Research Statement: Combinatorial Methods in Algebra and Geometry

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My research is at the intersection of algebra, geometry, and combinatorics. My dissertation work under Pramod Achar involved studying singularities of certain topological spaces via **perverse sheaves**. Concurrently, I began working with Greg Muller on **cluster algebras**, a subject that has an easier point of entry due to its computable examples, but which has deep connections throughout mathematics and physics.

This early blend of almost diametrically-opposed styles of math shaped my research direction. My work gravitated toward interpreting complicated topological or algebraic questions as combinatorial ones. Sometimes the information flows the other way as well. Recently, I am most excited about the following projects: my collaborators and I are developing an "**intersection cohomology**" theory for **matroids**, allowing for powerful algebro-geometric tools to be used for solving classical combinatorial problems about matroids (Section 1.1), are proving that various polynomials in **algebraic combinatorics** satisfy **log-concavity** properties both in a continuous and discrete way (Section 2), and are discerning that certain moduli spaces of connections on $\mathbb{P}^1$ that arise in the **geometric Langlands program** are (non)empty, the so-called Deligne–Simpson problem (Section 5.1).

Some of my research problems lend themselves to **explicit computations** which allow **undergraduates and beginning graduate students** to make early contributions to solving problems in various areas of mathematics. Moreover, experiments can often be done using **computer software** like Macaulay2, GAP, or SageMath. My research statement is divided into five broad sections: matroids (Section 1), Lorentzian polynomials in algebraic combinatorics (Section 2), cluster algebras (Section 3), representation theory of finite-dimensional algebras (Section 4), and geometric representation theory and the Langlands program (Section 5).[1]

Section 1.1 *(joint with Tom Braden (UMass), June Huh (Stanford), Nicholas Proudfoot (Oregon), and Botong Wang (Wisconsin))* We aim to introduce topological techniques into the field of matroid theory. As applications, we have proven the non-negativity of Kazhdan–Lusztig polynomials of matroids, as well as Dowling and Wilson's 1974 "Top-Heavy Conjecture" on the lattice of flats of a matroid.

Section 2 *(joint with June Huh (Stanford), Karola Mészáros (Cornell), and Avery St. Dizier (Cornell))* We prove that (normalized) Schur polynomials are strongly log-concave; as an application, we prove Okounkov's conjecture in the special case of Kostka numbers.

Section 3.1 *(joint with Chris Fraser (University of Minnesota) and Maitreyee Kulkarni (HIM))* We study postroid cells in partial flag varieties (known in physics as the one-loop Grassmannian). In particular, we develop an analog (called momentum-twistor diagrams in physics) of Postnikov's plabic graphs in a disk to serve as a combinatorial model for postroids in partial flag varieties.

Section 3.2 *(joint with Greg Muller (Oklahoma))* We develop an algebro-geometric algorithm which gives a presentation for upper cluster algebras in terms of generators and relations. This algorithm is computable in finite-time, and I have implemented it into Sage with my collaborators.

Section 4.1 *(joint with Alexander Garver (LaCIM), Kiyoshi Igusa (Brandeis), and Jonah Ostroff (Washington))* We introduce a combinatorial model for exceptional sequences of type A quiver representations.

Section 4.2 *(joint with Pramod Achar (Louisiana State) and Maitreyee Kulkarni (HIM))* We introduce a combinatorial model for computing the Fourier–Sato transform on type A quiver representation varieties.

[1] Each of the broad sections can be read independently, so the reader is invited to pick his or her favorite subject and go directly to that page.

Section 5.1 *(joint with Maitreyee Kulkarni (HIM), Neal Livesay, Bach Nguyen (Xavier University), and Daniel Sage (Louisiana State)* We solve the Deligne–Simpson problem to determine when the moduli space of Coxeter connections on $\mathbb{P}^1$ is (non)empty in type A. We are working to extend this to other classical Lie types.

Section 5.2 This was my dissertation work which gives a geometric, functorial relationship between representations of an algebraic group and representations of the corresponding Weyl group at the level of mixed, derived categories of sheaves on the affine Grassmannian and nilpotent cone of the Langlands dual group.

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1 Matroids

Definition 1.0.1. A matroid M on finite ground set E is a collection of subsets of E, called flats, such that

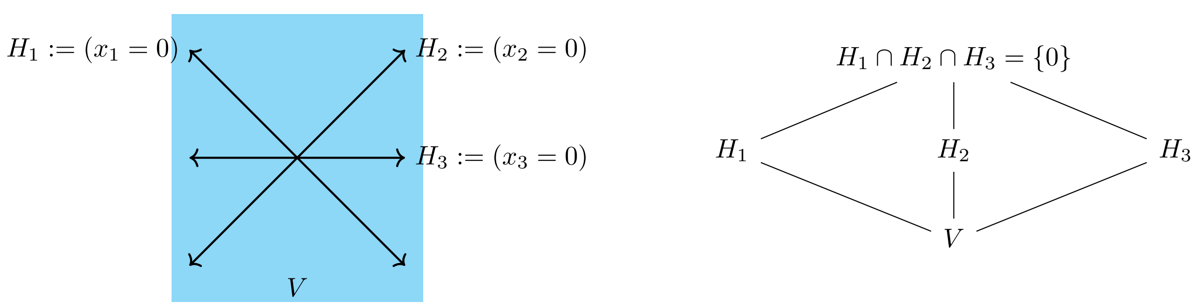
- The intersection of any two flats is a flat, and
- For any flat F, any element in $E \smallsetminus F$ is contained in exactly one flat that is minimal among the flats strictly containing F.

The set $\mathcal{L}(M)$ of all flats forms a graded lattice called the lattice of flats of M.

Remark 1.0.2. Throughout, it will be convenient to assume that M is a loopless matroid; that is, $\emptyset$ is a flat of M. This is harmless since removing loops gives an isomorphic lattice of flats.

We point to [Oxl11] for an excellent resource on matroids. One source of examples of matroids is those arising from hyperplane arrangements—here the ground set is the finite set of hyperplanes, the flats are all subspaces gotten by intersecting hyperplanes, and the rank of a flat is its codimension.

Example 1.0.3. The linear subspace $V := (x_1 + x_2 + x_3 = 0)$ in $\mathbb{C}^{E=\{1,2,3\}}$ defines a hyperplane arrangement $\mathcal{A}$ (hence a matroid) after intersecting it with the coordinate hyperplanes. We illustrate $\mathcal{A}$ together with $\mathcal{L}(M)$ below. Note that the partial order of $\mathcal{L}(M)$ is given by reverse inclusion of subspaces.

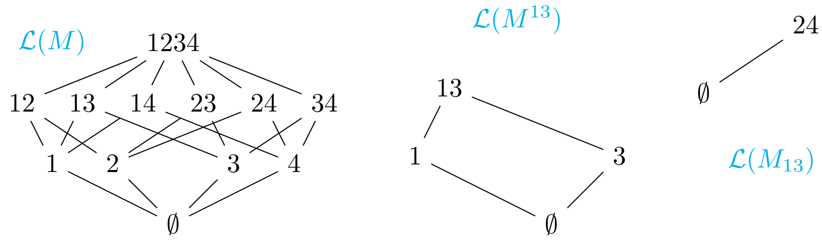


Definition 1.0.4. A matroid M is realizable over a field $\Bbbk$ if there is a $\Bbbk$-vector space and a hyperplane arrangement inside it whose associated matroid is M. A matroid M is realizable if it is realizable over some field.

For each flat $F \in \mathcal{L}(M)$, there are two new matroids:

- M^F is a certain matroid on F called the localization of M at F, and
- M_F is a certain matroid on $E \setminus F$ called the contraction of M at F.

The figure below shows an example (here M is the matroid on $E = \{1, 2, 3, 4\}$ realized by the hyperplane arrangement of four generic planes in $\mathbb{C}^3$ whose intersection is the origin) of these two constructions in terms of the lattice of flats $\mathcal{L}(M)$.



1.1 Two problems in matroid theory

1.1.1 The "Top-Heavy Conjecture"

Denote by $\mathcal{L}^k(M)$ the collection of flats of rank k in $\mathcal{L}(M)$. These are just the flats k levels up from the bottom of $\mathcal{L}(M)$ (starting with the bottom flat at rank 0).

Conjecture 1.1.1 ("Top-Heavy Conjecture", Dowling–Wilson 1974 [DW74, DW75])**.** *Let M be a rank d matroid. For any $k \leq d/2$, we have*

$$\#\mathcal{L}^k(M) \leq \#\mathcal{L}^{d-k}(M).$$

Despite being simple to state and understand, the Top-Heavy Conjecture remained unsolved for forty-three years. In 2017, June Huh and Botong Wang proved the conjecture in the case of realizable matroids.

Theorem 1.1.2 (Huh–Wang 2017 [HW17])**.** *Let M be a realizable matroid of rank d. For any $k \leq d/2$, we have*

$$\#\mathcal{L}^k(M) \leq \#\mathcal{L}^{d-k}(M).$$

The proof of this theorem uses the topology of a certain singular projective variety Y (see Section 1.1.3 for the definition of Y) defined from a hyperplane arrangement $\mathcal{A}$. When the matroid is not realizable, all geometric techniques are missing; however, one can emulate the topology in a purely combinatorial setting. We will explain some of the ingredients of the proof of following theorem in Section 1.1.4.

Theorem 1.1.3 (Braden–Huh–M.–Proudfoot–Wang [BHM+20b])**.** *Dowling and Wilson's Top-Heavy Conjecture (Conjecture 1.1.1) holds for all matroids.*

In the next section, we explain another result which can be formulated for all matroids, and whose proof in the non-realizable setting relies on using geometric inspiration to inform a combinatorial argument, when no geometry exists.

1.1.2 Kazhdan–Lusztig polynomials of matroids

In [EPW16], Elias, Proudfoot, and Wakefield associated to each matroid M a polynomial $P_M(t)$ called the KL polynomial of M. These polynomials share many analogies with the classical KL polynomials [KL79] for Coxeter groups, but exhibit some interesting differences (see the table at the end of this section). Both types of polynomials have a purely combinatorial definition—while KL polynomials for Coxeter groups are defined in terms of more elementary polynomials called R-polynomials, every matroid has a characteristic polynomial $\chi_M(t)$ which plays this role. In the classical setting, Polo showed that every polynomial with non-negative coefficients and constant term 1 occurs as a KL polynomial [Pol99] for some symmetric group $\mathfrak{S}_n$. In stark contrast, it is conjectured that the $P_M(t)$ are real-rooted [GPY16].

When the Coxeter group is a finite Weyl group, there is a geometric interpretation of the classical KL polynomials. They are the *dimensions* of intersection cohomology stalks of Schubert varieties, which implies non-negativity of their coefficients. In a similar way, when M is realizable, Elias, Proudfoot, and Wakefield identified $P_M(t)$ with the *dimensions* of intersection cohomology stalks of a certain singular projective variety Y. (We call Y the *Schubert variety of a hyperplane arrangement*—see Section 1.1.3 for the definition of Y.)[2]

Theorem 1.1.4 (Elias–Proudfoot–Wakefield [EPW16])**.** *If M is a realizable matroid, then*

$$P_M(t) = \sum_{i \geq 0} t^i \dim \mathrm{IH}^{2i}_{(\infty,\ldots,\infty)}(Y).$$

In this way, the $P_M(t)$ have non-negative coefficients when M is realizable. The question of non-negativity of the KL polynomials for arbitrary Coxeter groups was conjectured in [KL79], but remained unsolved for thirty-five years. In 2014, Elias and Williamson settled this question in the affirmative by using sophisticated diagrammatic combinatorics [EW17, EW16] to give an algebraic proof of the decomposition theorem [BBD82] via the Hodge-theoretic properties of Soergel bimodules [EW14]. Braden, Huh, Proudfoot, Wang, and I have proven the analogous conjecture for all matroids. We will explain some of the ideas of the proof in Section 1.1.4.

Theorem 1.1.5 (Braden–Huh–M.–Proudfoot–Wang [BHM+20b])**.** *For an arbitrary matroid M, the coefficients of the KL polynomial $P_M(t)$ are nonnegative.*

Despite the simplicity of this statement, just as in the classical setting, it has deep connections relating topology, combinatorics, and representation theory. Proving Theorem 1.1.5 requires producing a working Hodge theory purely combinatorially, when no geometry exists. A recent example of work in the same vein is that of Adiprasito, Huh, and Katz [AHK18]. They proved, using deep Hodge-theoretic arguments, that the coefficients of the characteristic polynomial $\chi_M(t)$ of an arbitrary matroid form a log-concave sequence; thereby settling a long-standing conjecture of Rota, Heron, and Welsh [Rot71, Her72, Wel76].

We include here, for convenience, a table summarizing the above discussion, and more.

KL theory for Coxeter groups	KL theory for matroids
Coxeter group W	matroid M
Weyl group W	realizable matroid M
Bruhat poset	lattice of flats $\mathcal{L}(M)$
R-polynomial	characteristic polynomial $\chi_M(t)$
Hecke algebra	?
Polo	real-rooted
Schubert variety	Schubert variety Y of a hyperplane arrangement
Nonneg. of KL polys of W (Elias–Williamson [EW14])	Nonneg. of KL polys of M (Theorem 1.1.5)

1.1.3 The realizable case

Because the topology of the realizable case informs our strategy for proving Theorems 1.1.3 and 1.1.5 in the general case, I will briefly explain the proof of the "Top-Heavy Conjecture" and the non-negativity of the KL polynomials for realizable matroids (Theorems 1.1.2 and 1.1.4 above).

Suppose that we have a hyperplane arrangement $\mathcal{A}$ in a $\mathbb{C}$-vector space V such that the intersection of the hyperplanes in the origin. Consider the maps

$$V \hookrightarrow \bigoplus_{H \in \mathcal{A}} V/H \cong \prod_{H \in \mathcal{A}} \mathbb{A}^1 \hookrightarrow \prod_{H \in \mathcal{A}} \mathbb{P}^1,$$

and define Y to be the closure of V inside $\prod_{H \in \mathcal{A}} \mathbb{P}^1$. We call Y the *Schubert variety of the hyperplane arrangement* $\mathcal{A}$ because it plays an analogous role in the KL theory of matroids that a Schubert variety in

[2] Note that in an open neighborhood of the most singular point $(\infty, \ldots, \infty)$, the projective variety Y is isomorphic to a well-studied affine conical variety called the reciprocal plane. (We point to [PS06, AB16] for more information on reciprocal planes.)

the flag variety plays in the KL theory of Coxeter groups. One of the most important analogies is that Y admits a stratification by affine spaces[3] [PXY18, Lemmas 7.5 and 7.6]:

$$Y = \coprod_{F\in\mathcal{L}(M)} Y_F \quad \text{with} \quad Y_F \cong \mathbb{C}^{\operatorname{rk} F}. \tag{1}$$

Thus we obtain the following result.

Proposition 1.1.6. *The odd-degree cohomology of Y vanishes, and* $\dim \mathrm{H}^{2k}(Y) = \#\mathcal{L}^k(M)$.

Thus, the proof of the "Top-Heavy Conjecture" in the realizable case reduces to producing an injective map $\mathrm{H}^{2k}(Y) \hookrightarrow \mathrm{H}^{2(d-k)}(Y)$. If Y were smooth, the Hard Lefschetz Theorem would provide such an injection. But the non-smoothness of Y requires moving to intersection cohomology.

Proof of Theorem 1.1.2 [HW17]. Let $L \in \mathrm{H}^2(Y)$ be the class of an ample line bundle. Since Y is a singular projective variety, the Hard Lefschetz Theorem asserts that there is an isomorphism $L^{d-2k}\colon \mathrm{IH}^{2k}(Y) \to \mathrm{IH}^{2(d-k)}(Y)$. The natural map $\mathrm{H}^\bullet(Y) \to \mathrm{IH}^\bullet(Y)$ making $\mathrm{IH}^\bullet(Y)$ a module over $\mathrm{H}^\bullet(Y)$ is an injection, since Y has a stratification by affine spaces [BE09]. Therefore, the map L^{d-2k} restricts to an injection $L^{d-2k}|_{\mathrm{H}^{2k}}(Y)\colon \mathrm{H}^{2k}(Y) \hookrightarrow \mathrm{H}^{2(d-k)}(Y)$. The proof follows from Proposition 1.1.6. □

1.1.4 The general case

The proofs of Theorems 1.1.2 and 1.1.4 inform our approach for arbitrary matroids (even though no geometry exists here).

Semi-wonderful geometry There is a resolution of singularities $\pi\colon \widetilde{Y} \to Y$, where the variety $\widetilde{Y}$ is given by blowing up Y at the point stratum $Y_\emptyset$, then blowing up the proper transforms of all one-dimensional strata Y_F ($\operatorname{rk} F = 1$), and so on[4]. It has a stratification indexed by chains of flats in the lattice $\mathcal{L}(M)$ ending at the maximal flat E (for example, the variety $\widetilde{Y}$ of Example 1.0.3 has eight strata).

Theorem 1.1.7 (Huh–Wang [HW17] for $\mathrm{H}^\bullet(M)$, Braden–Huh–M.–Proudfoot–Wang [BHM$^+$20a] for $\mathrm{CH}^\bullet(M)$)**.** *Let M be an arbitrary matroid. There exist graded rings $\mathrm{CH}^\bullet(M)$ and $\mathrm{H}^\bullet(M)$ with explicit presentations in terms of $\mathcal{L}(M)$ such that when M is a realizable matroid, there are isomorphisms of graded rings*

$$\mathrm{CH}^\bullet(M) \cong \mathrm{H}^{2\bullet}(\widetilde{Y}) \qquad \textit{and} \qquad \mathrm{H}^\bullet(M) \cong \mathrm{H}^{2\bullet}(Y).$$

Moreover, there is a natural inclusion $\mathrm{H}^\bullet(M) \hookrightarrow \mathrm{CH}^\bullet(M)$ which makes $\mathrm{CH}^\bullet(M)$ and algebra over $\mathrm{H}^\bullet(M)$.

Remark 1.1.8. In [FY04] the Chow ring of a matroid is introduced, and it is the main object of study in [AHK18]. This ring has a similar presentation to $\mathrm{CH}^\bullet(M)$, and geometrically it is the cohomology $\mathrm{H}^\bullet(\widetilde{\widetilde{Y}})$ of de Concini and Procesi's full wonderful model $\widetilde{\widetilde{Y}}$ (which is the fiber of $\pi\colon \widetilde{Y} \to Y$ over the most singular point $Y_\emptyset$ of Y). We prefer working with $\mathrm{CH}^\bullet(M)$ because $\pi\colon \widetilde{Y} \to Y$ is a stratified map.

Combinatorial intersection cohomology of matroids There is a canonical isomorphism $\mathrm{CH}^d(M) \cong \mathbb{C}$, and a perfect pairing, we call the *Poincaré pairing*,

$$\mathrm{CH}^k(M) \otimes \mathrm{CH}^{d-k}(M) \to \mathrm{CH}^d(M) \cong \mathbb{C}$$

for any $0 \le k \le d$.

Definition 1.1.9. The *intersection cohomology* $\mathrm{IH}^\bullet(M)$ *of a matroid* M is the unique indecomposable summand of $\mathrm{CH}^\bullet(M)$ as an $\mathrm{H}^\bullet(M)$-module which contains $\mathrm{CH}^d(M) \cong \mathbb{C}$.

[3] The closure $\overline{Y_F}$ of a stratum Y_F is isomorphic to a Schubert variety with underlying matroid the localization M^F, and a normal slice to Y at a point in Y_F is isomorphic to a Schubert variety with underlying matroid the contraction M_F.

[4] The reader familiar with the Weyl group setting (the left column of the table at the end of Section 1.1.2) should interpret this as an analog of a Bott–Samelson (BS) resolution of a Schubert variety in the flag variety; however, our resolution is *canonical*—it does not depend on any choices, whereas a BS resolution depends on a choice of reduced word.

Theorem 1.1.10 (Braden–Huh–M.–Proudfoot–Wang [BHM+20b])**.** *The intersection cohomology* $\mathrm{IH}^\bullet(M)$ *of a matroid* M *satisfies Poincaré duality with respect to the Poincaré pairing on* $\mathrm{CH}^\bullet(M)$*.*

Theorem 1.1.10 gives a (Björner–Ekedahl-type) inclusion $\mathrm{H}^\bullet(M) \hookrightarrow \mathrm{IH}^\bullet(M)$; indeed, $\mathrm{CH}^d(M) \subset \mathrm{IH}^\bullet(M)$ and by Poincaré duality of $\mathrm{IH}^\bullet(M)$, we know $\mathrm{CH}^0(M) \subset \mathrm{IH}^\bullet(M)$. Thus $1_{\mathrm{CH}^\bullet(M)} \in \mathrm{IH}^\bullet(M)$, and since $\mathrm{IH}^\bullet(M)$ is an $\mathrm{H}^\bullet(M)$-module, the claim follows. What remains to emulate the proof of Theorem 1.1.2 is Hard Lefschetz for $\mathrm{IH}^\bullet(M)$.

Theorem 1.1.11 (Braden–Huh–M.–Proudfoot–Wang [BHM+20b])**.** *The intersection cohomology* $\mathrm{IH}^\bullet(M)$ *of a matroid* M *satisfies the Hard Lefschetz Theorem; that is, there exists a class* $L \in \mathrm{H}^1(M)$ *such that for every* $k \leq d/2$*, the map* $L^{d-2k} : \mathrm{IH}^k(M) \to \mathrm{IH}^{d-k}(M)$ *is an isomorphism.*

Theorems 1.1.10 and 1.1.11 together give a proof of the "Top-Heavy Conjecture" (Theorem 1.1.3) for all matroids using the same argument as in the realizable case.

To prove Theorem 1.1.5, we interpret the coefficients of the KL polynomials of matroids as graded dimensions of the primitive part of $\mathrm{IH}^\bullet(M)$. (This is similar to the interpretation given in Theorem 1.1.4 for realizable matroids.)

Theorem 1.1.12 (Braden–Huh–M.–Proudfoot–Wang [BHM+20b])**.** *For an arbitrary matroid* M*, we have* $P_M(t) = \sum_{i\geq 0} t^i \dim(\mathrm{IH}^i(M)/\mathrm{H}^1(M) \cdot \mathrm{IH}^{i-1}(M))$*.*

Remark 1.1.13. The proofs of Theorems 1.1.10, 1.1.11, and 1.1.12 require a complicated induction in the style of [EW14, AHK18]. In the course of the proofs, we prove the entire Kähler package (Poincaré duality, Hard Lefschetz, and the Hodge–Riemann bilinear relations) for $\mathrm{IH}^\bullet(M)$ as well as for various intermediate objects we must define along the way.

2 Lorentzian polynomials in algebraic combinatorics and representation theory

2.1 (Normalized) Schur polynomials are Lorentzian

2.1.1 Motivation from representation theory

Let Λ be the integral weight lattice of $\mathfrak{sl}_n(\mathbb{C})$. For each $\lambda \in \Lambda$, the irreducible representation $V(\lambda)$ of highest weight λ decomposes into finite-dimensional weight spaces

$$V(\lambda) = \bigoplus_{\mu} V(\lambda)_\mu.$$

The dimensions of the weight spaces $V(\lambda)_\mu$ are called *weight multiplicities*. We show that if $\lambda \in \Lambda$ is a dominant weight, then the sequence of weight multiplicities we encounter is log-concave, as we walk along any root direction in the weight diagram of $V(\lambda)$.

Theorem 2.1.1 (Huh–M.–Mészáros–St. Dizier [HMMS19])**.** *For* $\lambda, \mu \in \Lambda$ *with* λ *dominant, we have*

$$(\dim V(\lambda)_\mu)^2 \geq \dim V(\lambda)_{\mu+e_i-e_j} \dim V(\lambda)_{\mu-e_i+e_j}$$

for any $i, j \in [n]$*.*

We note that Theorem 2.1.1 already fails for $\mathfrak{sp}_4(\mathbb{C})$ for the irreducible representation of highest weight $2\varpi_2$. In the case that $\lambda \in \Lambda$ is an antidominant weight, then the irreducible representation $V(\lambda)$ is the Verma module $M(\lambda)$; we prove that all Verma modules $M(\lambda)$ for $\lambda \in \Lambda$ enjoy the same log-concavity property.

Proposition 2.1.2 (Huh–M.–Mészáros–St. Dizier [HMMS19])**.** *For any* $\lambda, \mu \in \Lambda$*, we have*

$$(\dim M(\lambda)_\mu)^2 \geq \dim M(\lambda)_{\mu+e_i-e_j} \dim M(\lambda)_{\mu-e_i+e_j}$$

for any $i, j \in [n]$*.*

Proof sketch. It is known that weight multiplicities of Verma modules are evaluations of Kostant's partition function:

$$\dim M(\lambda)_\mu = p(\mu - \lambda),$$

which is the number of ways of writing $\mu - \lambda$ as a sum of negative roots. In turn, Kostant partition function evaluations are mixed volumes of Minkowski sums of polytopes [BV08], so the Alexandrov–Fenchel inequality for mixed volumes finishes the proof. □

We conjecture that this surprising log-concavity phenomenon holds not only for dominant (Theorem 2.1.1) and antidominant weights (Proposition 2.1.2), but also for all integral weights.

Conjecture 2.1.3 (Huh–M.–Mészáros–St. Dizier [HMMS19])**.** *For $\lambda, \mu \in \Lambda$, we have*

$$(\dim V(\lambda)_\mu)^2 \geq \dim V(\lambda)_{\mu+e_i-e_j} \dim V(\lambda)_{\mu-e_i+e_j}$$

for any $i, j \in [n]$.

2.1.2 Schur polynomials

Our proof of Theorem 2.1.1 goes via Schur polynomials. Schur polynomials were first studied as ratios of alternates by Cauchy in 1815. They are the characters of irreducible polynomial representations of $\mathbf{GL}_n(\mathbb{C})$. We point to [Ful97] for a history of Schur polynomials.

Definition 2.1.4. The *Schur polynomial* (in n variables) of a partition λ is

$$s_\lambda(x_1, \ldots, x_n) = \sum_{T \in \text{SSYT}(\lambda)} x^{\mu(T)}, \qquad x^{\mu(T)} := x_1^{\mu_1(T)} \cdots x_n^{\mu_n(T)},$$

where the sum is over all semi-standard Young tableaux of shape λ.

Grouping terms with the same μ gives

$$s_\lambda(x_1, \ldots, x_n) = \sum_\mu K_{\lambda\mu} x^\mu,$$

where $K_{\lambda\mu}$ is the *Kostka number.*

Example 2.1.5. When $\lambda = (2, 1)$, we have the two semi-standard Young tableaux

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \\ \cline{1-1}\end{array},$$

so the corresponding Schur polynomial is $s_{(2,1)}(x_1, x_2) = x_1^2 x_2 + x_1 x_2^2$.

The **normalization operator** is given by

$$N(x^\mu) = \frac{x^\mu}{\mu!} := \frac{x^{\mu_1} \cdots x^{\mu_n}}{\mu_1! \cdots \mu_n!}.$$

Continuous Theorem 1 (Huh–M.–Mészáros–St. Dizier [HMMS19])**.** *For any partition λ, we have*

$$N(s_\lambda(x_1, \ldots, x_n)) = \sum_\mu K_{\lambda\mu} \frac{x^\mu}{\mu!}$$

is either identically 0 or $\log(N(s_\lambda))$ is a concave function on $\mathbb{R}^n_{>0}$.

Discrete Theorem 1 (Huh–M.–Mészáros–St. Dizier [HMMS19])**.** *For any partition λ and $\mu \in \mathbb{N}^n$, we have*

$$K_{\lambda\mu}^2 \geq K_{\lambda,\mu+e_i-e_j} K_{\lambda,\mu-e_i+e_j}$$

for any $i, j \in [n]$.

The Discrete Theorem implies Theorem 2.1.1 weight multiplicities because

$$\dim V(\lambda)_\mu = K_{\lambda\mu}.$$

Littlewood–Richardson coefficients $c^\nu_{\lambda\kappa}$ are given by

$$V(\lambda) \otimes V(\kappa) \simeq \bigoplus_\nu V(\nu)^{c^\nu_{\lambda\kappa}}.$$

Motivated from statistical mechanics, Okounkov made the following conjecture in 2003; that is, that in the space of all triples of partitions of the same size, the logarithm of the associated Littlewood–Richardson coefficient is a concave function.

Conjecture 2.1.6 (Okounkov [Oko03]). *The discrete function*

$$(\lambda, \kappa, \nu) \longmapsto \log c^\nu_{\lambda\kappa}$$

is a concave function.

A counterexample to Conjecture 2.1.6 was found by Chindris, Derksen, and Weyman [CDW07]. However, using our Discrete Theorem 1, we immediately obtain a special case of Okounkov's Conjecture since Kostka numbers are examples of Littlewood–Richardson coefficients.

2.1.3 Lorentzian polynomials

The key ingredient to in the proof of Theorem 2.1.1 is to show that (normalized) Schur polynomials are Lorentzian. Lorentzian polynomials are a class of polynomials introduced by Brändén and Huh in 2019 which link discrete convex analysis, various log-concavity properties in combinatorics, volume polynomials in algebraic geometry, and stable polynomials in optimization theory [BH20].

Definition 2.1.7 (Brändén–Huh [BH20]). A degree d homogeneous polynomial $h(x_1, \ldots, x_n)$ is Lorentzian if

- all coefficients of h are nonnegative,
- $\mathrm{supp}(h)$ has the *exchange property*, and
- the quadratic form $\frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_{d-2}}}(h)$ has at most one positive eigenvalue for all $i_1, \ldots, i_{d-2} \in [n]$.

Example 2.1.8. The Schur polynomial $s_{(2,0)}(x_1, x_2) = x_1^2 + x_1x_2 + x_2^2$ is not Lorentzian because the eigenvalues of its associated quadratic form are $3/2$ and $1/2$, but the normalized Schur polynomial $N(s_{(2,0)}(x_1, x_2)) = \frac{x_1^2}{2} + x_1x_2 + \frac{x_2^2}{2}$ is Lorentzian because its associated quadratic form has eigenvalues 0 and 1.

Theorem 2.1.9 (Brändén–Huh [BH20]). *If $f = \sum_\alpha \frac{c_\alpha}{\alpha!} x^\alpha$ is a Lorentzian polynomial, then*

- *f is either identically 0 or $\log(f)$ is concave on $\mathbb{R}^n_{>0}$, and*
- *$c_\alpha^2 \geq c_{\alpha+e_i-e_j} c_{\alpha-e_i+e_j}$ for all α and for all $i, j \in [n]$.*

This immediately implies both Continuous Theorem 1 and Discrete Theorem 1, which in turn implies Theorem 2.1.1.

We note that we were unable to prove that normalized Schur polynomials are Lorentzian using a purely combinatorial argument. Instead, we showed that any normalized Schur polynomial is an example of a volume polynomial in algebraic geometry (and volume polynomials are known to be Lorentzian via [BH20]).

We conjecture that a host of polynomials occurring in algebraic combinatorics are Lorentzian polynomials. In the table below, we summarize our computer experiments in this direction.

Polynomial	**Tested for**
Schubert: $N(\mathfrak{S}_w(x_1,\ldots,x_n))$	$n \leq 8$
Skew Schur: $N(s_{\lambda/\mu}(x_1,\ldots,x_n))$	λ with ≤ 12 boxes and ≤ 6 parts
Schur P: $N(P_\lambda(x_1,\ldots,x_n))$	strict λ with $\lambda_1 \leq 12$ and ≤ 4 parts
homog. Grothendieck: $N(\widetilde{\mathfrak{G}}_w(x_1,\ldots,x_n,z))$	$n \leq 7$
Key: $N(\kappa_\mu(x_1,\ldots,x_n))$	compositions μ with ≤ 12 boxes and ≤ 6 parts

Lastly, we note that, together with Alejandro Morales and Jesse Selover, we conjectured that chromatic symmetric functions of Hessenberg graphs of Dyck paths are Lorentzian polynomials. We have some preliminary progress in this direction.

3 Cluster Algebras

3.1 A combinatorial model for positroids in partial flag varieties

Positroid cells are strata in a particularly well-behaved stratification of the (positive part of the) Grassmannian. In 2006, Alexander Postnikov developed a combinatorial model for these positroid cells which has led to a surge of study by a number of mathematicians. These positroid cells (and their associated combinatorial diagrams) have been linked to cluster algebras, representation theory, and string theory. Continuing work we started at IAS during discussions with Thomas Lam, Chris Fraser, Maitreyee Kulkarni, and I have studied positroid cells for partial flag varieties; moreover, motivated from physics, we have constructed a combinatorial diagram (called a momentum-twistor diagram in physics) in this more general setting.

A key tool in our results is the existence of a map

$$\phi\colon \mathrm{Fl}(1,3;n) := \{\mathbb{C} \subseteq \mathbb{C}^3 \subseteq \mathbb{C}^n\} \longrightarrow \mathrm{Gr}(4,2n).$$

In fact, this map lands in the cyclic shift ρ fixed point subvariety $\mathrm{Gr}(4,2n)^{\rho^n}$ of $\mathrm{Gr}(4,2n)$. We expect this map to send positroid cells into positroid varieties (the closures of postroid cells). Understanding the map ϕ has allowed us to associate a bounded affine permutation to each positroid cell in $\mathrm{Fl}(1,3;n)$. We are currently concentrating our efforts into describing the poset of strata of positroid cells for $\mathrm{Fl}(1,3;n)$, and plan to continue by extending our results to arbitrary partial flag varieties as the project progresses.

3.2 Computing upper cluster algebras

Cluster algebras are commutative unital domains generated by distinguished elements called **cluster variables** which are defined by a combinatorial process called mutation. Many notable spaces are equipped with cluster structures where certain regular functions play the role of cluster variables. For example, the coordinate ring of the space of $m \times n$ matrices is naturally a cluster algebra, and each matrix minor is a cluster variable. In this way, identities among matrix minors are a special case of the theory.

Cluster algebras are generally defined in terms of an infinite generating set; however, the most important cluster algebras may be finitely-generated. For this reason, computing generating sets of cluster algebras and the relations between those generators is an interesting problem. The cluster algebra $\mathcal{A}$ is the combinatorially defined object, but from a geometric perspective, there is a more natural algebra to consider: the **upper cluster algebra** $\mathcal{U}$, which was introduced in [BFZ05]. It is defined as an infinite intersection of Laurent polynomial rings, which makes $\mathcal{U}$ difficult to work with in general, as it is hard to write down any (even infinite) generating set. Despite this hurdle, all known examples of upper cluster algebras enjoy many nice properties, such as normality and being log Calabi–Yau. Studying $\mathcal{U}$ geometrically often gives information about the more intrinsic algebra $\mathcal{A}$, since $\mathcal{A} \subseteq \mathcal{U}$ by the Laurent phenomenon [FZ02].

One obstacle in the theory of cluster algebras has been an almost complete lack of examples in situations where $\mathcal{A} \neq \mathcal{U}$. In a recent paper [MM15], G. Muller (University of Michigan) and I exploited the algebraic geometry of $\mathcal{U}$ to obtain an algorithm for producing a presentation of $\mathcal{U}$ (when $\mathcal{A}$ is "totally coprime"—a mild technical condition) in terms of generators and relations.

Theorem 3.2.1 (M.–Muller 2015 [MM15])**.** *If $\mathcal{A}$ is a totally coprime cluster algebra, then a finite-time algorithm for presenting $\mathcal{U}$ exists whenever $\mathcal{U}$ is finitely generated.*

We used this technique to give presentations of several interesting upper cluster algebras where $\mathcal{A} \neq \mathcal{U}$. Also, together with Mills, Muller, and Williams, I implemented the algorithm in the computer algebra system Sage (see `https://trac.sagemath.org/ticket/18800`).

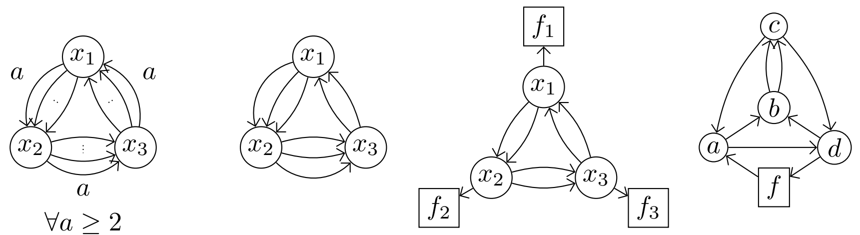


Fig. 1: Important cluster algebras for which our algorithm gave a presentation of $\mathcal{U}$.

Theorem 3.2.1 has been used by several other mathematicians in their research—I. Canakci, K. Lee, and R. Schiffler used our work to prove that $\mathcal{A} = \mathcal{U}$ for the dreaded torus (the last quiver in Figure 1) [CLS14], and more recently, M. Gross, P. Hacking, S. Keel, and M. Kontsevich used our computations to make conjectures about the cluster variety for the once-punctured torus (the first quiver in Figure 1 with $a = 2$) [GHKK14].

4 Representation theory of quivers and finite-dimensional algebras

4.1 Exceptional sequences and linear extensions

An **exceptional sequence** $(V_1, \dots, V_n)$ of quiver representations is a sequence of representations obeying certain strong homological constraints. Exceptional sequences were introduced in [GR87] to study exceptional vector bundles on $\mathbb{P}^2$. Since then, they have been shown to have many useful applications to several areas of mathematics. For example, they have applications in combinatorics because (complete) exceptional sequences are in bijection with maximal chains in the lattice of noncrossing partitions [IT09, HK13]. Furthermore, they have been shown to be intimately connected to acyclic cluster algebras as their dimension vectors appear as rows of **c**-matrices [ST13].

Even though they are pervasive throughout mathematics, very little work has been done to give a concrete description of exceptional sequences. Previously, A. Garver (University of Minnesota) and I extended work of T. Araya [Ara13] by classifying exceptional sequences of representations of the linearly-ordered quiver of type A using noncrossing edge-labeled trees in a disk with boundary vertices [GM15].

To extend this classification to type A quivers with any orientation, Igusa, Garver, Ostroff, and I used a more general combinatorial model called a **strand diagram**. We defined a bijective map Φ, which takes an indecomposable representation to its corresponding strand. It turns out that all of the homological information in the definition of an exceptional sequence is stored in a noncrossing diagram of strands.

Theorem 4.1.1 (Garver–Igusa–M.–Ostroff 2015 [GIMO15])**.** *Let Q be a type A quiver and let U and V be two distinct indecomposable representations of Q.*

a) The strands $\Phi(U)$ and $\Phi(V)$ intersect nontrivially if and only if neither (U, V) nor (V, U) are exceptional pairs.

b) The strand $\Phi(U)$ is clockwise from $\Phi(V)$ if and only if (U, V) is an exceptional pair and (V, U) is not an exceptional pair.

c) The strands $\Phi(U)$ and $\Phi(V)$ do not intersect at any of their endpoints and they do not intersect nontrivially if and only if (U, V) and (V, U) are both exceptional pairs.

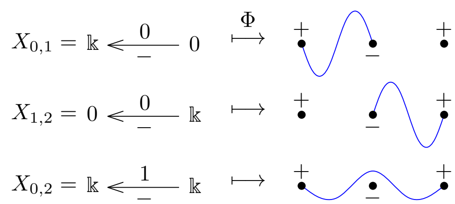


Fig. 2: The indecomposable representations of the type A_2 quiver $1 \leftarrow 2$ and their corresponding strands.

Theorem 4.1.1 has many immediate consequences. B. Keller proved that any two green-to-red sequences (sequences that are like MGS's except that mutation at red vertices is allowed) applied to Q with principal coefficients produce isomorphic ice quivers [Kel13]. His proof hinges on deep representation-theoretic and geometric methods, but the statement itself is purely combinatorial. During the FPSAC'13 conference, B. Keller asked for a combinatorial proof—our theorem solves this problem. Another application of Theorem 4.1.1 is that it allowed us to classify all c-matrices for type A cluster algebras in a completely combinatorial way using strand diagrams.

A **complete exceptional collection** is an unordered list of quiver representations which can be ordered to make a complete exceptional sequence. We have shown that the number of complete exceptional sequences arising from a given complete exceptional collection is equal to the number of linear extensions of a certain poset arising from the corresponding strand diagram. Counting the number of linear extensions of posets is a notoriously difficult problem in combinatorics; however, the class of posets arising from strand diagrams are very nice.

Theorem 4.1.2 (Garver–Igusa–M.–Ostroff 2015 [GIMO15])**.** *The class of posets arising from strand diagrams are exactly the posets $\mathcal{P}$ that satisfy the following properties.*

- *each $x \in \mathcal{P}$ has at most two covers and covers at most two elements,*
- *the underlying graph of the Hasse diagram of $\mathcal{P}$ has no cycles,*
- *the Hasse diagram of $\mathcal{P}$ is connected.*

In particular, zig-zag posets are contained in this class, and their linear extensions are counted by a determinant formula. Together with Alejandro Morales, I have been working towards a formula for the number of linear extensions of all posets in this class (see Section **??** for progress in counting linear extensions of a related class of posets).

Problem 4.1.3 (M.–Morales)**.** *Count the number of complete exceptional sequences arising from a given complete exceptional collection by developing a method for counting the number of linear extensions of posets arising from strand diagrams.*

4.2 Computing the Fourier–Sato transform combinatorially

In this section, I will describe a problem involving perverse sheaves on certain quiver varieties and discuss possible applications to representation theory of quantum loop algebras $\mathcal{U}_q(L\mathfrak{g})$ and cluster algebras.

The Fourier–Sato transform is a geometric version of the Fourier transform for functions from analysis. If V is a complex vector space, then the Fourier–Sato transform is a certain functor $\mathbb{T}$ which gives an equivalence of categories between derived categories of conical sheaves

$$\mathbb{T} : \mathcal{D}^{\mathrm{b}}_{\mathrm{con}}(V) \to \mathcal{D}^{\mathrm{b}}_{\mathrm{con}}(V^*),$$

where V^* is the dual vector space. The functor $\mathbb{T}$ has been used in various forms to much success for more than thirty years—it was used by Laumon in the mid-1980s to significantly shorten Deligne's proof of the Weil conjectures (see [KW01] for a nice treatment), it gives an alternative construction of the Springer correspondence [HK84, Bry86], and it plays a central role in the theory of character sheaves [Lus87, Mir04].

Despite its usefulness, in practice $\mathbb{T}$ is difficult to compute explicitly. In fact, it has only been explicitly computed in few settings—some examples include the nilpotent cone $\mathcal{N}$ of a semisimple Lie algebra $\mathfrak{g}$ [AM15] and various stratified spaces of $n \times n$ matrices [BG99].

P. Achar, M. Kulkarni, and I have developed a method for computing $\mathbb{T}$ combinatorially. Let Q be the type A_n quiver $\bullet \to \bullet \to \cdots \to \bullet$. Fix a dimension vector $\mathbf{w}$, and consider the space of representations of Q with this dimension vector. This is an affine space $E_{\mathbf{w}}$ which carries an action of a group $G_{\mathbf{w}}$, a product of general linear groups. This action gives a stratification of $E_{\mathbf{w}}$ into $G_{\mathbf{w}}$-orbits, and we have the Fourier–Sato transform

$$\mathbb{T} : \mathcal{D}^{\mathrm{b}}_{G_{\mathbf{w}}}(E_{\mathbf{w}}) \to \mathcal{D}^{\mathrm{b}}_{G_{\mathbf{w}^*}}(E_{\mathbf{w}^*})$$

where $\mathbf{w}^*$ is the reverse of $\mathbf{w}$. In this setting, $\mathbb{T}$ is t-exact for the perverse t-structure, and it sends (simple) perverse sheaves to (simple) perverse sheaves.

To each $G_{\mathbf{w}}$-orbit, we associate a triangular array of nonnegative integers satisfying certain conditions arising from dominant weights for $\mathbf{GL}_n$. These triangular arrays completely determine the $G_{\mathbf{w}}$-orbits.

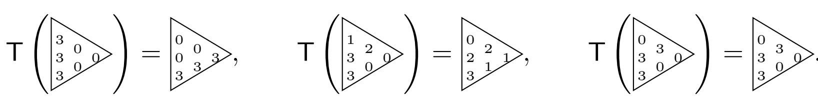
Theorem 4.2.1 (Achar–Kulkarni–M. 2016)**.** *There is a bijection between $G_{\mathbf{w}}$-orbits in $E_{\mathbf{w}}$ and a certain set $\mathbf{P}_{\mathbf{w}}$ of triangular arrays of nonnegative integers.*

The combinatorial set $\mathbf{P}_{\mathbf{w}}$ carries a wealth of information about the geometry and representation theory of $E_{\mathbf{w}}$—the dimension of $G_{\mathbf{w}}$-orbits and the partial order on them, as well as whether a representation is injective or projective, can be read off the diagram. Most notably, it allows for a combinatorial computation of $\mathbb{T}$.

Theorem 4.2.2 (Achar–Kulkarni–M. 2018 [AKM18])**.** *There exists a combinatorial algorithm $\mathsf{T} : \mathbf{P}_{\mathbf{w}} \to \mathbf{P}_{\mathbf{w}^*}$ that computes $\mathbb{T}$ for every simple perverse sheaf; i.e., such that $\mathbb{T}(\mathrm{IC}(\mathcal{O}_\lambda)) = \mathrm{IC}(\mathcal{O}_{\mathsf{T}(\lambda)})$, where $\mathrm{IC}(\mathcal{O}_\lambda)$ is the simple perverse sheaf supported on the closure $\overline{\mathcal{O}_\lambda}$ of the $G_{\mathbf{w}}$-orbit $\mathcal{O}_\lambda$ given by $\lambda \in \mathbf{P}_{\mathbf{w}}$.*

Corollary 4.2.3 (Achar–Kulkarni–M. 2018 [AKM18])**.** *The map T and its inverse T' both give* new algorithms *for computing the Knight–Zelevinsky multisegment duality (defined in [KZ96]).*

We give a few examples of T below for the A_3 quiver with dimension vector $\mathbf{w} = (3, 3, 3)$. Note that the first example is the combinatorial version of the geometric fact that $\mathbb{T}$ takes the unique zero-dimensional orbit to the unique dense orbit.



Recently, Nakajima used the Fourier–Sato transform on his graded quiver varieties to prove a character formula for representations of quantum loop algebras $\mathcal{U}_q(L\mathfrak{g})$, as well as to give a monoidal categorification of certain cluster algebras [Nak11]. In his proof, he computes the Fourier–Sato transform of only a single object. Theorem 4.2.2 allows for the computation of $\mathbb{T}$ for every object, at least for the quiver Q.

Problem 4.2.4. *Get information about representations of $\mathcal{U}_q(L\mathfrak{g})$ by using T, our combinatorial description of $\mathbb{T}$. What does our explicit computation of $\mathbb{T}$ say about the cluster algebras involved?*

This problem is particularly interesting to me as cluster algebras have been another important research interest of mine—I have results about their structure theory (see Section 3.2) and their relationship with quiver representations (see Section 4.1). Solving this problem would blend geometric representation theory, quivers, and cluster algebras (several of my research interests).

5 Geometric representation theory and the geometric Langlands program

5.1 Moduli spaces of Coxeter connections on $\mathbb{P}^1$

Recently Kamgarpour and Sage investigated rigidity for a class of connections on $\mathbb{P}^1$ that they call "Coxeter connections" [KS20]. This class of connections is particularly interesting from the perspective of the geometric Langlands program, as they should be examples of "cuspidal" Langlands parameters.

Together with Maitreyee Kulkarni, Neal Livesay, Bach Nguyen, and Daniel Sage, we have recently solved the Deligne–Simpson problem for the moduli space of framed Coxeter connections on $\mathbb{P}^1$ in type A. (Deligne–Simpson problems are concerned with determining whether a given moduli space is (non)empty.) Surprisingly, these results were obtained using only elementary linear algebra and various ad-hoc methods. More surprisingly, one key ingredient of our proof is the seemingly unrelated paper [KL95]. We are currently working to generalize [KL95] to other classical Lie types beyond type A, and we expect the resolution of the Deligne–Simpson problem in these settings to follow immediately afterward.

A tangential direction we are workng in is to develop a more technical approach, rather than working in an ad-hoc way. In particular, Crawley–Boevey solved the additive Deligne–Simpson problem for connections with regular singularities on $\mathbb{P}^1$ in 2003 by relating the moduli space of such connections to a certain quiver variety [CB03]. (This problem was very influential at the time; in fact, Crawley–Boevey was invited to the ICM to speak about this work.) Recently, Boalch [Boa12] and Hiroe–Yamakawa [HY14] have extended Crawley–Boevey's approach to a class of connections with irregular singularities. It would be desirable to solve the Deligne–Simpson problem for Coxeter connection on $\mathbb{P}^1$ by relating the moduli space of such objects to a quiver variety.

These results were initiated during an AIM (American Institute of Mathematics) SQuaRE workshop, where we all collaborated together for one week in San Jose, California during the month of February. We will continue to meet at AIM for the next two years.

5.2 Derived geometric Satake equivalence, Springer correspondence, and small representations

Let G be a semisimple complex algebraic group and T be a maximal torus. The Weyl group W acts on the zero weight space V^T of any representation V of G, giving a functor $\Phi_G : \mathrm{Rep}(G) \to \mathrm{Rep}(W)$.

Achar, Henderson, and Riche recently constructed a geometric lift of this functor [AH13, AHR15]. Its construction begins with a result of Lusztig [Lus81] that for $\mathbf{GL}_n(\mathbb{C})$ the nilpotent cone $\mathcal{N}$ (the variety of nilpotent $n \times n$ matrices) can be embedded in the affine Grassmannian $\mathrm{Gr}_{\mathbf{GL}_n(\mathbb{C})}$, an infinite-dimensional analog of the Grassmannian of k planes in n space. To generalize this result to other groups, one needs to restrict to a certain finite-dimensional closed subvariety $\mathrm{Gr}^{\mathrm{sm}}$ of Gr. There is a certain open subvariety $\mathcal{M} \subset \mathrm{Gr}^{\mathrm{sm}}$ and a finite map $\pi : \mathcal{M} \to \mathcal{N}$, which can be viewed as a generalization of Lusztig's embedding for other groups. The map π gives rise to a functor $\Psi_{\check{G}} : \mathrm{Perv}_{\check{G}(\mathfrak{O})}(\mathrm{Gr}_{\check{G}}) \to \mathrm{Perv}_{\check{G}}(\mathcal{N})$.

To explain in what sense $\Psi_{\check{G}}$ is a geometric lift of Φ_G, we invoke two major theorems in geometric representation theory: the geometric Satake equivalence and the Springer correspondence.

(a) There is an equivalence of categories $\mathcal{S} : \mathrm{Perv}_{\check{G}(\mathfrak{O})}(\mathrm{Gr}_{\check{G}}) \to \mathrm{Rep}(G)$, where $\mathrm{Perv}_{\check{G}(\mathfrak{O})}(\mathrm{Gr}_{\check{G}})$ is the category of $\check{G}(\mathfrak{O})$-equivariant perverse sheaves on the affine Grassmannian $\mathrm{Gr}_{\check{G}} := \check{G}(\mathfrak{K})/\check{G}(\mathfrak{O})$, $\check{G}$ is the Langlands dual group, $\mathfrak{K} = \mathbb{C}((t))$, and $\mathfrak{O} = \mathbb{C}[[t]]$ [Lus83, Gin95, MV07].

(b) There is a functor $\mathbb{S} : \mathrm{Perv}_{\check{G}}(\mathcal{N}) \to \mathrm{Rep}(W)$, where $\mathrm{Perv}_{\check{G}}(\mathcal{N})$ is the category of $\check{G}$-equivariant perverse sheaves on the nilpotent cone $\mathcal{N} \subset \check{\mathfrak{g}}$ [Spr76, Lus81, BM81].

Theorem 5.2.1 ([AH13, AHR15])**.** *The functor $\Psi_{\check{G}}$ is a geometric lift of Φ_G; i.e., there exists a commutative diagram:*

$$\begin{array}{ccc} \mathrm{Perv}_{\check{G}(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}}_{\check{G}}) & \xrightarrow[\sim]{\mathcal{S}^{\mathrm{sm}}} & \mathrm{Rep}(G)_{\mathrm{sm}} \\ \downarrow{\scriptstyle \Psi_{\check{G}}} & & \downarrow{\scriptstyle \Phi_G} \\ \mathrm{Perv}_{\check{G}}(\mathcal{N}) & \xrightarrow{\mathbb{S}} & \mathrm{Rep}(W) \end{array}$$

Parallel to this development, derived versions of (a) and (b) were developed—Bezrukavnikov and Finkelberg gave an equivalence $\mathrm{der}\mathcal{S} : \mathcal{D}^{\mathrm{b,mix}}_{\check{G}(\mathfrak{O})}(\mathrm{Gr}_{\check{G}}) \to \mathcal{D}^{\mathrm{b}}\mathrm{Coh}^{G \times \mathbb{G}_m}(\mathfrak{g}^*)$ [BF08], and Rider established the equivalence $\mathrm{der}\mathbb{S} : \mathcal{D}^{\mathrm{b,mix}}_{\check{G},\mathrm{Spr}}(\mathcal{N}_{\check{G}}) \to \mathcal{D}^{\mathrm{b}}\mathrm{Coh}^{W \times \mathbb{G}_m}(\mathfrak{h}^*)$ [Rid13]. Here, $\mathfrak{g}^*$ and $\mathfrak{h}^*$ are both affine varieties, so the categories on the right-hand side are graded modules over their coordinate rings. The word "mix" refers to a geometric analog of grading. My dissertation work involved producing a derived analog of Φ_G and showing that its derived lift is $\Psi_{\check{G}}$.

Theorem 5.2.2 (M. 2016 [Mat16])**.** *There exists a commutative diagram of derived categories:*

$$\begin{array}{ccc} \mathcal{D}^{\mathrm{b,mix}}_{\check{G}(\mathfrak{O})}(\mathrm{Gr}^{\mathrm{sm}}_{\check{G}}) & \xrightarrow[\sim]{\mathrm{der}\mathcal{S}^{\mathrm{sm}}} & \mathcal{D}^{\mathrm{b}}\mathrm{Coh}^{G\times\mathbb{G}_m}(\mathfrak{g}^*)_{\mathrm{sm}} \\ {\scriptstyle \Psi_{\check{G}}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{der}\Phi_G} \\ \mathcal{D}^{\mathrm{b,mix}}_{\check{G},\mathrm{Spr}}(\mathcal{N}_{\check{G}}) & \xrightarrow[\mathrm{der}\mathbb{S}]{\sim} & \mathcal{D}^{\mathrm{b}}\mathrm{Coh}^{W\times\mathbb{G}_m}(\mathfrak{h}^*) \end{array}$$

The solution to this problem involves two major steps.

1. Establish the result for all groups G of semisimple rank 1.
2. Show that each of the functors involved commutes with restriction to such a group.

Over fields $\mathbb{F}$ of positive characteristic, representation theory is much more difficult—for example, the categories involved are not semisimple in general. In his thesis, Mautner proved that for $G = \mathbf{GL}_n(\mathbb{F})$ the category $\mathrm{Perv}_G(\mathcal{N})$ is equivalent to the category of finitely-generated modules for a certain Schur algebra [Mau10]. Very recently, tremendous progress has been made toward producing analogs of Springer theory in positive characteristic [AHJR16, AHJR14, AHJR15]. It is motivating to consider a positive characteristic version of Theorem 5.2.2 for its possible applications to positive characteristic Springer theory and representations of Schur algebras.

Problem 5.2.3. *Develop an analog of Theorem 5.2.2 for sheaves in positive characteristic.*

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