Research Statement

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My research is, broadly, *applied mathematics*. This includes utilizing a mix of complex analysis in one variable, differential geometry, and functional analysis used for the purpose of obtaining explicit formulae for the reconstruction of integral ray transforms on Riemannian manifolds. I'm also quite interested in problems of mathematical physics and in the implementation of algorithms.

Contents

1 Past Work: The Method of Complexification

1.1 Short Background on AtRT

In several tomographic situations in engineering one encounters the inverse problem of having accessed data consisting of line integrals $I_\theta f$ of an unknown function f over a broad class of lines with the goal of using this data to reconstruct f or "inverting I_θ". For example, in the arena of medical imaging, this arises in positron emission tomography (PET), single photon emission computerized tomography (SPECT), and (originally) CT-scan tomography [9]. In other applications the line integral $I_\theta f$ is taken not over lines, but rather over a class of one-dimensional curves in either Euclidean space or more generally, a Riemannian manifold. In geophysics for instance, the problem can arise as the linearization of the problem of determining geophysical properties of the inner Earth based on travel-time measurements made at the surface [23].

Generally, *explicit* inversion formulae over curves other than lines tend to restrict focus to situations/manifolds with a strong amount of symmetry [10, 11, 12, 3, 21] and typically do not include the effects of absorption encountered during propagation. A recent exception to this statement can be found in [14, 15].

Quite often the physics dictates that the signal suffers absorption along its trajectory and is thereby attenuated. The resulting data is then called, not surprisingly, the **attenuated ray transform** (AtRT), and is often denoted $I_a f$. This arose first in SPECT and has recently been discussed in relation to EIT, in the form of the Calderón problem for inverse conductivity in anisotropic media in [6, 4]. Early results on the AtRT were presented first in [8], and then in [22, 16, 13, 19].

1.2 Some New Ground

In [18] I obtained a rather general prescription

$$f(z) = \frac{1}{4\pi}\int_0^{2\pi} P(\lambda_i,\theta)X_\theta^\perp H(I_\theta f)(s(ze^{-i\theta}), e^{i\theta})d\theta \tag{1}$$

for an explicit filtered backprojection formula in the reconstruction of a smooth enough function $f(z)$ given its integral $I_\theta f$ over a class of curves in the unit disc $D^+ \subset \mathbb{C}$, parameterized by angles θ. The method used to obtain (1) generalized the technique appearing in [19, 2] and is known as *the method of complexification*. Formula (1), once obtained, was used to give a **holomorphic integrating factor** method to get a similar, though more complicated, formula for the AtRT on the same class of curves, namely

Theorem 1.1 *If X_λ is a vector field of **type H**, $u(z,\lambda_i) = 0$ and $f \in C_0^\infty(D^+)$, then*

$$f(z) = \frac{1}{4\pi}\int_0^{2\pi} P(\lambda_i,\theta)X_\theta^\perp (e^{-(D_\theta a)(z)}H_a I_{a,\theta}f)(s(ze^{-i\theta}),\theta))d\theta$$

gives an exact reconstruction formula for the density f based on the data $I_{a,\theta}f$ of attenuated ray transforms of f over the integral curves of X_θ.

The technique used to obtain (1) rests on the complexification of a certain class of differential operators in $\mathbb{R}^2$, known as **type H**. The complexification takes the form of introducing a complex parameter in place of the angles $e^{i\theta}$. This allowed me to recast the inversion problem in terms of complex analysis in the unit disc. The terms λ_i appearing above are the zeroes of a complexified coefficient of the original vector field X_θ, over whose integral curves $I_\theta f$ are the traces of f.

2 Current Research

2.1 H-Ness and its Limitations

While formula 1 is quite general, its derivation required stringent conditions on the original vector field defining the transport, namely that it be of **type H**. I am currently improving upon this limitation by addressing the following

Conjecture 2.1 *The condition known as **type H** can be reduced to just one condition on the uncomplexified coefficient $\mu(z,\bar{z})$ of $\frac{\partial}{\partial z}$ in X_θ.*

To that end, I reduced H-ness and identified a wide class of polynomials $\Gamma(D^+)$ for which the following proposition is valid

Proposition 2.2 *If $\mu(z,\bar{z}) = \sum_{p+q=r} a_{pq}z^p\bar{z}^q \in \Gamma(D^+)$ and if for $z \neq 0$ and $c_r(z,\bar{z}) \doteq \sum_{p,qq-p=r} a_{pq}z^p\bar{z}^q$ we have*

$$\log|c_k(z)| < \log|c_l(z)|$$

then the first two conditions of H-ness are met for a rescaled field $\lambda_*(\frac{1}{w}X_\theta)$.

Using classical results on Beltrami equations (c.f. [20], Thm. 3.2) there is then a wide class of real-analytic functions, $\mathcal{H}_{k,l}(D^+)$, which are then *scalable* in the sense that while X_θ may fail H-ness, it can easily be reformulated to meet it. More precisely one has

Theorem 2.3 *If* $\mu \in \mathcal{H}_{k,l}(D^+)$, $\lambda_* \frac{\mu}{w}\big|_{\lambda_i(z)} = 0$, $X_\theta^\perp \doteq ie_*^{i\theta}(-\frac{\mu(z)}{w(z)}\frac{\partial}{\partial z} + \frac{\bar{\mu}(z)}{w(z)}\frac{\partial}{\partial \bar{z}})$ *and* $f \in C_0^\infty(D^+)$ *then*

$$f(z) = \frac{w(z)}{4\pi}\int_0^{2\pi} P(\lambda_i, \theta) X_\theta^\perp H(\tilde{I}_\theta f)(s(ze^{-i\theta}), e^{i\theta}) d\theta$$

gives an exact reconstruction formula for the density f *based on the data* $\tilde{I}_\theta f$ *of ray transforms of* f *over the integral curves of* $e_*^{i\theta}(\mu\partial + \bar{\mu}\bar{\partial})$*.*

I am currently working on constructing dense truncating approximants within $\mathcal{H}_{k,l}(D^+)$ to the original vector field coefficients to allow me to extend formula (1) to an even more general setting. This utilizes a combination of harmonic analysis, complex analysis of one variable, and some geometric function theory.

2.2 Some Fan-Beam Work

In examining related work by Pestov and Uhlmann, I noticed a remarkable similarity to my own result (1). The problem they considered in [14] concerns geodesic transport in the unit sphere bundle on a 2-dimensional, simple Riemannian manifold $\mathcal{M}$. The transport BVP is

$$\begin{aligned} (\xi^i \frac{\partial}{\partial x^i} - \Gamma^i_{jk}\xi^j\xi^k \frac{\partial}{\partial \xi^i})u(x,\xi) &= -f(x), \qquad (x,\xi) \in \Omega(\mathcal{M}) &(2)\\ u|_{\partial_-\Omega(\mathcal{M})} &= 0 &(3)\end{aligned}$$

with Γ^i_{jk} the local components of the Christoffel symbol defining parallel transport. Then the ray transform data addressed in [14] is

$$If(x,\xi) \doteq u^f(x,\xi)\Big|_{\partial_+\Omega(\mathcal{M})} \tag{4}$$

which is of fan-beam type. The main result the authors present relevant to my work is

Theorem 2.4 *Let* $(\mathcal{M}, g)$ *be a* 2*-dimensional simple manifold. Then*

$$f + W^2 f = \frac{1}{4\pi}\delta_\perp I_1^* \alpha^* \, H(I_0 f)^-|_{\partial_+\Omega(\mathcal{M})} \tag{5}$$

or a certain operator W

I became interested in this variation of the problem when I noticed the similarity of the Fredholm term W appearing in (5) since a very similar operator made an appearance in the derivation of (1). Though their result was derived via microlocal analysis, I noticed that it could be formally obtained by using a different variation on the method of complexification. Namely by producing, *ex nihilo*, local holomorphic functions via the fiberwise Cauchy transform $\mathcal{C} : C^\infty(\partial D^+) \to \mathcal{H}(D^+)$ one can recover their result. I applied this formal technique to predict the conjectural result

Conjecture 2.5

$$f + W^2 f = \frac{1}{2}(X^\perp \alpha^* \{(e^{-\frac{Ia}{2}} H_a I_a f \mid_{\partial_+\Omega(\mathcal{M})}\})_0 \tag{6}$$

to hold for the *attenuated* ray transform. In the interim, in [15], Uhlmann and Salo used a method almost identical to my conjectural formal calculations to produce an *algorithm* to invert the attenuated ray transform under similar circumstances. It stops short of producing (6) since the W operator now needed appears to no longer be compact. I am working on justifying the formal calculations needed to arrive at (6) and classifying the conditions on its validity. This requires a mix of geometry, complex analysis, and classical functional analysis.

3 A Glimpse Ahead

There are several areas I am strongly interested in pursuing postdoctoral research in. They extend upon my previous work as well as delve into new domains.

3.1 Numerical Implementation and Scientific Computation

The role of computation in the practice of both applied mathematics generally and ray transforms particularly can hardly be overstated. Traditional methods have used projection-slice type theorems in combination with the FFT to give high-order, efficient reconstructions in Euclidean settings, c.f. [17]. In [1] an algebraically exact *pseudopolar* FFT is introduced which resolves the Radon data in $O(NlogN)$ steps with $N = n^2$ pixels and skirts traditional problems associated with interpolating non-Cartesian gridpoints. These ideas are elaborated and extended upon in [25] for curvilinear ridge detection.

Regardless of the details of **condition H** and its sufficiency in producing formula (1) it is beneficial to understand its applicability in practice. This requires

- Efficient code to implement (1) numerically under ideal (**type H**) circumstances as a calibration
- Stability checks for the case of noisy data $(I_\theta f)(s(ze^{-i\theta}),\theta) + \epsilon(s(ze^{-i\theta}),\theta)$, for small Gaussian perturbation ϵ
- Checks on necessity, not sufficiency, of **condition H** for (1) to hold through broad numerical examples

The first of the above is not so easy to accomplish. First of all, in practical examples X_θ may be the local projection onto the base of a geodesic foliation of our domain. This would require solving many systems of the form

$$\dot{\theta}^i = -\Gamma^i_{jk}\theta^j\theta^k$$

which are generally quite numerically stiff. This being overcome, the challenge of an *efficient, adaptive* way to implement the necessary filtered integrations over a broad class of geometries is one that I am very eager to solve since this would definitively answer the pragmatic quandary that a formula like (1) necessarily presents.

3.2 Some Questions of Fields

In [7], Finch and Uhlmann looked at a problem concerning a Schrödinger-type equation in the presence of an external gauge field. The inverse problem they considered was that of recovering the matrix-valued unitary Yang-Mills connections on the bundle based on knowledge of the Dirichlet-to-Neumann map Λ_A, modulo the allowable gauge transformations induced by $U(m)$. Knowing Λ_A allows one to reformulate the problem as one of matrix transport

$$\theta \cdot \nabla C = \sum A_i\theta_i C(\mathbf{x},\theta), \qquad C(\mathbf{x},\theta) = Id \qquad on \qquad \mathbf{x}\cdot\theta < -R$$

where the equivalent problem asks whether knowledge of far-field $C(\mathbf{x},\theta)$ is sufficient to recover the connection modulo the allowable bundle transformations. The answer they give is in the affirmative provided one has stringent bounds placed on the Lie-group valued curvature form. They succeed by fixing the gauge, using an energy bound and applying some clever algebra. A related problem was attacked by Sharafudtinov in [24] where the goal was recovering a connection, up to automorphisms of the gauge group, based on known parallel transport measurements made at the boundary. Again, the affirmative answer depends on the associated curvature.

Since both of these problems use transport and beam transforms in answering an intrinsically geometrical question they are quite fascinating to me. I would like, in the future, to spend time working on generalizing the method of complexification to situations like the above, which would require introduction of *several* complex parameters and analysis in the associated polydisc. An interesting single variable complexification technique on $\mathbb{C}^3$ was used in [26] and may turn out to be useful in this endeavor.

3.3 Spectral Graph Theory

Lastly, I should mention that I have an interest in pursuing spectral graph theory [5] as a discrete inverse problem. I was exposed to this subject, as a prospective graduate student, by Dan Spielman `http://www-math.mit.edu/~spielman/eigs/` and have become increasingly interested in exploring computational graph theory and cluster analysis. Since Wigner's law on the density of states combines mathematical physics with graph invariants, spectral graph theory is a useful combination of two highly interesting areas of mathematics that I would like to explore. In the future, I would like to incorporate discrete networks, their invariants, and sound algorithms into my research in inverse theory.

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