Hindawi Publishing Corporation
International Journal of Differential Equations
Volume 2013, Article ID 210270, 4 pages
http://dx.doi.org/10.1155/2013/210270

Research Article

Behavior of the p-Laplacian on Thin Domains

Ricardo P. Silva

Universidade Estadual Paulista, Departamento de Matemática, Instituto de Geociências e Ciências Exatas, 13506-900 Rio Claro, SP, Brazil

Correspondence should be addressed to Ricardo P. Silva; rpsilva@rc.unesp.br

Received 23 July 2013; Revised 2 October 2013; Accepted 3 October 2013

Academic Editor: Tuncay Candan

We give the characterization of the limiting behavior of solutions of elliptic equations driven by the p-Laplacian operator with Neumann boundary conditions posed in a family of thin domains.

1. Introduction

The investigation of parabolic and elliptic equations on thin domains has received considerable attention over the last twenty years. Such equations can appear motivated by homogenization problems in thin structures as in [1–7], as well as in the parabolic counterpart, associated with the continuity of global attractors for dissipative equations as in [1, 8–15]. Whatever the motivations that appear, the key point in the study of any kind of perturbation problem is to find the limiting one. In this specific domain perturbation problem (thin domains), it means to find an equation posed in a lower dimensional domain in order to compare the perturbed problems with. Our contribution in this short note goes in this direction. We give the characterization of the limiting problem of a family of elliptic equations driven by the p-Laplacian operator. This can be used, for example, in the study of the asymptotic behavior (attractors) of dissipative equations governed by the p-Laplacian on thin domains, which is associated with localized large diffusion phenomena, see, for example, [16]. This is the first step in order to consider other aspects as the asymptotic dynamics (attractors). For the best of our knowledge this is an untouched topic in the literature and can be the starting point for investigation of quasi-linear parabolic equations on thin domains which is relevant in a variety of physical phenomena as non-Newtonian fluids as well as in flow through porous media.

In order to set up the problem, let ω be a smooth bounded domain in $\mathbb{R}^n$, $n \geq 1$, and $g \in C^2(\overline{\omega};\mathbb{R})$ a positive function; ϵ will represent a small positive parameter which will converge to zero. We consider the family of domains $\Omega^\epsilon \subset \mathbb{R}^{n+1}$ defined by

$$\Omega^\epsilon := \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : x \in \omega,\ 0 < y < \epsilon g(x)\}. \quad (1)$$

The aim of this paper is to characterize the limiting problem ($\epsilon = 0$) for the family of elliptic equations

$$\begin{aligned} -\Delta_p u + |u|^{p-2}u &= f^\epsilon, \quad \text{in } \Omega^\epsilon, \\ \frac{\partial u}{\partial \eta^\epsilon} &= 0, \quad \text{on } \partial\Omega^\epsilon, \end{aligned} \quad (2)$$

where $p > 2$, $f^\epsilon \in L^q(\Omega^\epsilon)$, $(1/p) + (1/q) = 1$, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ denotes the p-Laplacian operator and η^ϵ denotes the outward unitary normal vector field to $\partial\Omega^\epsilon$.

Definition 1. Given $f^\epsilon \in L^q(\Omega^\epsilon)$, $1 < q < 2$, one says that $u \in W^{1,p}(\Omega^\epsilon)$, $(1/p) + (1/q) = 1$ is a solution of (2) if

$$\int_{\Omega^\epsilon} \left(|\nabla u|^{p-2}\nabla u \cdot \nabla\varphi + |u|^{p-2}u\varphi\right) dx\,dy = \int_{\Omega^\epsilon} f^\epsilon \varphi\, dx\,dy, \quad (3)$$

for all $\varphi \in W^{1,p}(\Omega^\epsilon)$.

We recall that by [17, Theorem 2.1] and [17, Theorem 2.3] (3) has a unique solution $u^\epsilon \in W^{1,p}(\Omega^\epsilon)$.

In the analysis of the limiting behavior of u^ϵ, it will be useful to introduce the domain $\Omega := \omega \times (0, 1)$ which is independent of ϵ and is obtained from Ω^ϵ by the change of coordinates

$$\begin{aligned}\mathcal{T}^\epsilon : \Omega &\longrightarrow \Omega^\epsilon\\ (x, y) &\longmapsto (x, \epsilon g(x) y).\end{aligned} \tag{4}$$

Such change of coordinates induces an isomorphism from $W^{m,p}(\Omega^\epsilon)$ onto $W^{m,p}(\Omega)$ by

$$u \overset{\Phi^\epsilon}{\longmapsto} v := u \circ \mathcal{T}^\epsilon, \tag{5}$$

with partial derivatives related by

$$u_{x_i} = v_{x_i} - \frac{y g_{x_i}}{g} v_y, \quad i = 1, \ldots, n, \quad u_y = \frac{1}{\epsilon g} v_y. \tag{6}$$

In this new system of coordinates, (2) is written as

$$\begin{aligned}-\frac{1}{g}\operatorname{div}\left(\left|\mathscr{L}_\epsilon v\right|^{p-2} B_\epsilon v\right) + |v|^{p-2} v = h^\epsilon, &\quad \text{in } \Omega,\\ B_\epsilon v \cdot \eta = 0, &\quad \text{on } \partial\Omega,\end{aligned} \tag{7}$$

where $h^\epsilon := \Phi^\epsilon(f^\epsilon)$, $\mathscr{L}_\epsilon v = (v_{x_1} - (g_{x_1} y v_y/g), \ldots, v_{x_n} - (g_{x_n} y v_y/g), v_y/\epsilon g)$,

$$B_\epsilon v = \begin{bmatrix} g v_{x_1} - y g_{x_1} v_y \\ \vdots \\ g v_{x_n} - y g_{x_n} v_y \\ -\displaystyle\sum_{i=1}^n y g_{x_i} v_{x_i} + \frac{1}{\epsilon^2 g}\left(1 + \sum_{i=1}^n \left(\epsilon y g_{x_i}\right)^2\right) v_y \end{bmatrix}, \tag{8}$$

and η denotes the unit outward normal vector field to $\partial\Omega$.

Noticing that $u \in W^{1,p}(\Omega^\epsilon)$ is a solution of (2) if and only if $v := \Phi(u) \in W^{1,p}(\Omega)$ is a solution of (7), the rest of this paper is dedicated to the study of the limiting behavior of the solutions of (7); that is, functions $v \in W^{1,p}(\Omega)$ such that

$$\int_\Omega \left(\left|\mathscr{L}_\epsilon v\right|^{p-2} B_\epsilon v \cdot \nabla\varphi + g|v|^{p-2} v\varphi\right) dx\, dy = \int_\Omega g h^\epsilon \varphi\, dx\, dy, \tag{9}$$

for all $\varphi \in W^{1,p}(\Omega)$.

Due to the nature of this specific domain perturbation, solutions of (7) tend not to depend "so much" on the variable y as $\epsilon \approx 0$. This suggests comparing such solutions with solutions of the following equation:

$$\begin{aligned}-\frac{1}{g}\operatorname{div}\left(g|\nabla v|^{p-2}\nabla v\right) + |v|^{p-2} v = \widehat{h}, &\quad \text{in } \omega,\\ \frac{\partial v}{\partial \nu} = 0, &\quad \text{on } \partial\omega,\end{aligned} \tag{10}$$

for some appropriate $\widehat{h} \in L^q(\omega)$, where ν denotes the unit outward normal vector field to $\partial\omega$.

Again by [17, Theorem 2.1] and [17, Theorem 2.3] we can derive a unique solution $v^0 \in W^{1,p}(\omega)$ of (10).

The paper is organized as follows. In Section 2 we set up the appropriate functional framework which will be used to compare the problems (7) and (10), and in the subsequent Section 3, we formulate and prove the convergence results.

2. Preliminaries

Stressing for the fact that the domains Ω^ϵ vary in accordance with the small parameter ϵ, collapsing themselves to a lower dimensional subset as ϵ goes to 0, we perform a dilatation on the Lebesgue measure in $\mathbb{R}^{n+1}$ in order to preserve the relative capacity of measurable subsets of Ω^ϵ. Thus, we consider the Lebesgue space, $L^p(\Omega^\epsilon)$, endowed with the equivalent norm

$$|||u|||_{L^p(\Omega^\epsilon)} = \left[\epsilon^{-1}\int_{\Omega^\epsilon} |u|^p dx\, dy\right]^{1/p} \tag{11}$$

and the Sobolev space, $W^{1,p}(\Omega^\epsilon)$, endowed with the equivalent norm

$$|||u|||_{W^{1,p}(\Omega^\epsilon)} = \left[\epsilon^{-1}\int_{\Omega^\epsilon}\left(\left|\nabla_x u\right|^p + \left|\nabla_y u\right|^p + |u|^p\right) dx\, dy\right]^{1/p}. \tag{12}$$

We also consider equivalent norms in $L^p(\Omega)$ and in $W^{1,p}(\Omega)$ given, respectively, by

$$\begin{aligned}|||u|||_{L^p(\Omega)} &= \left[\int_\Omega g(x)|u|^p dx\, dy\right]^{1/p},\\ |||u|||_\epsilon &:= \left[\int_\Omega g(x)\left(\left|\nabla_x u\right|^p + \frac{1}{\epsilon^p}\left|\nabla_y u\right|^p + |u|^p\right) dx\, dy\right]^{1/p}.\end{aligned} \tag{13}$$

It is immediate from these definitions that

$$|||u|||_{L^p(\Omega^\epsilon)} = \left|\left|\left|\Phi^\epsilon(u)\right|\right|\right|_{L^p(\Omega)}, \tag{14}$$

and there exist positive constants c_1, c_2 such that

$$c_1|||u|||_{W^{1,p}(\Omega^\epsilon)} \le \left|\left|\left|\Phi^\epsilon(u)\right|\right|\right|_\epsilon \le c_2|||u|||_{W^{1,p}(\Omega^\epsilon)}. \tag{15}$$

Finally, since we need to compare functions defined in different domains, for example, Ω and ω, is natural to introduce the following operators.

Average projector:

$$\begin{aligned}M &: W^{1,p}(\Omega) \longrightarrow W^{1,p}(\omega),\\ M(u)(x) &= \int_0^1 u(x, y)\, dy.\end{aligned} \tag{16}$$

Extension operator:

$$\begin{aligned}E &: W^{1,p}(\omega) \longrightarrow W^{1,p}(\Omega),\\ E(u)(x, y) &= u(x).\end{aligned} \tag{17}$$

3. Convergence

Given $f^\epsilon \in L^q(\Omega^\epsilon)$, $1 < q < 2$, if $u^\epsilon \in W^{1,p}(\Omega^\epsilon)$ is a solution of (2) we have by Hölder's inequality that

$$\begin{aligned}\left\|u^\epsilon\right\|_{W^{1,p}(\Omega^\epsilon)}^p &= \int_{\Omega^\epsilon} f^\epsilon u^\epsilon dx\, dy\\ &\le \left\|f^\epsilon\right\|_{L^q(\Omega^\epsilon)}\left\|u^\epsilon\right\|_{L^p(\Omega^\epsilon)}\\ &\le \left\|f^\epsilon\right\|_{L^q(\Omega^\epsilon)}\left\|u^\epsilon\right\|_{W^{1,p}(\Omega^\epsilon)}.\end{aligned} \tag{18}$$

This shows that

$$|||u^\epsilon|||^{p-1}_{W^{1,p}(\Omega^\epsilon)} \leq |||f^\epsilon|||_{L^q(\Omega^\epsilon)}, \tag{19}$$

which gives us the following a priori estimate for solutions of (7):

$$|||v^\epsilon|||^{p-1}_\epsilon \leq c_2^{p-1}|||h^\epsilon|||_{L^q(\Omega)}, \tag{20}$$

where $v^\epsilon = \Phi^\epsilon(u^\epsilon)$ and $h^\epsilon = \Phi^\epsilon(f^\epsilon)$.

Such a priori estimate is the essence of the following lemma.

Lemma 2. *Let $h^\epsilon \in L^q(\Omega)$, $1 < q < 2$, such that $h^\epsilon \rightharpoonup h^0$ in $L^q(\Omega)$. If $v^\epsilon \in W^{1,p}(\Omega)$ is a solution of* (7)*, there exists $v^0 \in W^{1,p}(\omega)$ solution of* (10) *with $\widehat{f} = M(h^0)$, such that up to subsequence*

$$v^\epsilon \xrightarrow{\epsilon\to 0} Ev^0, \quad \textit{weakly-}W^{1,p}(\Omega) \textit{ and strongly-}L^p(\Omega). \tag{21}$$

Proof. It follows from (20) that $\|v^\epsilon\|^{p-1}_{W^{1,p}(\Omega)} \leqslant c$, for some constant c independent of ϵ. Since $W^{1,p}(\Omega)$ is a reflexive space and $L^p(\Omega) \hookrightarrow W^{1,p}(\Omega)$ compactly, taking subsequence if necessary, there exists $\widetilde{v}^0 \in W^{1,p}(\Omega)$ such that $v^\epsilon \xrightarrow{\epsilon\to 0} \widetilde{v}^0$, weakly in $W^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$. Moreover, noticing that $\{|||v^\epsilon|||_\epsilon\}_\epsilon$ is bounded, one has that $\|\nabla_y v^\epsilon\|_{L^p(\Omega)} = O(\epsilon)$. Therefore $\nabla_y v^\epsilon \xrightarrow{\epsilon\to 0} 0$ in $L^p(\Omega)$, which means that $\nabla_y \widetilde{v}^0 = 0$ almost everywhere in Ω. This implies the existence of $v^0 \in W^{1,p}(\omega)$, such that $\widetilde{v}^0(x,y) = v^0(x)$, almost everywhere in Ω.

On the other hand, noticing that $E(\varphi) \in W^{1,p}(\Omega)$ whenever $\varphi \in W^{1,p}(\omega)$ and since $\nabla_y E(\varphi) = 0$, it follows from the weak convergence $|v^\epsilon|^{p-2}v^\epsilon \rightharpoonup |v^0|^{p-2}v^0$ in $L^q(\Omega)$ that

$$\begin{aligned}
&\int_\omega gM(h^0)\varphi\, dx \xleftarrow{\epsilon\to 0} \int_\Omega gh^\epsilon E(\varphi)\, dx\, dy \\
&= \int_\Omega \left(|\mathscr{L}_\epsilon v^\epsilon|^{p-2} B_\epsilon v^\epsilon \cdot \nabla E(\varphi) + g|v^\epsilon|^{p-2}v^\epsilon\, E(\varphi)\right) dx\, dy \\
&\xrightarrow{\epsilon\to 0} \int_\Omega \Big(g|\nabla\widetilde{v}^0|^{p-2}\nabla\widetilde{v}^0 \cdot \nabla E(\varphi) \\
&\qquad + g|\widetilde{v}^0|^{p-2}\widetilde{v}^0\, E(\varphi)\Big) dx\, dy \\
&= \int_\omega \left(g|\nabla v^0|^{p-2}\nabla v^0 \cdot \nabla\varphi + g|v^0|^{p-2}v^0\varphi\right) dx,
\end{aligned} \tag{22}$$

for all $\varphi \in W^{1,p}(\omega)$. □

As a consequence of the following theorem and inspired by [8, 15], we obtain the convergence of the family of solutions v^ϵ in the norm $|||\cdot|||_\epsilon$.

Theorem 3. *Let h^ϵ, v^ϵ, h^0, and v^0 be as in Lemma 2. Then*

$$\lim_{\epsilon\to 0}\left\|v^\epsilon - Ev^0\right\|_{W^{1,p}(\Omega)} = 0. \tag{23}$$

Proof. It follows from the weak convergence $v^\epsilon \rightharpoonup \widetilde{v}^0$ in $W^{1,p}(\Omega)$ (obtained in Lemma 2) that

$$\begin{aligned}
&\int_\Omega g\left(|\nabla_x\widetilde{v}^0|^p + |\widetilde{v}^0|^p\right) dx\, dy \\
&\leqslant \liminf_{\epsilon\to 0}\int_\Omega g\left(|\nabla_x v^\epsilon|^p + |\nabla_y v^\epsilon|^p + |v^\epsilon|^p\right) dx\, dy \\
&\leqslant \limsup_{\epsilon\to 0}\int_\Omega g\left(|\nabla_x v^\epsilon|^p + |\nabla_y v^\epsilon|^p + |v^\epsilon|^p\right) dx\, dy \\
&\leqslant \lim_{\epsilon\to 0}\int_\Omega g\left(|\nabla_x v^\epsilon|^p + \frac{1}{\epsilon^p}|\nabla_y v^\epsilon|^p + |v^\epsilon|^p\right) dx\, dy \\
&= \lim_{\epsilon\to 0}\int_\Omega gh^\epsilon v^\epsilon\, dx\, dy = \int_\Omega gh^0\widetilde{v}^0 \\
&= \int_\Omega g\left(|\nabla\widetilde{v}^0|^p + |\widetilde{v}^0|^p\right) dx\, dy.
\end{aligned} \tag{24}$$

Recalling that for $\widetilde{v}^0$, $[\int_\Omega g(|\nabla_x\widetilde{v}^0|^p + |\widetilde{v}^0|^p)dx\,dy]^{1/p}$ is an equivalent norm in $W^{1,p}(\Omega)$, this proves the statement. □

Corollary 4. *Let v^ϵ and v^0 be as in Theorem 3. Then*

$$\lim_{\epsilon\to 0}\left|\left|\left|v^\epsilon - Ev^0\right|\right|\right|_\epsilon = 0. \tag{25}$$

Proof. According to Theorem 3

$$\begin{aligned}
&\int_\Omega g\left(|\nabla_x\widetilde{v}^0|^p + |\widetilde{v}^0|^p\right) dx\, dy \\
&= \lim_{\epsilon\to 0}\int_\Omega g\left(|\nabla_x v^\epsilon|^p + \frac{1}{\epsilon^p}|\nabla_y v^\epsilon|^p + |v^\epsilon|^p\right) dx\, dy,
\end{aligned} \tag{26}$$

which implies that $\lim_{\epsilon\to 0}(1/\epsilon^p)|\nabla_y v^\epsilon|^p = 0$. □

Remark 5. We would like to recall that Hale and Raugel in [12] obtained in the case $p = 2$ as the limiting problem for the similar equation

$$\begin{aligned}
-\frac{1}{g}\operatorname{div}(\mathscr{B}_\epsilon v) + v = h^\epsilon, &\quad \text{in } \Omega, \\
B_\epsilon v \cdot \eta = 0, &\quad \text{on } \partial\Omega,
\end{aligned} \tag{27}$$

the problem

$$\begin{aligned}
-\frac{1}{g}\operatorname{div}(g\nabla v) + v = \widehat{h}, &\quad \text{in } \omega, \\
\frac{\partial v}{\partial \nu} = 0, &\quad \text{on } \partial\omega.
\end{aligned} \tag{28}$$

After the previous considerations we point out the robustness of the structure of this limiting problem in the following sense: considering p as a parameter as well as ϵ, allowing $p \to 2$ and $\epsilon \to 0$, independent of the order of the convergence, we obtain the same limiting problem, namely, (28). We summarize that in the following commutative diagram:

$$\begin{array}{ccc}
\text{Equation (7)} & \xrightarrow{p\to 2} & \text{Equation (27)} \\
\epsilon \to 0 \downarrow & \circlearrowleft & \downarrow \epsilon \to 0 \\
\text{Equation (10)} & \xrightarrow[p\to 2]{} & \text{Equation (28)}
\end{array} \tag{29}$$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The author thanks Professor Olímpio Myagaki for stimulating conversations and useful remarks. The author also thanks the referees for their suggestions which led to an improvement of the text. This work was partially supported by FAPESP no. 2012/06753-8, FUNDUNESP no. 0135812, and FUNDUNESP-PROPe no. 0019/008/13, Brazil.

References

[1] J. M. Arrieta, A. N. Carvalho, M. C. Pereira, and R. P. Silva, "Semilinear parabolic problems in thin domains with a highly oscillatory boundary," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 74, no. 15, pp. 5111–5132, 2011.

[2] J. M. Arrieta and M. C. Pereira, "Elliptic problems in thin domains with highly oscillating boundaries," *Boletín de la Sociedad EspañoLa de Matematica Aplicada*, vol. 51, pp. 17–25, 2010.

[3] J. M. Arrieta and M. C. Pereira, "Homogenization in a thin domain with an oscillatory boundary," *Journal des Mathematiques Pures et Appliquees*, vol. 96, no. 1, pp. 29–57, 2011.

[4] J. M. Arrieta and M. C. Pereira, "Thin domain with extremely high oscillatory boundaries," *Journal of Mathematical Analysis and Applications*, vol. 404, no. 1, pp. 86–104, 2013.

[5] D. Cioranescu and J. S. Jean-Paulin, *Homogenization of Reticulated Structures*, Springer, New York, NY, USA, 1980.

[6] M. C. Pereira and R. P. Silva, "Rates of convergence for a homogenization problem in highly oscillating thin domains," *Proceeding of Dynamic Systems and Applications*, vol. 6, pp. 337–340, 2012.

[7] M. C. Pereira and R. P. Silva, "Error estimatives for a Neumann problem in highly oscillating thin domain," *Discrete and Continuous Dynamical Systems A*, vol. 33, no. 2, pp. 803–817, 2013.

[8] F. Antoci and M. Prizzi, "Reaction-diffusion equations on unbounded thin domains," *Topological Methods in Nonlinear Analysis*, vol. 18, pp. 283–302, 2001.

[9] T. Elsken, "Limiting behavior of attractors for systems on thin domains," *Hiroshima Mathematical Journal*, vol. 32, no. 3, pp. 389–415, 2002.

[10] T. Elsken, "A reaction-diffusion equation on a net-shaped thin domain," *Studia Mathematica*, vol. 165, no. 2, pp. 159–199, 2004.

[11] T. Elsken, "Continuity of attractors for net-shaped thin domain," *Topological Methods in Nonlinear Analysis*, vol. 26, pp. 315–354, 2005.

[12] J. K. Hale and G. Raugel, "Reaction-diffusion equations on thin domains," *Journal de Mathematiques Pures et Apliquees*, vol. 9, no. 71, pp. 33–95, 1992.

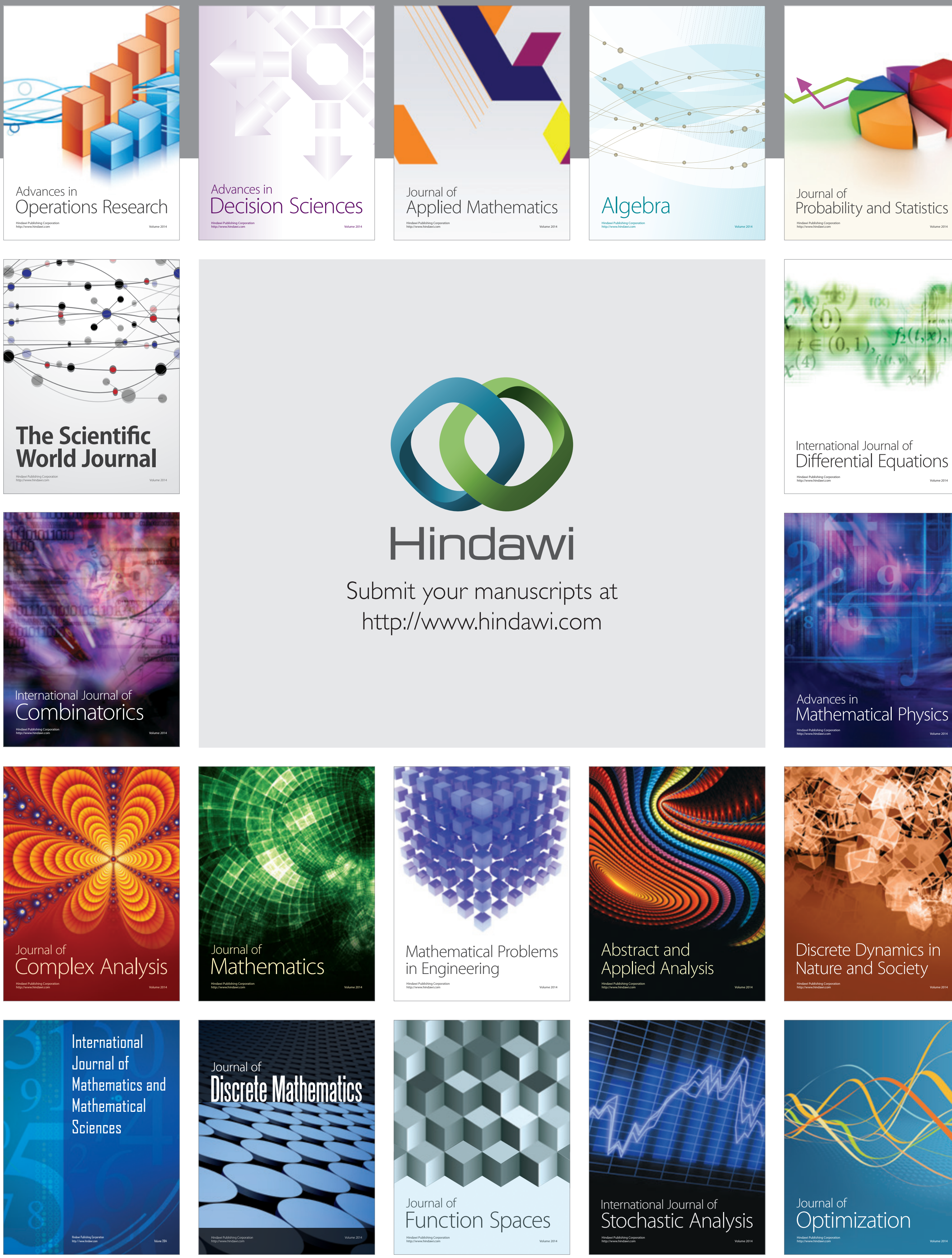
[13] M. Prizzi and K. P. Rybakowski, "The effect of domain squeezing upon the dynamics of reaction-diffusion equations," *Journal of Differential Equations*, vol. 173, no. 2, pp. 271–320, 2001.

[14] G. Raugel, "Dynamics of partial differential equations on thin domains," in *Dynamical Systems*, vol. 1609 of *Lecture Notes in Mathematics*, pp. 208–315, Springer, New York, NY, USA, 1995.

[15] R. P. Silva, "A note on resolvent convergence on a thin domain," *Bulletin of the Australian Mathematical Society*, 2013.

[16] V. L. Carbone, C. B. Gentile, and K. Schiabel-Silva, "Asymptotic properties in parabolic problems dominated by a p-Laplacian operator with localized large diffusion," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 74, no. 12, pp. 4002–4011, 2011.

[17] J. L. Lions, *Quelques Methodes de Resolution des Problemes aux Limites non Lineaires*, Dunod, Paris, France, 1969.


Advances in Operations Research
Advances in Decision Sciences
Journal of Applied Mathematics
Algebra
Journal of Probability and Statistics
The Scientific World Journal
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of Differential Equations
International Journal of Combinatorics
Advances in Mathematical Physics
Journal of Complex Analysis
Journal of Mathematics
Mathematical Problems in Engineering
Abstract and Applied Analysis
Discrete Dynamics in Nature and Society
International Journal of Mathematics and Mathematical Sciences
Journal of Discrete Mathematics
Journal of Function Spaces
International Journal of Stochastic Analysis
Journal of Optimization