

Research in Financial Mathematics

Sergey Nadtochiy

Department of Mathematics
University of Michigan

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Department of Mathematics, University of Michigan

Financial Mathematics

① Sell side

- Pricing
- Hedging
- Risk management

② Buy side

- Optimal investment
- Optimal execution: high frequency trading, optimal liquidation

③ Systemic risk

④ Financial Economics

Problems for Sell Side

- Large banks, and other market-makers, are in the business of **selling the financial products** to their clients (**pricing**).
- They do not (at least in theory) need to make profits from the changes in price – they only charge the commission to the buyers/sellers.
- In this case, the only thing they need is that **their books stay insensitive to changes in the market**:
 - If you sell a certain amount of asset A and buy the same amount of A – you are in a perfect world (making money from the commission).
 - The real world is not so simple: people want to buy A and sell B ...
 - Good thing is that A and B are often closely related through asset C .
 - Examples of this include options written on the same underlying stock price (or, FX rates, Exchange rates, etc.).
 - As a result, we can trade in C to offset the residual risk (**hedging**).

Example: pricing of standard options

- For an asset with the prices process $(S_t)_{t \geq 0}$, there are Call and Put options traded. Each of them is equipped with a strike K and maturity T . These options pay out to their holder at time T the amount of

$$(S_T - K)^+ \quad \text{or} \quad (K - S_T)^+,$$

for a **call** or a **put** option respectively.

- What is the "fair" price of such option?** We can answer this question by postulating a model for the evolution for S . There is a theory of **Arbitrage Pricing** that puts constraints on the admissible pricing models.
- One possibility is the **Black-Scholes** model:

$$S_t = S_0 \exp((r - \sigma^2/2)t + \sigma B_t)$$

My Research: Market-Based Models

- Theoretical restrictions on the models still leave enough room to reproduce the observed **stylized facts**.
- Modern markets include **derivatives (options)** as **liquidly traded assets**. So, we should incorporate the **information contained in their dynamics** (stylized facts) into the model.
- We incorporate the available information by constructing a flexible class of models, which
 - contains **free parameters**
 - and allows us to **calibrate** them to reproduce the desired phenomena.
- Designing a flexible class of models for **derivatives prices** is quite **challenging**: their prices, themselves, have to satisfy some complicated (nonlinear) relations.
 - For example, if the randomness is given by **Brownian motion** B , the call and put options have to satisfy

$$\begin{cases} dS_t = (\dots)dW_t, & dC_t(T, K) = (\dots)dW_t, \\ C_T(T, K) = (S_T - K)^+ \end{cases}$$

Calibration to Option prices

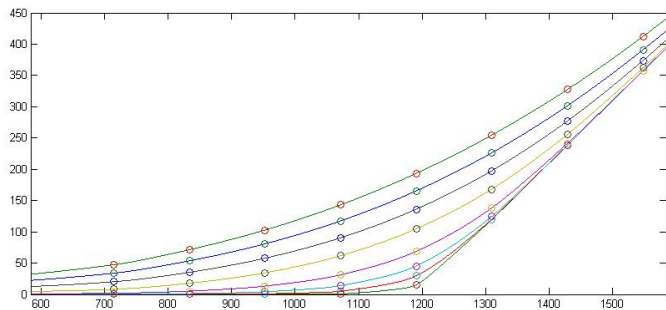


Figure : Calibrated model reproduces the market prices of standard options.

Example: hedging of standard options

- Imagine that after buying and selling the options, we still have some **outstanding position** left. In other words, you have some options, and you would like to get rid of the risk associated with them.
- Then, you hedge these options by trading the underlying asset.
 - For example, the **Call** option prices are given by a function of the underlying value and time:

$$C(S_t, t; T, K)$$

- Then the changes in price are given by

$$dC(S_t, t; T, K) = C'_S dS_t + (\dots)dt$$

- In particular, the above expression tells you how to replicate (or, offset) the changes on option prices by buying or selling the underlying asset (stock).

My Research: Static Hedging

- The above example of hedging is **dynamic**: it requires buying and selling the underlying asset at every moment in time. **static** portfolio of available (liquid) financial instruments, such that the price of this portfolio coincides with (or is the negative of) the price of the target 'exotic' derivative.
- In particular, we can hedge **barrier options** by buying and selling (**once!**) a certain number of standard (**call and put**) options.
- Barrier options have payoffs at the terminal time, which depend both on the value of the underlying and the maximum, or the minimum, of the underlying process. For example, and Up-and-Out Call has the payoff

$$(S_T - K)^+ \mathbf{1}_{\{\max_{t \in [0, T]} S_t \leq U\}}$$

- Hedges (both static and dynamic) are, usually, **model-dependent**.
- We have solved the static hedging problem in all models in which the underlying is a **diffusion (Markov) process**.

Static Hedge Payoff

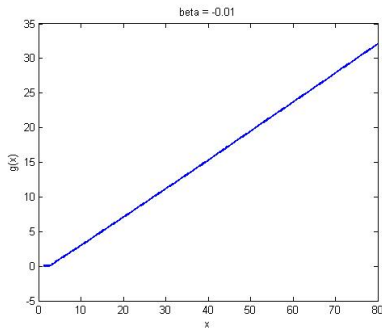
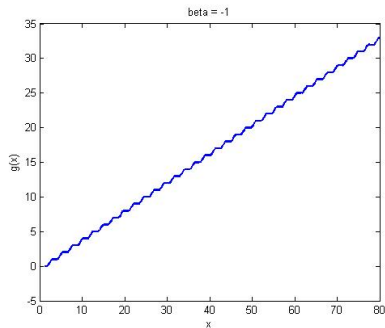


Figure : The payoff function of a European option which provides an exact static hedge for the target barrier option.

Risk Management

- **Risk Management** is a combination of **pricing**, **hedging**, and application of appropriate **measure of risk**.
- We will discuss some measures of risk in the context of optimal investment. However, there is a comprehensive mathematical theory of **Risk Measures**, due to Fölmer, Delbaen, etc.

Problems for Buy Side

- Hedge funds, mutual funds and private investors want to make money from the **changes in assets' prices**.
- The question is how to find the **optimal trade-off** between **return and risk**.
- The prevailing approach is to use an **expected utility function** to determine how good is the generated wealth at a given time horizon:

$$\mathbb{E}U(X_T^\pi)$$

- Solving an optimal investment problem for a given utility function can be rather challenging even in the simplest setting – this is due to the very nature of financial problems.

My Research: Optimal Investment for All Time Horizons

- The classical Optimal Investment Problem assumes that all we care about is the value of the generated wealth at a given time horizon.
- In reality, life doesn't stop at time T , so, we need to find investment criteria that **work for all time horizons**.
- These criteria should also have natural economic (axiomatic) justification.
- One possible solution is given by the **Forward Performance Processes**. Constructing such process and solving the resulting optimization problem may be a very challenging mathematical problem.

HFT and Optimal Execution

- With the advances in technology, it became tempting for many speculators to trade at a very **high frequency (nano-seconds)** and make profit from the **irrationalities of other market participants** over very small time periods.
- On the other hand, **long-term investors**, who need to buy or sell a big chunk of the asset, want to do it in an optimal way: to **avoid irrationalities** which can be exploited by the HF traders..
- Both of these problems involve solving **Stochastic Control** problems – a certain type of optimization problems.

Systemic Risk

- Systemic Risk is a new area of research in the intersection of Financial Mathematics and Economics.
- It is meant to provide insights into the **risks** associated with a **large interconnected system** of financial institutions, and ways to find a —bf tradeoff between this risk and the growth of the overall economy.
- This is a brand new field. Existing approaches include: **random graphs**, **interacting particle systems** and mixtures of these with special economic models.

Financial Economics

- This is a very rich field of research – border-line with Financial Mathematics..
- The subjects of Financial Economics are of a qualitative rather than quantitative nature: "specific models are not meant to be taken very seriously".
- Specific topics include: **trading exhaustable resources** (energy, carbon emissions, etc), **maturity mismatch**, **bubbles**, **insider trading**, etc.