CHAPTER 7

RESPONSE OF FIRST-ORDER RC AND RL CIRCUITS

CONTENTS

7.1 The Natural Response of an RC Circuit

Resistive Circuit => RC Circuit

algebraic equations => differential equations

Same Solution Methods (a) Nodal Analysis

(b) Mesh Analysis

7.1 The Natural Response of an RC Circuit

The solution of a linear circuit, called dynamic response, can be decomposed into

Natural Response + Forced Response

or in the form of

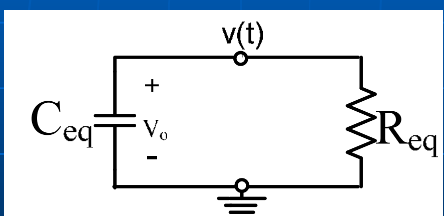
Steady Response + Transient Response

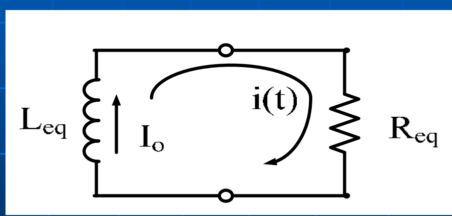
7.1 The Natural Response of an RC Circuit

- The natural response is due to the initial condition of the storage component (C or L).
- The forced response is resulted from external input (or force).
- In this chapter, a constant input (DC input) will be considered and the forced response is called step response.

7.1 The Natural Response of an RC Circuit

Example 1 : Two forms of the first order circuit for natural response

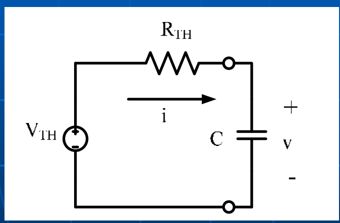




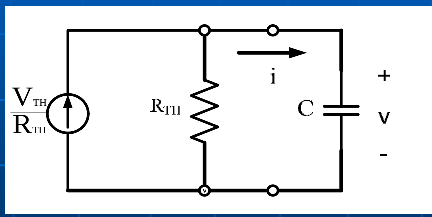
Find $v(t)$ Find $i(t)$

7.1 The Natural Response of an RC Circuit

Example 1 : (cont.) Four forms of the first order circuit for step response



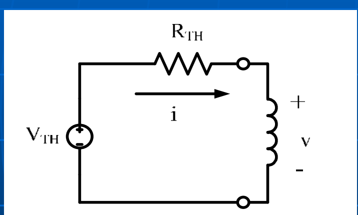
A capacitor connected to a Thevenin equivalent



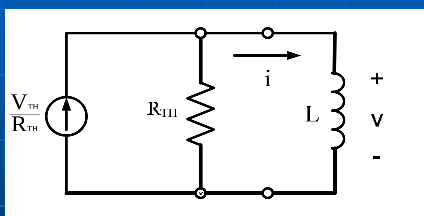
A capacitor connected to a Norton equivalent

7.1 The Natural Response of an RC Circuit

Example 1 : (cont.)



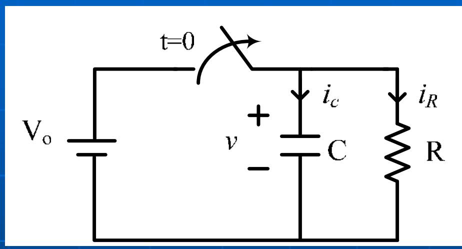
An inductor connected to a Thevenin equivalent



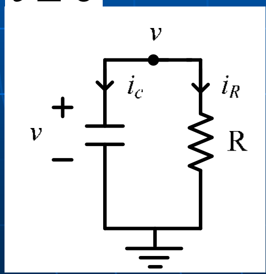
An inductor connected to a Norton equivalent

7.1 The Natural Response of an RC Circuit

Example 2



$t \geq 0$



nodal analysis

$$i_c + i_R = 0$$

$$C\frac{dv}{dt} + \frac{1}{R}v = 0$$

7.1 The Natural Response of an RC Circuit

Example 2 (cont.)

characteristic root S,

$$CS + \frac{1}{R} = 0$$

$$S = -\frac{1}{RC}$$

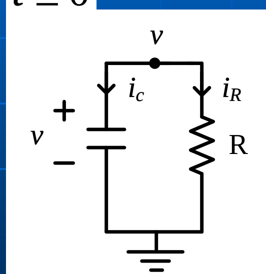
$$\therefore v(t) = Ke^{-\frac{1}{RC}t}, t \geq 0$$

From the initial condition

$$v(0^+) = v(0^-) = V_0$$

$$\therefore v(t) = V_0 e^{-\frac{t}{RC}}$$

$t \geq 0$



7.1 The Natural Response of an RC Circuit

Example 2 (cont.)

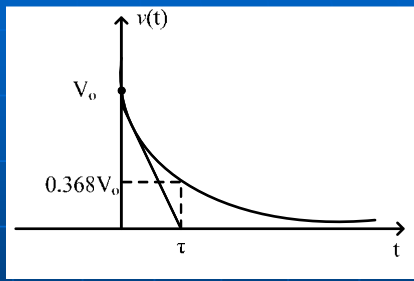
$t = RC$ *time constant*

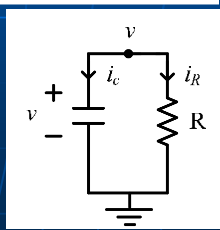
$v(\text{t}) = V_0 e^{-\frac{t}{t}}, t \geq 0$

$t = t \quad , \quad \frac{v(t)}{V_0} = 0.36788$

$t = 3t \quad , \quad 4.979\% < 5\%$

$t = 5t \quad , \quad 0.674\% < 1\%$



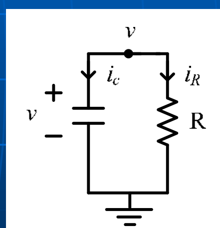


7.1 The Natural Response of an RC Circuit

Example 2 (cont.)

It is customary to assume that the capacitor is fully discharged after five time constants.

$$i_R(t) = \frac{v(t)}{R} = \frac{V_0}{R} e^{-\frac{t}{t}}, t \geq 0$$



The power dissipated in R is

$$p(t) = vi_R = \frac{V_0^2}{R} e^{-\frac{2t}{t}}$$

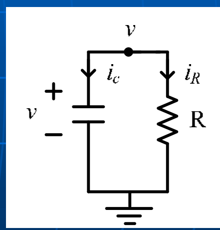
7.1 The Natural Response of an RC Circuit

Example 2 (cont.)

The energy dissipated in R is

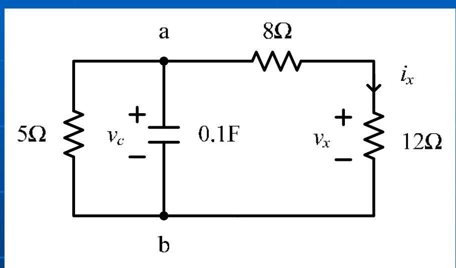
$$w_R(t) = \int_0^t p\,dt = \frac{1}{2} C V_0^{\,2}(1 - e^{-\frac{2t}{t}})$$

$$w_R(\infty) = \frac{1}{2} C V_0^{\,2}$$



7.1 The Natural Response of an RC Circuit

Example 3 : Find v_c , v_x , and i_x for t > 0



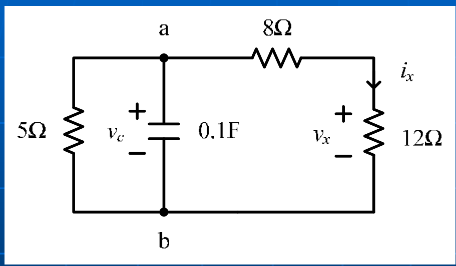
$v_c(0)$=15V

This circuit contains only one energy storage element.

Step 1. Use Thevenin theorem to find the equivalent R_{TH} looking into a-b terminals.

7.1 The Natural Response of an RC Circuit

Example 3 : Find v_c , v_x , and i_x for t > 0



$v_c(0)$=15V

Step 2. Find $v_c(t)$

Step 3. Replace $v_c(t)$ as a voltage source in the original circuit and solve the resistive circuit.

7.1 The Natural Response of an RC Circuit

Step 1. $R_{TH} = (8+12) \parallel 5 = 4\Omega$

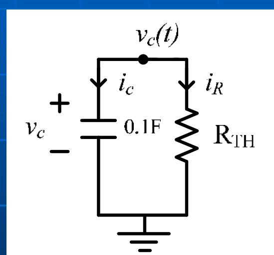
Step 2.

$$v_c(0) = 15\text{V}$$

$$0.1\frac{dv_c}{dt} + \frac{v_c}{4} = 0$$

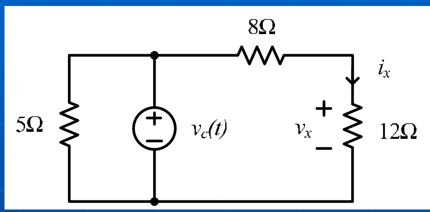
$$\tau = RC = 0.4\ \text{s}$$

$$v_c(t) = 15e^{-2.5t}\ \text{V}, t \geq 0$$



7.1 The Natural Response of an RC Circuit

Step 3.



By using voltage divider principle

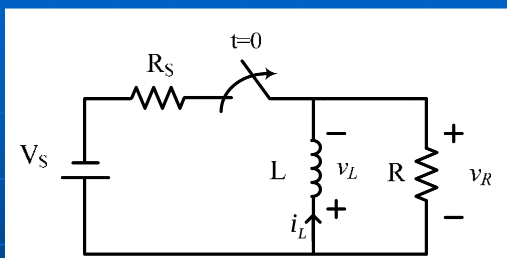
$$v_x(t) = \frac{12}{8+12} v_c(t)$$

$$= 9 \times e^{-2.5t} V, t \geq 0$$

$$i_x(t) = \frac{v_x(t)}{12} = 0.75e^{-2.5t} A$$

7.2 The Natural Response of an RL Circuit

Example 4

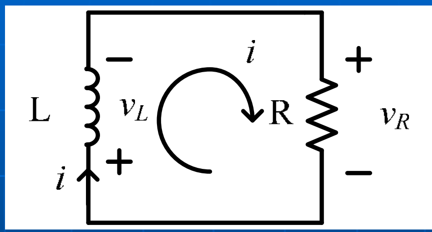


Remember that $v_c(t)$ and $i_L(t)$ are continuous functions for bounded inputs.

$$t \leq 0 \ , \ i = \frac{V_S}{R_S} = i(0^-) @ I_0$$

7.2 The Natural Response of an RL Circuit

$t \geq 0$



mesh analysis

$$v_L + v_R = 0$$

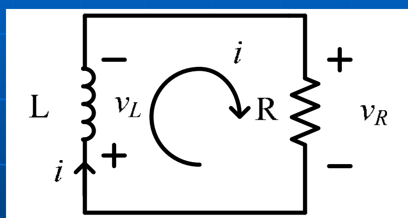
$$L\frac{di}{dt} + Ri = 0$$

7.2 The Natural Response of an RL Circuit

characteristic equation

$$LS + R = 0$$

$$\therefore S = -\frac{R}{L}$$



$$i(t) = Ke^{-\frac{R}{L}t}, t \geq 0$$

$$i(0^+) = i(0^-) = I_0$$

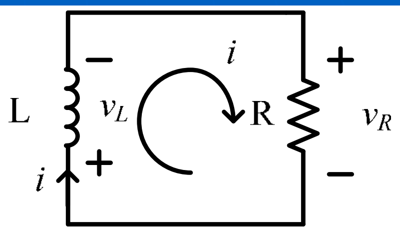
$$\therefore i(t) = I_0 e^{-\frac{t}{\tau}}, t \geq 0$$

$$\tau \triangleq \frac{L}{R}, \text{ time constant}$$

7.2 The Natural Response of an RL Circuit

$$v_R(t) = Ri = I_0 Re^{-\frac{t}{t}}$$

$$p_R = v_R i = I_0^{\,2} Re^{\frac{-2t}{t}}$$

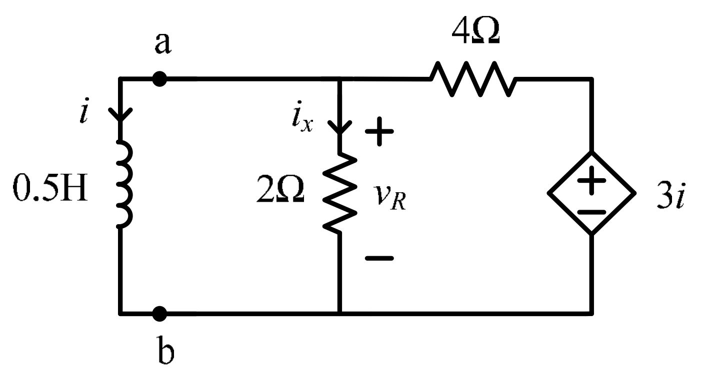


$$w_R(t) = \int_0^t p_R(t)dt = \frac{1}{2} LI_0^{\,2}(1 - e^{\frac{-2t}{t}})$$

$$w_R(\infty) = \frac{1}{2} LI_0^{\,2}$$

7.2 The Natural Response of an RL Circuit

Example 5 : $i(0)$=10A , find i(t) and i_x(t)



7.2 The Natural Response of an RL Circuit

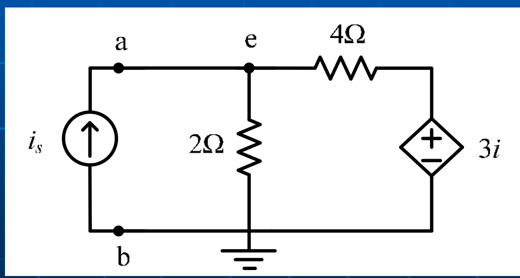
Step 1. Find the Thevenin equivalent circuit looking into a-b terminals.

Apply i_s , find e. Then $R_{TH} = e / i_s$

$$i = -i_s$$

$$\frac{e-3i}{4} + \frac{e}{2} = i_s$$

$$\frac{e}{4} + \frac{3}{4} i_s + \frac{e}{2} = i_s$$

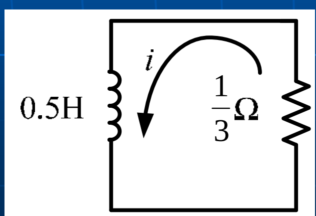


7.2 The Natural Response of an RL Circuit

$$\frac{3}{4} e = \frac{1}{4} i_s$$

$$\therefore \frac{e}{i_s} = \frac{1}{3} \Omega$$

Step 2.



mesh equation

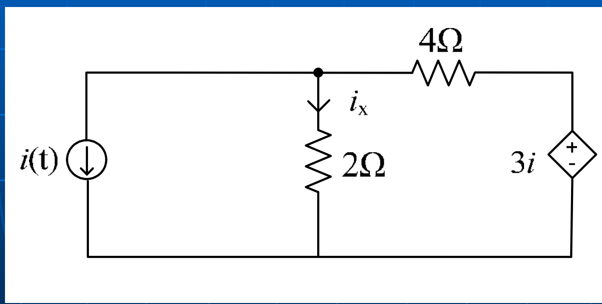
$$0.5 \frac{di}{dt} + \frac{1}{3} i = 0$$

$$t = \frac{L}{R} = \frac{0.5}{1/3} = 1.5 \text{ S}$$

$$\therefore i(t) = 10e^{-\frac{2}{3}t} \text{ A} , \ t \geq 0$$

7.2 The Natural Response of an RL Circuit

Step 3 Replace the inductor with an equivalent current source with i(t) and solve the resistive circuit.



$$i(t) = 10\,e^{-\frac{2}{3}t} \quad \text{A}$$

7.2 The Natural Response of an RL Circuit

For this problem, since 2Ω is in parallel with the inductor, it is trivial to get

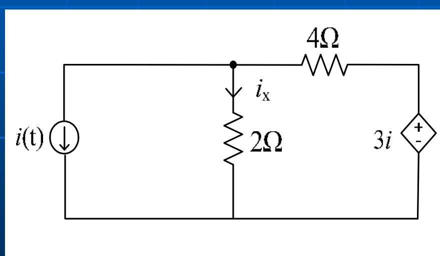
$$v_L(t) = L\frac{di}{dt}$$

$$= 0.5\,\frac{d}{dt}(10\,e^{-\frac{2}{3}t})$$

$$= 0.5 \times 10 \times (\frac{-2}{3})e^{-\frac{2}{3}t}$$

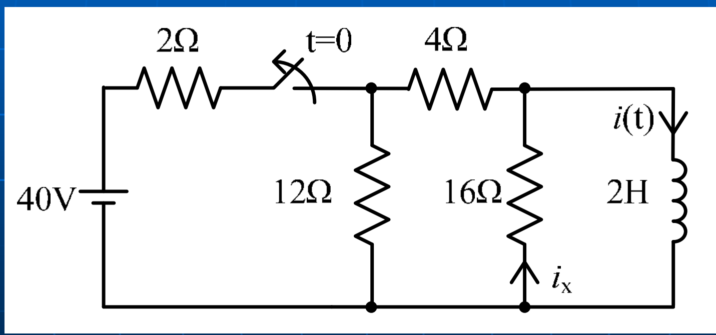
$$= \frac{-10}{3}e^{-\frac{2}{3}t} \quad \text{V}$$

$$\therefore\ i_x = \frac{v_L}{2\Omega} = \frac{-5}{3}\,e^{-\frac{2}{3}t} \quad \text{V}, \quad t \geq 0$$



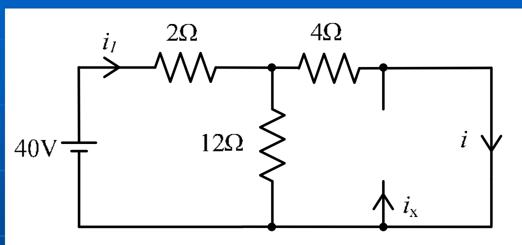
7.2 The Natural Response of an RL Circuit

Example 6 : The switch has been closed for a long time. At t=0, it is opened. Find i_x .



7.2 The Natural Response of an RL Circuit

t=0⁻



$$i_1(0^-) = \frac{40V}{2+(12//4)} = 8\text{ A}$$

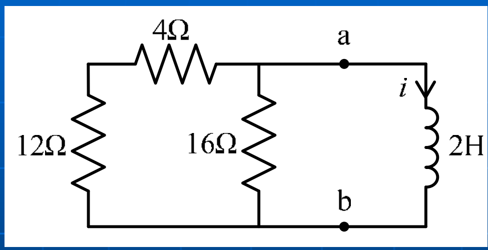
$$i_x(0^-) = 0$$

$$i(0^-) = i_1(0^-) \times \frac{12}{12+4} = 8\times\frac{3}{4} = 6\text{ A}$$

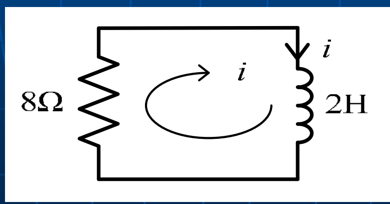
$\therefore$ Initial codition $i(0^+) = i(0^-) = 6$ A

7.2 The Natural Response of an RL Circuit

$t \geq 0^+$



Step 1 $R_{TH} = (4+12) \,//\, 16 = 8\ \Omega$



7.2 The Natural Response of an RL Circuit

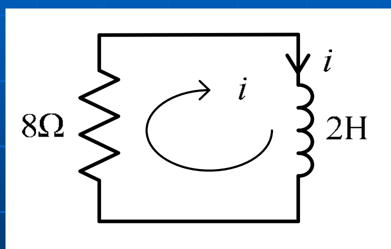
Step 2 mesh analysis

$$2\frac{di}{dt} + 8i = 0$$

$$i(t) = K\,e^{-4t}\ ,\quad t \geq 0$$

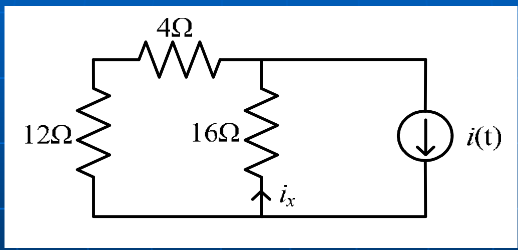
$$i(0^+) = 6\ A$$

$$\therefore\ i(t) = 6\,e^{-4t}\ A\ ,\quad t \geq 0$$



7.2 The Natural Response of an RL Circuit

Step 3 Replace L with an equivalent current source, and find i_x, solve the resistive circuit.



$$\therefore i_x(t) = \frac{4+12}{12+4+16} i(t)$$
$$= 3e^{-4t} \text{ A}, \quad t \geq 0$$

7.3 Singularity Functions

Switching functions are convenient for describing the switching actions in circuit analysis.

They serve as good approximations to the switching signals.

unit step function	$u(t)$
unit impulse function	$\delta(t)$
unit ramp function	$r(t)$

7.3 Singularity Functions

Definition

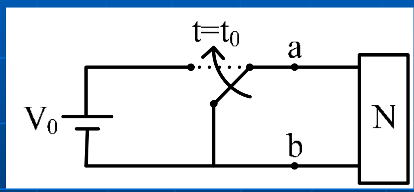
$$u(t) = \begin{cases} 0 \ , & t<0 \\ 1 \ , & t>0 \end{cases}$$

undefined at t=0

or more general

$$u(t-t_0) = \begin{cases} 0 \ , & t<t_0 \\ 1 \ , & t>t_0 \end{cases}$$

Example 7



$V_0\, u(t-t_0)$ is applied to a-b terminals

7.3 Singularity Functions

Definition

$$d(t) = \frac{d}{dt}u(t) = \begin{cases} 0 & , \ t<0 \\ \text{undefined} & , \ t=0 \\ 0 & , \ t>0 \end{cases}$$

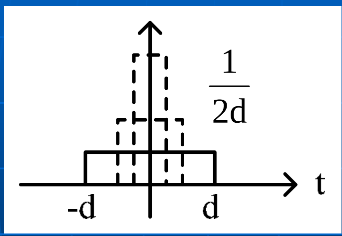
Also known as the delta function

or $\int_{0^-}^{0^+} d(t)\, dt = 1$

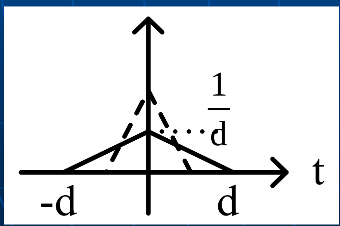
or $\int_{t_0^-}^{t_0^+} d(t-t_0)\, dt = 1$

7.3 Singularity Functions

The unit impulse function is not physically realizable but is a very useful mathematical tool.



An approximation as $d \rightarrow 0$



Another approximation as $d \rightarrow 0$

7.3 Singularity Functions

Shifting property, if $t_0 \in [a , b]$

$$\int_a^b f(t) d(t-t_0)dt = f(t_0)$$

$$Q \int_a^b f(t) d(t-t_0)dt = \int_a^b f(t_0) d(t-t_0)dt$$

$$= f(t_0)\int_a^b d(t-t_0)dt$$

$$= f(t_0)$$

7.3 Singularity Functions

Definition

$$r(t) = \int_{-\infty}^{t} u(t)dt = t\,u(t)$$

or

$$r(t) = \begin{cases} 0 & , \quad t \le 0 \\ t & , \quad t \ge 0 \end{cases}$$

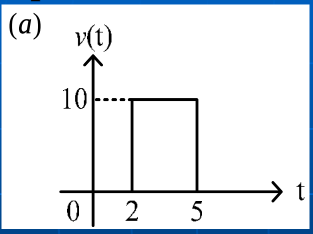
Note that

$$\delta(t) = \frac{d\,u(t)}{dt} \quad , \quad u(t) = \int^{t} \delta(t)dt$$

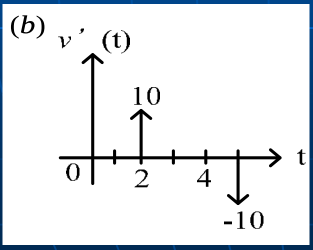
$$u(t) = \frac{d\,r(t)}{dt} \quad , \quad r(t) = \int^{t} u(t)dt$$

7.3 Singularity Functions

Example 8

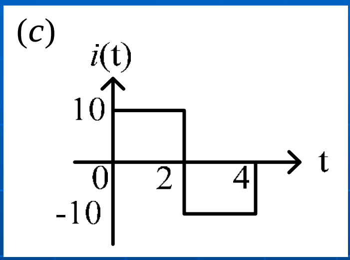


$$v(t) = 10\,[u(t-2) - u(t-5)]$$

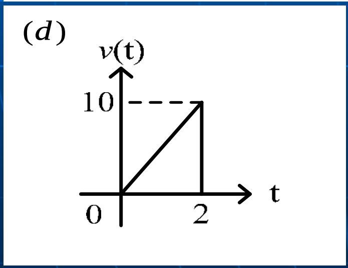


$$\frac{dv(t)}{dt} = 10\delta(t-2) - 10\delta(t-5)$$

7.3 Singularity Functions



$i(t) = 10\,u(t) - 20\,u(t-2) + 10u(t-4)$



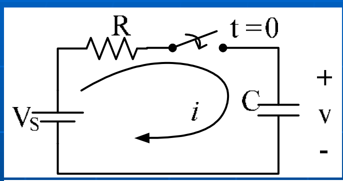
$v(t) = 5r(t) - 5r(t-2) - 10u(t-2)$

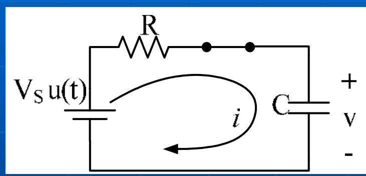
7.4 The Step Response of RC and RL Circuits

When a dc voltage (current) source is suddenly applied to a circuit , it can be modeled as a step function , and the resulting response is called step response .

7.4 The Step Response of RC and RL Circuits

Example 9





$$V(0^-) = V_0 \ , \ i = C\frac{dv}{dt} \ , \ choose\ v\ as\ unknown$$

Step 1 Mesh Analysis

$$RC\frac{dv}{dt} + v = V_S \ , \ t \geq 0$$

7.4 The Step Response of RC and RL Circuits

Step 2 Solving the differential equation

(a) homogeneous solution

$$RC\frac{dv_h}{dt} + v_h = 0$$

$$v_h(t) = Ke^{-\frac{t}{RC}} \ , \ t \geq 0$$

(b) particular solution

$$RC\frac{dv_p}{dt} + v_p = V_S$$

$$v_p = V_S$$

7.4 The Step Response of RC and RL Circuits

(c) complete solution + Initial condition

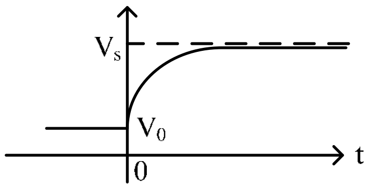
$$v(t) = v_h + v_p$$

$$= Ke^{-\frac{t}{RC}} + V_s$$

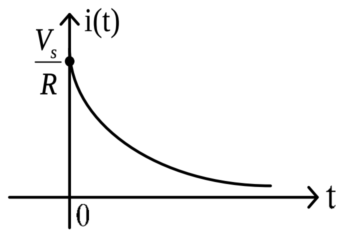
$$v(0^+) = v(0^-) = V_0$$

$$\therefore K = V_0 - V_s$$

$$v(t) = \begin{cases} V_0 \ , \ t < 0 \\ V_s + (V_0 - V_s)e^{-\frac{t}{t}} \ , \ t \geq 0 \end{cases}$$



7.4 The Step Response of RC and RL Circuits



$$i(t) = C\frac{dv}{dt} = C(V_o - V_s)(\frac{-1}{RC})e^{-\frac{t}{RC}}$$

$$= \frac{V_s - V_o}{R}e^{-\frac{t}{RC}} \quad , \quad t \geq 0$$

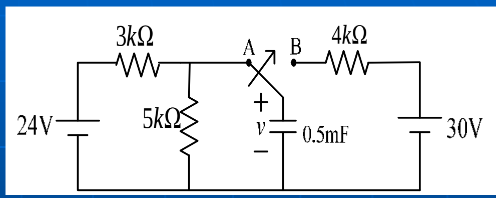
$$For \quad V_o = 0 \quad , \quad then \quad i(o^+) = \frac{V_S}{R}$$

C is initially short circuited.

7.4 The Step Response of RC and RL Circuits

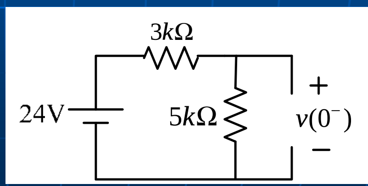
Example 10

Before t=0 , the circuit is under steady state . At t=0 , the switch is moved to B . Find v(t) , t > 0



For t<0 , the circuit shown:

$$v(0^-) = \frac{5}{3+5} \times 24\text{V} = 15\text{V}$$



7.4 The Step Response of RC and RL Circuits

For t>0

mesh analysis

$$4\times10^3 \times (0.5\times10^{-3})\frac{dv}{dt} + v = 30V$$

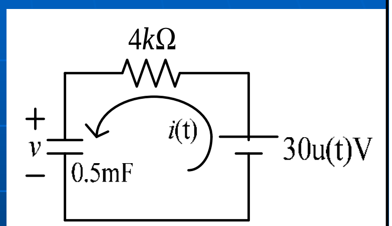
$$v(t) = v_p + v_h$$

$$= 30 + Ke^{-\frac{t}{t}} \quad , \ t > 0$$

$$t = RC = 4\times10^3 \times 0.5\times10^{-3} = 2 \ second$$

$$v(0^+) = v(0^-) = 15V$$

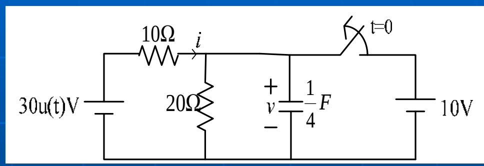
$$\therefore v(t) = 30 + (15-30)e^{-\frac{t}{t}} \ , \ t \ge 0$$



7.4 The Step Response of RC and RL Circuits

Example 11

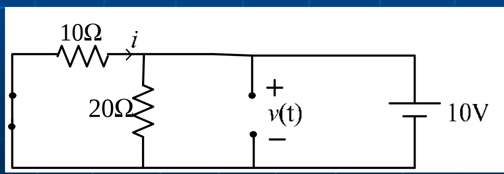
Before the switch is open at t=0, the circuit is in steady state.
Find i(t) & v(t) for all t.



For t<0 , then

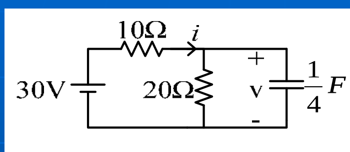
$$v(t)=10V\ ,\ t<0$$

$$i(t)=-\frac{10}{10}A=-1A\ ,\ t<0$$

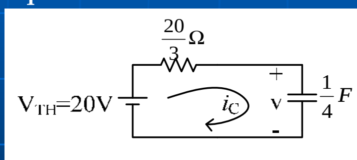


7.4 The Step Response of RC and RL Circuits

For t>0



Thevenin equivalent circuit



$$v(0^+)=v(0^-)=10V$$

$$i_c(t)=C\frac{dv}{dt}=\frac{1}{4}\frac{dv}{dt}$$

$$\frac{20}{3}\times\frac{1}{4}\frac{dv}{dt}+v=20V,t>0$$

$$v(t)=20+Ke^{-\frac{3}{5}t}=20+(10-20)e^{-\frac{3}{5}t},t>0$$

7.4 The Step Response of RC and RL Circuits

From the original circuit

$$KCL: i(t) = \frac{v}{20\Omega} + C\frac{dv}{dt}$$

$$= 1 + e^{-0.6t}, t > 0$$

$$v(t) = \begin{cases} 10V, t < 0 \\ \left(20 - 10e^{-0.6t}\right)V, t \geq 0 \end{cases}$$

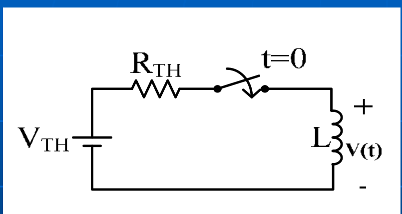
$$i(t) = \begin{cases} -1A, t < 0 \\ \left(1 + e^{-0.6t}\right)A, t \geq 0 \end{cases}$$

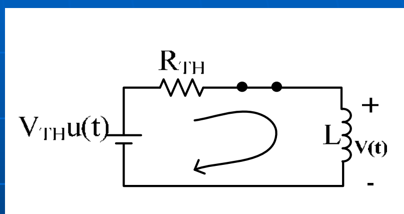
Note that

$$i(0^-) = -1A \neq i(0^+) = 2A$$

7.4 The Step Response of RC and RL Circuits

Example 12



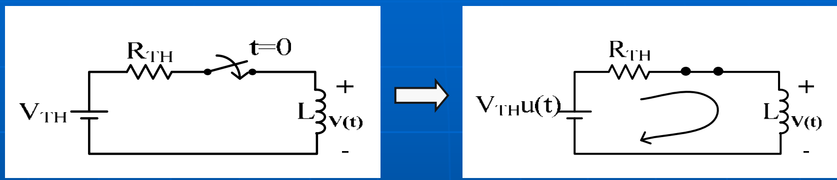


Mesh analysis

$$L\frac{di}{dt} + R_{TH}\ i = V_{TH}\ ,\ t \geq 0$$

$$i(0) = 0$$

7.4 The Step Response of RC and RL Circuits



Solution:

$i(t) = i_h(t) + i_p(t)$

characteristic equation

$LS + R_{TH} = 0$

$S = -\frac{R_{TH}}{L} = -\frac{1}{t}$

$i_h(t) = Ke^{-\frac{t}{t}}$

$i_p(t) = \frac{V_{TH}}{R_{TH}}$

$\therefore i(t) = \frac{V_{TH}}{R_{TH}} + Ke^{-\frac{t}{t}}$

$i(0^+) = i(0^-) = 0$

Note: L is equivalent to open circuit

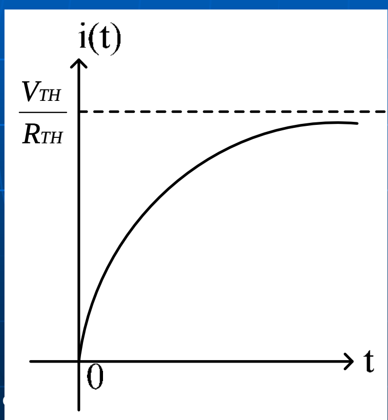
$i(t) = \frac{V_{TH}}{R_{TH}}(1 - e^{-\frac{t}{t}})\ ,\ t \geq 0$

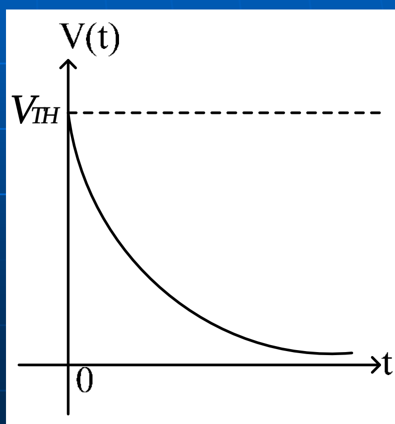
$V(t) = L\frac{di}{dt} = V_{TH}\ e^{-\frac{t}{t}}\ u(t)$

7.4 The Step Response of RC and RL Circuits

$i(t) = \frac{V_{TH}}{R_{TH}}(1 - e^{-\frac{t}{t}})\ ,\ t \geq 0$

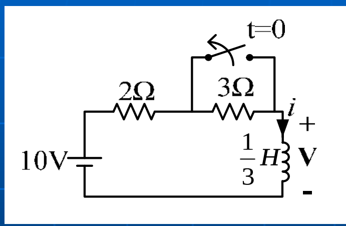
$V(t) = L\frac{di}{dt} = V_{TH}\ e^{-\frac{t}{t}}\ u(t)$





7.4 The Step Response of RC and RL Circuits

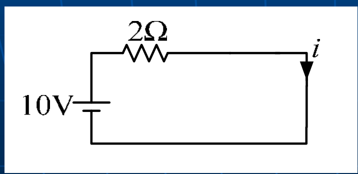
Example 13



$t < 0$, steady state

$t = 0$, switch is opened

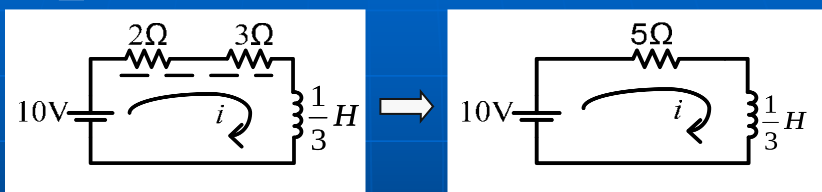
For t<0



$$i\,(0^{-}) = \frac{10}{2} = 5A$$

7.4 The Step Response of RC and RL Circuits

For t ≧ 0



Mesh Analysis

$$\frac{1}{3}\frac{di}{dt} + 5i = 10V \;,\; t \geq 0$$

$$s = -15$$

$$i_h(t) = Ke^{-\frac{t}{t}} \;,\; t = \frac{1}{15}$$

$$i_p(t) = 2A$$

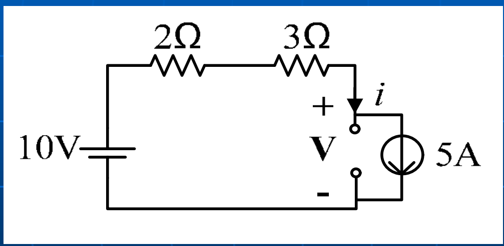
$$i(t) = i_h + i_p$$

$$= Ke^{-\frac{t}{t}} + 2A \;,\; t \geq 0$$

7.4 The Step Response of RC and RL Circuits

initial condition

$$i(0^+) = i(0^-) = 5A$$



$$\therefore i(t) = 2 + 3e^{-\frac{t}{\tau}} \quad , t \geq 0$$

7.4 The Step Response of RC and RL Circuits

Note that if V is chosen for solution:

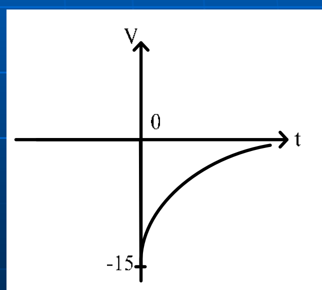
$$V\left(0^-\right) = 0V$$

$$KVL: V(0^+) = 10V - (2+3)5 = -15V$$

$$\text{or} \quad V = L\frac{di}{dt} = \frac{1}{3}\frac{d}{dt}(2 + 3e^{-\frac{t}{\tau}})$$

$$= -\frac{1}{\tau}e^{-\frac{t}{\tau}} \quad , t \geq 0^+$$

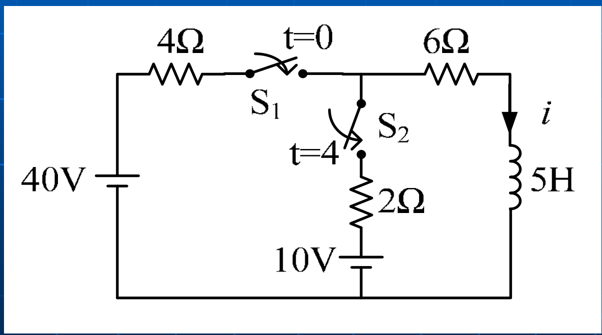
$$V(0^+) = -15V \text{ , same answer.}$$



7.4 The Step Response of RC and RL Circuits

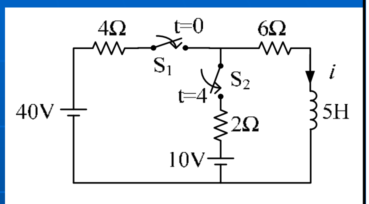
Example 14 : S_1 is closed at t=0 Find $i(2)$= ?

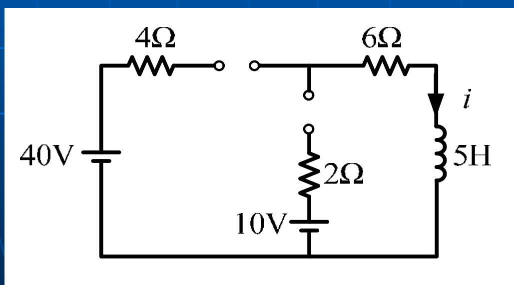
S_2 is closed at t=4 $i(5)$=?



7.4 The Step Response of RC and RL Circuits

For t < 0

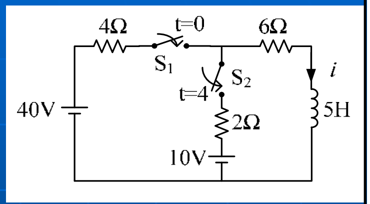


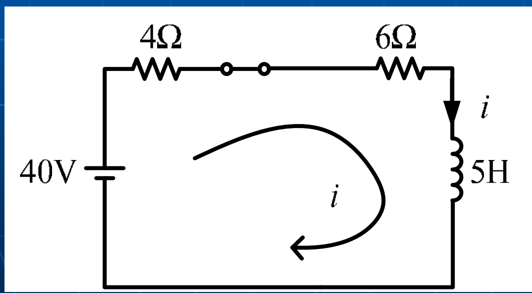


t < 0 , i=0 , open circuit

7.4 The Step Response of RC and RL Circuits

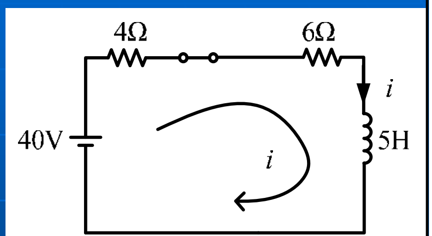
For $0^+ \leq t < 4S$





7.4 The Step Response of RC and RL Circuits

For $0^+ \leq t < 4S$



$$5\frac{di}{dt} + 10i = 40V \ , \qquad 0^+ \leq t \leq 4S$$

$$i(0^+) = i(0^-) = 0A$$

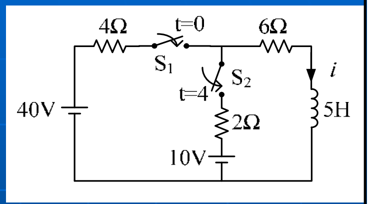
$$t = \frac{L}{R} = \frac{5}{10} = \frac{1}{2}$$

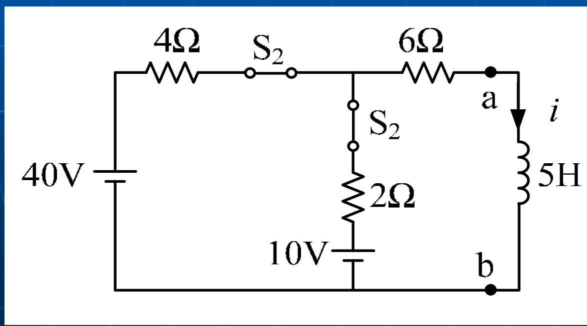
$$\therefore i(t) = 4(1 - e^{-2t})A \ , \ 0^+ \leq t \leq 4S$$

$$i(4^-) = 4(1 - e^{-8t}) \approx 4A$$

7.4 The Step Response of RC and RL Circuits

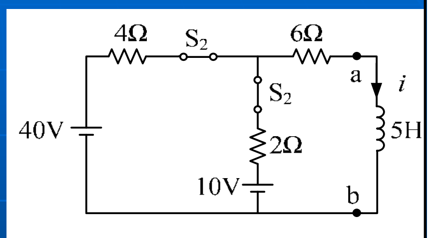
For $t \geqq 4^+$





7.4 The Step Response of RC and RL Circuits

For $t \geqq 4^+$



$$i(4^-) = 4A$$

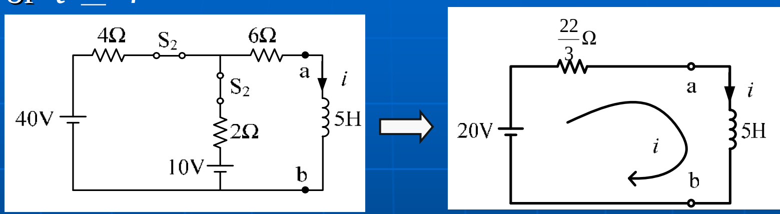
$$V_{TH} = \frac{40-10}{4+2} \times 2 + 10 = 20V$$

$$R_{TH} = 4\Omega \parallel 2\Omega + 6\Omega = \frac{4}{3} + 6 = \frac{22}{3}\Omega$$

$$t = \frac{L}{R_{TH}} = \frac{5}{22/3} = \frac{15}{22} S$$

7.4 The Step Response of RC and RL Circuits

For $t \geqq 4^+$



$$i(4^+) = i(4^-) = 4A$$

$$5\frac{di}{dT} + \frac{22}{3} i = 20V$$

where $T = t - 4$

$$i_h(t) = Ke^{-\frac{22}{15}(t-4)} \ , t \geq 4^+$$

$$i_p(t) = \frac{20}{22/3} = \frac{60}{22} = \frac{30}{11} = 2.727$$

$$i(t) = \frac{30}{11} + Ke^{-\frac{22}{15}(t-4)} \ , t \geq 4^+$$

$$i(4^+) = 4A = \frac{30}{11} + K$$

$$K = 4 - \frac{30}{11} = \frac{14}{11}$$

$$\therefore i(t) = \frac{30}{11} + \frac{14}{11} e^{-\frac{22}{15}(t-4)} \ , t \geq 4^+$$

7.4 The Step Response of RC and RL Circuits

In Summary

$$i(t) = \begin{cases} 0 \ , t \leq 0 \\ 4(1 - e^{-2t}) \ , 0 \leq t \leq 4 \\ \frac{30}{11} + \frac{14}{11} e^{-22(t-4)} \ , t \geq 4 \end{cases}$$

Summary

- Objective 1 : Be able to determine the natural response of both RC and RL circuits.
- Objective 2 : Be able to find the step response of both RC and RL circuits.
- Objective 3 : Know and be able to use the singularity functions.
- Objective 4 : Be able to analyze circuits with sequential switching.

Summary

Chapter Problems : 7.7
7.14
7.23
7.32
7.47
7.63

Due within one week.