Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2016, Article ID 6842521, 7 pages
http://dx.doi.org/10.1155/2016/6842521

Research Article

Global Character of a Six-Dimensional Nonlinear System of Difference Equations

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Received 25 November 2015; Accepted 2 June 2016

Academic Editor: Zhan Zhou

The aim of this paper is to study the dynamical behavior of positive solutions for a system of rational difference equations of the following form: $u_{n+1} = \alpha u_{n-1}/(\beta + \gamma v_{n-2}^{p})$, $v_{n+1} = \alpha_1 v_{n-1}/(\beta_1 + \gamma_1 u_{n-2}^{p})$, $n = 0, 1, \ldots$, where the parameters $\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1, p$ and the initial values u_{-i}, v_{-i} for $i = 0, 1, 2$ are positive real numbers.

1. Introduction

Higher-order fractional difference equations are of great importance in applications. Such equations also seem surely as discrete analogues and as numerical solutions of differential and delay differential equations which model various diverse phenomena in biology, ecology, physiology, physics, engineering, economics, and so on. For example, see [1, 2]. The theory of difference equations is the focus point in applicable analysis. That is, the importance of the theory of difference equations in mathematics as a whole will go on. Hence, it is very valuable to investigate the behavior of solutions of a system of fractional difference equations and to present the stability character of equilibrium points.

In the recent times, the behavior of solutions of various systems of rational difference equations has been one of the main topics in the theory of difference equations (see [3–7] and the references cited therein).

According to us, it is of great importance to investigate not only rational nonlinear difference equations and their systems, but also those equations and systems which contain powers of arbitrary positive numbers (see [8–12]).

In [13], Kurbanlı et al. studied the behavior of positive solutions of the following system of difference equations:

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} + 1}, \qquad y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} + 1}, \qquad n \in \mathbb{N}, \tag{1}$$

where the initial conditions are arbitrary nonnegative real numbers.

In [14], Papaschinopoulos and Schinas studied the system of two nonlinear difference equations:

$$x_{n+1} = A + \frac{y_n}{x_{n-p}}, \qquad y_{n+1} = A + \frac{x_n}{y_{n-q}}, \qquad n \in \mathbb{N}, \tag{2}$$

where $A \in (0, \infty)$ and p, q are positive integers.

In [15], Zhang et al. studied the dynamical behavior of positive solutions for a system for third-order rational difference equations:

$$x_{n+1} = \frac{x_{n-2}}{B + y_n y_{n-1} y_{n-2}}, \qquad y_{n+1} = \frac{y_{n-2}}{A + x_n x_{n-1} x_{n-2}}, \qquad n \in \mathbb{N}. \tag{3}$$

In [16], Din et al. studied the dynamics of a system of fourth-order rational difference equations:

$$x_{n+1} = \frac{\alpha x_{n-3}}{\beta + \gamma y_n y_{n-1} y_{n-2} y_{n-3}},$$
$$y_{n+1} = \frac{\alpha_1 y_{n-3}}{\beta_1 + \gamma_1 x_n x_{n-1} x_{n-2} x_{n-3}}, \tag{4}$$
$$n \in \mathbb{N}.$$

In [17], Touafek and Elsayed investigated the behavior of solutions of systems of difference equations:

$$x_{n+1} = \frac{x_{n-3}}{\pm 1 \pm x_{n-3} y_{n-1}},$$
$$y_{n+1} = \frac{y_{n-3}}{\pm 1 \pm y_{n-3} x_{n-1}}, \tag{5}$$
$$n \in \mathbb{N},$$

with a nonzero real number's initial conditions.

In [18], El-Owaidy et al. investigated the global behavior of the following difference equation:

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma x_{n-2}^p}, \quad n \in \mathbb{N}, \tag{6}$$

with nonnegative parameters and nonnegative initial values.

Motivated by all above-mentioned works, in the paper we investigate the equilibrium points, the local asymptotic stability of these points, the global behavior of positive solutions, the existence unbounded solutions, and the existence of the prime two-periodic solutions of the following system:

$$u_{n+1} = \frac{\alpha u_{n-1}}{\beta + \gamma v_{n-2}^p},$$
$$v_{n+1} = \frac{\alpha_1 v_{n-1}}{\beta_1 + \gamma_1 u_{n-2}^p}, \tag{7}$$
$$n = 0, 1, \ldots,$$

where the parameters $\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1, p$ and the initial values u_{-i}, v_{-i} for $i = 0, 1, 2$ are positive real numbers. Our results extend and complement some results in the literature.

Note that system (7) can be reduced to the following system of difference equations:

$$x_{n+1} = \frac{r x_{n-1}}{1 + y_{n-2}^p},$$
$$y_{n+1} = \frac{s y_{n-1}}{1 + x_{n-2}^p}, \tag{8}$$
$$n = 0, 1, \ldots$$

by the change of variables $u_n = (\beta_1/\gamma_1)^{1/p} x_n$ and $v_n = (\beta/\gamma)^{1/p} y_n$ with $r = \alpha/\beta$ and $s = \alpha_1/\beta_1$. So, in order to study system (7), we investigate system (8).

By using the induction and the equations in (8) we see that if x_{-i}, y_{-i} are positive real numbers for $i \in \{0, 1, 2\}$, then

$$\min\{x_n, y_n\} > 0, \quad n \geq -2, \tag{9}$$

which means that positive initial values generate positive solutions of system (8).

As far as we examine, there is no paper dealing with system (7). Therefore, in this paper we focus on system (7) in order to fill in the gap.

2. Preliminaries

For the completeness in the paper, we find it useful to remember some basic concepts of the difference equations theory as follows.

Let us introduce the six-dimensional discrete dynamical system:

$$x_{n+1} = f_1(x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2}),$$
$$y_{n+1} = f_2(x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2}), \tag{10}$$

$n \in \mathbb{N}$, where $f_1 : I_1^3 \times I_2^3 \to I_1$ and $f_2 : I_1^3 \times I_2^3 \to I_2$ are continuously differentiable functions and I_1, I_2 are some intervals of real numbers. Also, a solution $\{(x_n, y_n)\}_{n=-2}^{\infty}$ of system (10) is uniquely determined by initial values $(x_{-i}, y_{-i}) \in I_1 \times I_2$ for $i \in \{0, 1, 2\}$.

Definition 1. An equilibrium point of system (10) is a point $(\overline{x}, \overline{y})$ that satisfies

$$\overline{x} = f_1(\overline{x}, \overline{x}, \overline{x}, \overline{y}, \overline{y}, \overline{y}),$$
$$\overline{y} = f_2(\overline{x}, \overline{x}, \overline{x}, \overline{y}, \overline{y}, \overline{y}). \tag{11}$$

Together with system (10), if we consider the associated vector map $F = (f_1, x_n, x_{n-1}, x_{n-2}, f_2, y_n, y_{n-1}, y_{n-2})$, then the point $(\overline{x}, \overline{y})$ is also called a fixed point of the vector map F.

Definition 2. Let $(\overline{x}, \overline{y})$ be an equilibrium point of system (10).

(i) An equilibrium point $(\overline{x}, \overline{y})$ is said to be stable if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every initial value $(x_{-i}, y_{-i}) \in I_1 \times I_2$, with $\sum_{i=-2}^{0} |x_i - \overline{x}| < \delta$, $\sum_{i=-2}^{0} |y_i - \overline{y}| < \delta$, implying $|x_n - \overline{x}| < \varepsilon$, $|y_n - \overline{y}| < \varepsilon$ for $n \in \mathbb{N}$.

(ii) If an equilibrium point $(\overline{x}, \overline{y})$ is not stable, then it is said to be unstable.

(iii) If an equilibrium point $(\overline{x}, \overline{y})$ is stable and there exists $\gamma > 0$ such that $\sum_{i=-2}^{0} |x_i - \overline{x}| < \gamma$, $\sum_{i=-2}^{0} |y_i - \overline{y}| < \gamma$ and $(x_n, y_n) \to (\overline{x}, \overline{y})$ as $n \to \infty$, then it is said to be asymptotically stable.

(iv) If $(x_n, y_n) \to (\overline{x}, \overline{y})$ as $n \to \infty$, then an equilibrium point $(\overline{x}, \overline{y})$ is said to be a global attractor.

(v) If an equilibrium point $(\overline{x}, \overline{y})$ is both global attractor and stable, then it is said to be globally asymptotically stable.

Definition 3. Let $(\overline{x}, \overline{y})$ be an equilibrium point of the map F where f_1 and f_2 are continuously differentiable functions at $(\overline{x}, \overline{y})$. The linearized system of (10) about the equilibrium point $(\overline{x}, \overline{y})$ is

$$X_{n+1} = F(X_n) = BX_n, \tag{12}$$

where

$$X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ y_n \\ y_{n-1} \\ y_{n-2} \end{pmatrix} \quad (13)$$

and B is a Jacobian matrix of system (10) about the equilibrium point $(\overline{x}, \overline{y})$.

Definition 4. For the system $X_{n+1} = F(X_n)$, $n = 0, 1, \ldots$, of difference equations such that $\overline{X}$ is a fixed point of F, if no eigenvalues of the Jacobian matrix B about $\overline{X}$ have absolute value equal to one, then $\overline{X}$ is called hyperbolic. If there exists an eigenvalue of the Jacobian matrix B about $\overline{X}$ with absolute value equal to one, then $\overline{X}$ is called nonhyperbolic.

The following result, known as the Linearized Stability Theorem, is very practical in confirming the local stability character of the equilibrium point $(\overline{x}, \overline{y})$ of system (10).

Theorem 5. *For the system $X_{n+1} = F(X_n)$, $n = 0, 1, \ldots$, of difference equations such that $\overline{X}$ is a fixed point of F, if all eigenvalues of the Jacobian matrix B about $\overline{X}$ lie inside the open unit disk $|\lambda| < 1$, then $\overline{X}$ is locally asymptotically stable. If one of them has a modulus greater than one, then $\overline{X}$ is unstable.*

For other basic knowledge about difference equations and their systems, the reader is referred to books [19–22].

3. Main Results

In this section we prove our main results.

Theorem 6. *One has the following cases for the equilibrium points of (8):*

(i) $(\overline{x}_0, \overline{y}_0) = (0, 0)$ *is always the equilibrium point of system (8).*

(ii) *If $r > 1$ and $s > 1$, then system (8) has the equilibrium point* $(\overline{x}_1, \overline{y}_1) = ((s-1)^{1/p}, (r-1)^{1/p})$.

(iii) *If $r > 1$ and $s = 1$, then system (8) has the equilibrium point* $(\overline{x}_2, \overline{y}_2) = (0, (r-1)^{1/p})$.

(iv) *If $r = 1$ and $s > 1$, then system (8) has the equilibrium point* $(\overline{x}_3, \overline{y}_3) = ((s-1)^{1/p}, 0)$.

(v) *If $r \in (0, 1)$, $s = 1$, and $1/p$ is an even positive integer, then system (8) has the equilibrium point* $(\overline{x}_4, \overline{y}_4) = (0, (r-1)^{1/p})$.

(vi) *If $r = 1$, $s \in (0, 1)$, and $1/p$ is an even positive integer, then system (8) has the equilibrium point* $(\overline{x}_5, \overline{y}_5) = ((s-1)^{1/p}, 0)$.

(vii) *If $r, s \in (0, 1)$ and $1/p$ is an even positive integer, then it also has the positive equilibrium point* $(\overline{x}_6, \overline{y}_6) = ((s-1)^{1/p}, (r-1)^{1/p})$.

Proof. The proof is easily obtained from the definition of equilibrium point. □

Before we give the following theorems about the local asymptotic stability of the aforementioned equilibrium points, we build the corresponding linearized form of system (8) and consider the following transformation:

$$\begin{aligned}&(x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2}) \\ &\quad \longrightarrow (f, f_1, f_2, g, g_1, g_2),\end{aligned} \quad (14)$$

where $f = rx_{n-1}/(1+y_{n-2}^p)$, $f_1 = x_n$, $f_2 = x_{n-1}$, $g = sy_{n-1}/(1+x_{n-2}^p)$, $g_1 = y_n$, and $g_2 = y_{n-1}$. The Jacobian matrix about the fixed point $(\overline{x}, \overline{y})$ under the above transformation is as follows:

$$B(\overline{x}, \overline{y}) = \begin{pmatrix} 0 & \dfrac{r}{1+\overline{y}^p} & 0 & 0 & 0 & -\dfrac{rp\overline{x}\overline{y}^{p-1}}{(1+\overline{y}^p)^2} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\dfrac{sp\overline{y}\overline{x}^{p-1}}{(1+\overline{x}^p)^2} & 0 & \dfrac{s}{1+\overline{x}^p} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad (15)$$

where $r, s, p \in (0, \infty)$.

Theorem 7. *(i) If $r < 1$ and $s < 1$, then the zero equilibrium point $(\overline{x}_0, \overline{y}_0)$ is locally asymptotically stable.*

(ii) If $r > 1$ or $s > 1$, then the zero equilibrium point $(\overline{x}_0, \overline{y}_0)$ is locally unstable.

Proof. (i) The linearized system of (8) about the equilibrium point $(\overline{x}_0, \overline{y}_0)$ is given by

$$X_{n+1} = B(\overline{x}_0, \overline{y}_0) X_n, \quad (16)$$

where $X_n = (x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2})^T$ and

$$B(\overline{x}_0, \overline{y}_0) = \begin{pmatrix} 0 & r & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \quad (17)$$

The characteristic equation of $B(\overline{x}_0, \overline{y}_0)$ is as follows:

$$P(\lambda) = \lambda^6 - (r+s)\lambda^4 + rs\lambda^2 = 0. \quad (18)$$

The roots of $P(\lambda)$ are

$$\begin{aligned} \lambda_1 &= \sqrt{s}, \\ \lambda_2 &= -\sqrt{s}, \\ \lambda_{3,4} &= 0, \\ \lambda_5 &= \sqrt{r}, \\ \lambda_6 &= -\sqrt{r}. \end{aligned} \tag{19}$$

Since all eigenvalues of the Jacobian matrix B about $(\overline{x}_0, \overline{y}_0)$ lie inside the open unit disk $|\lambda| < 1$, the zero equilibrium point is locally asymptotically stable.

(ii) It is easy to see that if $r > 1$ or $s > 1$, then there exists at least one root λ of $P(\lambda)$ such that $|\lambda| > 1$. Hence by Theorem 5, if $r > 1$ or $s > 1$, then $(\overline{x}_0, \overline{y}_0)$ is unstable. Thus, the proof is complete. □

Theorem 8. *(i) If $r > 1$ and $s > 1$, then the positive equilibrium point $(\overline{x}_1, \overline{y}_1)$ is locally unstable.*

(ii) If $r < 1$, $s < 1$, and $1/p$ is an even positive integer, then the positive equilibrium point $(\overline{x}_6, \overline{y}_6)$ is locally unstable.

Proof. (i) The linearized system of (8) about the equilibrium point $(\overline{x}_1, \overline{y}_1) = ((s-1)^{1/p}, (r-1)^{1/p})$ is given by

$$X_{n+1} = B(\overline{x}_1, \overline{y}_1) X_n, \tag{20}$$

where $X_n = (x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2})^T$ and

$$B(\overline{x}_1, \overline{y}_1) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & -\dfrac{p(s-1)^{1/p}(r-1)^{(p-1)/p}}{r} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\dfrac{p(r-1)^{1/p}(s-1)^{(p-1)/p}}{s} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{21}$$

The characteristic equation of $B(\overline{x}_1, \overline{y}_1)$ is as follows:

$$P(\lambda) = \lambda^6 - 2\lambda^4 + \lambda^2 - \frac{p^2}{rs}(r-1)(s-1) = 0. \tag{22}$$

It is clear that $P(\lambda)$ has a root in the interval $(1, \infty)$ since

$$\begin{aligned} P(1) &= -\frac{p^2}{rs}(r-1)(s-1) < 0, \\ \lim_{\lambda\to\infty} P(\lambda) &= \infty. \end{aligned} \tag{23}$$

This completes the proof.

(ii) The proof is similar to the proof of (i), so it will be omitted. □

Theorem 9. *(i) If $r > 1$ and $s = 1$, then the equilibrium point $(\overline{x}_2, \overline{y}_2)$ is nonhyperbolic point.*

(ii) If $r = 1$ and $s > 1$, then the equilibrium point $(\overline{x}_3, \overline{y}_3)$ is nonhyperbolic point.

(iii) If $r < 1$, $s = 1$, and $1/p$ is an even positive integer, then the equilibrium point $(\overline{x}_4, \overline{y}_4)$ is nonhyperbolic point.

(iv) If $r = 1$, $s < 1$, and $1/p$ is an even positive integer, then the equilibrium point $(\overline{x}_5, \overline{y}_5)$ is nonhyperbolic point.

Proof. (i) The linearized system of (8) about the equilibrium point $(\overline{x}_2, \overline{y}_2) = (0, (r-1)^{1/p})$ is given by

$$X_{n+1} = B(\overline{x}_2, \overline{y}_2) X_n, \tag{24}$$

where $X_n = (x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2})^T$ and

$$B(\overline{x}_2, \overline{y}_2) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{25}$$

The characteristic equation of $B(\overline{x}_2, \overline{y}_2)$ is as follows:

$$P(\lambda) = \lambda^6 - (s+1)\lambda^4 + s\lambda^2 = 0. \tag{26}$$

The roots of $P(\lambda)$ are

$$\begin{aligned} \lambda_1 &= \sqrt{s}, \\ \lambda_2 &= -\sqrt{s}, \\ \lambda_{3,4} &= 0, \\ \lambda_5 &= 1, \\ \lambda_6 &= -1. \end{aligned} \tag{27}$$

Thus, the equilibrium point $(\overline{x}_2, \overline{y}_2) = (0, (r-1)^{1/p})$ is nonhyperbolic point.

(ii) The proof is similar to the proof of (i), so it will be omitted.

(iii) The proof is easily seen from the proof of (i).
(iv) The proof is easily seen from the proof of (i). □

Now, we will study the global behavior of zero equilibrium point.

Theorem 10. *If $r < 1$ and $s < 1$, then the zero equilibrium point is globally asymptotically stable.*

Proof. We know by Theorem 7 that the equilibrium point $(\overline{x}_0, \overline{y}_0)$ of system (8) is locally asymptotically stable. Hence, it suffices to show for any $\{(x_n, y_n)\}$ solution of system (8) that

$$\lim_{n\to\infty}(x_n, y_n) = (\overline{x}_0, \overline{y}_0). \tag{28}$$

Since

$$\begin{aligned} 0 \le x_{n+1} &= \frac{rx_{n-1}}{1+y_{n-2}^p} < rx_{n-1}, \\ 0 \le y_{n+1} &= \frac{sy_{n-1}}{1+x_{n-2}^p} < sy_{n-1}, \end{aligned} \tag{29}$$

we have by induction

$$\begin{aligned} x_{2n-1} &< r^n x_{-1}, \\ x_{2n} &< r^n x_0, \\ y_{2n-1} &< s^n y_{-1}, \\ y_{2n} &< s^n y_0. \end{aligned} \tag{30}$$

Thus, for $r < 1$ and $s < 1$, we obtain

$$\lim_{n\to\infty}(x_n, y_n) = (0, 0). \tag{31}$$

This completes the proof. □

Theorem 11. *Consider system (8) and suppose that*

$$\begin{aligned} r &> 1, \\ s &> 1. \end{aligned} \tag{32}$$

Then, we obtain the following invariant intervals, for $i \in \{0, 1, 2\}$:

(i) $(x_{-i}, y_{-i}) \in (0, (s-1)^{1/p}) \times ((r-1)^{1/p}, \infty) \Rightarrow (x_n, y_n) \in (0, (s-1)^{1/p}) \times ((r-1)^{1/p}, \infty)$ *for* $n \ge 1$.

(ii) $(x_{-i}, y_{-i}) \in ((s-1)^{1/p}, \infty) \times (0, (r-1)^{1/p}) \Rightarrow (x_n, y_n) \in ((s-1)^{1/p}, \infty) \times (0, (r-1)^{1/p})$ *for* $n \ge 1$.

Proof. (i) Let $(x_{-i}, y_{-i}) \in (0, (s-1)^{1/p}) \times ((r-1)^{1/p}, \infty)$ for $i \in \{0, 1, 2\}$. From system (8), we have

$$\begin{aligned} x_1 &= \frac{rx_{-1}}{1+y_{-2}^p} < \frac{r\overline{x}_1}{1+\overline{y}_1^p} = \overline{x}_1 = (s-1)^{1/p}, \\ y_1 &= \frac{sy_{-1}}{1+x_{-2}^p} > \frac{s\overline{y}_1}{1+\overline{x}_1^p} = \overline{y}_1 = (r-1)^{1/p}. \end{aligned} \tag{33}$$

We prove by induction that

$$(x_n, y_n) \in \left(0, (s-1)^{1/p}\right) \times \left((r-1)^{1/p}, \infty\right), \quad \forall n > 1. \tag{34}$$

Suppose that (34) is true for $n = k > 1$. Then from system (8), we have

$$\begin{aligned} x_{k+1} &= \frac{rx_{k-1}}{1+y_{k-2}^p} < \frac{r\overline{x}_1}{1+\overline{y}_1^p} = \overline{x}_1 = (s-1)^{1/p}, \\ y_{k+1} &= \frac{sy_{k-1}}{1+x_{k-2}^p} > \frac{s\overline{y}_1}{1+\overline{x}_1^p} = \overline{y}_1 = (r-1)^{1/p}. \end{aligned} \tag{35}$$

Therefore, (34) is true for all $n \ge -2$. This completes the proof of (i). Similarly, we can obtain the proof of (ii) which will be omitted. □

Corollary 12. *Consider system (8) and suppose that*

$$\begin{aligned} r &< 1, \\ s &< 1, \\ \frac{1}{p} &\in 2\mathbb{Z}^+. \end{aligned} \tag{36}$$

Then, we obtain the following invariant intervals, for $i \in \{0, 1, 2\}$:

(i) $(x_{-i}, y_{-i}) \in (0, (s-1)^{1/p}) \times ((r-1)^{1/p}, \infty) \Rightarrow (x_n, y_n) \in (0, (s-1)^{1/p}) \times ((r-1)^{1/p}, \infty)$ *for* $n \ge 1$.

(ii) $(x_{-i}, y_{-i}) \in ((s-1)^{1/p}, \infty) \times (0, (r-1)^{1/p}) \Rightarrow (x_n, y_n) \in ((s-1)^{1/p}, \infty) \times (0, (r-1)^{1/p})$ *for* $n \ge 1$.

Theorem 13. *Assume that $r, s \in (1, \infty)$; then there exist unbounded solutions of system (8).*

Proof. From Theorem 11, we can assume without loss of generality that the solution $\{x_n, y_n\}$ of system (8) is such that

$$\begin{aligned} x_n &< \overline{x}_1 = (s-1)^{1/p}, \\ y_n &> \overline{y}_1 = (r-1)^{1/p}, \\ &\text{for } n \ge -2. \end{aligned} \tag{37}$$

Then

$$\begin{aligned} x_{n+1} &= \frac{rx_{n-1}}{1+y_{n-2}^p} < \frac{rx_{n-1}}{1+(r-1)} = x_{n-1}, \\ y_{n+1} &= \frac{sy_{n-1}}{1+x_{n-2}^p} > \frac{sy_{n-1}}{1+(s-1)} = y_{n-1}, \end{aligned} \tag{38}$$

from which it follows that

$$\begin{aligned} \lim_{n\to\infty} x_n &= 0, \\ \lim_{n\to\infty} y_n &= \infty. \end{aligned} \tag{39}$$

This completes the proof. □

Corollary 14. *Assume that $r, s \in (0,1)$ and $1/p \in 2\mathbb{Z}^+$; then there exist unbounded solutions of system (8).*

Theorem 15. *If $r = s = 1$, then system (8) possesses the prime period two solution which is one of the following forms:*

$$\begin{gathered} \ldots,(0,b),(0,d),(0,b),(0,d),\ldots \\ \ldots,(0,b),(c,0),(0,b),(c,0),\ldots \\ \ldots,(a,0),(c,0),(a,0),(c,0),\ldots \\ \ldots,(a,0),(0,d),(a,0),(0,d),\ldots \end{gathered} \tag{40}$$

with $a, b, c, d > 0$.

Proof. Let

$$\ldots,(a,b),(c,d),(a,b),(c,d),\ldots \tag{41}$$

be a period two solution of system (8). Then, we have

$$\begin{aligned} a &= \frac{ra}{1+d^p}, \\ b &= \frac{rb}{1+c^p}, \end{aligned} \tag{42}$$

$$\begin{aligned} c &= \frac{sc}{1+b^p}, \\ d &= \frac{sd}{1+a^p} \end{aligned} \tag{43}$$

such that $a \neq b$ and $c \neq d$. Firstly, we consider the case $a \neq 0$, $b \neq 0$. Then, we obtain from (42) that $c = d = (r-1)^{1/p}$ which is a contradiction. Similarly, the case $c \neq 0, d \neq 0$ is impossible with (43). Thus, one of them must be equal to zero. Now we assume that $a = 0, b \neq 0, c = 0, d \neq 0$. In this sense, we obtain from (42) and (43) that $r = s = 1$. Therefore,

$$\ldots,(0,b),(0,d),(0,b),(0,d),\ldots \tag{44}$$

is a two-periodic solution of system (8) with $b, d > 0$. The other cases are similar and the proofs of them are omitted. □

4. Conclusion and Future Work

System (7) is the extension of the third-order equation in [18]. Here, we studied some dynamics of a six-dimensional discrete system. We investigated the local asymptotic behavior of solutions of system (7) using linearization method. We also obtained the global asymptotic stability result for the zero equilibrium point by using analytical techniques. At the end of the paper, the existence of prime two-periodic solutions and the existence of unbounded solutions were proved.

Fractional difference equations and their systems which contain powers of arbitrary positive degrees will next our aim to study.

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

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