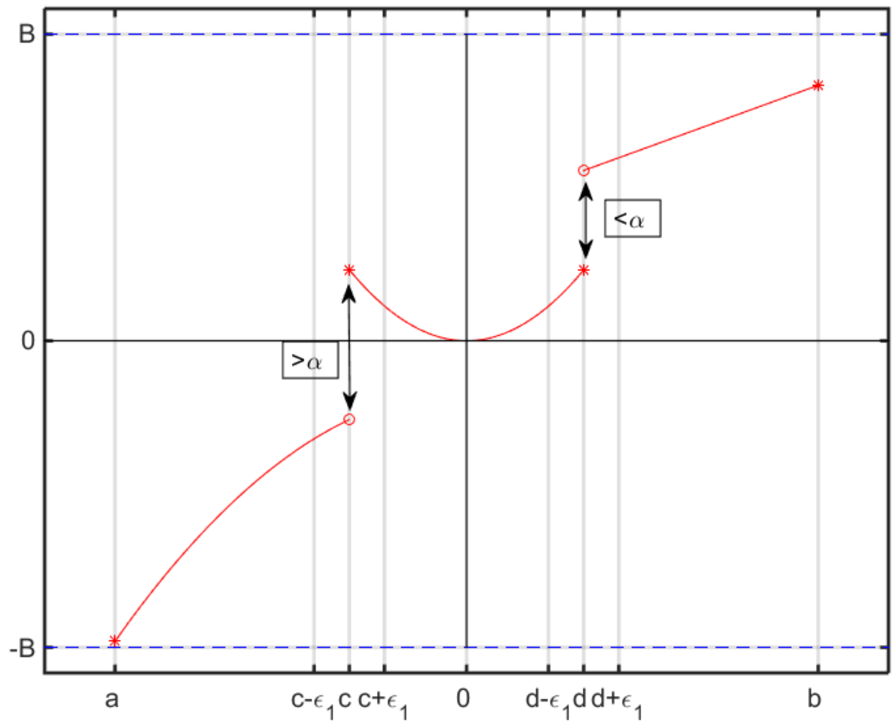

Figure 4: Figure illustrating the definition of D_α

We split the proof of Theorem 3.1 into the proof for necessary and sufficient conditions for Riemann integrability. These proofs were inspired from [4] and [6].

4.1 Some useful lemmas for the proof of Theorem 3.1

Consider the following lemmas:

Lemma 4.1. $\sup_{x \in E} f(x) - \inf_{x \in E} f(x) = \sup_{x,y \in E} |f(x) - f(y)|$ *where E is a subset of the domain of f.*

Proof. For $x, y \in E$, $f(x) - f(y) \leq |f(x) - f(y)|$. From definitions of sup and inf, we can say that $\sup_{x,y \in E} |f(x) - f(y)|$ is equivalent to $\sup_{x \in E}\left(\sup_{y \in E}(f(x) - f(y))\right) = \sup_{x \in E} f(x) - \inf_{x \in E} f(x)$ since $\sup(-E) = -\inf(E)$. □

Lemma 4.2. *D_α is a closed and compact set.*

Proof. Let x_0 be a limit point of D_α. By definition of a limit point (Definition 6.4.1 in [1]), $\exists$ a sequence $\{x_i\}_{i=1}^{\infty}$ in D_α which has its limit as x_0. Hence, for any $r > 0$, we have all but finitely many points of the sequence in $B(x_0, r)$. But this means $\forall\ r > 0$, $\left(\sup_{B(x_0,r)} f(x) - \inf_{B(x_0,r)} f(x)\right) > \alpha$ and hence, $x_0 \in D_\alpha$. Since D_α contains all its limit points, D_α is closed by Definition 9.1.15 in [1].

Since D_α is a subset of a bounded interval, I, it is bounded from Definition 9.1.22 in [1]. From Heine-Borel theorem, we conclude that since $D_\alpha \subseteq \mathbb{R}$ is closed and bounded in $\mathbb{R}$, it is compact in $\mathbb{R}$. □

Lemma 4.3. *f is continuous at x_0 if and only if given $\epsilon > 0$, $\exists\ r > 0$ such that* $\sup_{B(x_0,r)} f(x) - \inf_{B(x_0,r)} f(x) < \epsilon$.

Proof. The necessary condition for the continuity is proved as follows:

The hypothesis is that given $\epsilon > 0$, $\exists\ r > 0$ such that $\sup_{B(x_0,r)} f(x) - \inf_{B(x_0,r)} f(x) < \epsilon$. We know that for $x, y \in B(x_0, r)$, $|f(x) - f(y)| \leq \sup_{x,y \in B(x_0,r)} |f(x) - f(y)| < \epsilon$ as given by Lemma 4.1 and hypothesis. We have met the condition for continuity of f at x (Proposition 9.4.7 in [1]).

The sufficient condition for the continuity is proved by proving its contrapositive: *f is not continuous at x_0 if $\exists\ \epsilon$ such that $\forall\ r > 0$,* $\sup_{B(x_0,r)} f(x) - \inf_{B(x_0,r)} f(x) > \epsilon$.

The hypothesis is that $\exists\ \epsilon$ such that $\forall\ r > 0$, $\sup_{B(x_0,r)} f(x) - \inf_{B(x_0,r)} f(x) > \epsilon$. From Lemma 4.1, we know that $\forall\ r > 0$, $\sup_{x,y \in E} |f(x) - f(y)| > \epsilon$. By definition of sup, $\forall\ r > 0$, $\exists\ x, y \in B(x_0, r)$, $|f(x) - f(y)| > \epsilon$. We have met the condition for discontinuity of f at x (Proposition 9.4.7 in [1]). □

Lemma 4.4. $S = \bigcup_{\alpha > 0} D_\alpha$.

Proof. If $x \in \bigcup_{\alpha > 0} D_\alpha$, then $\exists\ \alpha > 0$ such that $\forall\ r > 0$, $\sup_{B(x,r)} f(x) - \inf_{B(x,r)} f(x) > \alpha$. From Lemma 4.3, it is clear that the function is not continuous at x and hence $x \in S$. Therefore, $\bigcup_{\alpha > 0} D_\alpha \subseteq S$.

If $x \in S$, then by definition, f is discontinuous at x. Hence, from Lemma 4.3, we know that $\exists\ \alpha > 0$ such that $x \in D_\alpha$. Hence, $x \in \bigcup_{\alpha > 0} D_\alpha$ implying $S \subseteq \bigcup_{\alpha > 0} D_\alpha$ or $\bigcup_{\alpha > 0} D_\alpha = S$. □

4.2 Proof for necessary condition for Riemann integrability

In this subsection, the necessary condition is proven: *If $f(x)$ is a bounded function defined on $[a, b]$ and if the set of points on which f is discontinuous, say S, is a set of measure zero, then f is Riemann integrable.*

Proof. We would like to show f is Riemann integrable. We know,

1. f is bounded in $[a, b]$.

2. S is of measure zero.

From definition of a set of measure zero, given $\epsilon_1 > 0$, $\exists$ a collection of open intervals $\{I_n\}_{n=0}^{\infty}$ such that $S \subseteq \bigcup_{n=0}^{\infty} I_n$ and $\sum_{n=0}^{\infty} |I_n| < \epsilon_1$. Given $\alpha > 0$, $D_\alpha \subset S$ from Lemma 4.4. Hence, from Lemma 4.2 and open cover definition of compact sets (Theorem 12.5.8 in [7]), there exists a finite sub-cover $\{I_{n_k}\}_{k=0}^{N_1}$ for D_α.

We can extract out from this finite sub-cover, a sub-cover comprising of disjoint open sets G_n. This new sub-cover is generated using the algorithm of fusing overlapping open sets I_{n_k}. Hence, $D_\alpha \subseteq \bigcup_{n=0}^{N} G_n$ and we define a closed set $K = [a,b] \setminus \bigcup_{n=0}^{N} G_n$. By definition,

$$\sum_{n=0}^{N} |G_n| \leq \sum_{n=0}^{\infty} |I_n| < \epsilon_1 \tag{2}$$

and from Remark 2.3, we have $m([a,b]) = m(K) + m(\bigcup_{n=0}^{N} G_n) \leq m(K) + \epsilon_1$. Hence, if $\epsilon_1 < m([a,b])$,

$$m(K) \geq m([a,b]) - \epsilon_1 > 0 \tag{3}$$

Remark 4.1. *Note that this approach is very similar to the proof for Theorem 2.3.*

Since all points in $[a,b]$ which have a discontinuous jump greater than α have been included in $\bigcup_{n=0}^{N} G_n$, $\exists$ a partition $\boldsymbol{P}_K$ of K such that for all $x, y \in J_i \in \boldsymbol{P}_K$, $|f(x) - f(y)| < \alpha$. But, $f(x) - f(y) \leq |f(x) - f(y)|$, $\forall\ x, y \in J_i$. Hence, $\forall\ J_i \in \boldsymbol{P}_K$, we have, from Lemma 4.1,

$$\sup_{x \in J_i} f(x) - \inf_{x \in J_i} f(x) = \sup_{x,y \in J_i} |f(x) - f(y)| \leq \alpha \tag{4}$$

We define a partition $\boldsymbol{P}$ for $[a,b]$ defined as $\boldsymbol{P} = \boldsymbol{P}_K \cup \{\{G\}_{n=0}^{N}\}$. For brevity, we denote $\left(\sup_{x \in I_i} f(x) - \inf_{x \in I_i} f(x)\right)$ as $(M-m)_i$ for each $I_i \in \boldsymbol{P}$. Since f is bounded, $\exists\, B \in \mathbb{R}$ such that $\forall\ x \in [a,b]$,

$$|f(x)| \leq B \tag{5}$$

Given $\epsilon > 0$, (chosen ϵ must be less than[3] $2B \times m([a,b])$), consider $U(f,\boldsymbol{P}) - L(f,\boldsymbol{P})$. If f was not bounded in $[a,b]$, then $U(f,\boldsymbol{P}) - L(f,\boldsymbol{P})$ would not be well-defined. We have

$$\begin{aligned}
U(f,\boldsymbol{P}) - L(f,\boldsymbol{P}) &= \sum_{I_i \in \boldsymbol{P}} (M-m)_i |I_i| \\
&= \left[\text{By definition, } \boldsymbol{P} = \boldsymbol{P}_K \cup \{\{G\}_{n=0}^{N}\}\right] \\
&= \sum_{J_i \in \boldsymbol{P}_K} (M-m)_i |J_i| + \sum_{i=0}^{N} (M-m)_i |G_i| \\
&= \left[\text{From (4), } (M-m)_i \leq \alpha,\ \forall\ J_i \in \boldsymbol{P}_K \text{ and by (5), } (M-m)_i \leq 2B,\ \forall\ \{G_i\}_{i=0}^{N}\right] \\
&\leq \alpha \sum_{J_i \in \boldsymbol{P}_K} |J_i| + 2B \sum_{i=0}^{N} |G_i| \\
&\leq \left[\text{Use (2), and Remark 2.3 and } K = \bigcup_{J_i \in \boldsymbol{P}_K} J_i \text{ gives } m(K) = \sum_{J_i \in \boldsymbol{P}_K} |J_i|\right] \\
&\leq \alpha m(K) + 2B\epsilon_1 \\
&= \left[\text{Choosing } \epsilon_1 = \tfrac{\epsilon}{4B} > 0 \text{ and } \alpha = \tfrac{\epsilon}{2m(K)} > 0 \text{ which is allowed by (3)}\right] \\
&\leq \epsilon
\end{aligned}$$

Hence, by Definition 2.2, we conclude that f is Riemann integrable. □

[3] This constraint comes from (3) and the subsequent usage of ϵ in determining ϵ_1.

4.3 Proof for sufficient condition for Riemann integrability

We prove the sufficient condition by proving the contrapositive: *If $f(x)$ is a bounded function defined on $[a, b]$ and if the set of points on which f is discontinuous, say S, is not a set of measure zero, then f is not Riemann integrable.*

Proof. We would like to show f is not Riemann integrable. We know,

1. f is bounded in $[a, b]$.

2. S is not of measure zero.

From Lemma 4.4 and Remark 2.4, we know that since $m(S) > 0$, then $\exists\ \alpha > 0$ such that $m(D_\alpha) > 0$. By density of reals, $\exists\ \epsilon_\alpha > 0$ such that $m(D_\alpha) > \epsilon_\alpha$.

Let $\boldsymbol{P} = \{J_i\}_{i=0}^{N}$ for some $N \in \mathbb{N}$ be a partition of $[a, b]$. We can define an index set A as $A = \{i : J_i \cap D_\alpha \neq \phi\}$. Therefore, A has the indices of those intervals in the partition $\boldsymbol{P}$ which contain at least one $x \in D_\alpha$. Hence, following the convention in Subsection 4.2, we have $(M - m)_i > \alpha\ \forall\ i \in A$. Also, by Remark 2.5, we have $m(\bigcup\limits_{i \in A} J_i) \geq m(D_\alpha) \geq \epsilon_\alpha$ since by definition, $D_\alpha \subseteq \bigcup\limits_{i \in A} J_i$.

Given $\epsilon > 0$, consider $U(f, \boldsymbol{P}) - L(f, \boldsymbol{P})$. If f was not bounded in $[a, b]$, then $U(f, \boldsymbol{P}) - L(f, \boldsymbol{P})$ would not be well-defined. We have

$$
\begin{aligned}
U(f, \boldsymbol{P}) - L(f, \boldsymbol{P}) &= \sum_{i \in A} (M - m)_i |J_i| + \sum_{i \notin A} (M - m)_i |J_i| \\
&= \left[\sum_{i \notin A} (M - m)_i |J_i| \geq 0 \text{ and using Proposition 5.4.7}(d) \text{ in [1]}\right] \\
&\geq \sum_{i \in A} (M - m)_i |J_i| \\
&\geq \alpha \epsilon_\alpha \\
&= \left[\text{Choosing } \epsilon_\alpha = \tfrac{\epsilon}{\alpha} > 0\right] \\
&\geq \epsilon
\end{aligned}
$$

We have thus produced an $\epsilon > 0$ such that for every partition $\boldsymbol{P}$, $U(f, \boldsymbol{P}) - L(f, \boldsymbol{P}) > \epsilon$. Using Definition 2.2, we conclude that f is not Riemann integrable. □

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