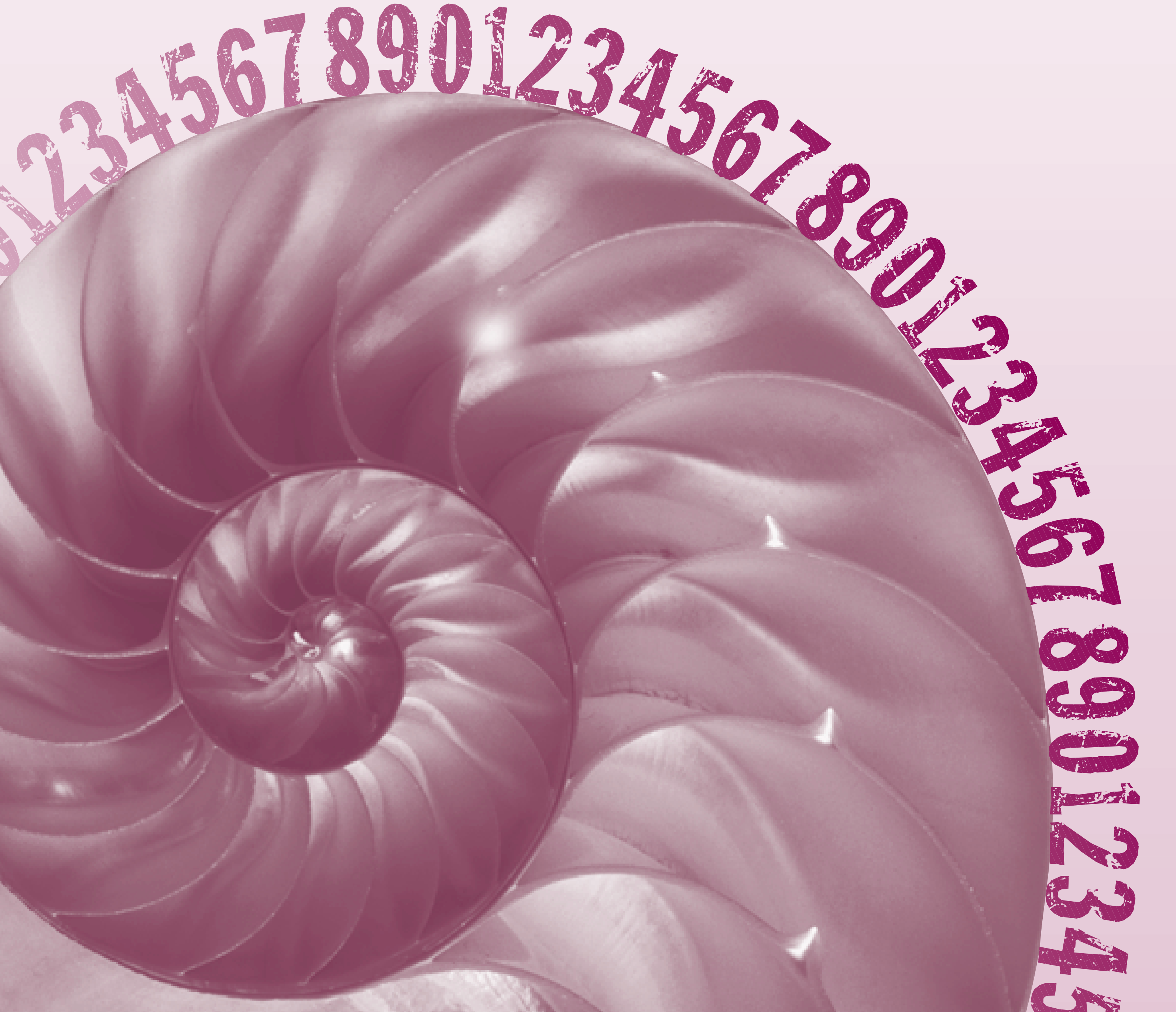
MINISTRY OF EDUCATION
Te Tāhuhu o te Mātauranga

Book 7
Teaching Fractions, Decimals, and Percentages

Revised Edition 2008: Draft



Numeracy Professional Development Projects

EFFECTIVE MATHEMATICS TEACHING

The Numeracy Professional Development Projects (NDP) assist teachers to develop the characteristics of effective teachers of numeracy.

Effective teachers of numeracy demonstrate some distinctive characteristics.[1] They:

- have high expectations of students' success in numeracy;
- emphasise the connections between different mathematical ideas;
- promote the selection and use of strategies that are efficient and effective and emphasise the development of mental skills;
- challenge students to think by explaining, listening, and problem solving;
- encourage purposeful discussion, in whole classes, in small groups, and with individual students;
- use systematic assessment and recording methods to monitor student progress and to record their strategies for calculation to inform planning and teaching.

The focus of the NDP is number. A key component of the projects is the Number Framework. This Framework provides teachers with:

- the means to effectively assess students' current levels of thinking in number;
- guidance for instruction;
- the opportunity to broaden their knowledge of how students acquire number concepts and to increase their understanding of how they can help students to progress.[2]

The components of the professional development projects allow us to gather and analyse information about students' learning in mathematics more rigorously and respond to their learning needs more effectively. While, in the early stages, our efforts may focus on becoming familiar with the individual components of the projects, such as the progressions of the Framework or carrying out the diagnostic interview, we should not lose sight of the fact that they are merely tools in improving our professional practice. Ultimately, the success of the projects lies in the extent to which we are able to synthesise and integrate their various components into the art of effective mathematics teaching as we respond to the individual learning needs of the students in our classrooms.

1 Askew et al (2000) *Effective Teachers of Numeracy.* London: King's College.

2 See also the research evidence associated with formative assessment in mathematics: Wiliam, Dylan (1999) "Formative Assessment in Mathematics" in *Equals*, 5(2); 5(3); 6(1).

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This book is also available on the New Zealand Maths website, at www.nzmaths.co.nz/Numeracy/2008numPDFs/pdfs.aspx

Teaching Fractions, Decimals, and Percentages

Teaching for Number Strategies

The activities in this book are specifically designed to develop students' mental strategies. They are targeted to meet the learning needs of students at particular strategy stages. All the activities provide examples of how to use the teaching model from *Book 3: Getting Started*. The model develops students' strategies between and through the phases of *Using Materials*, *Using Imaging*, and *Using Number Properties*.

Each activity is based on a specific learning outcome. The outcome is described in the "I am learning to ..." statement in the box at the beginning of the activity. These learning outcomes link to the planning forms online at www.nzmaths.co.nz/numeracy/Planlinks/

The following key is used in each of the teaching numeracy books. Shading indicates which stage or stages the given activity is most appropriate for. Note that CA, Counting All, refers to all three *Counting from One* stages.

Strategy Stage

E	→	Emergent
CA	→	Counting All (One-to-one Counting, Counting from One on Materials or by Imaging)
AC	→	Advanced Counting
EA	→	Early Additive Part-Whole
AA	→	Advanced Additive–Early Multiplicative Part-Whole
AM	→	Advanced Multiplicative–Early Proportional Part-Whole
AP	→	Advanced Proportional Part-Whole

The table of contents below details the main sections in this book. These sections reflect the strategy stages as described on pages 15–17 of *Book One: The Number Framework*. The Knowledge and Key Ideas sections provide important background information for teachers in regard to the development of students' thinking in fractions, decimals, and percentages.

Key Ideas in Fractions,

The need for fractions comes from measurement and sharing situations where one units are not accurate enough to do the job, e.g., 7 cakes shared among 4 people.

The numerator (top number) in a fraction tells how many times the unit fraction is counted.
The denominator (bottom number) tells the number of splits of the one units that were used to create the unit fraction, e.g., $\frac{3}{7}$ means three iterations of one-seventh.

Fractions are most commonly used in real life as operators, e.g., Find $\frac{1}{4}$ (one-quarter) of a number or quantity.
A fraction as a number is the special case where it operates on one, e.g., $\frac{1}{4}$ is the unit created when one is split into four equal parts.

Fractions are found in many mathematical models that can be described as constructs. These constructs include part–whole comparison, measurement, operator, sharing, rates and ratios, and probability. These constructs form a connected body of ideas that students must understand to become generalised proportional thinkers.

Finding a multiplicative relationship between two numbers often involves finding an unknown fraction as an operator.
The multiplier that maps a number, a, onto another number, b, is $\frac{b}{a}$, e.g., $4 \times \frac{5}{4} = 5$.

Rates describe a relationship between two different measurements. For example, 90 kilometres per hour describes a relationship between distance and time. In science, measurements such as km/h are described as derived units because they are derived by combining two measures.
Rate problems frequently involve quantity changes and measurement changes, e.g., 90 km/h equates to 270 kilometres in 3 hours (quantity change) and 25 metres per second (25 m/s) (measurement change).

In sharing and measurement situations, the result of the division can be anticipated. In general, $a \div b = \frac{a}{b}$, e.g., If 7 pizzas are shared among 3 girls, each girl gets $\frac{7}{3}$ or $2\frac{1}{3}$ lots of one pizza. Similarly, the reciprocal $\frac{b}{a}$ gives the share when b is shared among a; $b \div a = \frac{b}{a}$, e.g., $3 \div 7 = \frac{3}{7}$.

Probability is a difficult construct because it involves variability. The theoretical probability of some events can be found by counting all the possible outcomes. For example, tossing a coin has two outcomes, a head or a tail. The chance of a head on one toss is one-half, which can be described using $\frac{1}{2}$, 0.5, or 50%.
The results of real coin tossing will vary from this expectation e.g., In 10 tosses, heads came up 3 times.
In situations where the probability cannot be found theoretically, it must be estimated by taking large samples and allowing for variability.

Decimals, and Percentages

Any fraction or rational number has an infinite number of names. These are called equivalent fractions and name the same number, e.g., $\frac{2}{3} = \frac{66}{99}$.
Equivalent fractions occupy the same position on the number line.

Fractions are an extension of the number system that includes whole numbers, e.g., $\frac{10}{5}$, $\frac{24}{12}$, $\frac{100}{50}$ are the same quantity as 2.
Rational numbers are an extension of the number system that includes fractions and integers, e.g., $\frac{-10}{5}$, $\frac{-24}{12}$, $\frac{100}{-50}$ are the same number as -2.

Decimals and percentages are symbols that name a restricted set of equivalent fractions. Decimal fractions have 10, 100, 1000, and other powers of ten as their denominator, e.g., $\frac{456}{100} = 4.56$.
Percentages only have 100 as their denominator, e.g., $\frac{3}{4} = \frac{75}{100} = 75\%$.

It is always possible to find a fraction between two fractions by subdividing one into smaller splits, e.g., $\frac{17}{24}$ lies between $\frac{2}{3}$ and $\frac{3}{4}$ because $\frac{2}{3} = \frac{8}{12} = \frac{16}{24}$ and $\frac{3}{4} = \frac{9}{12} = \frac{18}{24}$.

Linear proportional relationships that come from ratios and rates can be shown on a number plane. The ordered pairs created lie on a straight line. This is because a constant rate exists between the x and y values for each pair.
For example, T earns \$12.00 in 2 hours, so he earns \$24.00 in 4 hours and \$0.00 if he does no work. The ordered pairs (0,0), (2,6), (4,24) lie on a straight line. The constant unit rate is \$6/h.

Understanding of unit size and relationships is critical with fractions. In addition and subtraction of fractions, only like units can be combined or compared.
If two fractions $\frac{a}{b}$ and $\frac{c}{d}$ are to be added or compared, a difficulty occurs if b and d are different numbers because $\frac{a}{b}$ means a counts of $\frac{1}{b}$ and $\frac{c}{d}$ means c counts of $\frac{1}{d}$. In general, one or both of the fractions need/s to be changed into equivalent form with a common denominator. The common denominator may be the lowest common multiple of b and d, e.g., $\frac{3}{8} + \frac{5}{6} = \frac{9}{24} + \frac{20}{24} = \frac{29}{24}$.

Decimal fractions were created to avoid the difficulties of carrying out the four operations with fractions. Decimals only have denominators of 10, 100, 1000, etc., so the place value system can be used. For example, 4.7 + 3.8 = □ is 4 ones + 3 ones and 7 tenths + 8 tenths.
The same “ten for one” canon that applies to whole numbers applies to decimals, e.g., 7 tenths + 8 tenths = 15 tenths = one and 5 tenths or 1.5.

Ratios describe a part-to-part or (sometimes) a part-to-whole comparison. *a:b* usually describes *a* equal parts of one “thing” combined with *b* parts of a different “thing”. The “things” are always of the same attribute, such as mass or capacity, e.g., 2:3 might mean 2 litres of blue paint mixed with 3 litres of yellow paint.
Fractional relationships exist in comparing parts with the whole, e.g., The mixture is $\frac{2}{5}$ blue or $\frac{3}{5}$ yellow. Fractional relationships also exist in comparing parts with parts, e.g., There is $\frac{2}{3}$ times as much blue as yellow and $\frac{3}{2}$ or $1\frac{1}{2}$ times as much yellow as blue.

Multiplication and division of fractions is an extension of what happens with whole numbers. In general, $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$, e.g., two-thirds of three-quarters is six-twelfths ($\frac{2}{3} \times \frac{3}{4} = \frac{6}{12}$).
In general, $\frac{a}{b} \div \frac{c}{d} = \frac{a \times d}{b \times c}$, e.g., two-thirds measured with three-quarters gives eight-ninths of a measure ($\frac{2}{3} \div \frac{3}{4} = \frac{8}{9}$).
These results underpin sensible calculation with decimal fractions, e.g., $0.8 \times 0.2 = 0.16$ because $\frac{8}{10} \times \frac{2}{10} = \frac{16}{100}$ and $0.8 \div 0.2 = 4$ because $\frac{8}{100} \times \frac{100}{2} = \frac{800}{200} = \frac{8}{2} = 4$.

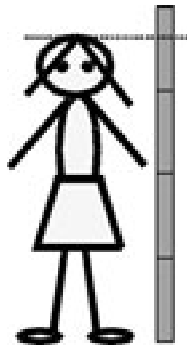
Percentages provide a uniform operator or fractional comparison of different amounts. In situations such as discounts and interest rates, percentages act as uniform operators, e.g., 30% off whatever price. In situations such as goal shooting and lambing, percentages act as uniform comparisons through changing fractions to equivalent fractions with denominators of 100.
Shooting percentages are a part-whole comparison, so they cannot be more than 100 percent, e.g., 26 out of 40 is 65% because $\frac{26}{40} = \frac{65}{100}$. Lambing percentages are a within ratio comparison, so they can exceed 100%, e.g., 90 lambs from 60 ewes is 150% because $\frac{90}{60} = \frac{150}{100}$.

Rational Numbers and Proportional Reasoning: An Introduction

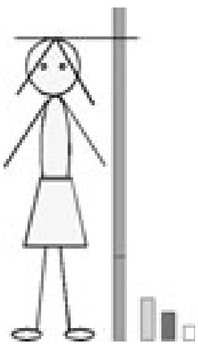
In *Book 6: Teaching Multiplication and Division*, the distinction was made between division as measuring and division as sharing. The need for fractional numbers (fractions) arises from these two situations. Fractions are needed when ones, i.e., whole units, are inadequate to get a job done. Consider these situations:

1. Division as measuring

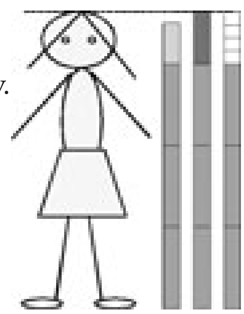
How tall is Heidi in dark green rods?



If we want to be more accurate than saying "between three and four rods", we will need to create a smaller measure that is a fracture of a dark green (Cuisenaire®) rod.



So we might create new measures that are a two-split (light green), three-split (red), and six-split (white) of our original unit. In English, the words for these measures are confusing and disguise the fracturing. We say "half, third, and sixth", but it would be more useful to say "twoth, threeth, and sixth". The suffix "-th" indicates splitting.



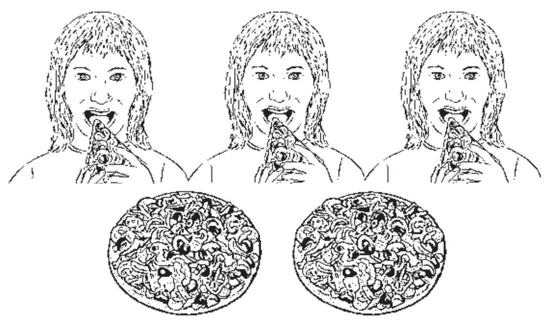
Using the new two-, three- and six-split measures, we can now measure Heidi's height more accurately.

So Heidi is three whole rods and two three-splits tall, which is the same as three whole rods and four six-splits tall. We write this as $3\frac{2}{3} = 3\frac{4}{6}$.

2. Division as sharing

Three girls share two pizzas equally. How much pizza does each girl get?

In this situation, you can't create equal shares by dealing out whole pizzas.



Students' early strategies are usually to give out halves and then share out what remains. They may describe the share for each girl as one-half and one-third of a half ($\frac{1}{2} + \frac{1}{6}$).

The generalisation we want eventually is that a pizzas shared among b people results in each person getting $a \div b$ ths of a pizza, e.g., $2 \div 3 = \frac{2}{3}$ ($a \div b = \frac{a}{b}$).

From the above examples, we see that any fraction can be seen as iteration (repetition) of a unit. For example, viewing $\frac{4}{5}$ as "four out of five" may be applicable in some situations, but viewing it as four iterations of one-fifth of one (a whole) is more transferable. So $\frac{4}{5}$ is shorthand for $\frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5}$ or $4 \times \frac{1}{5}$. The numerator (top number) counts the number of units, and the denominator (bottom number) describes the size of the units as splits of one. This view accommodates improper fractions such as $\frac{13}{5}$ (thirteen-fifths) as $2\frac{3}{5}$ and relates multiplying by a fraction as sharing then iterating, e.g., $\frac{4}{5}$ of 20 as four iterations of one-fifth of 20 or as $\frac{1}{5}$ of $20 = 4$ so $\frac{4}{5}$ of $20 = 4 \times 4 = 16$.

Clarification of number vocabulary

A rational number is any number that can be expressed in the form $\frac{a}{b}$, where *a* and *b* are integers (integers are positive or negative whole numbers). So $\frac{2}{3}$, $\frac{11}{4}$, and $\frac{-2}{3}$ are rational numbers. Fractions are usually classified as rational numbers in which both *a* and *b* are positive integers. So $\frac{3}{4}$ is a fraction, whereas $\frac{-3}{4}$ is not. A fraction describes a quantity and has a place on the number line. It also has an infinite number of equivalent forms, e.g., $\frac{3}{5} = \frac{6}{10} = \frac{9}{15}$, etc. Note that integers are rational numbers, e.g., $7 = \frac{7}{1} = \frac{14}{2} = \frac{21}{3}$, etc. For students, fractions represent a significant shift in their thinking, from working with ones and groupings of one to working with parts of one whole.

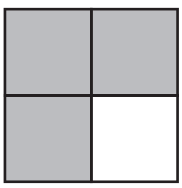
Proportional reasoning

Proportional reasoning is about solving problems with rational numbers. These problems may involve determining equality, e.g., $\frac{4}{7} = \frac{n}{28}$ and 4:11 = *n*:44, or comparing size, e.g., $\frac{4}{9}$ <, =, > $\frac{1}{3}$. The fractions can be expressed using different numbers or symbols, such as decimals and percentages, or represented graphically as ordered pairs on a number plane. Proportional reasoning is the pinnacle of number understanding and is critical to the understanding of other mathematical ideas such as scaling, predicting probabilities, metric measures, and algebra.

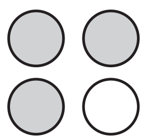
Understanding fractions

An understanding of fractions requires students to connect several constructs:

1. **Part–Whole** comparisons involve comparing a part of an object (continuous) or set of objects (discrete) to a whole. Here are two examples:

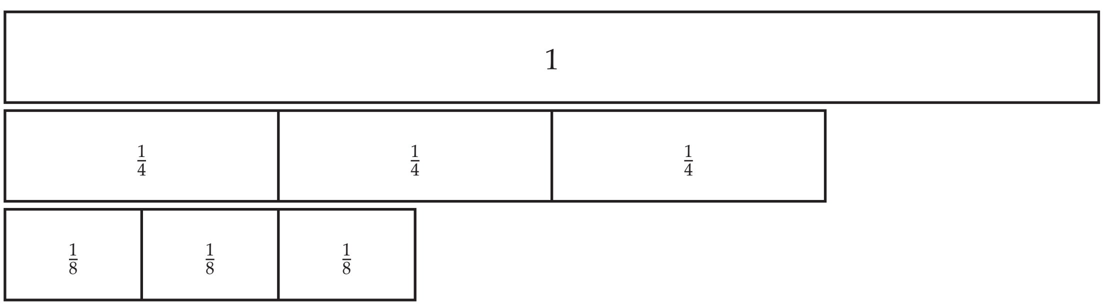


What fraction of the square is shaded? (Continuous)



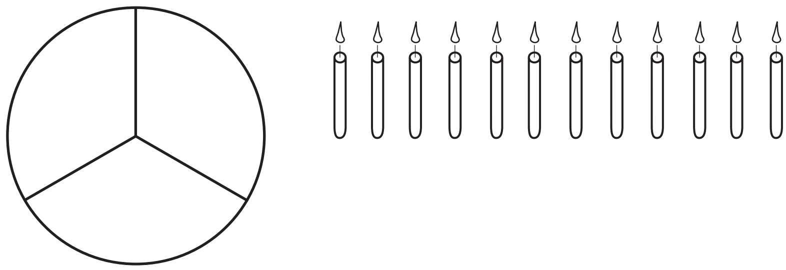
What fraction of the set is shaded? (Discrete)

2. **Measures** involve the idea of finding out how many times a fraction or ratio fits into a given fraction or ratio. A simple example of this is $\frac{3}{4} \div \frac{3}{8}$.



In this problem, three-quarters measures two lots of three-eighths.

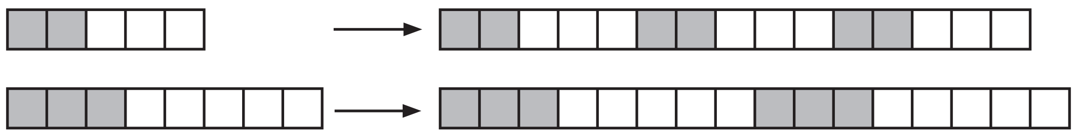
3. **Operators** refer to situations in which a fraction is used to act on another number. In the example below, one third is operating on 12 ($\frac{1}{3}$ of 12 or $\frac{1}{3} \times 12$).



Shelley has twelve candles on her birthday cake. There is the same number of candles on each third. How many candles on one-third?

4. **Quotients** are the answers to sharing division problems such as the pizza-sharing problem on page 4.

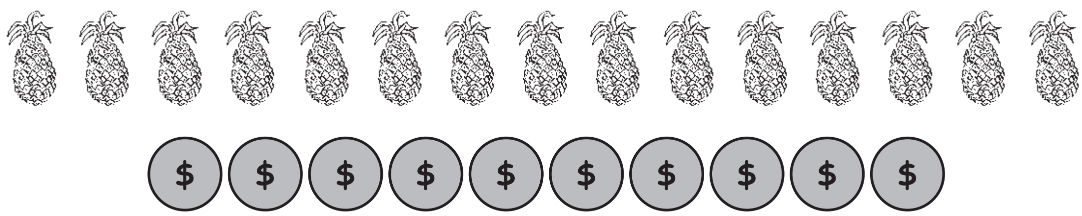
5. **Rates and ratios** are about part-to-part comparisons. In a ratio problem, the measures are in the same units. For example:



Which is the darker shade of blue to white mixture, 2:3 or 3:5?

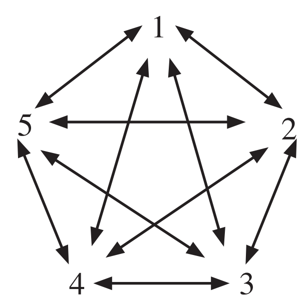
In this problem, the units are the same, measures of paint, so it is a ratio problem.

In the next problem, the units are different, so it involves a rate. The mathematical relationships within both ratios and rates are multiplicative. Some researchers say ratios are a special case of rates.



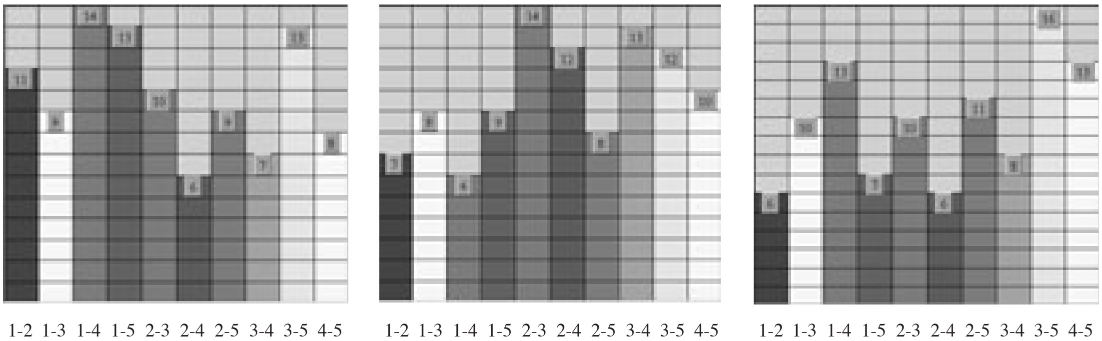
Fifteen pineapples cost ten dollars. At the same rate, how much would you pay for six pineapples?

6. **Probability** concerns the chance of particular outcomes occurring compared to all the possible outcomes that might occur in a given situation. For example, in Little Lotto, you must choose two numbers from a list containing 1, 2, 3, 4, and 5. Suppose you choose 2 and 4. This diagram shows all the outcomes when two numbers are drawn from the list.



There are ten possible number pair outcomes for this situation shown by each double arrow. So the probability of 2-4 occurring is one in ten, which can be expressed as $\frac{1}{10}$, 10%, or 0.1.

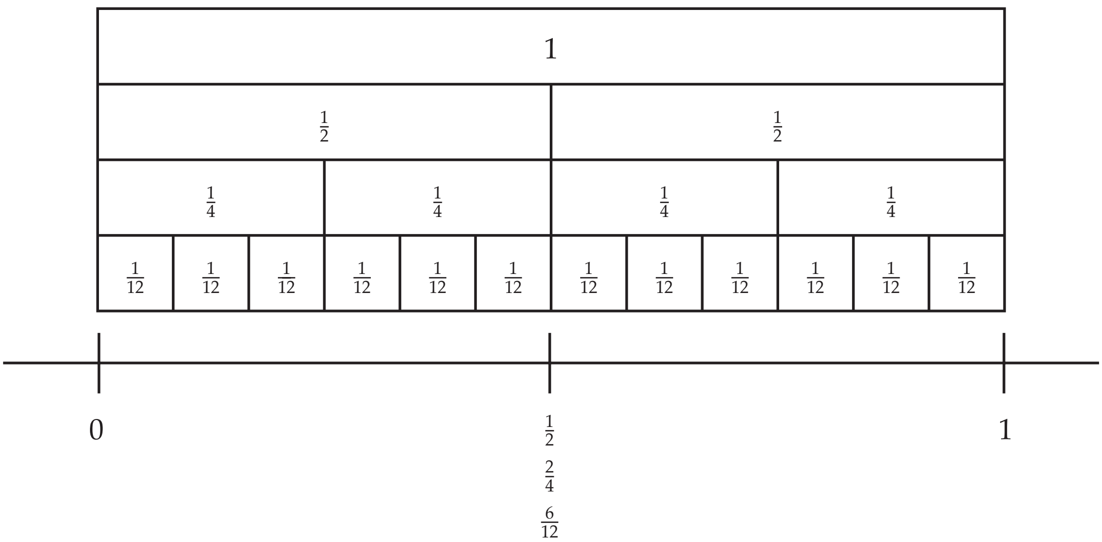
Probability can be seen as a special case of the part-whole construct because the outcome 2-4 is a one-tenth part of all the possible outcomes (the whole). However, variation occurs from theoretical expectations when situations are trialled. This makes probability a difficult context for proportional thinking. Suppose the results from 100 games of Little Lotto are recorded on a bar graph. If this was done on three occasions, the results might look like this:



Variation occurs between samples of the situation because all the distributions (shapes of the graphs) are different. Variation also occurs within samples because, although each outcome has a one-tenth theoretical chance of occurring, rarely in trialling does each outcome occur ten percent of the time.

Equivalence and comparison

The nature of equivalence and comparison of fractions varies in subtle ways within and between the constructs. In the part-whole and measures constructs, one (whole) remains constant. This makes it the easiest context for equivalence, but it is limited in application. A one unit can be split into smaller equal units, so many names for the same quantity emerge.



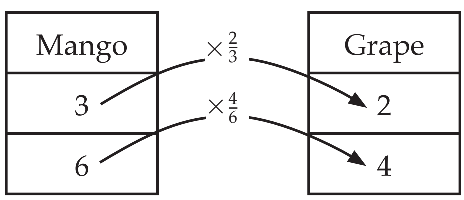
Transfer to the operator and measure constructs involves recognising that equivalent fractions operating on the same amount give the same result (e.g., $\frac{1}{2}$ of 18 = $\frac{3}{6}$ of 18) and the same quantity measured with equivalent fractions gives the same result (e.g., $6 \div \frac{3}{4} = 6 \div \frac{9}{12}$). In the quotient construct, certain situations result in the same equal shares. For example, two pizzas shared among three girls gives the same result as four pizzas shared among six girls (i.e., $2 \div 3 = 4 \div 6$ because $\frac{2}{3} = \frac{4}{6}$).

The rates/ratios construct presents the most conceptually difficult setting for equivalence because the referent whole (one) changes in ratios and the units change in rates. Consider these two ratios of grape and mango juice mix:

When we are comparing parts to the whole, we can see that the top mixture is the ratio 6:4, which is six-tenths mango and four-tenths grape. The bottom mixture is in the ratio 3:2, which is three-fifths mango and two-fifths grape. Recognising that these mixtures have the same taste because they involve equivalent part-whole relationships (i.e. , $\frac{6}{10} = \frac{3}{5}$) is foundational to understanding percentages. Both mixtures contain 60% mango because $\frac{6}{10} = \frac{3}{5} = \frac{60}{100}$. So percentages are just equivalent fractions with one hundred as the denominator and are most frequently used to compare fractions that have different denominators or as common operators on different amounts (e.g., 30% off sale).

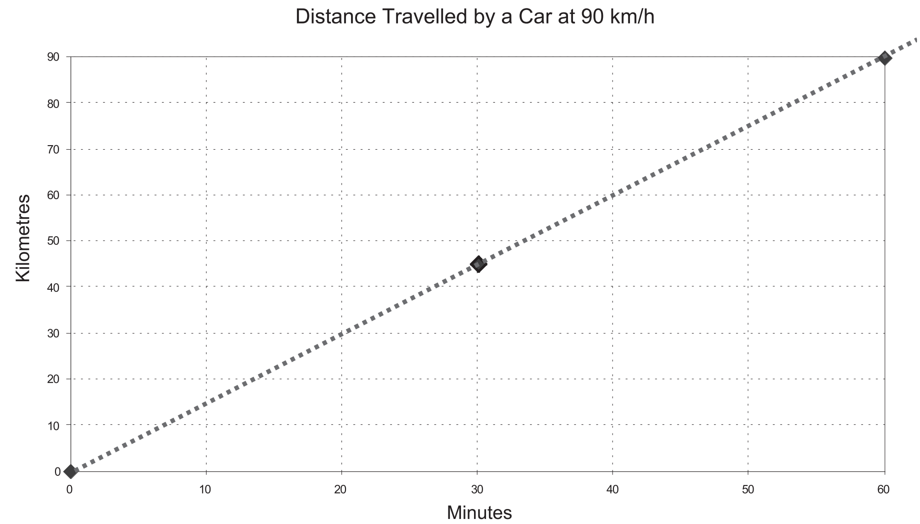
Part-to-part relationships in ratios and unit-to-unit relationships in rates involve fractions as operators. Consider the 3:2 mixture again. It has two-thirds as much grape as mango and three-halves as much mango as grape. Some researchers call this the rates within a ratio, e.g., 1 part mango per $\frac{2}{3}$ part grape and $\frac{3}{2}$ parts mango per 1 part grape. Note that $\frac{2}{3}$ and $\frac{3}{2}$ are reciprocals (inversions of one another) and the units are derived from the relationships between the parts, e.g., parts mangos per parts grape.

For equivalent ratios, these part-to-part rates are conserved (i.e., do not change), even though they appear in equivalent forms.



Rates involve another complexity. Consider the rate "ninety kilometres per hour". The rate can be written as $\frac{90\ kilometres}{hour}$ (90 km/h). Solving problems such as "At that rate, how many kilometres will the car travel in 7 hours?" involve replicating the rate, seven times in this case. An assumption of constant rate, like the assumption of uniform mixing in the juice ratios, must be made. These situations become conceptually harder when the rate units change, e.g., "At that speed, how far does the car travel in ten minutes?" ($\frac{90\ kilometres}{hour} = \frac{1.5\ kilometres}{minute} = \frac{15\ kilometres}{10\ minutes}$).

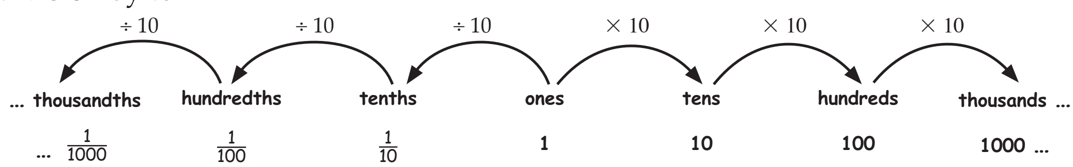
Rates can be shown graphically on a number plane. Consider the distance traveled by a car at a constant speed of ninety kilometres per hour. The axes of the graph represent the two measures kilometres (distance) and hours (time). For simplicity, we will convert hours to minutes on the graph.



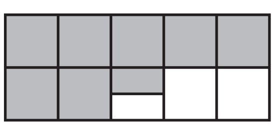
Constant rates result in ordered pairs that are co-linear when represented as points on the number plane. The slope of the line connecting these points is the unit rate 1 minute per $1\frac{1}{2}$ kilometres. For all points on the line, this relationship within the number pairs is conserved, e.g., in 20 minutes, the car travels 30 kilometres ($1\frac{1}{2} \times 20 = 30$). Note that with equivalent ratios such as the juice mixture, a common part-to-part rate can be shown in the same way, e.g., parts mango on the x-axis and parts grape on the y-axis, or vice versa.

Decimals

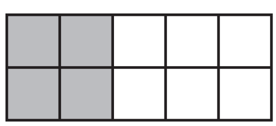
The most common representation of equivalent fractions in real life is decimals. Decimals were created to reduce the difficulty of operating on fractions, especially with addition. They have become commonplace because of their application to the standard international system of measures, e.g., 1.8 metres, 1.450 tonnes. Decimals are constrained equivalent fractions in that the denominators are restricted to tenths, hundredths, thousandths, etc. Any fraction can be expressed in equivalent form to a desired degree of accuracy. The place value system is positional in that the total value of a digit is the multiplication of its face value by its place value (position), e.g., the 7 in 23.479 represents $7 \times \frac{1}{100} = \frac{7}{100}$. The system is infinite in both left and right directions, with the connection between consecutive places being multiplication or division by ten.



The power of the place value system is in the simplicity of calculation. To add $\frac{3}{4} + \frac{2}{5}$ requires conversion of both fractions to equivalent forms with the same denominator, $\frac{3}{4} = \frac{15}{20}$, $\frac{2}{5} = \frac{8}{20}$, so $\frac{3}{4} + \frac{2}{5} = \frac{15}{20} + \frac{8}{20} = \frac{23}{20} = 1\frac{3}{20}$. If the decimals are known, the corresponding calculation is 0.75 + 0.4 = 1.15. Note that the decimals involved are themselves equivalent fractions.



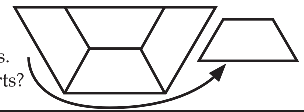
$\frac{3}{4} = \frac{75}{100} = 0.75$



$\frac{2}{5} = \frac{4}{10} = 0.4$

Decimals can also be used to compare fractions. For example, $\frac{1}{3} = 0.\dot{3}$, so $\frac{1}{3}$ is less than $\frac{2}{5}$ (0.4).

The chart on page 10 presents a variety of problem types involving fractions, ratios/rates, decimals, and percentages. The problems are examples of how changing the place of the unknown can influence the task difficulty. Equations are used to show how the problems could be recorded symbolically.

Construct	Result Unknown	Change Unknown	Start Unknown
Part–Whole Continuous	Cut this shape into quarters so that each piece is the same shape and size. 	This shape was cut up into equal parts like this. What fraction are these parts? 	This is one-quarter of a shape. What is the shape? 
Part–Whole Discrete	There are 12 marbles. You get one-quarter of them. How many marbles do you get? $\frac{1}{4} \times 12 = \square$	There are 12 marbles. You take three of them. What fraction of all the marbles is that? $\square \times 12 = 3$ or 3 out of $12 = \frac{3}{12} = \frac{1}{4}$	You take out three marbles. That was one-quarter of your marbles. How many marbles do you have? $\frac{1}{4} \times \square = 3$
Operator (Fractions)	You have \$24. You spend three-quarters of it. How much do you spend? $\frac{3}{4} \times 24 = \square$	You have \$24 and spend \$18 of it. What fraction of your money is that? $\square \times 24 = 18$ or $18 \div 24 = \square$	You spend \$18. That is three-quarters of your money. How much money do you have at first? $\frac{3}{4} \times \square = 18$ or $18 \div \square = \frac{3}{4}$
Operator (Decimals)	Salmon costs \$24 per kilogram. You buy a fillet that weighs 0.750 kilograms. How much do you pay? $0.750 \times 24 = \square$	Salmon costs \$24 per kilogram. You pay \$18 for a fillet. What is the weight of the fillet in kilograms? $\square \times 24 = 18$	You pay \$18 for a 0.750 kilogram fillet of salmon. What is the price of the salmon in dollars per kilogram? $0.750 \times \square = 18$
Operator (Percentages)	You get 75% of 24 netball shots in. How many of your shots go in? $75\% \times 24 = \square$	You take 24 shots and get 18 of them in. What is your shooting percentage? $\square\% \times 24 = 18$ or $18 \div 24 = \square\%$	You get 18 shots in. That gives you a percentage of 75%. How many shots did you take altogether? $75\% \times \square = 18$ or $18 \div \square = 75\%$
Measure (Fractions)	A car uses one-eighth of a tank of petrol for each trip. How many trips can it make on three-quarters of a tank? $\frac{3}{4} \div \frac{1}{8} = \square$ or $\square \times \frac{1}{8} = \frac{3}{4}$	A car makes six trips on three-quarters of a tank of petrol. How much of a tank is used on each trip? $\frac{3}{4} \div \square = 6$ or $6 \times \square = \frac{3}{4}$	A car makes exactly six trips on part of a tank of petrol. Each trip takes one-eighth of a tank. What fraction of the tank was full? $\square \div \frac{1}{8} = 6$ or $6 \times \frac{1}{8} = \square$
Measure (Decimals)	A bottle holds 0.750 litres of fruit juice. Each glass holds 0.125 litres. How many glasses can be filled? $0.750 \div 0.125 = \square$ or $\square \times 0.125 = 0.750$	Six glasses are filled from a bottle to empty it. The bottle holds 0.750 litres of fruit juice. How much does each glass hold? $0.750 \div \square = 6$ or $6 \times \square = 0.750$	Six glasses are filled to empty a bottle. Each glass holds 0.125 litres. How much juice does the bottle hold? $\square \div 0.125 = 6$ or $6 \times 0.125 = \square$
Quotient	Four fruit strips are shared equally between seven students. How much of one fruit strip does each student get? $4 \div 7 = \square$	Four fruit strips are shared equally between some students. Each student gets four-sevenths of a strip. How many students are there? $4 \div \square = \frac{4}{7}$ or $\frac{4}{7} \times \square = 4$	Six students share some fruit strips equally. Each student gets four-sevenths of a fruit strip. How many fruit loops are shared? $\square \div 7 = \frac{4}{7}$ or $7 \times \frac{4}{7} = \square$
Rates/Ratios	Six oranges cost four dollars. What would fifteen oranges cost? $6{:}4 = 15{:}\square$	Fifteen oranges cost ten dollars. What is the cost of one orange? $15{:}10 = 1{:}\square$ (Unit rate)	Fifteen oranges cost ten dollars. What is the cost of six oranges? $15{:}10 = 6{:}\square$ How many oranges cost four dollars? $15{:}10 = \square{:}4$

Teaching Fractions, Decimals, and Percentages

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Learning Experiences to Move Students from Counting from One by Imaging to Advanced Counting

Required Knowledge

Before attempting to develop their ideas about fractions, decimals, and percentages, check that Counting All students have the following knowledge.

Key Knowledge	Tasks for Key Knowledge
• Forwards and backwards number sequences up to and back from 20 at least	Use a number strip to walk up and back through the whole numbers to 20 at least, with the students saying the sequence backwards and forwards. Develop imaging of the numbers by turning the strip over. Walk up and back again saying the sequences without the numbers being visible. Point to individual numbers and ask the students to say the number. Note difficulties with discriminating "teen" from "ty" in the students' pronunciation. For more ideas about teaching number sequences, refer to *Book 4: Teaching Number Knowledge.*

Learning Experiences

The early experiences with fractions explored in Fair Shares will require a number of teaching sessions.

Fair Shares

I am learning to find halves, quarters, and other fractions of sets and shapes.

Key Mathematical Ideas

- Students must come to know common vocabulary for fractions, particularly halves, thirds, quarters (fourths), fifths, and tenths. The language for the most common fractions does not necessarily indicate how many pieces, e.g., while the connection between tenths and ten is evident, this is not the case with halves and two or quarters and four.
- Initially, the emphasis is on **unit** fractions, such as $\frac{1}{2}$ (one-half) and $\frac{1}{4}$ (one-quarter), that have **one** as a numerator. However, it is important to introduce non-unit fractions, such as $\frac{3}{4}$ (three-quarters) and $\frac{2}{5}$ (two-fifths), when learning opportunities present themselves.
- At this stage, students are introduced to the symbols and words related to models of fractions, e.g., the word one-half is recorded as well as the symbol $\frac{1}{2}$.
- Students need to understand what the numbers represent in a fraction symbol. The bottom number (denominator) indicates how many pieces make up the whole. The top number (numerator) tells us how many pieces there are. It counts the number of pieces.

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- Students need to read fractions such as $\frac{1}{4}$ as **one-quarter** and not as one out of four or one over four. This helps students to see the numerator (top number) as a count and the denominator (bottom number) as the size of the units.
- Students need to experience both **continuous** models, e.g., length and regions (shapes), and **discrete** models, e.g., using sets of objects.
- Students need to understand that fractional parts are equal shares or equal-sized portions of a whole.
- Students need to have experiences with finding both a **part of a whole**, e.g., $\frac{1}{2}$ (one-half) of 12 or half of a region, and determining the **whole, given a part** of it, e.g., "If □ □ □ is one-half of my set, how many □s are in the whole set?"

Key Mathematical Knowledge

Knowledge of doubles to 20 will assist the students to solve problems involving halving and doubling in the imaging and number properties phases of this series of learning experiences.

Diagnostic Snapshot

There are important key mathematical ideas related to understanding fractions that are explored in this series of learning experiences. Indicators that students are ready to move on are: an understanding of equal partitioning of sets and shapes, fractions as repeated addition, e.g., three-quarters equals one-quarter plus one-quarter plus one-quarter ($\frac{3}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$), and connecting language to symbols, particularly in counting sequences, e.g., $\frac{1}{4}, \frac{2}{4}, \frac{3}{4}$ $\frac{4}{4}$, $1\frac{1}{4}$ (one-quarter, two-quarters, and so on).

*The students achieving success proceed to **Using Imaging**. Otherwise, they proceed to the next activity at some later time.*

Equipment: Plastic glasses, play dough, paper circles, a length of paper, sliced bread (to make sandwiches), paper bags of 20 cubes, plastic jars (for example, peanut butter jars), plastic teddies, rubber bands

Using Materials

Continuous (lengths and regions)

Problem: Show the students the length of paper and tell them that it is a strap of liquorice.

"I have this big liquorice strap, and I'm going to cut it in half to share it between Tyler and Briar (name two students from the group). I'm going to run my scissors along the strip, and Tyler and Briar will tell me when to stop."

Run the scissors slowly along the strip until one student calls, "Stop." Cut the strip at that place and give the caller the passed over part and the other student the remainder.

Discuss whether the sharing has been fair (it may not be). Ask for other ways to cut the strip in half so that each half is the same length. Folding will be a likely reply. The two pieces can be placed beside or on top of each other to check that they are equal.

Record one-half using both words and the symbol, $\frac{1}{2}$. Discuss why the number two is the denominator. The denominator gives the number of equal parts that one (whole) is divided into.

Tell the students that you want them to find one-half of some other things. Set up the following examples:

"Cut the blob of play dough in half."

"Pour one-half of the water into one glass, leaving half in the other." (Start with two glasses, one full of water.)

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"Cut this pizza in half." (Start with a paper circle.)

"Cut this sandwich in half." Explore the different possibilities for how the sandwich could have been partitioned into halves. This can lead to further exploration of other shapes. Students need to justify that the pieces are equal through folding and mapping one side onto the other.

Using a line to represent a map of the students' distance from school to home, ask the students questions such as " Where would I be if I was halfway (half of the way) to school?"

Discrete (sets of objects)

Students need experiences in finding a fraction of a set through sharing objects into equal groups. For example:

"Give one-half of these lollies to Tyler and the other half to Briar." (Start with a bag of cubes.)

"One-half of the teddies go to the picnic, the other half stay behind. How many went to the picnic?"

Prior to sharing, ask the students "How many do you think one-half would be?"

After the students have shared the cubes out, a discussion needs to occur on whether the share was fair. Students will often count each person's share to determine whether the share was fair.

Record the fraction of the share using both the words (one-half) and the symbol ($\frac{1}{2}$). Discuss what the bottom number (the denominator) represents (it tells us how many groups make up the whole set).

The above experiences involve the students finding halves of shapes and sets of objects. The discussion needs to be broadened, either during a particular session or as a part of a series of learning experiences, to initially include finding quarters, and then thirds and fifths.

Note that finding thirds and fifths of shapes and lengths can be difficult because it involves concepts of angle, area, and volume. Students' inability to share equally may be due to these factors rather than the fraction concept itself.

The students achieving success proceed to ***Using Number Properties****. Otherwise, they proceed to the next activity at some later time.*

Using Imaging

For one-half and one-quarter, have a variety of circle fraction shapes in front of the students. Show the students a circle (one whole). Pose the problem "If I fold this circle into quarters, how many equal parts will I have? Show me what one-quarter would look like." Have the students check their choice of fraction piece by mapping it onto the folded circle.

Predicting: Fill a plastic jar to the top with equal-sized plastic teddies. Count out the teddies to see how many the jar holds when full.

Provide each group of students with a plastic jar, a thick rubber band, and one teddy. Tell them that they are to work out how far up the jar half of the teddies would come and mark where that level would be. Putting the rubber band around the jar and sliding it up and down is a good way to show the level.

Discuss their predictions, looking for the use of number knowledge, such as simple doubles, to estimate half of the teddies and for equalising in their prediction of the level.

The jar can be filled and the teddies shared into two equal sets, counted, and put into the jar to confirm the students' predictions.

Pose the same problem requiring quarters, thirds, or fifths to broaden the students' knowledge of fractions. (See note above re difficulty.)

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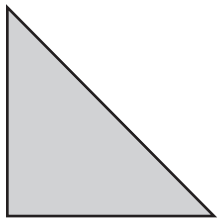
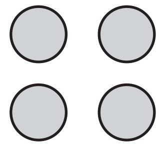
The experiences with fractions so far have focused on finding **part of a whole**, e.g., "Find one-half of this shape" or "What is one-half of this set of objects?"

Students also need opportunities to find the **whole, given a part**.

For example:



"This is one-quarter of a shape. What is the shape?"



"This is one-third of a set. How many objects are in the set?"

"This line shows one-quarter of my trip from my house to town. Draw a line that shows how far it is from town to my house."

Using Number Properties

The students' lack of strategies for division will limit their ability to apply number properties. Examples provided should include simple halving and quartering. For example: one-half of six ($\frac{1}{2}$ of 6), one-quarter of eight ($\frac{1}{4}$ of 8).

Learning Experiences to Move Students from Advanced Counting to Early Additive Part-Whole

Required Knowledge

Before attempting to develop their ideas about fractions, decimals, and percentages, check that Advanced Counting students have the following knowledge.

Key Knowledge	Tasks for Key Knowledge
• Forwards and backwards number sequences up to and back from 100 at least	Use a hundreds board or a number strip to walk up and back through the whole numbers to 100 at least, with the students saying the sequence backwards and forwards.
• Skip-counting sequences in twos, fives, and tens up to and back from 100 at least	Use a hundreds board or a number strip to jump up and back through the whole numbers to 100 in twos, fives, and tens. Transparent counters can be used to mark the numbers that are landed on. Use the addition and subtraction constant function on a calculator to generate skip-counting sequences. Keying in + 5 = = = = = (+ + 5 = = = for some calculators) produces the counting sequence in fives. Encourage the students to predict how many times = must be pressed to get to target numbers, such as 35.
• Symbols for halves, thirds, quarters, and fifths at least	Play games like Bingo with game cards that contain fractions instead of whole numbers. The caller can either name the fractions or show diagrams of the fractions. The students cover the fractions with counters. Provide students with fraction kits (circles or lengths). Use a dice labelled with fractions such as $\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{1}{8}$, $\frac{3}{8}$, $\frac{5}{8}$. Students take turns to roll the dice and take the pieces that match the fraction. The aim of the "Make a Whole" game is to construct as many ones (wholes) as possible.
• Doubles to 20 and the corresponding halves	Use the top two rows of a Slavonic abacus to show doubles, like four and four. The students name the total and then find half of eight.

Knowledge to be Developed

- Symbols for halves, quarters, thirds, fifths, and tenths
- Symbols for improper fractions, such as $\frac{5}{3}$ (five-thirds)
- Counting in fractions with the same denominator, e.g., $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$ (one-quarter, two-quarters, three-quarters)
- Ordering simple fractions with the same denominator, e.g., $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$
- Multiplication facts for the two, five, and ten times tables at least.

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Key Ideas

The students must realise that the symbols for fractions tell how many parts the whole has been divided into (the bottom number or denominator) and how many of those parts have been chosen (the top number or numerator). For example, $\frac{2}{3}$ (two-thirds) shows that one (a whole) is divided into three equal parts (thirds) and that two of those parts are chosen. Note that the terminology is not as significant as the idea, although the students will acquire the correct terms if they are used consistently.

Students also need to appreciate that the most common context for fractions is division where the numbers do not divide evenly. For example, when four people share 14 things, each person will get three things but two things will remain to be shared. These two things must be divided into halves to make the equal sharing possible.

The English language presents a barrier to the students generalising the meaning of fractions. Halves, thirds, and quarters (fourths) are special words, and it is not until fifths (five-ths), sixths, sevenths, and so on are encountered that the "ths" suffix code becomes evident.

Learning Experiences

Wafers

I am learning to find fractions of lengths, including seeing when a fraction is greater than one.

Key Mathematical Ideas

- Students are exploring how to solve equal sharing problems by partitioning regions.
- Students may use a variety of strategies to partition the regions. Strategies that involve repeatedly halving a portion can lead to difficulties in determining how much each person has received at the end of the sharing process.
- Problems involving working with 2, 4, and 8 people to share with are easier than problems with 3 and 6 people.
- Students need to come to realise that the number of people that you are sharing the items with will determine how the item is partitioned, e.g., sharing 3 bars with five people leads to the bars being divided into fifths.

Key Mathematical Knowledge

- Students need to have a good understanding about what the numerator (top number) and the denominator (bottom number) represent in a fraction symbol.
- Check that the students can read the symbols for the common fractions.

Diagnostic Snapshot

Ask the students questions such as:

- "How much chocolate will each person get if we share five chocolate bars with four people?"
- "If I have three liquorice straps to share with eight people, how much liquorice strap will each person get?"

Students who are able to solve these problems need to work on the Using Number Properties section and be extended by exploring related connections.

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Equipment: Wafer biscuits (rectangles made from grid paper can be used). These are chosen because the grid pattern makes them easy to cut into equal parts.

Using Materials

Problem: "I want you to work in pairs. You will get three wafer biscuits. Think about how you might share the wafers so that each person gets one-half of the wafers."

Tell the students to discuss how they will cut up the wafers so that they can be shared equally. Some students will suggest cutting each wafer in half, but others will suggest giving each person a wafer and halving the remaining one. Discuss how these methods compare, that is, three halves make one and one-half wafers.

Ask, "How can we check that each person will get the same number of wafers?"

The wafers can be cut and each person's share of wafers put end on end. Ask the students how many wafers each person is getting. Develop vocabulary like "one and one-half" and "three halves". Use symbols to record their findings:

*The students achieving success proceed to **Using Imaging**. Otherwise, they proceed to the next activity at some later time.*

$$\tfrac{1}{2} \quad + \quad \tfrac{1}{2} \quad + \quad \tfrac{1}{2} \quad = \quad \tfrac{3}{2} \quad = \quad 1\tfrac{1}{2}$$

One-half plus one-half plus one-half equals three halves, which is the same as one and one-half.

Generalise the ideas further with similar problems, such as:

"This time you will be in groups of four, but I will still give you three wafers. Find a way to share the wafers equally. How much wafer will each person get?"

Ask the students to predict how much they will get. If necessary, say "It will be less than a whole wafer because four people would need four wafers for that."

"We cut each wafer into four pieces. Each person gets three pieces. What are the pieces called?" (quarters) "So each person gets three-quarters of a wafer."

Provide other examples for exploration with the materials, such as:

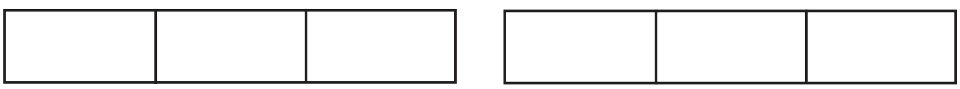
Three people share four wafers. Four people share six wafers.

Using Imaging

Third Person: Pose this problem: "Suppose there were three people, and they had to share two wafers. How much would each person get? I want you to think about that and draw a picture to show your ideas."

Discuss the diagrams the students draw, using the language of fractions.

*The students achieving success proceed to **Using Number Properties**. Otherwise, they proceed to the next activity at some later time.*



"You have cut each wafer into three pieces. What could you call each piece?" (one-third) "How many of those pieces will each person get?"

If the students do not have the language of thirds, teach them about it.

For example, "One-third plus another one-third is called two-thirds."

Record the students' findings using symbols and words, e.g., $\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$; one-third plus one-third equals (is the same as) two-thirds.

Pose other similar problems to see if the students can image a way to anticipate the sharing of biscuits.

"There are five people at the party and six wafers. How much wafer will each person get?"

Using Number Properties

The students have a good understanding of equal sharing when they can anticipate the result using the properties of the numbers involved rather than relying on images. The number size is increased to promote generalisation.

"Suppose I put you in groups of six people. Each group would get four biscuits. How much biscuit would each person get?"

Look for responses such as:

"They will get more than one-half but less than one whole."

(Ask: "How do you know?")

"It takes three biscuits to give each person one-half, but to give them each one whole biscuit would take six."

"That would be the same as three people sharing two biscuits."

Independent Activity

Animals

I am learning to find fractions of a set.

Key Mathematical Ideas

- The students will be moving from sharing by dealing out in ones to sharing by grouping to find the fraction of a set. For example, to find one-third of nine, students may use repeated addition, 3 + 3 + 3. This can also involve trial and improvement, e.g., they could start with a trial of groups of 2, 2 + 2 + 2, and then realise that they do not have enough and trial 3.
- Both models for teaching fractions, continuous (the farm paddock) and discrete (the cows), are used in the lesson.
- To support the students to move from sharing in ones to grouping, you need to direct them towards anticipating the result of the sharing.

Key Mathematical Knowledge

- Students need to have a good understanding about what the numerator (top number) and the denominator (bottom number) represent in a fraction symbol.
- Check that the students:
  - can read the symbols for the common fractions
  - have quick recall of the doubles to 20 and the corresponding halves
  - can skip count forwards and backwards in threes, fives, and tens.

Diagnostic Snapshot

Ask the students questions such as:

"The farmer has 25 animals and five paddocks. She wants to put the same number of animals in each paddock. What fraction will that be? How many animals will there be in each paddock?"

Students who are able to solve these problems using multiplication need to work on the Using Number Properties section and be extended by exploring related connections involving multiplication.

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Equipment: Animal counters or unilink cubes as a substitute

Using Materials

Problem: "Here is a farm (draw a farm divided into two fields on a piece of paper). The farmer uses an electric fence to make her farm into two paddocks. She has 10 animals. (Get a student to count out 10 animals.) She wants to put one-half of the animals in one paddock and one-half in the other. How many animals do you think will be in each paddock?"

Allow the students access to the animal counters to work out one-half of 10.

Look to see if the students can use a dealing strategy to find equal parts.

Ask, "Could we have worked out the number of animals in each paddock without sharing them out?" Some students may realise that they could apply their doubles knowledge (5 + 5 = 10).

Pose similar problems and allow the students to use equipment to solve them. For example: "The farmer fences her farm into four paddocks. She has 12 animals. Put one-quarter (a fourth) of the animals in each paddock."

*The students achieving success proceed to **Using Imaging**. Otherwise, they proceed to the next activity at some later time.*

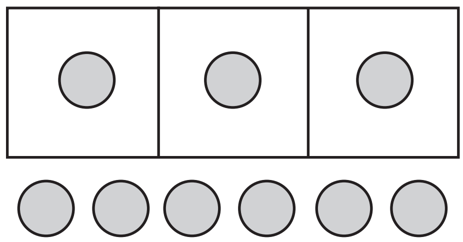
Using Imaging

Prediction: Set up an animals and paddocks problem, such as "On this farm, there are nine animals and three paddocks. One-third of the animals have to be put in each paddock. How many animals will be in each paddock?"

*The students achieving success proceed to **Using Number Properties**. Otherwise, they proceed to the next activity at some later time.*

Ask the students to progressively anticipate the results of sharing:

Put one animal in each paddock.



"How many animals are in the paddocks at the moment?"

"How many animals are left to be put in the paddocks?"

"If you put one more in each paddock, how many animals will you have left outside the paddocks?"

"How many animals do you think will be in each paddock when you have finished sharing out the animals?"

Record the final result using symbols, i.e., $\frac{1}{3}$ of 9 is 3 (in full language, one-third of nine is three).

Pose similar problems, such as:

"Twenty animals. One-fifth in each paddock (five paddocks)."

"Eight animals. One-quarter in each paddock (four paddocks)."

Using Number Properties

The number size is increased to promote generalisation.

"The farmer has 40 animals and 10 paddocks. She wants to put the same number of animals in each paddock. What fraction will that be?" (one-tenth) "How many animals will be in each paddock and why?"

Look for responses like:

"There will be four in each paddock. One in each paddock is 10, two is 20, three is 30, four is 40."

"If you put two animals in each paddock, that would be 10 and 10. That's 20, so four in each paddock must be 40."

Note that the students' application of strategies will be dependent on their skip-counting and addition knowledge. Restrict the examples used to numbers for which students will have counting sequences, like twos, fives, and tens, or doubles knowledge, in the case of halves and quarters.

For example:

"One hundred animals. Two paddocks."

"Thirty-five animals. Five paddocks."

Independent Activity

Fraction Circles

I am learning to put fractions in order from smallest to largest.

Key Mathematical Ideas

- This lesson reviews the understanding that a fraction is equal parts.
- Students need to realise that the size of a fraction involves co-ordinating both the numerator (top number) and the denominator (bottom number).
- Students need to see fractions as numbers and develop an understanding of the "home" of fractions among the whole numbers. The number line is a model that supports this understanding.
- Students need to recognise that if the numerator and the denominator are equal, the fraction is a whole number, e.g., four-quarters equals one whole ($4 \times \frac{1}{4} = 1$ or $\frac{4}{4} = 1$).

Key Mathematical Knowledge

- Students need to have a good understanding about what the numerator (top number) and the denominator (bottom number) represents in a fraction symbol.
- Check that the students can read the symbols for the common fractions.
- Students need to be able to count fractions with the same denominator going beyond one.

Diagnostic Snapshot

Draw a number line from 0 to 5 in your modelling book. Ask the students to locate the place of given improper fractions on the number line, e.g., $\frac{5}{2}$, $\frac{7}{4}$. If the students are able to solve the given problems, ask them to place $\frac{4}{1}$ on the number line. Students who can do this task and provide an explanation can be introduced to the independent work that follows Using Number Properties in the lesson.

Equipment: Paper circles or commercial fraction kits. These are chosen because it's easy to show when one whole unit (circle) is formed.

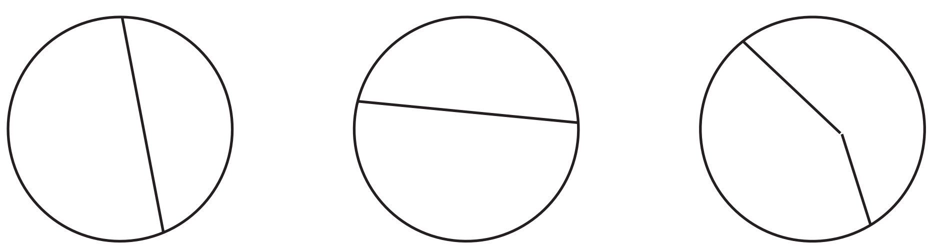
The students achieving success proceed to ***Using Imaging****. Otherwise, they proceed to the next activity at some later time.*

Using Materials

Problem: "Let's pretend that the circle you have is a pizza. Show me how you could cut your pizza into halves."

Get the students to cut their circles into halves in any way they wish. Many will choose to fold the circle. Show them some circles that have been folded into two pieces unequally.

Ask, "Are these pizzas cut in half?" "How do you know?"



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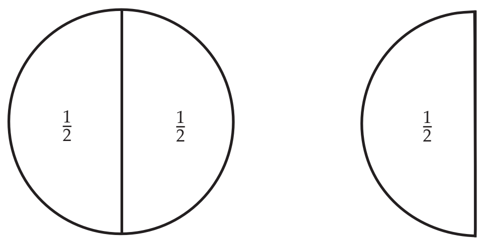
Look for the students to note that the halves must be equal or the cutting is unfair. Record the symbol $\frac{1}{2}$. Say "This is a way to write one-half. Where do you think the two on the bottom comes from?" The students may make the connection that "2" is the number of pieces that one circle is divided into. Get the students to label their halves, using symbols, and to cut out the pieces (across the circles).

Ask, "What do you think would happen if I put three halves together?"

Invite the students to give their ideas and then confirm these by joining halves.

Ask how they think the symbols for this new fraction could be written.



$\frac{1}{2} + \frac{1}{2} + \frac{1}{2}$ or $\frac{3}{2}$ or $1\frac{1}{2}$

(One-half plus one-half plus one-half equals three halves or one whole and one-half)

Ask the students what the three and the two mean in the fraction $\frac{3}{2}$ and relate this to the number of parts in one whole (pizza) and how many parts are chosen.

Using circles of the same size as models, get the students to cut another pizza into quarters. Label these, using the symbol for one-quarter ($\frac{1}{4}$). Ask them why four is the bottom number. Point out that quarters are sometimes called fourths.

Get the students to predict what three-quarters and seven-quarters will look like. Check their predictions by putting together the circle pieces. Record the symbols ($\frac{3}{4}$, $\frac{7}{4}$). Next, fold the circles into eighths. Get the students to fold quarters and then predict how many parts will be formed if they fold the fractions in half again. Have the students record their answers using the symbols, cut out the pieces, and form fractions with more than "1" as the top number.

Using Imaging

Shielding: Model comparing the size of fractions by placing some pizza pieces under paper plates. For example, place one-half under one plate and three-quarters under another. Label the plates by writing the symbols $\frac{1}{2}$ and $\frac{3}{4}$ on top. Ask the students which plate has the most pizza under it.

*The students achieving success proceed to **Using Number Properties**. Otherwise, they proceed to the next activity at some later time.*

Listen for the students to use benchmarks for comparison. Often these involve equivalent fractions, such as "Two-quarters is one-half, so three-quarters will have more".

Examples: $\frac{1}{2}$ and $\frac{3}{8}$, $\frac{4}{4}$ and $\frac{7}{8}$, $\frac{3}{4}$ and $\frac{3}{8}$, $\frac{3}{2}$ and $\frac{5}{4}$

Using Number Properties

The students have a good understanding of co-ordinating the numerator and denominator of fractions when they demonstrate that they do not need materials or images to make comparisons. Their justifications will relate to relationships within the symbols. For example, "Three halves are one and a half. Five-quarters are one and a quarter. A half is bigger than a quarter."

Note that halves, quarters, and eighths are particularly suited to the students at the early stages because they easily relate to doubling and halving.

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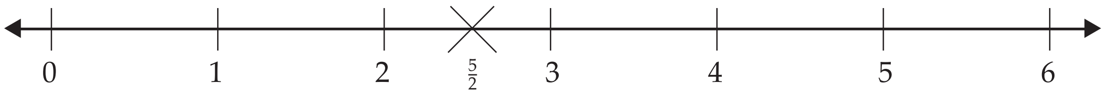
Independent Activity

Dotty Pairs Game

The students play in pairs. One student takes dots, the other takes crosses. Place the cards (cards 1–6, two lots, see Material Master 4-1) face down in a pile. The players take turns turning over two cards. The numbers are used to form a fraction, e.g., 2 and 5 are turned over, so $\frac{5}{2}$ or $\frac{2}{5}$ can be made. One fraction is chosen, made with the fraction pieces, if necessary, and marked on a 0–6 number line with the player's identifying mark (dot or cross).



Players take turns. The aim of the game is to get three of their marks uninterrupted by their opponent's marks on the number line. If a player chooses a fraction that is equivalent to a mark that is already there, they miss that turn.

NB: A fraction such as $\frac{4}{1}$ can be made using the cards. Students may not be familiar with fractions in this form and the meaning of the numerator and denominator will need to be explored with the fraction circles.

Hungry Birds

I am learning to use addition and subtraction to work out fractions of a set.

Key Mathematical Ideas

- The two models for fractions, continuous (the length of the blocks) and discrete (the number of blocks), are used in the lesson.
- Problems involving fractions like halves, quarters, fifths, and eighths will help the students in applying their addition knowledge and the use of strategies will encourage them to progress multiplicatively.

Key Mathematical Knowledge

Check that the students:

- have quick recall of the doubles to 20 and the corresponding halves
- can skip count forwards and backwards in threes, fives, and tens.

Diagnostic Snapshot

Ask the students questions such as:

"$\frac{1}{6}$ of 24?", "$\frac{2}{3}$ of 18?"

Students who are able to solve these problems using multiplication need to work on the Number Properties section and be extended by exploring related connections involving multiplication.

Equipment: Unilink cubes, preferably in sets of five cubes of each colour

These are chosen for the ease with which they can be joined and separated.

*The students achieving success proceed to **Using Imaging**. Otherwise, they proceed to the next activity at some later time.*

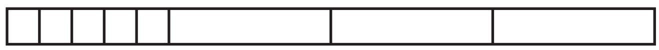
Using Materials

Problem: "Two birds, Bertie and Beatrice, pulled a large worm out of the ground. They measured it to be 10 cubes long. Here it is." Show the students a length of 10 cubes joined in five breaks. "If each bird gets one-half of the worm, how much will they get?"

At this stage, the students should be able to apply their doubles knowledge (5 + 5) to solve the problem because halving is similar to adding two equivalent sets. Ask them how the answer could be checked using the cubes (by splitting off the halves and aligning them).

Pose related problems so that the students can generalise the equal sharing nature of fractional numbers and how they can apply additive strategies. For example:

"Four birds shared a 20-cube worm. They each got one-quarter.
How much of the worm was that?" (Vary the number of birds.)



"Three birds caught a 15-cube worm. They ate one-third each.
How much was that?"

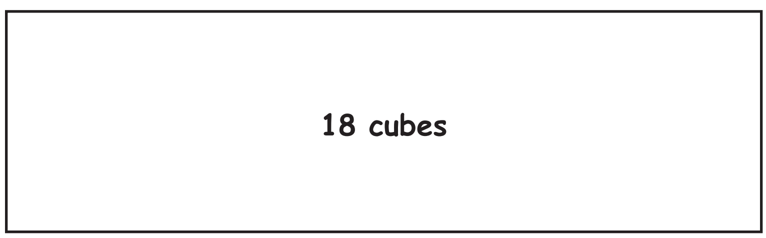
"Two birds caught a 12-cube worm. One bird got two-thirds. The other bird got the rest. How much worm did each bird get?" (Vary the unit fractions.)

Record the answers using symbols, for example, $\frac{1}{3}$ of 15 is 5, or in words, for example, one-third of fifteen is five.

Using Imaging

The students achieving success proceed to ***Using Number Properties****. Otherwise, they proceed to the next activity at some later time.*

Shielding and predicting: Make a worm of 18 cubes in five breaks. Show it to the students and ask them how long the worm is. Expect them to use fives and tens to shorten the counting. Hide the worm under a sheet of paper.



"Along come three birds, and they pull this worm out of the hole. If each bird gets one-third, how much will they eat?"

Look for the students to use additive strategies such as:

"Five and five is 10. Another five is 15. So they can have five cubes each. That leaves three, so they can have one more each. That's six altogether."

"Three and three and three are nine. Nine and nine is 18. So each bird will get three and three. That equals six cubes."

Students' predictions about the equal parts can be tried by splitting the worm under the masking paper and revealing the parts one at a time.

Record the answer as symbols, i.e., $\frac{1}{3}$ of 18 is 6 or $\frac{1}{3} \times 18 = 6$.

Pose similar problems with the variations from **Using Materials** above.

For example:

"There are three birds. The hidden worm is 12 cubes long. One bird gets one-half. Another bird gets one-third. The other bird gets the rest. How much does each bird get?"

Using Number Properties

Provide problems that involve finding fractions of a whole number. This encourages the use of number properties.

These can be presented in story and symbol form, e.g., word stories for $\frac{1}{5}$ of 20 is $\square$, $\frac{1}{6}$ of 24 is $\square$, $\frac{2}{3}$ of 12 is $\square$.

Extensions

For students who make strong progress towards using multiplicative strategies on the worm and birds problems, provide examples that have remainders.

For example, "Three birds share a worm that is 16 cubes long. Each bird gets one-third of the worm. How much is that?"

Students will need to recognise that each bird gets five whole cubes of worm and the remaining cube is cut into thirds. So $\frac{1}{3}$ of 16 is $5\frac{1}{3}$ (one-third of sixteen is five wholes and one-third).

This will strengthen the place of fractions as both operators and as numbers among the whole numbers.

Other examples might be:

"Four birds each get one-quarter of an 18-cube worm."

"Five birds each get one-fifth of a 23-cube worm."

"Three birds each get one-third of a 14-cube worm."

Learning Experiences to Move Students from Early Additive to Advanced Additive–Early Multiplicative Part-Whole

Required Knowledge

Before attempting to develop their ideas about fractions, decimals, and percentages, check that Early Additive students have the following knowledge.

Key Knowledge	Tasks for Key Knowledge
• Symbols for halves, quarters, thirds, fifths, and tenths, including improper fractions, e.g., $\frac{5}{4}$	Record fractions such as $\frac{2}{3}$, $\frac{4}{5}$, $\frac{3}{2}$. Give the students access to a circle fraction kit (Material Master 4-19) and tell them to make each fraction as you show it. Get the students to draw these fractions and rename them as mixed numerals, e.g., $\frac{7}{3} = 2\frac{1}{3}$ (seven-thirds is the same as two wholes and one-third).
• Ordering of fractions with like denominators, e.g., $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$, $\frac{4}{4}$, $\frac{5}{4}$, ...	Use length fraction kits (see Material Master 7–7) to build up counting sequences in halves, quarters, thirds, fifths, etc. Stress the renaming of fractions as whole numbers, such as $\frac{1}{4}$, $\frac{2}{4}$ ($\frac{1}{2}$), $\frac{3}{4}$, $\frac{4}{4}$ (1), $\frac{5}{4}$, ... (one-quarter, two-quarters [one-half], and so on). Ask the students why, when the denominator is held constant, the fraction gets bigger as the numerator is increased.
• Skip-counting sequences in twos, threes, fours, fives, and tens up to and back from 100 at least	Use the hundreds board and calculator as described for Advanced Counting students. Mark fixed jumps on a number line using pegs. For example, Wally Weka starts on zero and jumps three numbers each time. Will he land on 9, 16, 21, 36, etc? Use the Slavonic abacus as a skip-counting vehicle. For example, move four beads across each time and record the total number as the sequence grows. This will link nicely to place value. At first, allow the students to see the sequence of numbers but gradually erase them.
• Groupings of two, five, and ten, at least, in numbers to 50	Provide both sharing and "how many sets of" problems for the students to solve. For example, "Here are 20 seeds. There are four people. How many seeds do they get each?" Focus on number patterns. For example, "Two fives are in every ten, so there must be four fives in two tens (20)."
• Multiplication facts for the two, five, and ten times tables at least	Focus on number patterns and connections to addition. For example, the "-ty" numbers are the multiples of ten. Six tens are sixty (60), so "-ty" means tens. The two times tables are the doubles: two times eight is eight plus eight or double eight.

Knowledge to be Developed

- Multiplication facts for all times tables to 10
- Reading decimals to three places
- Rounding decimals with up to two decimal places to the nearest whole number

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- Saying the decimal number word sequences forwards and backwards in tenths and hundredths
- Knowing the number of tenths and hundredths in decimals to two places e.g., the number of tenths in 7.2 is 72
- Identifying symbols for any fraction including improper fractions
- Counting forwards and backwards in halves, quarters, thirds, fifths, and tenths at least
- Ordering unit fractions.

While the focus of the lessons at this stage is on teaching fractions, students also need to be taught the above decimals number knowledge.

Key Ideas

Fractions involve a significant mental jump for the students because units of one, which are the basis of whole-number counting, need to be split up (partitioned) and repackaged (re-unitised). It is crucial that the students have significant opportunities to split up ones through forming unit fractions with materials and are required to recombine several of these new units to form fractions like two-thirds and five-quarters. In this way, the students are required to co-ordinate the link between the numerator and denominator in fraction symbols.

Early additive students are progressing towards multiplicative thinking. Fraction contexts offer opportunities for these students to appreciate the links between addition and multiplication. This can be achieved through requiring the students to anticipate the results of equal sharing and involving fractions like thirds, fifths, and tenths, where addition methods are less efficient.

It's important that you require the students to construct the whole unit from given parts. For example, if a student is given a Cuisenaire® rod or pattern block and told that it's one-quarter of a length or shape, they should be able to reconstruct the whole. Similarly, given two red cubes and told that the cubes are one-fifth of a stack, students should be able to make the whole stack.

Understanding the relationship between fractions, as numbers, and the number one is critical for further learning. Ensure that the students know that one whole, as referred to in fraction work, means the same as the number one.

Learning Experiences

Birthday Cakes

I am learning to use multiplication to find a fraction of a set.

Key Mathematical Ideas

- The students will be moving from using addition strategies to find the fraction of a set to using multiplication strategies.
- Within the lesson, students find a fraction of an amount, e.g., $\frac{2}{3}$ of 18. They find the **part of a whole**. They also work with problems that involve finding the **whole if they know the part**, e.g., "If □ □ □ □ □ □ is $\frac{2}{3}$ of my set, how many □s are in the whole set?"
- Initially, problems involve finding a unit fraction of a set and progress to finding non-unit fractions.

Key Mathematical Knowledge

- Check that the students have quick recall of the 2, 3, 5, and 10 times tables.
- Students need to be able break up the one unit into different fractions using materials, e.g., $\frac{1}{4} + \frac{3}{4} = 1$ or $\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$.

Diagnostic Snapshot

Ask the students questions such as:

"Two-thirds of the cake has eight candles on it. How many candles are there on the whole cake? If $\frac{2}{3}$ of $\square$ is 8, what is $\square$?"

Students who are able to solve these problems using multiplication need to work on the independent work in the Using Number Properties section.

Equipment: Paper circles (to represent cakes) and counters (to represent candles)

Using Materials

Problem: "The four people at Carla's birthday will get one-quarter (one-fourth) of the cake each. Carla puts 16 candles on the cake so that each person gets the same number of candles on their piece of cake. How many candles will each person get on their piece of cake?"

*The students achieving success proceed to **Using Imaging**. Otherwise, they proceed to the next activity at some later time.*

Give the students 16 counters and a paper circle to fold into quarters.

Ask, "How many candles do you think each person will get?" "How do you know?"

Look for the students to use adding-related strategies, such as halving and halving again, like "Eight and eight is 16, so one-half of the cake has eight candles. Four and four is eight, so one-quarter has four candles."

Confirm the answers by equally sharing the counters onto the paper circle. **There is no need to use materials if the students are able to use partitioning strategies.**

Pose similar problems, using materials if required, such as:

"Five people are at the party. There are 25 candles."

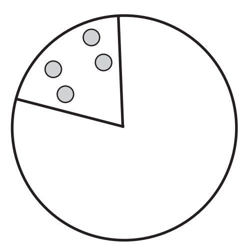
"Three people are at the party. There are 21 candles."

Record the answers using symbols, for example, $\frac{1}{5}$ of 25 is 5.

Vary the problems by using non-unit fractions, such as, "At the party the cake is cut into quarters (fourths). Twelve candles are put on the cake. Greedy Greg eats three-quarters of the cake. How many candles does he get?" Record the results using symbols, that is, $\frac{3}{4}$ of 12 is 9.

Using Imaging

*The students achieving success proceed to **Using Number Properties**. Otherwise, they proceed to the next activity at some later time.*



Problem: (Show the students one-fifth of a paper circle with four counters on it.)

"Here is a piece of Rongopai's birthday cake. Each piece of cake has the same number of candles. How old is Rongopai?"

Reconstructing: Listen to the students' ideas about how many candles were on the whole cake. Fifths have been deliberately chosen because they are easily confused with quarters. Ask the students to justify their answers.

"I think the cake was cut into five pieces."

"What would each piece be called?" (one-fifth)

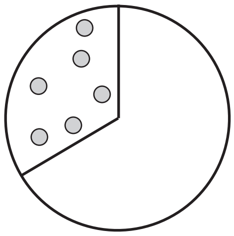
"If there are five pieces, then there are 4 + 4 + 4 + 4 + 4 or 5 × 4 candles on it. That's 20."

"How did you work out that was 20?"

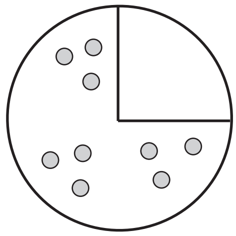
The whole cake can be constructed, if necessary, to confirm the answers.

Record the answer as: $\frac{1}{5}$ of $\square$ is 4, so $\square$ is 20. This means that Rongopai is 20 years old.

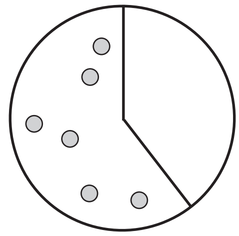
Pose similar word problems and ask the students to work out the number of candles on the whole cake.



$\frac{1}{3}$ of $\square$ is 6



$\frac{3}{4}$ of $\square$ is 9



$\frac{3}{5}$ of $\square$ is 6

Using Number Properties

Ask similar word problems and record them using symbols.

For example:

"Two-thirds of the cake has eight candles on it. How many candles are on the whole cake?" $\frac{2}{3}$ of $\square$ is 8, so $\square$ is 12.

"Three-quarters of the cake has nine candles on it. How many candles are on the whole cake?" $\frac{3}{4}$ of $\square$ is 9, so $\square$ is 12.

Independent Work

The students will benefit from playing the games Chocolate Chip Cheesecake (see Material Master 7–1) and Mystery Stars (see Material Master 7–8).

Fractional Blocks

I am learning to use patterns to find fractions of shapes and sets.

Key Mathematical Ideas

- Students will use symmetry to find fractions of continuous shapes; at the materials stage, they will fold or map one fraction onto another.
- The problems posed will progress from using a simple unit fraction to a more difficult fraction, e.g., one-quarter to one-eighth to one-sixteenth.

Key Mathematical Knowledge

- Check that the students have quick recall of the 2, 3, 5, and 10 times tables.

Diagnostic Snapshot

Ask the students questions such as:

" $\frac{2}{5}$ of 25, $\frac{4}{5}$ of 100", etc.

Students who are able to solve these problems using multiplication need to work on the Using Number Properties section and be extended by exploring related connections involving multiplication.

Equipment: Square tiles and/or square grid paper and scissors

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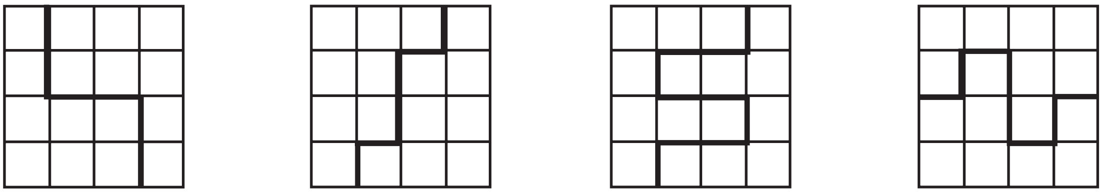
Using Materials

Provide the students with a four-by-four square of paper. Tell them to divide the square in half so that each part has the same size and shape. Challenge them to come up with four other ways to cut the same square in half.

Share the methods. Ask them what the methods all have in common. Most of the paths cutting the square are likely to have rotational symmetry about the centre of the square. The students need to understand that different cuts can be classified as identical, for example, diagonals.



*The students achieving success proceed to **Using Imaging**. Otherwise, they proceed to the next activity at some later time.*

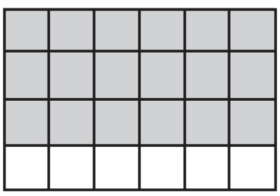
Ask the students what checks there could be to make sure that each of these ways of cutting results in two exact halves.

Focus the discussion on the fact that the two pieces map onto each other by reflection and/or rotation and are therefore of similar shape and size.

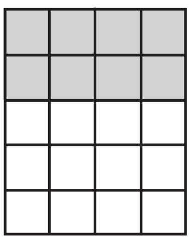
Discuss the idea that the square could be cut into two pieces that have the same area but different shapes. Choose an irregular half of the square and ask the students what the area of the shape will be in unit squares (eight, because that is one-half of 16).

Challenge the students to work out the area of quarters for the 16-unit square. They will probably use addition-based strategies such as 8 + 8 = 16, so eight is one-half; 4 + 4 = 8, so four is one-quarter.

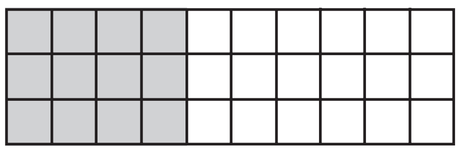
Provide other examples of finding fractions of blocks. Vary the numbers to include non-unit fractions. Encourage the students to use symmetry to simplify the problems. Each block can be made with square tiles, and the tiles can be dealt into equal sets.



$\frac{3}{4}$ of 24

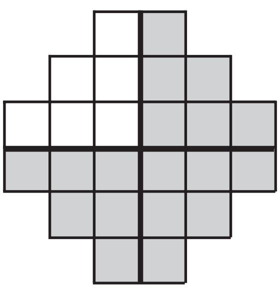


$\frac{2}{5}$ of 20

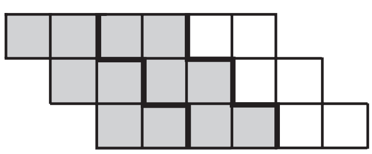


$\frac{4}{10}$ of 30

Provide some blocks made from squares and require the students to divide them into given fractions where use of symmetry might be advantageous. For example:



$\frac{3}{4}$



$\frac{2}{3}$

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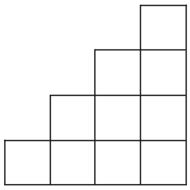
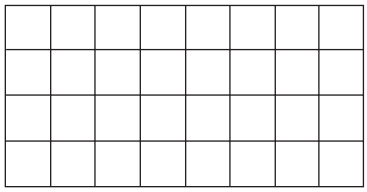
Using Imaging

Prediction: Allow the students to see and build the blocks shown below but ask them to visualise how they will cut them into the required fraction.

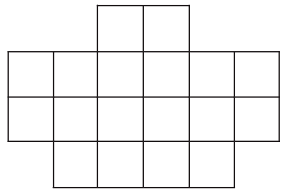
Have them describe their cutting to a partner to develop geometric vocabulary. For example:



Find $\frac{2}{5}$



Find $\frac{3}{8}$



Find $\frac{1}{6}$

*The students achieving success proceed to **Using Number Properties**. Otherwise, they proceed to the next activity at some later time.*

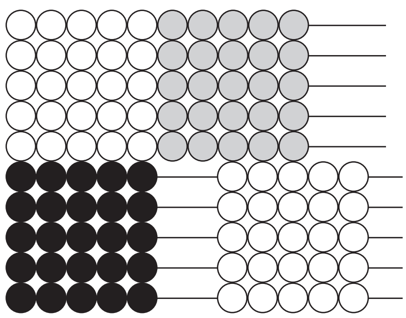
Focus the students on using multiplicative rather than additive strategies. For example, to find $\frac{1}{6}$ of the 18-unit square shape, the students might divide 18 by six.

Record the answers using equations such as $\frac{3}{8}$ of 32 is 12 ($\frac{3}{8} \times 32 = 12$).

Using Number Properties

Increase the number size while staying with fractions like halves, quarters, fifths, sixths, eighths, and tenths that make symmetrical partitioning effective. Provide examples that easily suggest the representation of the number as an array of square tiles. One hundred is of particular significance owing to its link with decimals and percentages. Such examples are easily modelled on the Slavonic abacus.

For example:
To find three-quarters of 100 ...



Other problems to pose:

$\frac{2}{5}$ of 25 $\frac{1}{4}$ of 80 $\frac{3}{8}$ of 40 $\frac{4}{5}$ of 100 $\frac{3}{10}$ of 100 $\frac{1}{6}$ of 30

Seed Packets

I am learning to solve simple ratio problems by repeated copying.

Key Mathematical Ideas

- Students will be using ratio tables to "build up" ratios through repeated copying; initially, students may use an additive strategy to determine the next ratio. Students need to be provided with examples that are more complex in order to encourage them to use multiplicative strategies.
- The use of the double number line to solve ratio problems encourages multiplicative thinking.

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Key Mathematical Knowledge

- Check that the students have quick recall of the 2, 3, 5, and 10 times tables.

Diagnostic Snapshot

Ask the students questions such as:

"Each bag has two red and three yellow beans. There are 16 bags in the can. How many beans of each colour are in the can?"

Students who are able to solve problems such as these using multiplication need to work on the Using Number Properties section and be extended by exploring related connections involving multiplication and division.

Equipment: Beans, counters, cubes or similar individual counters

Using Materials

Problem: Make up some packets of make-believe seeds. Similarly, candy bars could be made with cubes. Begin with a 1:2 ratio, for example, one blue seed and two yellow seeds in each packet. Get the students to make up similar packets.

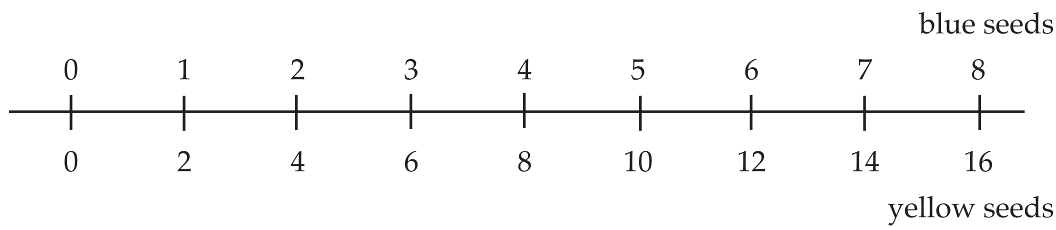
*The students achieving success proceed to **Using Imaging**. Otherwise, they proceed to the next activity at some later time.*

"Suppose we wanted a box with eight packets in it. How many blue seeds and how many yellow seeds would be in the box?"

Allow the students to use materials to solve the problem. Reflect on their strategies. The students will commonly put together copies of one packet until eight packets are formed.

Record this "build up" in a ratio table or a double number line:

Packets	1	2	3	4	5	6	7	8
Blue seeds	1	2	3	4				8
Yellow seeds	2	4	6	8				16



Pose predictive problems like:

"If we had 14 packets, how many yellow seeds would there be?"

Since doubles are involved, the students will find it easy to apply addition methods to this situation.

To encourage multiplication, make up more complex ratios. Some examples are:

"Three red and two yellow seeds are in each packet. In the box are nine packets. How many seeds of each colour are in the box?"

"One blue seed and three red seeds are in each packet. Ten packets are in the box. How many seeds of each colour are there in the box?"

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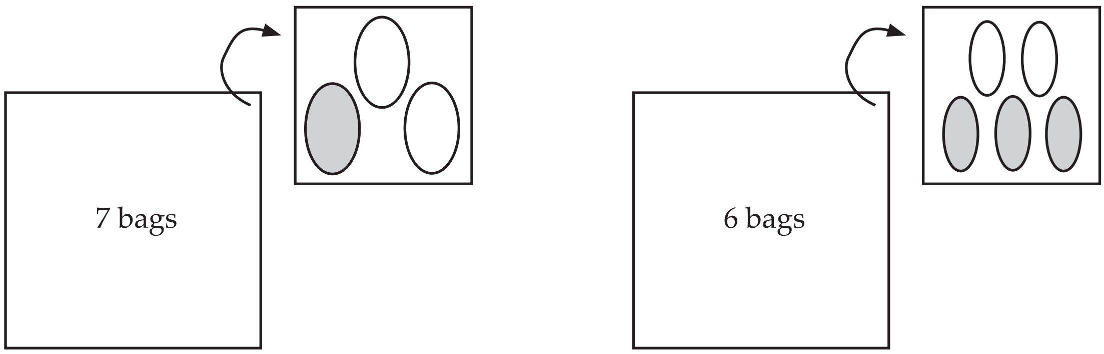
The students achieving success proceed to ***Using Number Properties****. Otherwise, they proceed to the next activity at some later time.*

Using Imaging

Shielding: Create examples of ratio problems using bags of beans or connected cubes. Put several of these bags or stacks into an empty tin can. The outside of the can may be labelled with how many bags are in the can. The students are shown one or two bags and asked to work out how many beans or cubes of each colour are in the can.



Encourage the students to use double number lines and ratio tables to show their strategies.

The students can make up mystery-can problems for others to solve.

Using Number Properties

Increase the number of bags involved while keeping the ratios simple (for example, 1:3, 2:3, 2:1, 6:4). This will make the addition strategies less efficient and motivate the students towards multiplicative thinking.

For example: "Each bag has two red and four yellow beans. There are 20 bags in the can. How many beans of each colour are in the can?"

Recording

Students can work through written examples, such as, "There are two green beans and three red beans in each bag. How many red beans would go with 14 green beans?"

Trains

I am learning to find where fractions live amongst whole numbers.

Key Mathematical Ideas

- The double number line is used to indicate where fractions live amongst whole numbers.
- Students need to have experiences that require interpretation of a fraction in relation to a changing one (whole). That way, a given fractional part does not get identified with a special shape or colour but with the relationship of the part to the designated whole.
- Students should know when they have sufficient fractional parts to make a whole, e.g., five-fifths makes one whole.

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Key Mathematical Knowledge

Check that the students:

- have quick recall of the 2, 3, 5, and 10 times tables.
- can recognise that improper fractions such as $\frac{14}{3}$ can be larger than one and can be renamed, using multiplication, as mixed numbers, e.g., $4\frac{2}{3}$ (four wholes and two thirds).

Diagnostic Snapshot

Ask the students questions such as:

"Rename these fractions as mixed numbers: $\frac{9}{2}$, $\frac{14}{3}$, $\frac{16}{5}$, etc."

Students who are able to solve the above problems using multiplication, e.g. $4 \times 3 = 12$, so $\frac{14}{3} = 4\frac{2}{3}$, need to work on the Using Number Properties section and be extended by exploring related connections involving multiplication.

Equipment: Cuisenaire® rods

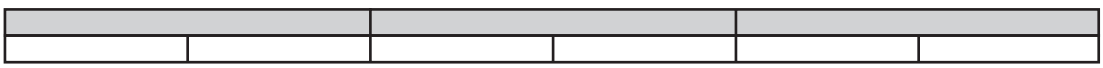
Using Materials

*The students achieving success proceed to **Using Imaging**. Otherwise, they proceed to the next activity at some later time.*

Problem: "Suppose that I make a train from these rods. (Hold up the orange rod.) How long do you think a train of three carriages would be? Use your hands to show me."

"If this orange rod is a full carriage, how big is this rod?" (Hold up a yellow rod.) The students should say it's half a carriage because two of them make a whole carriage.

Ask, "How many half carriages would I need to make three whole carriages?"



Record the answer as "six half carriages make three whole carriages".

Ask how this could be written using numerals. ($6 \times \frac{1}{2} = \frac{6}{2} = 3$)

"Suppose that you could make up whole carriages with these rods." (Red)

"What fraction of a carriage is each rod?" (One-fifth) "How do you know?" (Five of them make a whole carriage.)

"Suppose you had 12 of these one-fifth carriages.
How many whole carriages could you make?"

Ask the students how they could record this using numerals or words.

"Twelve-fifths is the same as two wholes and two-fifths ($\frac{12}{5} = 2\frac{2}{5}$)."

Give the students other experiences that involve modelling with the materials, for example, redefining the dark green rods as the whole carriages of a different railway:

"Red rods will be one-third of a carriage, and light green rods will be one-half.
How many whole carriages could you make with 18 one-third carriages?"
($18 \times \frac{1}{3} = \frac{18}{3} = 6$)

"How many whole carriages could you make with seven one-half carriages?"
($7 \times \frac{1}{2} = \frac{7}{2} = 3\frac{1}{2}$, so three whole carriages.)

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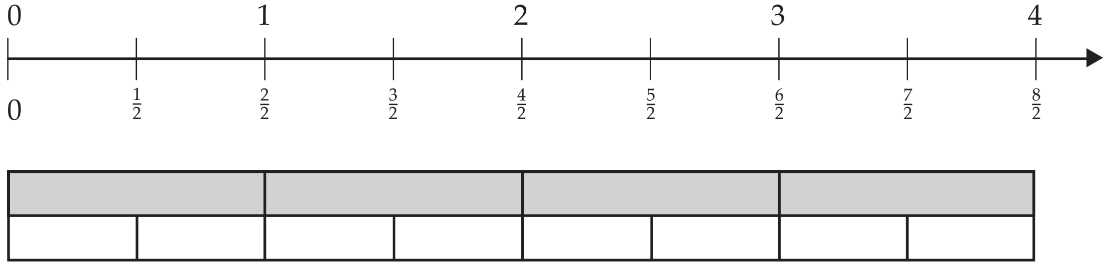
*The students achieving success proceed to **Using Number Properties**. Otherwise, they proceed to the next activity at some later time.*

Record these results using a double number line by putting the rods end-on-end to establish the correct position of numbers. For example:



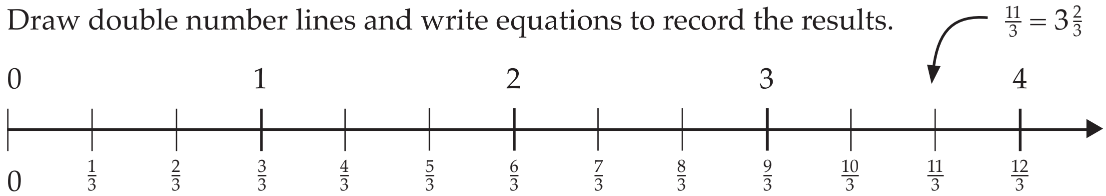
Using Imaging

Predicting: Pose problems for the students to solve where they must anticipate the answer by imaging. For example:

Choose the dark blue Cuisenaire® rod to be the length of one carriage. The light green rod will be one-third of a carriage.

Ask, "If you had 11 one-third carriages, how many whole carriages would you have?" Place the rods alongside each other to help the students to visualise the answer.

Draw double number lines and write equations to record the results.



Other examples might be:

Define a brown rod as one carriage.

"How many carriages can you make with nine red rods?" (quarters)

"How many carriages can you make with 17 crimson rods?" (halves)

Using Number Properties

The students solve problems involving renaming improper fractions (numerator larger than the denominator) as mixed numbers (whole numbers and fractions), e.g., $\frac{9}{4} = 2\frac{1}{4}$. Pose problems that are difficult to visualise so that the students are required to attend to the numbers.

For example: "Rename these fractions as mixed numbers:

$\frac{14}{3}$ ($4\frac{2}{3}$), $\frac{15}{4}$ ($3\frac{3}{4}$), $\frac{28}{7}$ (4), $\frac{46}{10}$ ($4\frac{6}{10}$), $\frac{75}{100}$"

(Note that the last fraction is less than one, so it can't be renamed as a mixed number.)

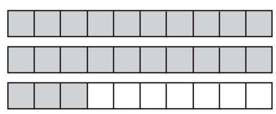
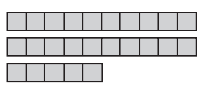
Recording

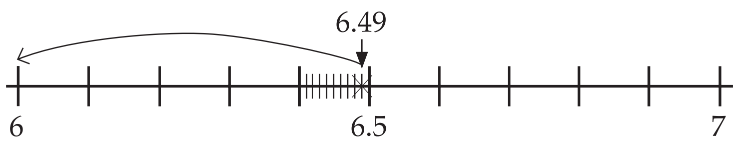
Students can record their answers as equations and on number lines.

Learning Experiences to Move Students from Advanced Additive–Early Multiplicative to Advanced Multiplicative–Early Proportional Part-Whole

Required Knowledge

Before attempting to develop their ideas about fractions, decimals, and percentages, check that Advanced Additive–Early Multiplicative students have the following knowledge.

Key Knowledge	Tasks for Key Knowledge
• Read decimals to three places	Use decimal arrow cards (see Material Master 7–2) to make decimal numbers and partition them by place value, e.g., 6.25 partitioned as 6 + 0.2 + 0.05. Develop counting by decimals by using the constant function of a calculator. For example, keying in 3.46 + 0.01 = = = ... will have the display increasing by one-hundredth each time equals is pressed. Students can anticipate and say each number before equals is keyed and then check. Ask them to explain which of 3.49 or 3.5 is larger. (Note: 3.49 + 0.01 = 3.5)
• Identify symbols for any fraction, including tenths, hundredths, thousandths, and improper fractions, e.g., $\frac{56}{10}$	Practise saying written fractions and model them using bars, unilink cubes, decipipes, or decimats. For example, $\frac{23}{10}$ as:  Place each fraction on a number line showing whole numbers, using a single bar as the unit of one. Form fractions involving tenths, hundredths, and thousandths using place value blocks, decimats (see Material Master 7–3), or decipipes. What fraction is  of a flat? ($\frac{25}{100}$ or 25 hundredths)
• Count forwards and backwards in halves, thirds, quarters, fifths, and tenths at least	Mark the location of fractions with the same denominator on a number line showing whole numbers. For example: $\frac{1}{2}$ $\frac{2}{2}$ $\frac{3}{2}$ $\frac{4}{2}$ $\frac{5}{2}$ $\frac{6}{2}$ $\frac{7}{2}$ ... Count through the fraction sequence backwards and forwards, interchanging whole numbers like 1, 2, and 3 for $\frac{2}{2}$, $\frac{4}{2}$, $\frac{6}{2}$. Work towards saying the sequence without seeing the number line.

• Order unit fractions (top number of one) such as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$	Fold paper strips of the same length into halves, thirds, quarters, fifths, etc. Use these strips to form a number line of unit fractions.  A key issue is why fractions get smaller as the denominator increases. One (the whole) is shared into more parts, therefore each part is smaller.
• Say the decimal number word sequences forwards and backwards in tenths and hundredths.	Use a flip chart. Starting with tenths first, make numbers and ask the students to count forwards and backwards in tenths, e.g., 0.8, 0.9, 1.0, 1.1, etc. This can also be explored using calculators. Encourage the students to predict the answer. These activities can be extended to counting in hundredths. Use a double number line and have the students place fractions (tenths) and equivalent decimal fractions (tenths) in their correct position. 
• Round decimals with up to two decimal places to the nearest whole number e.g., round 6.49 to 6, 19.91 to 20	Review students' ability to round whole numbers to the nearest ten, hundred on a number line. Extend this to decimals with one decimal place first, followed by two decimal places. 
• Know tenths and hundredths in decimals to two places e.g., tenths in 7.2 is 72.	Use decimats to model the link between how many tenths in one whole. This can then be explored for other decimals, e.g., 3.2 is the same as 32 tenths.

Knowledge to be Developed

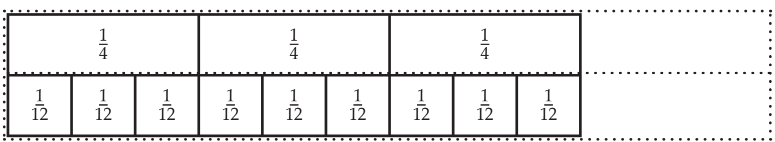
- Saying the forwards and backwards number sequences in tenths, hundredths, and thousandths, e.g., 74.193 as seven tens, four ones and one-tenth, nine-hundredths, and three-thousandths, and anticipating the number if one place is increased or decreased
- Ordering decimals to three places at least
- Ordering fractions including halves, thirds, quarters, fifths, and tenths
- Rounding whole numbers and decimals to the nearest whole number or tenth
- Knowing equivalent fractions for halves, quarters, thirds, fifths, and tenths with denominators up to 100, 1000, e.g., one in four is equivalent to 25 in 100 or 250 in 1000
- Converting between fractions, decimals, and percentages with halves, quarters, fifths, and tenths

- Knowing the divisibility rules for 2, 3, 5, and 10, e.g., 471 is divisible by 3 because 4 + 7 + 1 = 12 (see Nines and Threes, pages 70–72, *Book 6: Teaching Multiplication and Division, Revised draft 2007*)
- Identifying the whole-number factors of numbers to 100, e.g., factors of 36 = (1, 2, 3, 4, 6, 9, 12, 18, 36).

Key Ideas

Students who are at the Advanced Additive–Early Multiplicative stage have learned some of the properties of multiplication and division, particularly commutativity (e.g., 4 x 6 = 6 x 4), associativity (e.g., 4 x 7 = 2 x 2 x 7), and distributivity (e.g., 14 x 8 = 10 x 8 + 4 x 8). Their ability to work with fractions will be enhanced by the development of these properties as they relate them to division, their grasp of the two division situations, sharing and measuring, and their understanding of multiplication and division as inverse operations. In the transition from Advanced Additive–Early Multiplicative to Advanced Multiplicative–Early Proportional, these ideas about multiplication and division must develop concurrently alongside the ideas about fractions given below.

The key fraction idea is equivalence. Students need to see fractions as both **numbers** and **operators** simultaneously. Three-quarters ($\frac{3}{4}$) represents the same quantity as nine-twelfths ($\frac{9}{12}$), as long as the same referent whole (one) is used. This assumption of a uniform one whole underpins the concept of fractions as numbers. The equivalence of $\frac{3}{4}$ and $\frac{9}{12}$ is based on a measurement principle that is fundamental to the metric system of measures (e.g., kilometres, metres, centimetres, millimetres). Twelfths are one-third the size of quarters, so three times as many of them fit into the same measure as three-quarters. This explains the multiplicative relationships between the numerators and denominators of equivalent fractions:



$$\frac{3}{4} \overset{\text{x3}}{\underset{\text{x3}}{=}} \frac{9}{12}$$

Fractions can **operate** on other numbers multiplicatively. For example, three-quarters of another number can be found, such as three-quarters of twelve ($\frac{3}{4}$ x 12 = $\square$).

Decimals and percentages are special cases of equivalent fractions in which the denominators are restricted to powers of ten (… thousandths, hundredths, tenths, ones). For example, eighteen out of twenty-four is 75% because $\frac{18}{24} = \frac{3}{4} = \frac{75}{100} = 75\%$. The symbol % is derived from /, meaning "out of", and the two zeros from one hundred. The decimal 2.36 equals $2 + \frac{3}{10} + \frac{6}{100}$, which can be renamed as a single equivalent fraction, $\frac{236}{100}$. Decimals are a way of extending the whole number place value system to include units that are parts of one.

Fraction equivalence is critical to understanding the order of numbers when the set of whole numbers is expanded to include integers and fractions/rational numbers. With whole numbers, a unique number exists at intervals of one on the number line, i.e., 0, 1, 2, … In the fraction world, there are infinite names for a given point on the number line. These names are equivalent fractions, so $\frac{3}{4}$ and $\frac{9}{12}$ occupy the same point, along with an infinite number of other names. Whole numbers have fraction names as well, e.g., $3 = \frac{12}{4} = \frac{36}{12}$, etc. Between any two fractions exist an infinite number of other fractions, e.g., $\frac{17}{24}$ exists halfway between $\frac{2}{3}$ and $\frac{3}{4}$ because $\frac{2}{3} = \frac{16}{24}$ and $\frac{3}{4} = \frac{18}{24}$. Finding the size difference between two fractions also involves equivalence. For example, three-quarters ($\frac{18}{24}$) is two-twenty-fourths ($\frac{2}{24}$) greater than two-thirds ($\frac{16}{24}$), and $\frac{2}{24} = \frac{1}{12}$.

Learning Experiences

Pipe Music with Decimals

I am learning to add and subtract decimals.

Key Mathematical Ideas

- Students need to understand that the place value system extends to the left and to the right based on whether you are multiplying or dividing by ten.
- The decimal point is used to mark the ones place.
- The symbols 0.5, $\frac{1}{2}$, and $\frac{5}{10}$ represent the same quantity, i.e., are equivalent fractions. Decimal numbers are another way of writing fractions.
- Students need to be able to use whole number addition and subtraction strategies to solve decimal problems, with the support of empty number lines.

Key Mathematical Knowledge

Students need to:

- be able to order decimals to two places
- know how many tenths and hundredths there are in decimals to two places
- be able multiply and divide by ten
- be able to read a decimal in three ways, e.g., 0.75 as “ zero point seven five”, “seven-tenths and five-hundredths”, and “seventy-five hundredths”
- be able to round decimals with up to two decimal places to the nearest whole number.

Diagnostic Snapshot

Check that the students have the required decimal number knowledge, identified on page 35, by asking the following questions:

- “Read the following decimal fractions: 0.325, 0.8, 0.49.”
- “What is the decimal for eight-tenths?”
- “If I added one-tenth to 1.9, what would the answer be?”
- “What is 0.21 x 10?”
- “How many tenths in all of the number 4.2?”
- “How many hundredths in all of the number 0.36?”

For students who have very limited knowledge, it will be necessary to take time to teach the decimal knowledge required for this stage as outlined on page 35. Students who have one or two gaps in their knowledge will need additional time to work through the initial material in this lesson.

For students who are confident in answering these questions, the initial stage of this lesson will act as a quick review and an opportunity to gain familiarity with the decimal pipes prior to using them to solve addition and subtraction problems.

Equipment: Decipipes, or metre rulers and adding machine tape

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Using Materials

Show the students a "one" pipe on the floor and say you want them to consider this as "one" for today's lesson. Hold the pipe up to a whiteboard and use it to make a number line between zero and one.

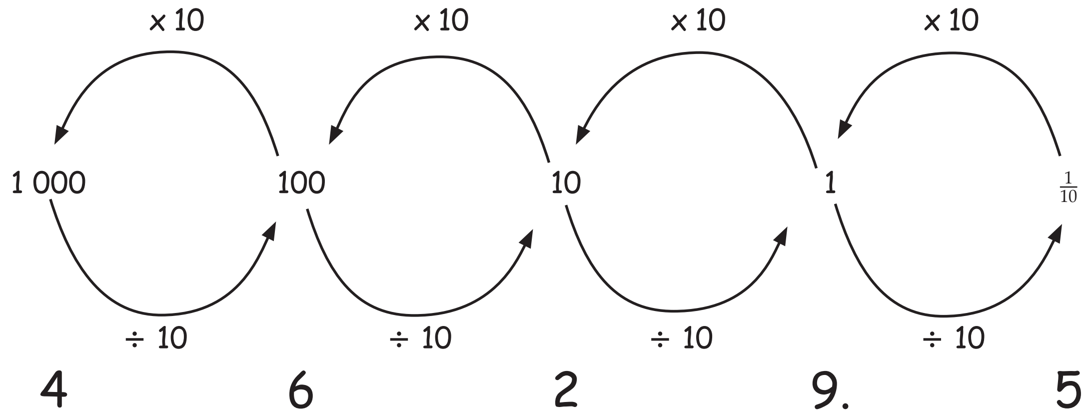
Write the numbers 3 and $^{-}2$ on the board and ask the students where three and negative two would be on this number line. This is to help students to recognise how this number line fits with their existing view of whole numbers and integers and to consolidate the acceptance of the pipe as "one", the referent whole. Students should recognise that three (3) will be three pipe lengths to the right of zero.

*The students achieving success proceed to **Using Imaging**. Otherwise, they proceed to the next activity at some later time.*

Lay the "one" pipe down and show the students a "one-tenth" pipe. Comparing the two pipes, ask them, "If this is 'one', how much do you think this is?" Let the students discuss what fraction they think the newest pipe is. For those who think it is one-tenth, ask them, "What is the smallest number of these pipes I would need to join to confirm that they are one-tenth pipes?"

Some students will say that five one-tenth pipes should be half of the length of the one pipe. This can be checked by threading five one-tenth pipes onto a piece of dowelling and aligning it to the one.

Ask the students, "What is the decimal for one-half?" Most will know this as 0.5, but many will not be able to explain what the five represents. Write a four-digit whole number on the board to develop the idea of decimal places to the right of the ones (for example, 4629.5). The next place to the right is found by dividing the previous place by 10 and a decimal point is used to mark the ones place.



So the place to the right of the ones is the tenths ($1 \div 10 = \frac{1}{10}$). Discuss the extension of further places to the right, hundredths, thousandths, etc. Note that the system is infinite.

Return to the pipe model. Ask a student to show you what one-quarter would look like if made with the pipes. Look for the students to realise that one-quarter cannot be made with a whole number of tenths. Often they will say that one-quarter is two and a half tenths, which is correct. Comment that the calculator does not show $0.2\frac{1}{2}$. Ask them what the calculator does show. Some will know that the decimal for one-quarter is 0.25.

Use this to extend the place value diagram to hundredths. Ask the students to show you with their forefinger and thumb how long one-hundredth will be.

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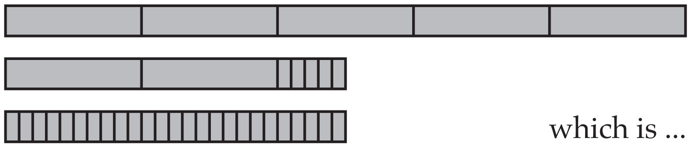
Produce the hundredths pipes and ask a student to show one-quarter as a decimal by threading pipe pieces onto a length of dowelling. This can be compared to the one-half already made.



5 tenths

2 tenths and 5 hundredths

which is ... 25 hundredths

Note that 0.25 is two-tenths and five-hundredths, which is also 25 hundredths. Ask the students what the decimals for $\frac{2}{4}$ (0.5), $\frac{3}{4}$ (0.75), $\frac{4}{4}$ (1.0), $\frac{5}{4}$ (1.25), and $\frac{6}{4}$ (1.5) would be, noting that equivalent fractions have the same decimal and that, like fractions, decimals can be greater or less than one.

Put the students into pairs and have each pair build one of the decimals below using pipes:

0.37 0.4 1.2 0.365 2.09

Each decimal brings out different considerations of the decimal system.

Write 0.37 and 0.4 on the board and ask the students which decimal they think is larger. Look out for whole-number thinking, as in, "0.37 is bigger because 37 is bigger than 4."

Also be aware that students may say 0.4 is bigger on the basis of the relative size of tenths and hundredths without attending to the place value. They may say, "0.4 is bigger because it has 4 tenths and 0.37 only has 3, and tenths are way bigger than hundredths."

Align the pipe models of these numbers and ask the students how much needs to be added to 0.37 to make 0.4 (three-hundredths). The students should recognise that ten-hundredths make one-tenth.

Making 1.2 and 2.09 will further consolidate the place of one because the students will need to access ones pipes to model these numbers. It is interesting to ask the students where these decimals lie in the original 0–3 number line. Focus on potential problems with 2.09 because students may use nine-tenths instead of nine-hundredths and highlight the significance of zero in holding the tenths place.

0.365 shows the need for thousandths, and the students should be asked to show how large they think one-thousandth might be by indicating with a gap between thumb and forefinger.

Using Imaging

*The students achieving success proceed to **Using Number Properties**. Otherwise, they proceed to the next activity at some later time.*

Pose addition and subtraction problems that can be modelled with the pipes. Fold back to using the pipes where students need support. Otherwise encourage them to image what the models might look like. Problem examples might be:

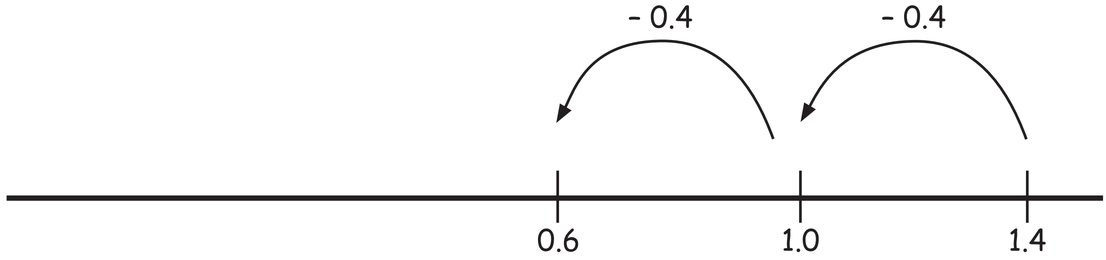
0.4 + 0.13 = 0.53
Note that whole-number thinking students might get 0.17 rather than adding tenths then hundredths.

1.6 + 2.7 = 4.3
Note that whole-number thinkers might get 3.13 rather than recognising six-tenths and seven-tenths gives 13 tenths, which is one-and-three-tenths.

0.56 – 0.3 = 0.26
Note that whole-number thinkers might get 0.53.

1.4 – 0.8 = 0.6
Look for application of whole-number strategies across the decimal point, such as reversibility, e.g., $0.8 + \square = 1.4$, $\square = 0.6$, or decomposition, e.g., 1.4 is 14 tenths, 14 tenths less 8 tenths is 6 tenths.

Capture students' responses on empty number lines. For example:
1.4 – 0.8 as 1.4 – 0.4 = 1.0, and then 1.0 – 0.4 = 0.6.



Using Number Properties

Pose more complex addition and subtraction problems with decimals. Look for the students to apply the same mental strategies to decimals that they use for whole numbers. Encourage the students to use empty number lines to describe their mental strategies. The examples below suggest possible strategies:

2.7 + 3.09 = 5.79
Using place value: 2 + 3 = 5, 5 + 0.7 = 5.7, 5.7 + 0.09 = 5.79

4.6 + 1.95 = 6.55
Using tidy numbers: 4.6 + 2 = 6.6, 6.6 – 0.05 = 6.55

4.18 – 2.9 = 1.28
Using tidy numbers: 4.18 – 3 = 1.18, 1.18 + 0.1 = 1.28

7.36 – 6.8 = 0.56
Using reversing: 6.8 + 0.2 = 7.0, 7.0 + 0.36 = 7.36, 0.2 + 0.36 = 0.56

0.675 – 0.49 = 0.185
Using tidy numbers: 0.675 – 0.5 = 0.175, 0.175 + 0.01 = 0.185

Decimats

I am learning to identify equivalent fractions and to name fractions as decimals.

Key Mathematical Ideas

- The successive folding of the decimats to create equivalent fractions encourages the students to recognise the multiplicative relationships between fractions.
- Using folding tasks helps students to see that while the one whole has been preserved, repeated folding changes the divisions, resulting in a new fraction name for the same fraction.
- Common fractions such as $\frac{1}{2}$, $\frac{3}{4}$, and $\frac{2}{3}$ are initially converted to decimals using the decimats.
- Students will be encouraged to work between and within fractions when renaming equivalent fractions because this will help with understanding ratio conversions in the next stage.

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Key Mathematical Knowledge

Check that the students:

- can identify symbols for any fraction including improper fractions
- read decimals to three places
- have quick recall of the times tables.

Diagnostic Snapshot

Ask the student questions such as:

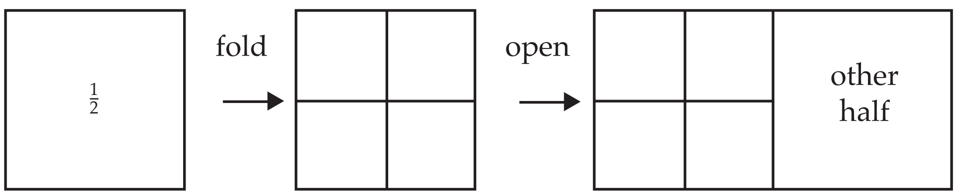
- "Convert the following fractions to decimals: $\frac{7}{5}$ and $\frac{5}{4}$"
- "Name an equivalent fraction for the following fractions: $\frac{5}{6}$, $\frac{2}{3}$, and $\frac{54}{81}$"
- Students who are able to solve these problems can start with the independent activity on page 44 and related activities in *Book 8: Teaching Number Sense and Algebraic Thinking.*

Equipment: Paper copies of decimats (see Material Master 7–3) are recommended, scissors, calculators

Using Materials

*The students achieving success proceed to **Using Imaging**. Otherwise, they proceed to the next activity at some later time.*

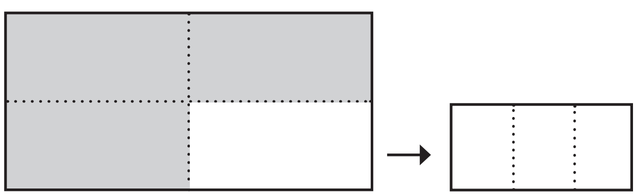
Give the students a copy of the whole decimat with no divisions. Tell them that they have been given one decimat each. Ask them to draw a line that cuts their mat in half and to cut along the line. Tell them that you now want them to fold the one-half into quarters.



Ask them what fraction of one these new pieces will be (one-eighth). Their answers can be verified by going back to one decimat.

Ask, "How many eighths make one half?" (four). Record this as $\frac{4}{8} = \frac{1}{2}$.

Carry out similar actions on other fractions. For example:



Shade $\frac{3}{4}$

Fold the piece of paper in half, then in half again, and then fold it in thirds.

Open the paper up. What fraction of the whole piece is shaded?

Record this as $\frac{9}{12} = \frac{3}{4}$.

Provide students with the whole decimat with no divisions. (For students who require further practice with equivalent fractions, see *Book 8: Teaching Number Sense and Algebraic Thinking*, page 16, Equivalent Fractions, and *Book 4: Teaching Number Knowledge*, page 30, The Same but Different.)

Ask the students to find out how many sixths are the same as two-thirds. Record this as $\frac{4}{6} = \frac{2}{3}$.

Look for the students to establish relationships between the numbers in the equations and relate them back to the decimat model.

For example, "If you cut quarters into three equal parts you get twelfths. So three-quarters will be nine-twelfths, three times as many."

Remind the students that the decimal system only allows tenths, hundredths, thousandths, etc. to be used in renaming fractions as decimals.

... thousands hundreds tens ones.tenths hundredths thousandths ...

Provide them with a new decimat and ask them to use the marks to draw on the mat lines that cut it into tenths.

Ask them, "If each tenth were cut into 10 equal parts, how many parts would that make?" (100) "What would each part be called?" (one-hundredth) Get the students to use the marks on the decimat to draw the hundredth partitions.

Tell the students to cut one-hundredth of their decimat into ten equal pieces. Ask them: "How many of these tenths of one-hundredth fit into the whole (one)?" (1000) "What will we call these parts?" (thousandths)

Challenge them to work co-operatively to find out how many tenths, hundredths, and thousandths are equivalent to the following fractions: $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{8}$, $\frac{1}{10}$.

Record their findings in a chart to highlight the patterns, for example:

fraction	tenths	hundredths	thousandths
$\frac{1}{2}$	5		
	5	0	
	5	0	0

Record the equivalence of fractions as equations, e.g., $\frac{1}{2} = \frac{5}{10} = \frac{50}{100} = \frac{500}{1000}$.

Check the students' predictions using a calculator, e.g., $\frac{1}{8} = 1 \div 8 = 0.125$. Match the pieces of decimat to the digits in each decimal.

Challenge the students to explain why it is not possible to divide the decimat into thirds using the lines. (One-third is not equivalent to an exact number of tenths, hundredths, or thousandths.)

Using Imaging

Cut out some non-unit fractions using photocopies of the decimat. For example, cut out $\frac{3}{5}$ (6 tenths). Turn the fraction piece over so that the lines are not visible.

Place it on top of a whole decimat. Label the paper piece with the fraction symbol.

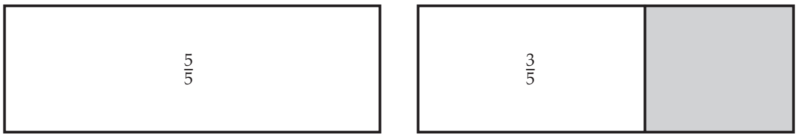
Challenge the students to tell you how many tenths, hundredths, and thousandths the fraction is equivalent to. Record their solutions, using both tables and equivalent fraction statements.

*The students achieving success proceed to **Using Number Properties**. Otherwise, they proceed to the next activity at some later time.*

fraction	tenths	hundredths	thousandths
$\frac{3}{5}$	6		
	6	0	
	6	0	0

Check the students' ideas on subdividing the decimat into hundredths, thousandths, etc.

Provide other similar fractions for the students to image, e.g., $\frac{3}{4}$, $\frac{7}{8}$, $\frac{9}{10}$, and $\frac{4}{5}$. Include fractions that have simple equivalents, such as $\frac{4}{8}$ and $\frac{8}{10}$, and fractions greater than one, such as $\frac{5}{4}$ and $\frac{8}{5}$, so that the students generalise the concept. For example: $\frac{8}{5}$ (eight-fifths) would be:



one whole and three-fifths

Record an equivalence statement and ask the students if they notice any pattern in the fractions. For example: $\frac{3}{4} = \frac{75}{100} = \frac{750}{1000}$

Students should notice multiplicative relationships across the equations such as, "To get 75 hundredths, you multiply both numbers by 25." Ask them why both numbers are multiplied by the same number. Refer back to the materials if necessary with questions, such as "How many parts is each quarter cut into to form hundredths?" (25).

Using Number Properties

Give the students problems that require them to rename fractions as equivalent forms or decimals. For example:

$\frac{3}{4} = \frac{\square}{8}$ $\quad \frac{7}{5} = \frac{\square}{100}$ $\quad \frac{2}{3} = \frac{\square}{9}$ $\quad \frac{12}{8} = \frac{\square}{2}$ $\quad \frac{125}{1000} = \frac{\square}{8}$ $\quad \frac{375}{1000} = \frac{\square}{8}$ $\quad \frac{5}{6} = \frac{15}{\square}$

$\frac{7}{5} = \square . \square$ $\quad 1.75 = \frac{\square}{\square}$ $\quad \frac{\square}{1000} = \frac{5}{8}$ $\quad \frac{\square}{8} = 2.25$ $\quad \frac{6}{4} = \square . \square$

Look for the students to be able to work between and within the fractions. For example:

$\times 3 \quad \frac{2}{6} = \frac{?}{12} \quad \times 3$ (within fractions) $\quad \overset{\times 2}{\frac{2}{6} = \frac{?}{12}}$ (between strategies) $\times 2$

Independent Activity

Create Three

Playing the game of Create Three (see Material Master 7–9) will reinforce students' knowledge of simple equivalent fractions.

How Can Two Decimals So Ugly Make One So Beautiful?

I am learning to add decimals.

Key Mathematical Ideas

- A common student misconception is that both parts of a decimal (before and after the decimal point) are whole numbers. This leads to difficulties with ordering and operating on whole numbers. For example, a student may say that 0.8 is smaller than 0.75 because 75 is bigger than 8. For 3.4 + 1.8, this student may arrive at the answer of 4.12.
- Addition and subtraction with decimals is based on a sound understanding of place value. Students have to understand the value of the unit that they are working with, e.g., 0.4 having the value of four-tenths and when 0.7 (seven-tenths) is added, it will require the renaming of ten-tenths to one whole, with one-tenth remaining.
- Underpinning decimal fractions is the concept of equivalent fractions, e.g., $\frac{3}{7}$ and $\frac{6}{14}$ are the same number. Decimal fractions are special cases of equivalent fractions, with denominators that are powers of ten (ten, hundred, thousand, etc.). For example, 0.6 is 6 tenths or 60 hundredths or 600 thousandths, etc.
- The canon of place value is that one unit can be split into ten units of the next place to the right. For example, one hundred can be split into ten tens, one-hundredth can be split into ten thousandths.

Key Mathematical Knowledge

- Check that the students know tenths and hundredths in decimals to two places, e.g., tenths in 7.2 is 72.
- Students need to be able to round decimals with up to two decimal places to the nearest whole number, e.g., rounding 6.49 to 6, 19.91 to 20.

Diagnostic Snapshot

Ask the students questions such as:

" 2.45 + 3.65 = $\square$ 0.111 + 1.889 = $\square$ "

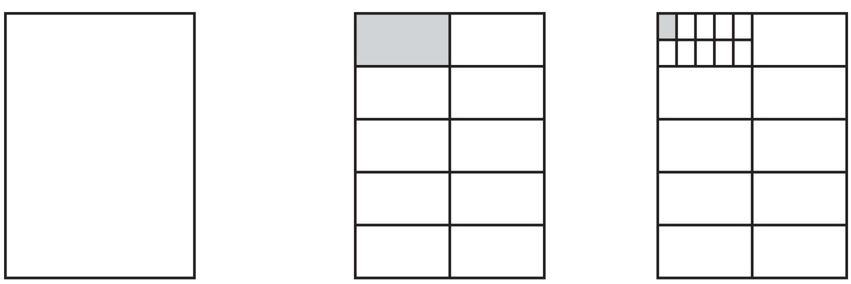
Students who are able to solve these problems need to work on the Using Number Properties section and be extended by exploring activities in *Book 8: Teaching Number Sense and Algebraic Thinking.*

Equipment: Decimats (Material Master 7–3), scissors, calculators

Using Materials

Introduce the students to the conventions of using decimats. You may like to use a context of chocolate blocks that will only break into a certain number of parts through dividing into ten equal parts, as follows:

The students achieving success proceed to ***Using Imaging****. Otherwise, they proceed to the next activity at some later time.*



One (whole) tenths ($\frac{1}{10}$ or 0.1) hundredths ($\frac{1}{100}$ or 0.01)

Extend the system so that the students consider the size of one-tenth of a hundredth (one-thousandth) and one-tenth of a thousandth (one ten-thousandth).

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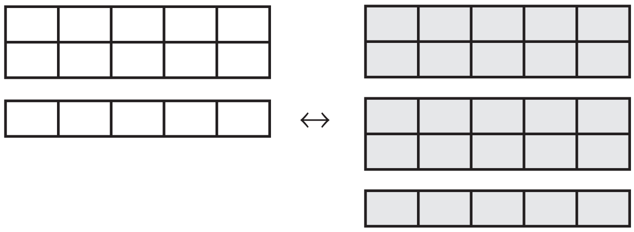
Problem: "Lisa has 1.5 bars of cooking chocolate; Leon has 2.5 bars. How many bars do they have altogether?"

Get the students to perform the calculation, 1.5 + 2.5, on a calculator. Challenge them to explain why the answer is 4 and what has become of the decimal point. Use decimats to model the calculation. The decimat divided into tenths is ideal for this:



Both 0.5's join to make another one. This gives four ones, which makes the decimal point redundant because there are no parts of one in the answer.

Provide similar examples where longer decimals added together produce shorter decimals as answers. Get the students to model the operations with decimals. Possible problems, in suitable story contexts, are:

2.3 + 2.7 = 5   1.25 + 1.75 = 3   0.26 + 1.74 = 2   0.375 + 0.125 = 0.5

The students' knowledge of fraction to decimal conversions will help in explaining the answers, for example: $1.25 = 1\frac{1}{4}$ and $1.75 = 1\frac{3}{4}$; $1\frac{1}{4} + 1\frac{3}{4} = 3$.

Using Imaging

*The students achieving success proceed to **Using Number Properties**. Otherwise, they proceed to the next activity at some later time.*

Provide addition and subtraction problems that require the students to think about how decimal ones, tenths, hundredths, etc. are partitioned and/or recombined. Fold back to using decimats when the students have difficulty either calculating or explaining the answer. Suitable problems, in story context, might be:

7.1 + 2.9 = 10   6.45 + 1.55 = 8   0.35 + 2.15 = 2.5   1.555 + 0.445 = 2

Number Properties

The students can demonstrate their understanding of the number properties involved by calculating and explaining the following problems:

2.31 + 1.99 = 4.3   0.348 + 1.652 = 2   1.0001 + 9.9999 = 11   0.3276 + 0.5724 = 0.9

0.333 + 0.333 + 0.001 = 1

The problems can be extended to subtraction where a decimal "so beautiful" is made "ugly" by subtraction of another decimal. This may involve the students in breaking up decimal place values, e.g., breaking five ones into 4 ones and 10 tenths. Some students might recognise that subtraction can be seen as difference. For example, an easy way to solve 2 – 1.97 is to see the answer as the difference between two and one point nine seven. Both numbers can be made with decimats of different colours and 1.97 overlaid on top of 2.

Suitable problems are:

1 – 0.02 = 0.98   2.5 – 0.9 = 1.6   8 – 3.333 = 4.667   100 – 0.009 = 99.991

Hot Shots

I am learning to work out percentages of amounts.

Key Mathematical Ideas

- The percentage sign means "out of 100". It comes from the "out of" symbol (/) and the two zeros from 100.
- Percent is another name for hundredths, so by converting a fraction to a fraction involving hundredths, it can be read, for example, as $\frac{45}{100}$ or 45%.
- Double number lines are used to show the mapping from a fraction to a percentage.

Key Mathematical Knowledge

Check that the students:

- are familiar with the conventions of recording percentages (%)
- have quick recall of the times tables
- can identify the factors of numbers to 100, e.g., factors of 36 = (1, 2, 3, 4, 6, 9, 12, 18, 36)
- know equivalent fractions for halves, quarters, thirds, fifths, and tenths with denominators up to 100, 1000, e.g., one in four is equivalent to 25 in 100 or 250 in 1000.

Diagnostic Snapshot

Ask the students questions such as:

"In a game of netball, Tracey gets in 37 out of her 50 shots. Katherine takes 24 shots and gets in 18. Who is the better shot?"

Students who can solve questions such as these need to work on the independent activity on page 49 and be extended by exploring activities in *Book 8: Teaching Number Sense and Algebraic Thinking*.

Equipment: A set of percentage strips (see Material Master 7–4), calculators, paper clips

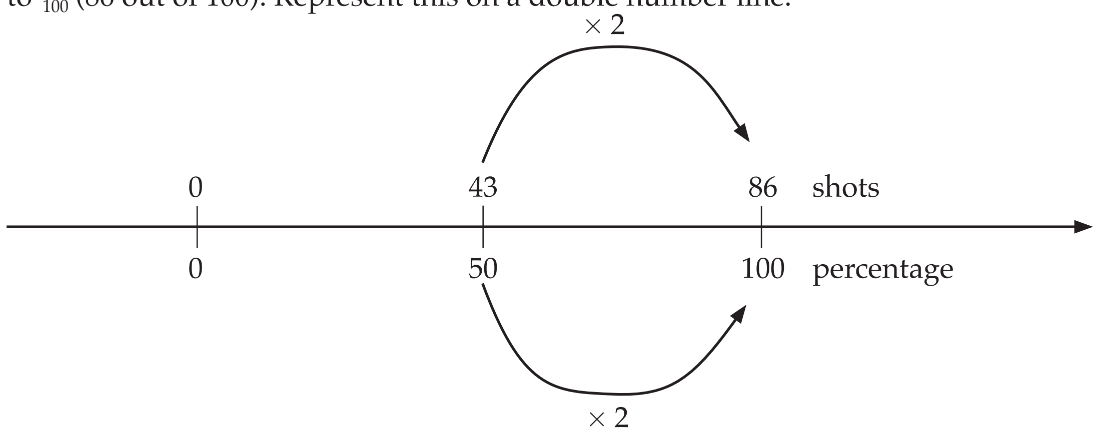
Using Materials

Discuss a situation where the students have encountered percentages in their daily life. They will often suggest sports (for example, shooting for goal), shopping (such as discounts or GST), and, in country areas, calving or lambing percentages. Tell the students that the % sign comes from the "out of" symbol, /, and the two zeros from 100. It means, "out of 100".

The students achieving success proceed to ***Using Imaging****. Otherwise, they proceed to the next activity at some later time.*

Problem: "In a game of netball, Irene gets in 43 out of her 50 shots. Sharelle takes 20 shots and gets in 17. Who is the better shot?"

Tell the students that percentages are used to compare fractions. In Irene's case, the fraction is $\frac{43}{50}$. Doubling 43 calculates the shooting percentage because $\frac{43}{50}$ is equivalent to $\frac{86}{100}$ (86 out of 100). Represent this on a double number line.



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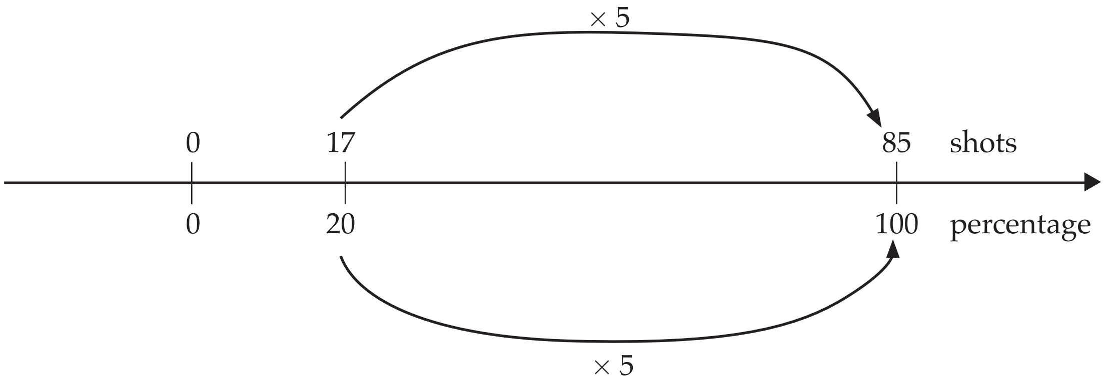
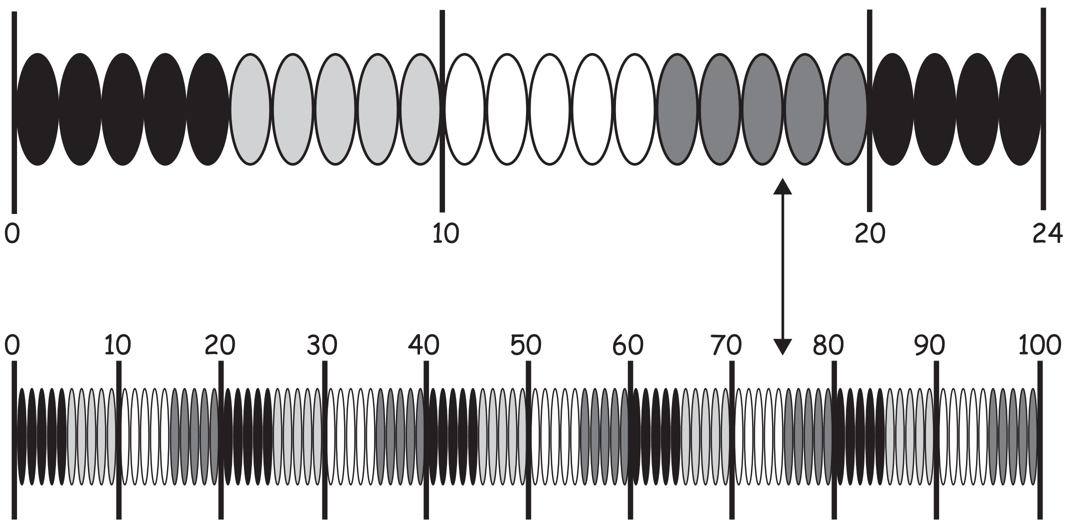
Ask the students to work out what Sharelle's shooting percentage was for the same game. Represent this on a double number line to show that finding a percentage is like mapping a proportion onto a base of 100.



Pose the students a percentage problem that can be modelled with the percentage strips. For example: "Tony got in 18 out of his 24 shots. What percentage did he shoot?"

Mapping 18 out of 24 onto a base of 100 gives 75%.



Pose similar problems that the students can solve by aligning differently based strips with the 100-base strip. Examples might be:

16 out of 32 (50%)	9 out of 36 (25%)	10 out of 25 (40%)
12 out of 16 (75%)	12 out of 40 (30%)	4 out of 20 (20%)

Using Imaging

*The students achieving success proceed to **Using Number Properties**. Otherwise, they proceed to the next activity at some later time.*

Show the students the base strip, but have the percentage strip aligned to it and turned over so they can't see the beads. Give the students "out of" problems and have them estimate the percentage by visualising.

For example, pose six out of 16. Mark six with a paper clip. The students should estimate the percentage as just below 40% or greater than 33.3% (one-third). A calculator can be used to work out the exact percentage by keying in 6 ÷ 16%. The percentage strip can then be turned over to check the estimate. Ask how else they could have estimated the percentage if there had been no strips.

Look for ideas like "Six out of 16 is the same as three out of eight, and that is half of three out of four" or "There are over six sixteens in 100. Six times six is 36, so it will be more than 36 percent."

Pose similar imaging problems like:

8 out of 20 (40%)	15 out of 25 (60%)	4 out of 16 (25%)
32 out of 40 (80%)	20 out of 32 (62.5%)	14 out of 36 (39%)

Focus on strategies based on the numbers involved that could have been used to estimate the percentages.

The students can play the game of Percents (see Material Master 7–5) to consolidate their visualisation.

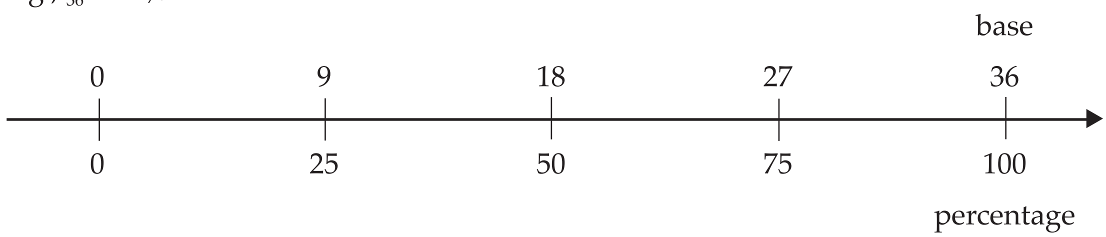
Using Number Properties

Give the students percentage problems to solve. Pose these problems in contexts of sports scores, shopping discounts or mark-ups, or lambing percentages. Pose some problems where duplication of the base onto 100 is not easy. For example, 25 is easily mapped onto 100 through multiplying by four, whereas 40 is not so easily mapped (although students should be encouraged to recognise that $2.5 \times 40 = 100$). Examples that involve percentages greater than 100 should also be used.

Some examples might be:

$\frac{18}{24} = \square\%$? (75%)	$\frac{25}{40} = \square\%$? (62.5%)	$\frac{18}{27} = \square\%$? ($66.\dot{6}\%$)
$\frac{8}{32} = \square\%$? (25%)	$\frac{24}{16} = \square\%$? (150%)	$\frac{55}{20} = \square\%$? (275%)

Get the students to record their thinking using double number lines or ratio tables, e.g., $\frac{27}{36} = 75\%$.



base	36	18	9	27
percentage	100	50	25	75

Independent Activity

Use brochures from local retailers. Tell the students that one shop has a "25% off" sale, another has a "40% off" sale, and a third has a "one-third off" sale. Give the students an arbitrary budget to spend at the three shops.

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Mixing Colours

I am learning to compare ratios and proportions.

Key Mathematical Ideas

- A ratio is a comparison of two quantities.
- Ratios can be written as proportions, e.g., 2:5 is equivalent to $\frac{2}{7}$ or $\frac{5}{7}$, depending on which number is being focused on.
- The verbal interpretation of these symbols (*a*:*b*) can be "**for every *a* there are *b***" or "**for every *b* there are *a***", e.g., for 1:3, for every one part of blue paint there are three yellow parts.
- Students can use a variety of strategies to solve comparison of ratio problems e.g., equivalent fractions, building up the ratio, and mapping (these are elaborated on in the Using Materials part of the lesson as points (i), (ii), and (iii)).

Key Mathematical Knowledge

Check that the students:

- can order fractions
- recognise equivalent fractions.

Diagnostic Snapshot

Ask the students questions such as:

"Which combination makes the strongest flavoured orange drink (orange concentrate to water)?

2:5 or 3:4 7:15 or 3:6"

Students who are able to solve these problems need to work on the Using Number Properties section and be extended by exploring activities in *Book 8: Teaching Number Sense and Algebraic Thinking.*

Equipment: Unilink cubes, plastic beans or similar, acrylic paint, rotating regions (see Material Master 7–6)

*The students achieving success proceed to **Using Imaging**. Otherwise, they proceed to the next activity at some later time.*

Using Materials

Ask the students if they know about the primary colours, red, yellow, and blue. Tell them that mixing the primary colours can create other colours. This can be demonstrated by mixing yellow and blue acrylic paint to get green.

Problem: "Picasso has three recipes for mixing green paint. They are:

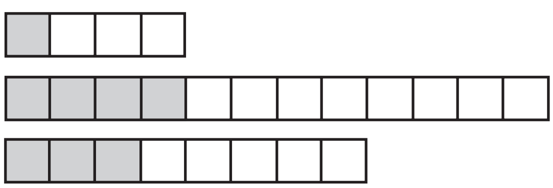
A: One part of blue with three parts of yellow (1:3)

B: Four parts of blue with eight parts of yellow (4:8)

C: Three parts of blue with five parts of yellow (3:5)

Which recipe will give him the darkest shade of green?" (It is the relative quantity of blue to yellow that makes the green darker.)

Get the students to build unilink cube or beans models of the three recipes:



Ask the students how the different ratios could be compared. Their responses might include:

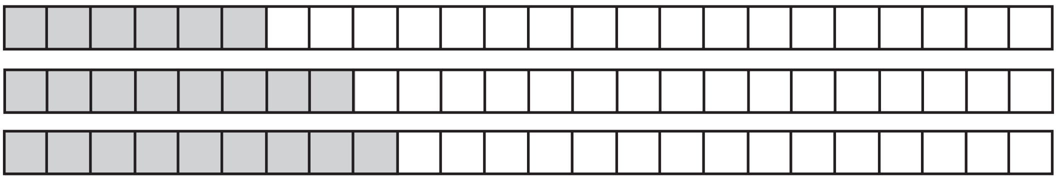
(i) Relating the recipes to equivalent fractions, e.g., 1:3 is the same as $\frac{1}{4}$, 4:8 is the same as $\frac{4}{12} = \frac{1}{3}$, 3:5 is the same as $\frac{3}{8}$.

(ii) Equalising the lengths of the cube models by building up or breaking down, for example, 1:3 makes up to 6:18 (six times), which is a smaller ratio than 4:8, which makes up to 8:16 (two times). Both stacks have 24 cubes.

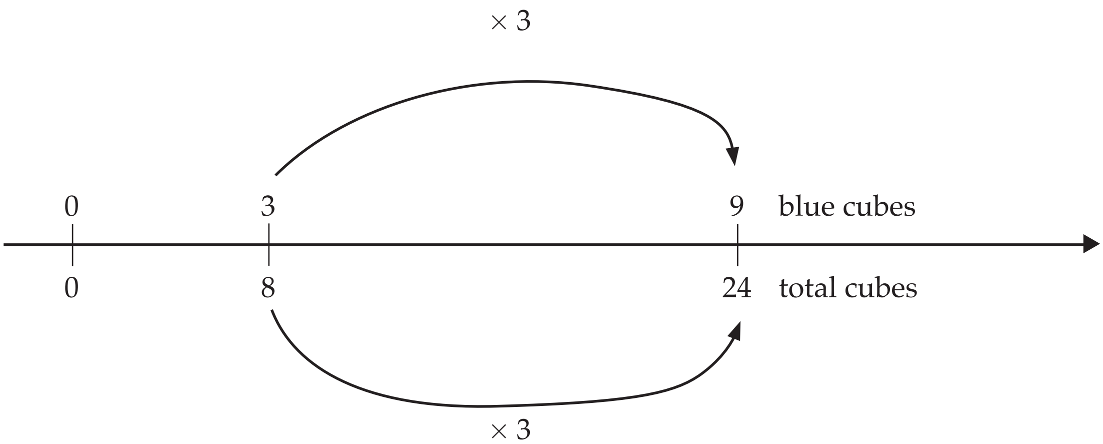
(iii) Mapping one of blue onto so many yellows, for example, 3:5 means that one blue is mapped onto one and two-thirds yellows ($1:1\frac{2}{3}$), 4:8 means that one blue is mapped onto two yellows (1:2).

Considering method (ii), the students might make duplicates of the recipes until the lengths are equalised. The yellows and blues are then collected so that the ratios can be compared.

A.
B.
C.



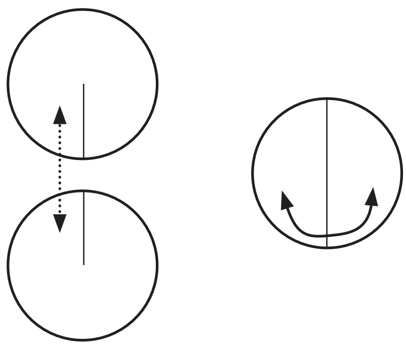
This is easily drawn on a double number line:



The students achieving success proceed to ***Using Number Properties****. Otherwise, they proceed to the next activity at some later time.*

Challenge the students to represent each ratio using the rotating region.

This provides an excellent link to percentages because each circle on the material master has one hundred divisions around its circumference.



These circles can be rotated to show any two fractions that add to one.

Pose similar problems for the students to explore using cubes or beans.

Recipes for orange:

X: 3 parts red to 7 parts yellow (3:7 or $\frac{3}{10}$)

Y: 5 parts red to 15 parts yellow (5:15 or $\frac{5}{20}$)

Z: 1 part red to 4 parts yellow (1:4 or $\frac{1}{5}$)

"Which recipe gives the darkest orange?" (It is the relative quantity of red to yellow that makes the colour darker.)

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The students achieving success proceed to ***Using Number Properties****. Otherwise, they proceed to the next activity at some later time.*

Using Imaging

Shielding: Pose problems that make comparison easy. For each recipe, make a stack with cubes. Trace around each length of cubes and mark and shade the colour break. Ensure that the ratios are recorded so that the students can refer to them.

Problem: Recipe for purple:

P: 6 parts red with 14 parts blue

Q: 4 parts red with 6 parts blue

R: 7 parts red with 8 parts blue

"Which recipe gives the darkest purple?" (It is the relative quantity of blue to red that makes the colour darker.)

Using Number Properties

Pose problems of ratio comparison using only numbers. Record these problems using symbols:

"Which blue-to-yellow ratio gives a darker shade of green?

2:5 or 3:4 7:15 or 3:6 1:4 or 18:80 4:7 or 9:18 7:8 or 8:9"

(Note: Once again, it is the quantity of blue that determines the darkness of the colour.)

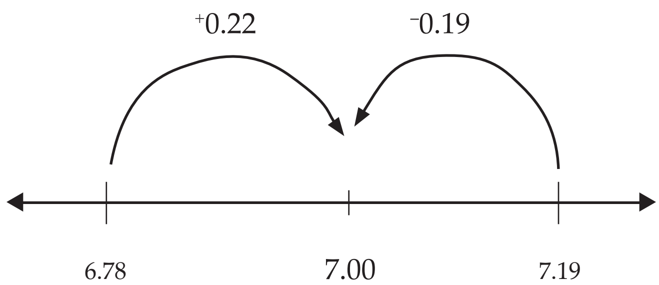
The students will need to consider these ratios as proportions, for example, 2:5 is equivalent to $\frac{2}{7}$ or $\frac{5}{7}$, depending on the colour concerned.

Learning Experiences to Move Students from Advanced Multiplicative–Early Proportional to Advanced Proportional Part-Whole

Required Knowledge

Before attempting to develop their ideas about fractions, decimals, and percentages, check that Advanced Multiplicative–Early Proportional students have the following knowledge.

Key Knowledge	Tasks for Key Knowledge
• Say the forwards and backwards number sequence in tenths, hundredths, and thousandths • Say the number one-thousandth, one-hundredth, one-tenth, one, ten, etc, before and after any given whole number.	Use two number flip charts. Re-label the places of one chart as tenths, hundredths, and thousandths. Use a sticky dot to create the decimal point. Make numbers and ask the students to say them. For example, 274.193 as "two hundreds, seven tens, four ones, one-tenth, nine-hundredths, and three-thousandths". Ask them to anticipate the number if one place is increased or decreased by one, for example, 23.964 plus one-tenth. Key decimal numbers into a calculator and ask the students to zap or change digits in the number using subtraction. For example, change 7.23 to 7.03. Note that this operation can be modelled with decimal compact numeral cards. When 7.23 is made, the 2 is a card showing 0.2.
• Order decimals to three places at least	Use measurement contexts to obtain sets of decimals. For example, measure the students' heights in metres, such as 1.27, 1.6. Get the students to put the decimals onto a number line to show the correct order. Compare the order given with the students measured.
• Order fractions including halves, thirds, quarters, fifths, and tenths at least	Choose sets of fractions that enable the students to compare their size with reference to benchmark numbers and equivalent fractions, for example, $\frac{3}{5}$, $\frac{1}{3}$, $\frac{3}{4}$, where $\frac{1}{2}$ is a key benchmark. Get the students to predict the order and then confirm it by forming each fraction with circle kits or lengths kits.
• Round whole numbers and decimals to the nearest whole number or tenth • Know equivalent fractions for halves, thirds, quarters, fifths, and tenths with denominators up to 100, 1000, e.g., 1 in 4 is equivalent to 25 in 100 or 250 in 1000.	Play the game of Target with a 1–6 dice. Students take turns to roll the dice and nominate whether the number represent ones, tenths, or hundredths. So if a 4 is thrown, it could be worth 4, 0.4, or 0.04. This is added to the player's total. Each player has eight throws to get as close to 7 as possible. They can go over 7. They keep their own running total as they go. At the end, the winner is the player with a total closest to 7, for example, scores of 6.78 and 7.19. *(see over page)*

Key Knowledge	Tasks for Key Knowledge
	(continued)  From the number line above, it can be seen that 7.19 is 0.19 away from 7.00 and is closer than 6.78, which is 0.22 away. Use distance on the number line as a model for developing rounding.
• Convert between fractions, decimals, and percentages with halves, quarters, fifths, and tenths at least	Use double number lines to support the students in converting between fractions, decimals, and percentages. For example, to rename $\frac{3}{4}$ (three-quarters): 
• Know the divisibility rules for 2, 3, 5, 9, and 10, e.g., 471 is divisible by 3 because 4 + 7 + 1 = 12 (see Nines and Threes, pages 70–72, *Book 6: Teaching Multiplication and Division, Revised draft 2007*)	Identify the factors of numbers to 100, e.g., factors of 36 = (1, 2, 3, 4, 6, 9, 12, 18, 36)

Knowledge to be Developed

- Recalls fraction ↔ decimal ↔ percentage conversions for given fractions and decimals
- Knows the number of tenths, hundredths, that are in numbers of up to three decimal places e.g., there are 456 tenths in 45.6
- Knows what happen when a whole number or decimal is multiplied or divided by a power of 10, e.g., 4.5 x 100 = 450
- Rounds decimals to the nearest hundred, ten, one, one-tenth, or one-hundredth.
- Says the number one-thousandth, one-hundredth, one-tenth, one, ten, etc. before and after any decimal number
- Orders fractions, decimals, and percentages.

Key Ideas

Students at the Advanced Multiplicative–Early Proportional stage are expected to understand the concept of equivalence as it relates to fractions used either as numbers or operators. They should understand that any fractional number has an infinite number of equivalent names that all represent the same quantity, e.g., $\frac{4}{7}$ is the same quantity as $\frac{12}{21}$. Advanced Multiplicative–Early Proportional students should also understand that fractions can operate on other numbers in a multiplicative way, e.g., $\frac{4}{7}$ of 63 means four-sevenths of sixty-three ($\frac{4}{7} \times 63$). They should also understand that equivalent fractions used as operators on the same number give the same result, e.g., $\frac{4}{7}$ x 63 is the same as $\frac{12}{21} \times 63$.

To become generalised proportional thinkers (levels 5 and 6 of the mathematics and statistics learning area of *The New Zealand Curriculum*), students must apply fractional numbers to a wide range of contexts as diverse as finding best deals, enlarging and reducing, establishing probabilities, equal sharing, and applying transmission factors to pulleys and gear trains.

In doing so, students must connect at least six constructs: part-whole, measurement, quotient, operator, rates and ratios, and probability (see Introduction, page 4). Advanced Proportional students are able to apply equivalence across all of these constructs. For example, they are able to understand that only common units can be added, so to add $\frac{3}{4}$ and $\frac{2}{5}$, the fractions must be converted to equivalent forms with the same common denominator: $\frac{3}{4} + \frac{2}{5} = \frac{15}{20} + \frac{8}{20} = \frac{23}{20}$. They are also able to use fractions as operators on other fractions, e.g., $\frac{2}{3}$ of $\frac{5}{8}$ ($\frac{2}{3} \times \frac{5}{8} = \frac{10}{24}$), and measure fractions with other fractions, e.g., "How many three-quarters measure two-thirds?" ($\frac{2}{3} \div \frac{3}{4} = \frac{8}{9}$).

In ratio and rate situations, Advanced Proportional students can co-ordinate pairs of numbers and calculate the multiplicative operators between the numbers to find missing values and make comparisons. For example, "If the bakery conveyor oven cooks 63 pies in 28 minutes, how long does it take to cook 27 pies?" The multiplicative operator between 63 and 28 is $\frac{28}{63} = \frac{4}{9}$, which is the unit rate of one pie cooked every $\frac{4}{9}$ minutes. So for 27 pies, $\frac{4}{9} \times 27 = 12$ minutes is the time required.

Advanced Proportional thinkers recognise that constant rates can be represented as ordered pairs on a number plane and that the unit rate, e.g., $63 \div 28 = \frac{9}{4}$ pies per minute, gives the slope of the line connecting the ordered pairs. They can distinguish between situations in which it is reasonable to apply a constant rate and those situations in which it is not, e.g., if person can sprint 100 metres in 14 seconds, this does not mean that 1000 metres will take 140 seconds.

Advanced Proportional thinkers can solve inverse rate problems by recognising that the product of measures is constant. For example, to balance a see-saw, an 80 kilogram person must be half the distance from the fulcram (balance point) as a 40 kilogram person is, i.e., $\frac{1}{2} \times 80 = 1 \times 40$.

Advanced Proportional students connect their understanding of fractions, decimals, and percentages and see these numbers as equivalent forms. For example, in solving $0.8 \times 0.15 = \square$, they can see this calculation as $\frac{8}{10} \times \frac{15}{100} = \frac{120}{1000} = \frac{12}{100} = 0.12$.

They transfer this understanding to operating with fractions and decimals. Students must modify the principles that work for whole numbers but do not always hold for decimals and fractions. For example, many students believe that multiplication makes bigger while division makes smaller. However, with fractions and decimals between zero and one, the opposite is the case, although for decimals greater than one, the whole number generalisations hold.

For example, $0.5 \times 8 = 4$ (smaller) and $8 \div 0.5 = 16$ (bigger), yet $1.4 \times 6 = 8.4$ (bigger) and $6.6 \div 1.1 = 6$ (smaller).

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Advanced Proportional students extend their multiplicative understanding of place value to decimals and have a sound understanding of multiplication and division. They recognise the importance of unit size.

For example, $\frac{1}{10} \times \frac{1}{10} = \frac{1}{100}$ $(0.1 \times 0.1 = 0.01)$, $100 \times \frac{1}{100} = 1$ $(100 \times 0.01 = 1)$ and $\frac{1}{10} \div \frac{1}{100} = 10$ (How many hundredths are in one-tenth?), $10 \div \frac{1}{10} = 100$ (How many tenths are in ten ones?)

Advanced Proportional thinkers can see percentages as specialised fractions and co-ordinate the part-whole and whole-to-whole relationships involved. For example, if Millie gets in 27 shots and that is 82 percent, each shot is worth about $81 \div 27 = 3\%$. So Millie takes $100 \div 3 = 33$ shots and 82% is a rounded figure. If a farmer gets 243 lambs from 138 ewes, that lambing percentage is $\frac{243}{138} = 176\%$ (co-ordinating whole with whole).

Extending Hotshots

*The students achieving success proceed to **Using Imaging**. Otherwise, they proceed to the next activity at some later time.*

I am learning to solve different kinds of percentage problems.

Begin with the lesson sequence described in Hotshots (page 47) under **Using Materials.**

Key Mathematical Ideas

- Percentage problems are in this form, $a\%$ of $b = c$, in which either a, b, or c is not known, e.g., 25% of 64 = ☐. The most difficult problems require re-construction of the whole, b, given a and c. For example, Joe got 24 questions right. His mark was 75%. How many questions were in the test?, i.e., 75% of ☐ = 24.
- The numbers make a big difference to the difficulty of percentage problems. Numbers that are easy to scale up to one hundred make problems relatively easy, e.g., 19 out of 25, $\frac{20}{25} = \frac{80}{100} = 80\%$. Problems where the numbers are untidy require students to think structurally about what each calculation step is doing, e.g., 31 out of 43 is ☐%, $100 \div 43 = 2.33$, which calculates what each question is worth as a percentage. So 31 out of 43 is $31 \times 2.33\% = 72\%$.
- Percentage problems can be seen as ratio problems involving the comparison of part to whole. Joe's test question above can be shown in a ratio table like this:

Correct Questions (Part)	Total Questions (Whole)
24	?
75	100

- Percentages can be greater than one hundred when two separate measurements are compared. For example, if a house cost \$140,000 in 2000 and \$420,000 in 2008, it is worth $\frac{420}{140} = 3$ times its previous price. So it is worth 300% of its previous price.

Diagnostic Snapshot

Students need access to a calculator to solve these problems:

1. Celia got in 83% of her netball shots. She took 36 shots. How many went in? (30)
2. Todd took 63 shots at goal during the rugby season. 46 shots went over. What was his shooting percentage? (73%)
3. Terina got 39 first serves in during her tennis match. That was about 55% of her first serves. How many first serves did she hit? (71)

Students who answer these questions correctly and can explain why they do each calculation step need to be further extended.

Using Materials

Begin with the lesson sequence described in Hotshots (page 47).

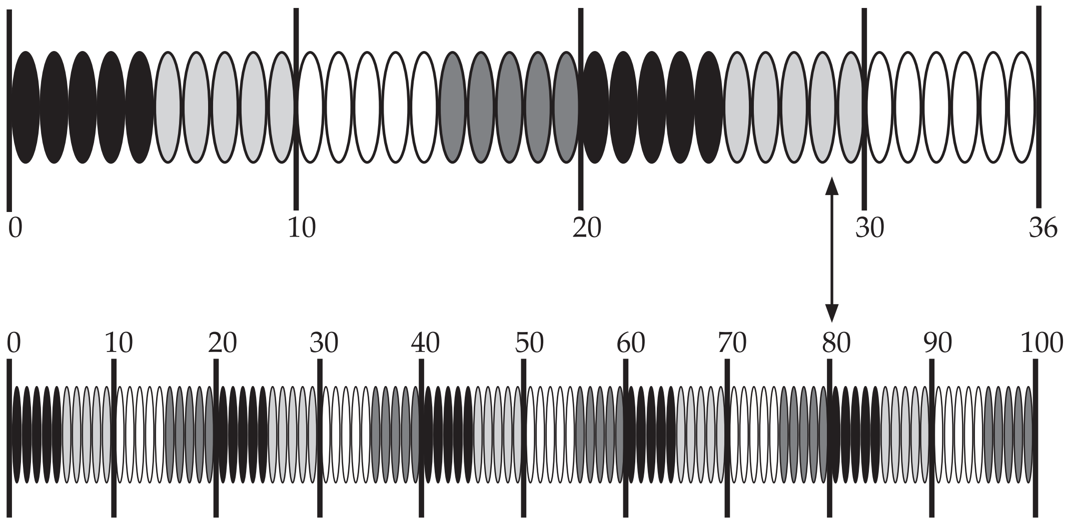
The aim of this lesson is for students to gain an algebraic understanding of what is occurring with the quantities in a percentage problem as they perform calculation steps. For this purpose, it is important to use problems in which scaling up to 100 or common factor strategies do not work easily. This lesson also extends the location of the unknown in such problems.

It is important that students have access to calculators for this lesson.

Pose three different types of problem using the percentage strips for support:

1. Percentage unknown

"Jordan takes 36 free throws and lands 29 of them. What is his shooting percentage?" ($\frac{29}{36} = \square\%$)



Encourage the students to estimate the percentage visually or by using their common factor strategies, e.g., "30 out of 36 is five-sixths …" or "three times 35 is a bit more than 100, so 29 out of 36 is a bit less than three times 29 is 87". Require the students to make qualitative adjustments to their estimates by asking questions such as "Is your estimate too high or too low? Why?"

Ask the students to represent the problem as an equation, i.e., $29 \div 36 = \square\%$. As students select calculation steps, which they will perform using a calculator, ask them to justify their calculations, referring to the strips as they do so. They will often select calculations that are not useful, e.g., "divide 100 by 29", and access to the strips helps their self-regulation. Where calculations are productive, question the students about what each step is doing:

Student: I divided 100 by 36.

Teacher: What did $100 \div 36 = 2.\dot{7}$ (recurring) tell you?

Student: The percentage each shot was worth

Or:

Student: I divided 29 by 36.

Teacher: What did that tell you?

Student: It told me the percentage as a decimal fraction (0.80555 …)

Other problems that might be modelled with the strips are:

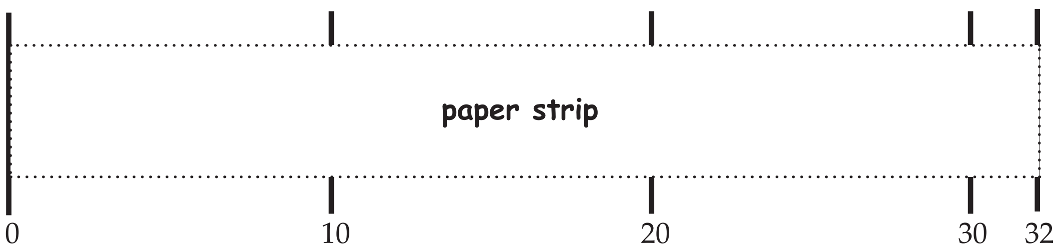
"Daniel kicked 16 shots at goals. He got 11 kicks over. What was his kicking percentage?"

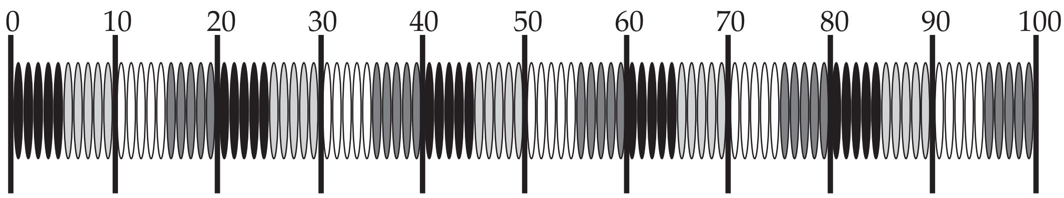
"Ronaldo took 32 spot kicks this season. He scored from 25 of those kicks. What was his penalty shooting percentage?"

2. Part unknown

"Cindy had to answer 32 questions for her drivers' licence. She scored 81%. How many questions did she get right?"

In modelling this problem, it is important that the 32 strip is aligned with the 100 strip but is turned over or masked with a paper strip so that it cannot be seen.





As with problem type 1, ask the students to estimate their answer first, represent the problem with an equation, and then perform calculation steps to find the answer.

Equation: $\square \div 32 \approx 81\%$

Student: I found 100 ÷ 32 = 3.125.

Teacher: What does that tell you?

Student: Each question is worth 3.125 percent.

Teacher: What calculation will you do then?

Student: 81 ÷ 3.125 = 25.92.

Or:

Student: 81% is 0.81, so I found $\square \div 32 = 0.81$. I did 0.81 x 32 = 25.92.

Teacher: So did Cindy sit 25.92 questions?

Student: She sat 26 questions. The percentage was rounded.

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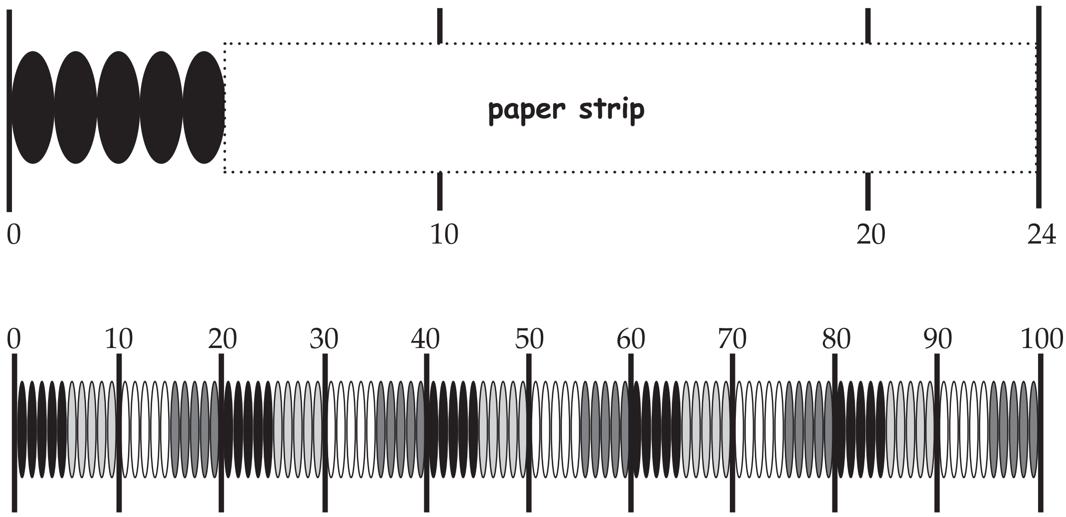
3. Whole unknown

"Zoe noticed that five of the lambs were black. That was about 21 percent of all the lambs in the paddock. How many lambs were in the paddock?"

In modelling the problem, the 24 and 100 strips need to be aligned, but the part of the 24 strip from 5 to the end is masked with a paper strip.



Equation: $5 \div \square \approx 21\%$

Student: 21 percent is about 20 percent or one-fifth. So I estimated that five is one-fifth of the total. So there must be about 25 lambs.

Teacher: Will the number of lambs be more or less than 25?

Note that whole unknown problems require students to have an algebraic understanding of equality, order of operations, and inverse. This is revealed through the calculation steps as actions on the equation:

$5 \div \square = 21 \div 100$	or	$21\% \times \square = 5$
$\rightarrow 5 \times 100 \div \square = 21$		$21 \div 100 \times \square = 5$
$\rightarrow 500 \div \square = 21$		$21 \times \square = 5 \times 100$
$\rightarrow 500 = 21 \times \square$		$21 \times \square = 500$
$\rightarrow 500 \div 21 = \square$		$\square = 500 \div 21$

Using Imaging

Provide the students with problems of all three types, percentage, part, and whole unknown. Use a double number line as the image of each problem and encourage the students to create their own equations and diagrams.

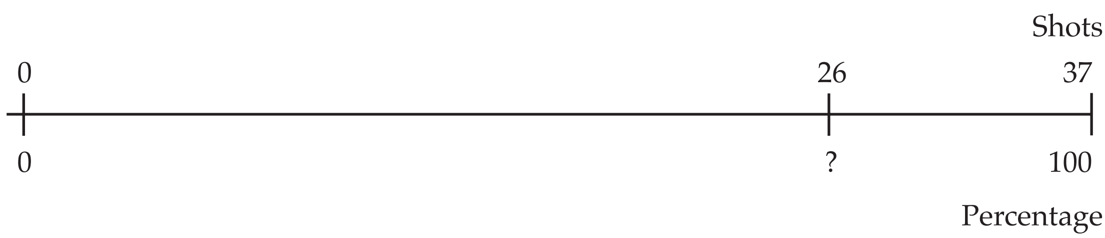
*The students achieving success proceed to **Using Number Properties**. Otherwise, they proceed to the next activity at some later time.*

1. Percentage unknown

"Mani took 37 shots and landed 26 of them. What was her shooting percentage?"

Equation: $26 \div 37 = \square\%$

Image:



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2. Part unknown

"Mani took 43 shots at goal. She got in 79% of them. How many of her shots went in?"

Equation: □ ÷ 43 ≈ 79% or 79% x 43 = □

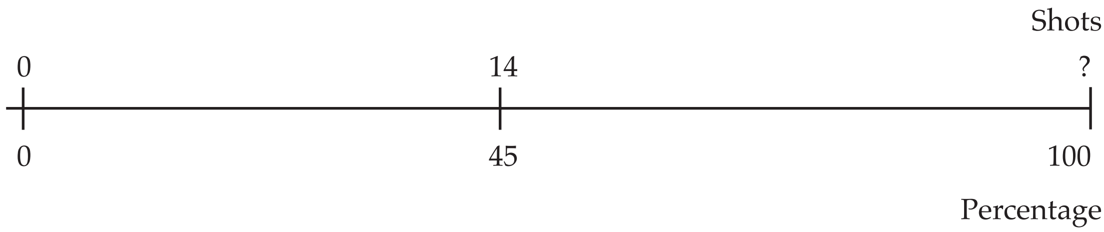
Image:

3. Whole unknown

"Mani took a number of shots at goal. She got in 14 of them. That was about 45%. How many shots did she take?"

Equation: 14 ÷ □ ≈ 45% or 45% x □ = 14

Image:



Also provide problems of these types that require a whole-to-whole comparison. This allows percentages that are over 100%.

"Kim scored 23 goals, and Todd scored 38 goals. Compare Todd's number of goals with Kim's as a percentage." (38 ÷ 23 = 165%)

"Shangzhi bought a gold coin. It is now worth $73, which is 155.3% of the price he paid. What did Shangzhi first pay for the coin?" (73 ÷ □ = 155.3%, □ = 47)

"On Kylie's farm, there was a 147% lambing percentage. If there were 87 ewes, how many lambs were born?" (□ ÷ 87 =147%, □ = 128)

Using Number Properties

Students will demonstrate a generalised understanding of how to solve percentage problems when they can perform correct calculation steps and justify their steps in terms of the quantities involved. Equations and diagrams such as double number lines or strips are still important in representing problems and for easing cognitive load.

Suitable problems might be:

"Tui got 89 bins of apples from her orchard this year compared to only 33 bins last year. By what percentage did her production ***increase*** this year?" (89 – 33 = 56. 56 ÷ 33 × 100 = 169.7%)

"This year, 53% of the babies born were boys. If 42 000 boys were born, how many babies in total were born?" (53% × □ = 42 000; □ = 42 000 ÷ 53%; □ = 79 245)

"77% of the jobs advertised in the newspaper required computer skills. There was a total of 483 jobs advertised. How many did not mention any need for computer skills?" (□ ÷ 483 = 23%, so 23% x 483 = □, □ = 111)

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Extending Mixing Colours

I am learning to solve difficult ratio problems.

Key Mathematical Ideas

- Ratios show a multiplicative relationship between two quantities of the same measure. For example, 2:7 might mean 2 litres of blueberry to 7 litres of orange or 2 shovels of cement to 7 shovels of shingle.
- Fractions can be used to represent the part-to-whole relationships in a ratio. For example, a ratio of 2:7 (cement:shingle) is made up of $\frac{2}{9}$ cement and $\frac{7}{9}$ shingle by volume.
- Comparing ratios involves finding a point of comparison. For example, to compare two blue:yellow mixtures, Mixture A 2:3 and Mixture B 3:5, might involve replicating both mixtures to get Mixture A 6:9 and Mixture B 6:10. This shows Mixture A to have the highest fraction ($\frac{6}{15}$) of blue.
- Converting the part-whole relationships to percentages provides a common denominator of 100 that is useful in comparison contexts. For example, 2:3 is $\frac{2}{5}$ blue = 40% and 3:5 is $\frac{3}{8}$ blue = 37.5%.

Diagnostic Snapshot

Students need access to a calculator to solve these problems:

1. You work for the Fruity Juicy Company. Here are three mixtures of orange and mango juice. Which mixture will taste most strongly of orange?

   Recipe A: 3 litres orange:5 litres mango (37.5%)

   Recipe B: 4 litres orange:7 litres mango (36.4%)

   Recipe C: 5 litres orange:8 litres mango (38.5%)

Students who answer this question correctly and can explain why they do each calculation step need to be further extended.

Equipment: Unilink cubes (red, blue, and yellow), rotating regions (Material Master 7–6), calculators

Using Materials

In Mixing Colours (pages 50–52), a sequence was provided for introducing the part-whole relationships in ratios and expressing them as fractions and percentages. The lesson for Extending Mixing Colours should initially follow the same sequence. The notes below show how the Number Properties phase of the lesson can be made more complex.

It is important for students at the Advanced Proportional stage to have multiplicative strategies to compare ratios without needing to replicate the ratios until a common part or whole is found, e.g., at earlier stages, to compare 2:4 with 3:5, both are duplicated (2:4 = 4:8 = 6:12 … and 3:5 = 6:10 …) until a common part of 6 measures of A is found.

*The students achieving success proceed to **Using Imaging**. Otherwise, they proceed to the next activity at some later time.*

Using Imaging

Follow the steps on the next page to create a computer spreadsheet that allows ratios to be displayed simultaneously as a table and connected graph.

This spreadsheet can be accessed online by downloading Material Master 7–10 from nzmaths (www.nzmaths.co.nz).

*The students achieving success proceed to **Using Number Properties**. Otherwise, they proceed to the next activity at some later time.*

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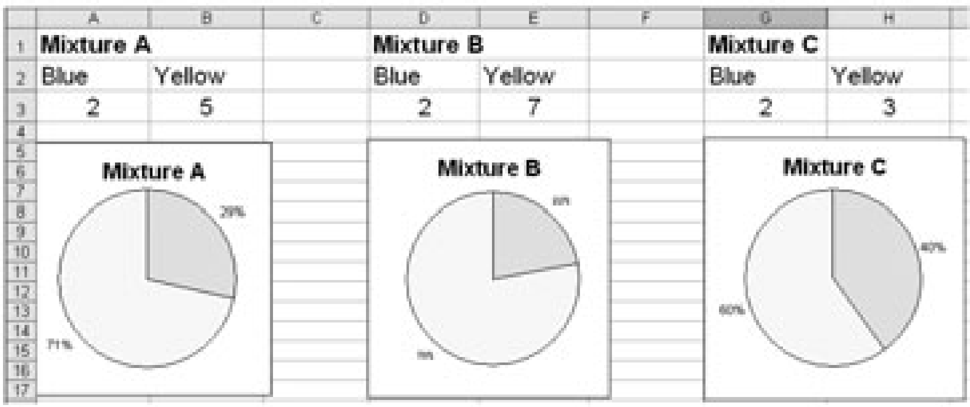
1. Create a table like this:

	A	B	C	D	E	F	G	H
1	**Mixture A**			**Mixture B**			**Mixture C**	
2	Blue	Yellow		Blue	Yellow		Blue	Yellow
3	2	5		2	7		2	3

2. For each mixture, create a pie graph and embed it in the spreadsheet rather than save it as a separate file. This means that any changes made to the cell entries are transferred to the pie chart.



The spreadsheet can then be used to compare mixtures visually and numerically. In the example above, the proportion of blue changes while the actual amount of it stays constant (two units).

Ask the students how the percentage of each colour was calculated and why this made the comparison easier. Percentages provide a common denominator of 100 for the equivalent fractions. In Mixture A, there are 2 units of blue out of a total number of 2 + 5 = 7 units. $\frac{2}{7} = 2 \div 7 = 0.2857 = 28.57\%$.

The students achieving success proceed to ***Using Imaging****. Otherwise, they proceed to the next activity at some later time.*

Pose other problems requiring students to compare ratios. Ask the students to anticipate which mixture gives the darkest shade of green before the spreadsheet is used to confirm. (A blue and yellow mixture produces green. The greater the proportion of blue, the darker the green will be.)

Good examples are (blue:yellow):

Mixture A	Mixture B	Mixture C	Darkest
3:5	4:6	6:12	Mixture B (40% blue)
5:6	3:4	7:8	Mixture C (46.67% blue)
4:9	3:7	5:8	Mixture C (38.46% blue)

Using Number Properties

Provide similar problems but allow students access only to their calculators rather than to any visual display. If confusions arise, the spreadsheet can be used to check solutions. Expect students to justify their calculation steps and encourage estimation and use of leveraging from known benchmark fractions, e.g., $\frac{5}{11}$ is slightly less than one-half or 50%.

Further problems are:

Mixture A	Mixture B	Mixture C	Darkest
4:2	7:4	5:3	Mixture A (66.67% blue)
3:7	2:5	4:12	Mixture A (30% blue)
7:9	5:7	2:3	Mixture A (43.75% blue)

Folding Fractions and Decimals

I am learning to multiply fractions and decimals.

Key Mathematical Ideas

- When multiplying with a fraction or decimal less than one, the product is always less than either factor.
- The commutative property allows the order of the factors to be transposed to make the problem simpler, e.g., 4.8 x 0.25 = 0.25 x 4.8
- Multiplying two fractions involves finding a fraction of another fraction, e.g., $\frac{1}{2}$ x $\frac{3}{4}$ is interpreted as $\frac{1}{2}$ of $\frac{3}{4}$.

Key Mathematical Knowledge

Check that the students:

- can convert fractions to decimals and vice versa
- multiplication facts to 10 x 10 and corresponding division facts.

Diagnostic Snapshot

Ask the students questions such as:

" 2.4 x 0.75"

"$\frac{4}{5}$ x $\frac{1}{4}$"

"0.8 x 0.3".

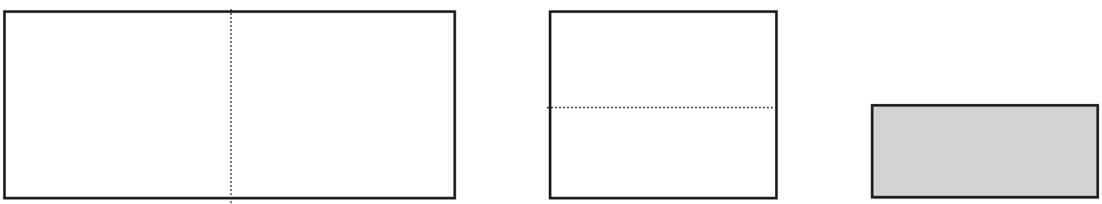
Students who are able to solve these problems need to work on the number properties section.

Equipment: Paper copies of the decimat (Material Master 7–3), laminated copies of the whole decimat, scissors, highlighters (or crayons or felt-tip pens)

Using Materials

Give the students a paper copy of the whole decimat. Tell them to fold their mat in half and then fold the half in half again. Get them to shade the last folded area with a crayon or felt-tip pen.

*The students achieving success proceed to **Using Imaging**. Otherwise, they proceed to the next activity at some later time.*



Ask, "What is one-half of one-half?" (one-quarter). Students can use a whole decimat to check how much of the original one was shaded.

Ask, "How do we write that as an equation?" ($\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$), and, "What would this equation look like recorded as decimals?" ($0.5 \times 0.5 = 0.25$).

Get the students to discuss patterns they can see in the equations and why they occur. Look for responses such as: "The denominators of the fractions are multiplied, like $2 \times 2 = 4$." Ask how we could predict the denominator from our folding. (Halving created two parts that were each divided into two parts.)

Students might comment that the decimals behave like whole numbers, as in $5 \times 5 = 25$. Ask, "What does the decimal point do?" (defines where the ones place is)

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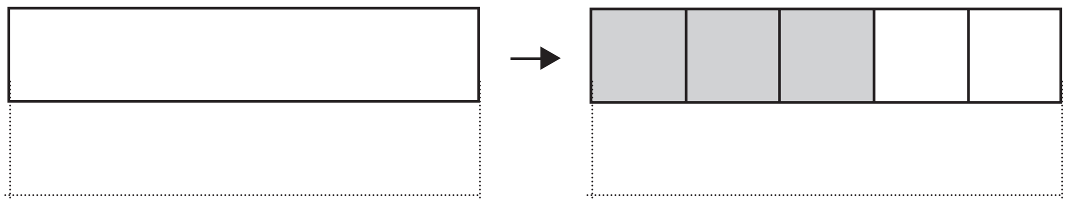
Use paper folding to further develop students' understanding of what happens to the numerators and denominators of fractions when they are multiplied and how the correct position of the decimal point can be determined in decimal multiplication by understanding the answer size. Focus on how the numerator and denominator in the answer can be predicted from folding and shading.

Good examples are:

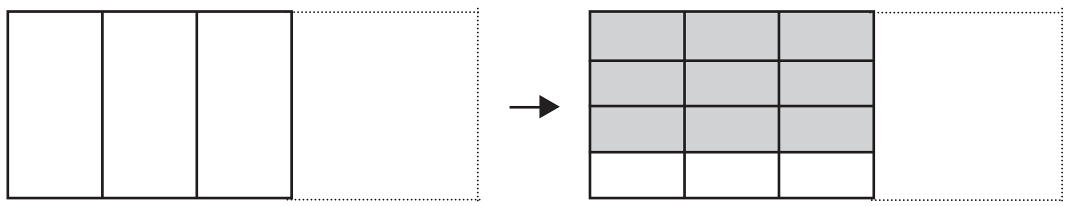
1.



$\frac{3}{5} \times \frac{1}{2} = \frac{3}{10}$ or $0.6 \times 0.5 = 0.3$ (three-fifths of one-half)

Each of the two parts (halves) was divided into five parts (fifths), creating $5 \times 2 = 10$ parts (tenths).

2.



$\frac{3}{4} \times \frac{3}{5} = \frac{9}{20}$ or $\frac{45}{100}$ or $0.75 \times 0.6 = 0.45$ (three-quarters of three-fifths)

Students should note that the numerators are multiplied because the selected shaded area has an area of nine (3×3). 75 hundredths of 6 tenths gives an answer of 450 thousandths (0.45).

Using Imaging

*The students achieving success proceed to **Using Number Properties**. Otherwise, they proceed to the next activity at some later time.*

Provide the students with other examples that could be solved by folding, cutting, and shading. Expect the students to image the process on materials and justify their answers. Suitable problems are:

$0.3 \times 0.3 = 0.09$ ($\frac{3}{10} \times \frac{3}{10} = \frac{9}{100}$)	$\frac{1}{4} \times \frac{2}{5} = \frac{2}{20} = \frac{1}{10}$ ($0.25 \times 0.4 = 0.1$)
$\frac{3}{4} \times \frac{1}{2} = \frac{3}{8}$ ($0.75 \times 0.5 = 0.375$)	$1.5 \times 0.5 = 0.75$ ($\frac{3}{2} \times \frac{1}{2} = \frac{3}{4}$)
$0.6 \times 0.02 = 0.012$ ($\frac{3}{5} \times \frac{2}{100} = \frac{6}{500} = \frac{12}{1000}$)	$\frac{8}{1} \times \frac{3}{4} = \frac{24}{4}$ ($8 \times 0.75 = 6.0$)

Using Number Properties

Provide students with problems that are difficult to image, ensuring that one fraction or decimal is well known. Suitable examples might be:

$0.25 \times 4.8 = 1.2$	$2.4 \times 0.75 = 1.8$	$3.5 \times 0.6 = 2.1$	$0.9 \times 1.9 = 1.71$
$6.4 \times 0.125 = 0.8$	$40 \times 0.4 = 16.0$	$0.2 \times 15.5 = 3.1$	$0.9 \times 0.3 = 0.27$
$\frac{3}{4} \times \frac{5}{2} = \frac{15}{8}$ or $1\frac{7}{8}$	$\frac{4}{5} \times \frac{1}{4} = \frac{4}{20} = \frac{1}{5}$	$\frac{4}{5} \times \frac{9}{100} = \frac{36}{500} = \frac{72}{1000}$	$\frac{6}{8} \times \frac{1}{2} = \frac{6}{16}$

Comparing Apples with Apples

I am learning to add and subtract fractions with unlike denominators.

Key Mathematical Ideas

- The addition and subtraction of fractions with like denominators can be linked to the students' knowledge of addition and subtraction of whole numbers, e.g., two-fifths + one-fifth = three-fifths, and 2 tens + 3 tens = 5 tens.
- Estimation and the use of benchmarks prior to calculating encourages students to develop a conceptual understanding of addition and subtraction of fractions, e.g., for $\frac{7}{8} + \frac{3}{4}$, because both fractions are close to 1, the answer therefore must be under 2 and above $1\frac{1}{2}$.
- Prior to working with the addition of fractions with unlike denominators, the students need to understand that equivalent fractions are other names for the same quantity.
- Addition of fractions with unlike denominators involves changing a fraction or fraction to like denominators (comparing apples with apples).

Key Mathematical Knowledge

Check that the students:

- recognise equivalent fractions
- can convert improper fractions to mixed numerals
- know that when adding and subtracting fractions with the same denominator, the denominator stays the same
- know that when the fractions being added or subtracted have unrelated denominators (that is, not a multiple of the other), both fractions need to be converted to equivalent fractions with a common denominator
- know the multiplication facts to 10 x 10 and corresponding division facts
- can rename improper fractions as mixed numerals
- know common factors
- know that fractions are made up of duplications (repeats) of a unit fraction.

Diagnostic Snapshot

Ask the students questions such as:

"$\frac{3}{4} + \frac{2}{5}$"

"$\frac{8}{6} - \frac{3}{8}$".

If the students are able to solve these problems, they need to work on the number properties section.

Equipment: Fraction strips

Note that the students will need to be familiar with the structure of the fraction strips, particularly ideas about equivalence, e.g., $\frac{3}{12} = \frac{1}{4}$.

Using Materials

Ask the students to model the following action with the fraction strips:

"Make four-sixths and five-sixths separately. Now join the two fractions together, end on end. How many sixths is that altogether? How should we record that operation using an equation?"

The students achieving success proceed to ***Using Imaging****. Otherwise, they proceed to the next activity at some later time.*

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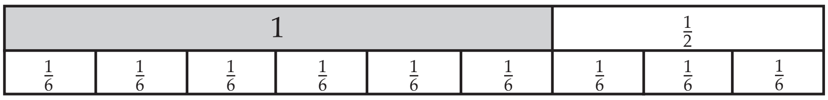
The students achieving success proceed to ***Using Number Properties****. Otherwise, they proceed to the next activity at some later time.*

Students may suggest that the equation is $\frac{4}{6} + \frac{5}{6} = \frac{9}{6}$. If not, record it for them. Ask them how else nine-sixths might be written ($1\frac{3}{6}$ or $1\frac{1}{2}$).



Pose other similar problems for the students to model with the fraction strips, such as:

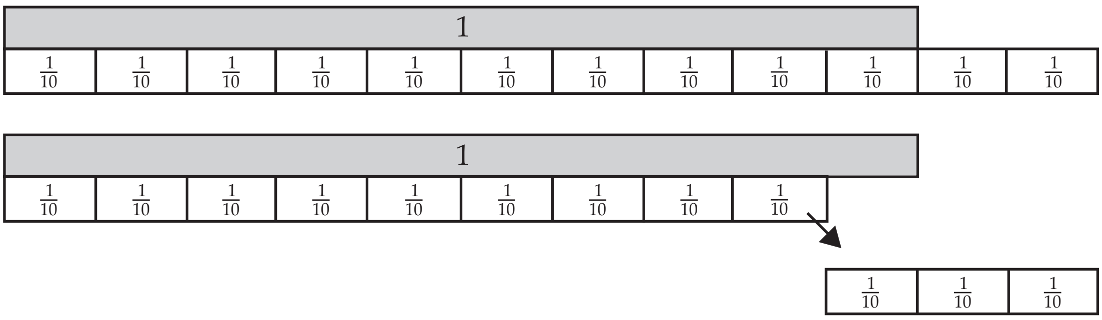
$\frac{3}{4} + \frac{5}{4}$ $\frac{1}{5} + \frac{4}{5}$ $\frac{7}{8} + \frac{4}{8}$

Record the answers to each problem using both fractions (e.g., $\frac{11}{8}$) and mixed numbers (e.g., $1\frac{3}{8}$). Challenge the students to explain the link between the model of the operation and what occurs with the symbols.

Ask why the numerators are being added but not the denominators. Link this to addition with equivalent units that the students already know, such as six tens plus five tens, seven hundreds plus eight hundreds. Develop a generalised rule for adding fractions that have the same denominator:

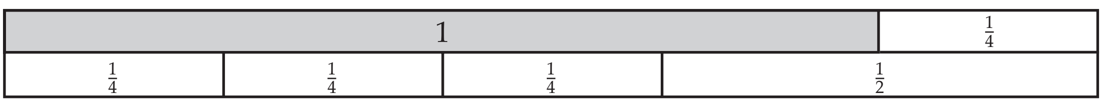
I am learning to add fractions with the same denominator: add the numerators and keep the denominator the same.

Ask the students to use the materials to confirm that this holds for subtraction as well. Get them to model problems such as $\frac{12}{10} - \frac{3}{10} = \frac{9}{10}$.



Pose other related problems, such as: $\frac{7}{5} - \frac{3}{5}$ $\frac{13}{8} - \frac{7}{8}$ $\frac{7}{2} - \frac{4}{2}$

Tell the students that subtracting fractions with common denominators is similar to adding fractions with the same denominators. Investigate adding and subtracting fractions with different denominators, using problems such as $\frac{3}{4} + \frac{1}{2}$.



Students may use the equivalence of one-half and two-quarters to realise that the answer must be $\frac{5}{4} = 1\frac{1}{4}$. The key idea in these examples is that one or more of the fractions must be renamed so that the denominators are the same. Provide other problems, such as: $\frac{1}{2} + \frac{2}{3}$ $\frac{7}{6} - \frac{2}{3}$ $\frac{7}{8} - \frac{3}{4}$ $\frac{3}{4} + \frac{3}{8}$ $\frac{4}{5} - \frac{4}{10}$

Record the answers using fractions and mixed numbers. Discuss the links between the numerators and denominators and the answers. Point out that the choice of common denominator is influenced by the denominators of the fractions that are being added or subtracted. Note that the common denominator is the lowest common multiple of both denominators.

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In the example of $\frac{1}{2} + \frac{2}{3}$, both halves and thirds can be renamed as sixths:
$\frac{1}{2} + \frac{2}{3} = \frac{3}{6} + \frac{4}{6} = \frac{7}{6} = 1\frac{1}{6}$

1						$\frac{1}{6}$
$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
$\frac{1}{2}$			$\frac{1}{3}$		$\frac{1}{3}$	

Using Imaging

Encourage the students to image or draw to show what fraction strips would be used to solve the following problems:

$\frac{3}{5} + \frac{1}{2}$ $\quad$ $\frac{3}{4} + \frac{2}{3}$ $\quad$ $\frac{7}{8} + \frac{3}{4}$ $\quad$ $\frac{5}{6} + \frac{2}{3}$

$\frac{2}{3} - \frac{1}{2}$ $\quad$ $\frac{6}{5} - \frac{1}{2}$ $\quad$ $\frac{3}{4} - \frac{1}{3}$ $\quad$ $\frac{11}{8} - \frac{3}{4}$

If necessary, fold back to using the materials to check the students' ideas.

*The students achieving success proceed to **Using Number Properties**. Otherwise, they proceed to the next activity at some later time.*

Using Number Properties

Use the following examples to generalise how fractions are added or subtracted:

$\frac{3}{4} + \frac{2}{5}$ $\quad$ $\frac{2}{3} + \frac{4}{5}$ $\quad$ $\frac{1}{6} + \frac{3}{8}$ $\quad$ $\frac{5}{7} + \frac{2}{3}$ $\quad$ $\frac{5}{6} + \frac{3}{4}$

$\frac{7}{8} - \frac{3}{4}$ $\quad$ $\frac{9}{10} - \frac{4}{5}$ $\quad$ $\frac{8}{6} - \frac{3}{8}$ $\quad$ $\frac{11}{12} - \frac{1}{3}$ $\quad$ $\frac{5}{4} - \frac{5}{10}$

Note that in these examples, the common denominators will require students to go outside the pieces available in the fraction strips. This requires students to generalise the number properties rather than rely on images of the materials.

Independent Activities

Students will enjoy playing the game Create 3 (See Material Master 7–9) to consolidate the addition of fractions. The game can be adapted to subtracting fractions if each student's score starts on three and the students aim to subtract the fractions they land on until they reach zero.

Feeding Pets

I am learning to find fractions between two other fractions.

Key Mathematical Ideas

- Between any two fractions, there is an infinite number of other fractions.

Key Mathematical Knowledge

Check that the students:

- recognise equivalent fractions
- know multiplication facts to 10 x 10 and corresponding division facts
- know common factors
- can convert fractions to decimals to percentages and vice versa
- can order fractions
- can use common fractions, e.g., $\frac{1}{2}$, as a benchmark to roughly order fractions.

Diagnostic Snapshot

Ask the students questions such as:

"Find fractions between these pairs of fractions; $\frac{3}{8}$ and $\frac{4}{10}$, and $\frac{2}{3}$ and $\frac{9}{13}$."

Students who are able to solve problems such as these need to be extended.

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*The students achieving success proceed to **Using Imaging**. Otherwise, they proceed to the next activity at some later time.*

Equipment: Fraction strips, calculators

Using Materials

Ask the students to make the fractions $\frac{2}{3}$ and $\frac{3}{4}$ with fraction strips. Use the scenario that the strips represent sections of dog jerky strip (dried meat). Tell them that two-thirds of a strip leaves Woof, the dog, hungry but three-quarters is too much. Challenge them to find a fraction between $\frac{2}{3}$ and $\frac{3}{4}$ that could tell you how much of a strip to feed Woof today.

Discuss their methods, which are likely to include:

(i) Change both fractions to decimals, $\frac{2}{3} = 0.\dot{6}$ or 0.67 (2 dp) and $\frac{3}{4} = 0.75$, and find a decimal fraction between these, e.g., 0.7 or $\frac{7}{10}$.

(ii) Change both fractions to equivalent fractions with a common denominator, in this case 12 because it is the least common multiple of three and four. $\frac{2}{3} = \frac{8}{12}$ and $\frac{3}{4} = \frac{9}{12}$, so $\frac{8\frac{1}{2}}{12}$ will work. This needs to be renamed as $\frac{17}{24}$.

"How many different fractions could we find between $\frac{2}{3}$ and $\frac{3}{4}$?" (Answer: An infinite number)

"How could you use your strategy to find others?" (Answer: Any decimal between $0.\dot{6}$ and 0.75 will work, and any fractional number of between eight-twelfths and nine-twelfths will work.)

Pose other "between" problems such as: "Find three fractions between these pairs of fractions:

$\frac{1}{2}$ and $\frac{3}{5}$ $\frac{1}{2}$ and $\frac{5}{8}$ $\frac{2}{3}$ and $\frac{4}{5}$ $\frac{6}{10}$ and $\frac{3}{4}$ $\frac{7}{8}$ and 1"

Using Imaging and Number Properties

Pose problems that cannot be modeled easily with the fractions strips and invite students to generalise their strategies. Good problems might be: "Find fractions between these pairs of fractions:

$\frac{3}{4}$ and $\frac{6}{7}$ $\frac{2}{5}$ and $\frac{5}{11}$ $\frac{3}{8}$ and $\frac{4}{10}$ $\frac{2}{3}$ and $\frac{9}{13}$ $\frac{1}{2}$ and $\frac{51}{100}$"

Compare the usefulness of the decimal conversion and equivalent fraction methods. In what ways are they similar? (Both are equivalent fractions.) When is one method easier to use than the other? (When the fractions are easy to convert to decimals)

Brmmm! Brmmm!

I am learning to solve problems in which a fraction measures another fraction.

Key Mathematical Ideas

- Most measurement involves finding how many copies of a given unit fit into a particular space. For example, when finding the length of a line, we find how many times a unit length tessellates along the line without gaps or overlaps. The units used to measure can be ones (e.g., 1 metre), multiples of a unit (e.g., kilometres) or parts of ones (e.g., centimetres).
- When a fraction measures another number, we find out how many times the fraction fits into that number. This can be expressed as either division or multiplication statements. For example, how many two-thirds fit into five can be recorded as $5 \div \frac{2}{3} = \square$ or $\square \times \frac{2}{3} = 5$.

Key Mathematical Knowledge

- Fractions as quotients, e.g., $3 \div 5 = \frac{3}{5}$ and $\frac{3}{5} = 3 \div 5$.
- Division of whole numbers as measures, e.g., $72 \div 4 = 18$ as a representation of "How many fours in seventy-two?"
- Equivalent fractions, e.g., $\frac{3}{8} = \frac{9}{24}$.
- Improper fractions to whole numbers, e.g., $\frac{23}{6} = 3\frac{5}{6}$.

Diagnostic Snapshot

Ask the students questions such as:

- Three people share seven pizzas equally. How much of one pizza does each person get?
- Four scones go in each bag. You have 72 scones. How many bags can you fill?
- Fold a rectangle of paper into eighths. Shade three-eighths. Fold the paper back into eighths, then fold the eighths in thirds (three parts). What fraction are each of the smaller parts? How many of them are shaded?
- Here is one-sixth (fraction strip). If you had twenty-three-sixths, how many ones could you make and how many extra sixths would you have?

Equipment: Fraction Strips (Material Master 7–7)

Using Materials

*The students achieving success proceed to **Using Imaging**. Otherwise, they proceed to the next activity at some later time.*

Present the students with this problem:

"Toline has just filled up the petrol tank of her car. She drives to and from work each day. Each return trip takes three-tenths of a tank. How many trips can Toline make before she needs to fill up?"

Encourage the students to draw representations of the problem such as strip diagrams. As the students provide solutions, link their ideas to a visual or equipment model. For example, fraction strips might look like this:

<table>
<tr><td colspan="10">1</td><td></td><td></td></tr>
<tr><td>$\frac{1}{10}$</td><td>$\frac{1}{10}$</td><td>$\frac{1}{10}$</td><td>$\frac{1}{10}$</td><td>$\frac{1}{10}$</td><td>$\frac{1}{10}$</td><td>$\frac{1}{10}$</td><td>$\frac{1}{10}$</td><td>$\frac{1}{10}$</td><td>$\frac{1}{10}$</td><td>$\frac{1}{10}$</td><td>$\frac{1}{10}$</td></tr>
</table>

Note that a conceptual obstacle is that the students must express the answer in terms of a new measurement unit, three-tenths. So the problem has two important units, the original one and the unit of measure, three-tenths. The number of trips is the answer to the question, "How many three-tenths measure one?" (Three whole trips and one-third of a trip)

This can be recorded symbolically as $1 \div \frac{3}{10} = 3\frac{1}{3}$ trips. Ask the students what $3\frac{1}{3}$ is as an improper fraction ($\frac{10}{3}$). Write the equation as $1 \div \frac{3}{10} = \frac{10}{3}$.

Students may notice that $\frac{3}{10}$ and $\frac{10}{3}$ are reciprocals. Ask, "Why is the answer $\frac{10}{3}$?" (There are ten-tenths in one, and three are used each time to make a unit of three-tenths). Note the important connection to $a \div b = \frac{a}{b}$, e.g., $10 \div 3 = \frac{10}{3}$.

Provide the students with two other examples in which one whole is divided by a non-unit fraction. For example:

1. [Name]'s car has a full tank of petrol. Each trip takes five-eighths of a tank.

How many trips can he/she do on a full tank? [$1 \div \frac{5}{8} = \frac{8}{5}$ ($1\frac{3}{5}$)]

<table>
<tr><td colspan="8">1</td><td></td><td></td></tr>
<tr><td>$\frac{1}{8}$</td><td>$\frac{1}{8}$</td><td>$\frac{1}{8}$</td><td>$\frac{1}{8}$</td><td>$\frac{1}{8}$</td><td>$\frac{1}{8}$</td><td>$\frac{1}{8}$</td><td>$\frac{1}{8}$</td><td>$\frac{1}{8}$</td><td>$\frac{1}{8}$</td></tr>
</table>

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2. [Name]'s car has a full tank of petrol. Each trip takes two-ninths of a tank. How many trips can he/she do on a full tank?

Extend the concept of division by a fraction by varying the amount of petrol available. For example, "It takes two-fifths of a tank to make one trip. How many trips can [Name] make on three full tanks of petrol?" ($3 \div \frac{2}{5} = \frac{15}{2} = 7\frac{1}{2}$).

Connect the answer to each problem to the answer for one full tank divided by the fraction consumed for one trip, e.g., $1 \div \frac{2}{5} = \frac{5}{2}$ ($2\frac{1}{2}$) so $3 \div \frac{2}{5} = \frac{15}{2}$ ($7\frac{1}{2}$).

Students should notice the scaling effect of how much petrol is available.

Another problem might be:

"It takes one-sixth of a tank to make one trip. How many trips can [Name] make on three-quarters of a tank of petrol?"($\frac{3}{4} \div \frac{1}{6} = \frac{18}{4}$ because it is three-quarters of the answer to $1 \div \frac{1}{6} = \frac{6}{1} = 6$) Note the connection to fractions as operators, e.g., $\frac{3}{4} \times \frac{6}{1} = \frac{18}{4}$, and to any whole number as a fraction with a denominator of one, e.g., $\frac{8}{1} = 8$.

Using Imaging

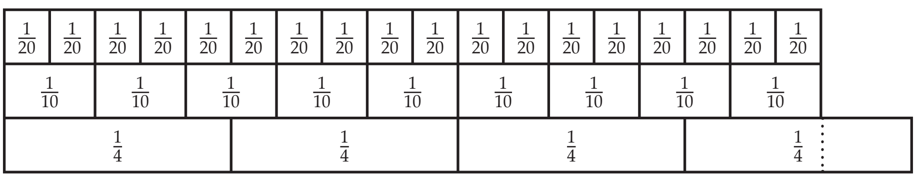
*The students achieving success proceed to **Using Number Properties**. Otherwise, they proceed to the next activity at some later time.*

Pose similar problems in context using students' names from the teaching group. The emphasis should be on students anticipating the answer to a problem and, where necessary, checking their prediction using a diagram or equipment. Recording the problems as division equations is vital in helping the students to generalise the process of solving such problems.

Be aware of other useful strategies that students may develop. Using equivalence is another way to solve division problems with fractions.

For example, [Name] has nine-tenths of a tank of petrol. Each trip takes one-quarter of a tank. How many trips can [Name] make? ($\frac{9}{10} \div \frac{1}{4} = \square$)

Both nine-tenths and one-quarter can be converted to fractions with the same denominator (e.g., twentieths), so the same units are involved.



$\frac{9}{10} \div \frac{1}{4} = \frac{18}{20} \div \frac{5}{20}$ (How many $\frac{5}{20}$ measured $\frac{18}{20}$? Answer: $\frac{18}{5} = 3\frac{3}{5}$ trips.)

Examples of other problems might be:

$\frac{3}{4} \div \frac{3}{5} = \frac{15}{12} = \frac{5}{4}$ $\quad \frac{11}{12} \div \frac{1}{6} = \frac{66}{12} = \frac{11}{2}$ $\quad \frac{7}{8} \div \frac{3}{5} = \frac{35}{24}$ $\quad \frac{4}{5} \div \frac{1}{2} = \frac{8}{5}$

$\frac{2}{3} \div \frac{3}{4} = \frac{8}{9}$ $\quad \frac{5}{12} \div \frac{2}{3} = \frac{15}{24} = \frac{5}{8}$

Note that the two problems on the second line may cause confusion because the divisor is larger than the dividend. This will mean that the car can make only a fractional part of a trip, not a full trip, e.g., $\frac{2}{3} \div \frac{3}{4} = \frac{8}{9}$.

Using Number Properties

Encourage students who are able to anticipate the answers to fraction division problems by considering the numerators and denominators to generalise the operation algebraically. This can be facilitated by asking them to describe their common procedure given three similar problems from Using Imaging.

For example, "To solve $\frac{15}{2} \div \frac{2}{3} = \square$, I made the numerator of the answer 5 x 3 and the denominator 12 x 2."

This can lead to expression of the common method using letters as variables to represent any integers that are chosen, i.e., $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$.

Give the students connected problems in which they must apply the properties of multiplication and division, particularly the associative property and multiplication and division as inverse operations. Examples might be:

- $\frac{3}{4} \div \frac{2}{3} = \frac{9}{8}$ so $\frac{3}{4} \div \frac{4}{3} = \square$ ($\square = \frac{9}{16}$ because $\frac{4}{3}$ is twice $\frac{2}{3}$)
- $\frac{3}{5} \times \frac{1}{2} = \frac{3}{10}$ so $\frac{3}{10} \div \frac{1}{2} = \square$ ($\square = \frac{3}{5}$ by inverse operation)
- $\frac{5}{9} \div \frac{3}{4} = \frac{20}{27}$ so $\frac{5}{3} \div \frac{3}{4} = \square$ ($\square = \frac{60}{27}$ because $\frac{5}{3}$ is three times $\frac{5}{9}$).

Rates of Change

I am learning to solve rate problems.

Key Mathematical Ideas

- Rates are a multiplicative comparison between two different measures. For example, 100 kilometres per hour is a multiplicative comparison meaning that 100 kilometres are travelled every hour if that constant rate is maintained.
- Rate problems take two basic forms, equalising or comparing. In equalising problems, three measures are given and the fourth must be found, e.g., "The car travels at 99 kilometres per 60 minutes. How far does it travel in 20 minutes?" Comparison problems involve finding which rate of change is greater or less, e.g., "Which rate of pay is better, earning \$64 in 6 hours or \$94 in 9 hours?"
- Rate problems can be solved using two main strategies, within or between. Within strategies involve scaling a rate, e.g., 99 kilometres in 60 minutes equals 33 kilometres in 20 minutes (multiplying by $\frac{1}{3}$). Between strategies involve creating a derived measure that is a combination of the two original measures, e.g., \$64 per 6 hours is a rate of $\frac{64}{6}$ = \$10.67 per hour.

Key Mathematical Knowledge

- Fractions as operators, e.g., $\frac{4}{5} \times 35 = 28$ ($\frac{1}{5} \times 35 = 7$). In this lesson, students must find a missing operator, e.g., $\square \times 36 = 24$ so $\square = \frac{24}{36} = \frac{2}{3}$.
- Rates can be written as x (measure one)/y (measure two), e.g., 99 kilometres per sixty minutes can be written as 99 km/60 min = 99 km/h (h means hours).
- The measures of the standard international system are based on powers of ten, e.g., 1 cubic metre = 1 000 000 cubic centimetres (1 000 000 = 10^6).

Diagnostic Snapshot

- What is $\frac{3}{8}$ of 40 ($\frac{3}{8} \times 40 = \square$)? (15)
- If you pay \$6 for 8 avocadoes, how much do you pay for 24 avocadoes? (\$18)
- How many metres are in 1 kilometre (1000)? How many centimetres are in 1 metre (100)? So how many centimetres are in 1 kilometre (100 000)?

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The students achieving success proceed to ***Using Imaging****. Otherwise, they proceed to the next activity at some later time.*

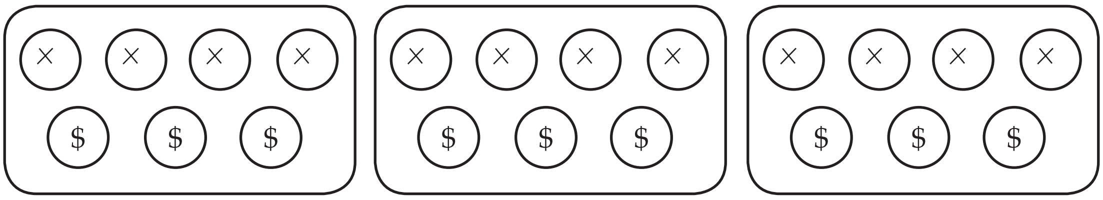
Using Materials

Pose this problem to the students:

"You can buy twelve oranges for nine dollars. How much will you pay for eight oranges?"

Represent the problem in these four ways:

1. Equipment



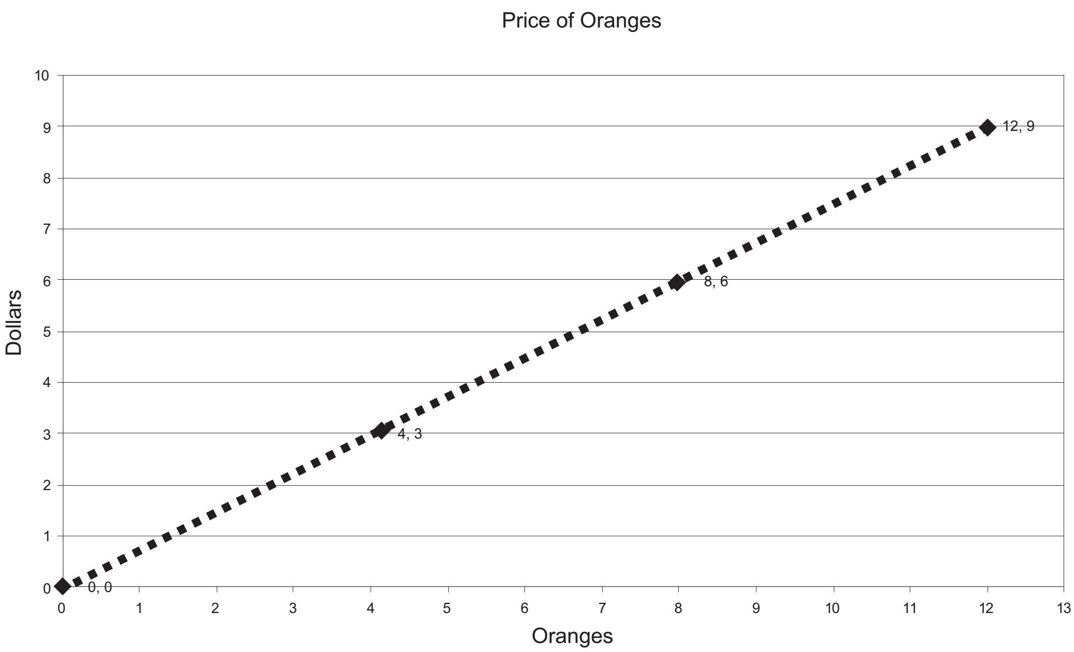
12 oranges cost $9, so 4 oranges cost $3, so 8 oranges cost $6.

2. Table

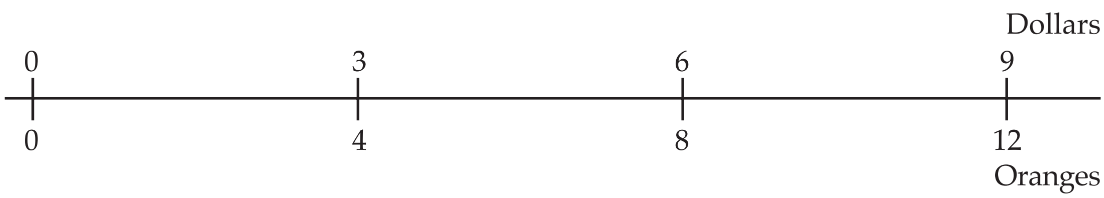
	Oranges	Dollars	
÷ 3	12	9	÷ 3
× 2	4	3	× 2
	8	6	

3. Graph

Note that in situations of constant rate, the points lie on a line of constant slope.



4. Double number line



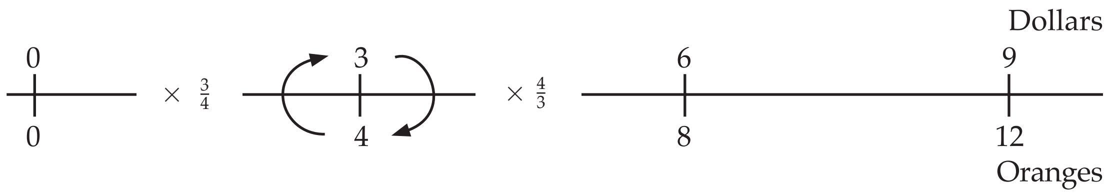
The most common strategy employed by students is likely to be based on common factors (see the table on the previous page) because it avoids having to work between the measurements. It is important for students to have a variety of approaches for solving problems such as these. Ask them "How many oranges do you get for one dollar?" This gives the unit rate of oranges per dollar. Some students may observe that this unit rate can be found by dividing 12 by 9 because there are nine dollars. They may be surprised to find that the rate obtained on the calculator ($1.\dot{3}$ oranges per dollar) is $1\frac{1}{3}$, which is $\frac{4}{3}$. Remind them that in algebra, $12 \div 9 = \frac{12}{9} = \frac{4}{3}$.

Ask the students, "What is the price of one orange in dollars?" Some students may work out that this can be found by dividing \$9 by 12 because there are 12 oranges in the given rate. Remind them that $9 \div 12 = \frac{9}{12} = \frac{3}{4}$. Ask them to predict the calculator result (0.75 dollars per orange). Ask how this rate could have been used to find the cost of eight oranges ($8 \times 0.75 = 6$ or $8 \times \frac{3}{4} = 6$). Connect the dollars per orange rate to the slope of the line connecting the ordered pairs on the graph. For movement of one unit to the right (one more orange), the line increases 0.75 units up ($\frac{3}{4}$ of a dollar).

Tell the students that $\frac{3}{4}$ and $\frac{4}{3}$ are known as reciprocals, i.e., fractions with transposed numerators and denominators. Ask, "What is the answer to $\frac{3}{4} \times \frac{4}{3}$?" (one).

Show these operators on the table and the double number line:

Oranges	Dollars
12 $\xrightarrow{\times \frac{3}{4}}$	9
4	$\xleftarrow{\times \frac{4}{3}}$ 3



Pose similar problems with the expectation that students will connect the equipment, table, graph, and double number line representations. Suitable problems might be:

"Lisa worked at the bookshop. She earned a total of \$76 for working 8 hours. How much would she earn if she worked for 5 hours?" (Answer: \$47.50)

"The machine made 30 gadgets in 18 minutes. How many gadgets would it make in 12 minutes?" (Answer: 20 gadgets)

Discuss what the problems have in common. (They are all rate problems because they involve comparing two different types of measurement.) Connecting the tables will help the students to see this structure.

Oranges	Dollars
12	9
8	?

Dollars	Hours
76	8
?	5

Gadgets	Minutes
30	18
?	12

Note that the unknown may appear in any cell within the table and this, as well as the numbers involved, may make a big difference to the difficulty of the problem.

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The students achieving success proceed to ***Using Number Properties****. Otherwise, they proceed to the next activity at some later time.*

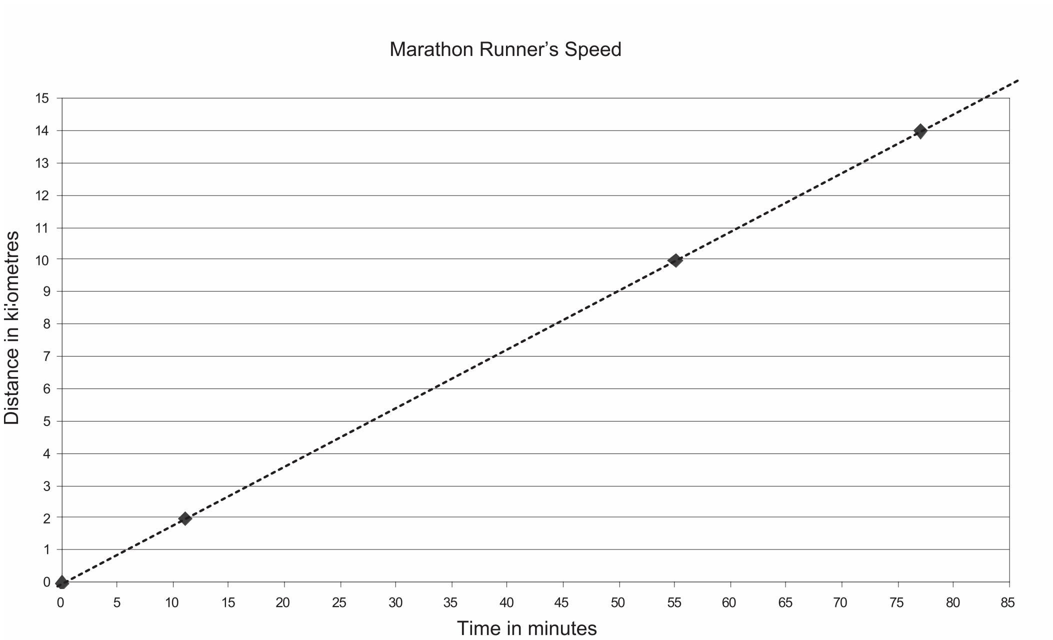
Using Imaging

In this phase of the lesson sequence, students are required to anticipate how each representation of a given rate will appear before they construct that representation.

For example, ask the students, "A marathon runner tries to travel at the same speed to save energy and avoid overheating. In 77 minutes, she runs 14 kilometres. What is her time for 10 kilometres?"

Let the students chose the representation they prefer to solve the problem but require them to image what it will look like. Compare their predictions before connecting the representations. For example, the table and graph appear like this:

	Minutes		Kilometres	
	77	$\times \frac{2}{11}$	14	
$\times \frac{5}{7}$	11		2	$\times \frac{5}{7}$
	55	$\times \frac{11}{2}$	10	



Ask the students, "What is the runner's speed, in kilometres per minute?" The unit rate in this case is $14 \div 77 = 2 \div 11 = \frac{2}{11}$ km/m (two-elevenths of a kilometre per minute). This is the slope of the graph line (increase in distance per increase in time).

"What is the number of minutes she takes to run 1 kilometre?" This is the reciprocal of $\frac{2}{11}$, which is $\frac{11}{2} = 5\frac{1}{2}$ m/km (minutes per kilometre). Multiplying this rate by ten, for ten minutes, to get $10 \times 5\frac{1}{2} = 55$ minutes, solves the problem.

Pose other problems that involve finding missing values in rate problems. Suitable problems are:

"Tracey uses 42 apples to make 30 pies. How many apples would she need to make 20 pies of the same kind?" (42:30, so 7:5, so 28:20)

"Kahu uses 36 litres of paint to spray-paint 16 similar cars. How many cars could he paint with 27 litres?" (36:16, so 9:4, so 27:12)

Emphasise the use of between strategies, such as working out the unit rate or other rates connecting the measures, and within strategies, where the given rate is scaled to match the other value given. For example, in the paint problem, the given rate of 36 litres per 16 cars can be scaled down to 27:12 through multiplying by $\frac{27}{36} = \frac{3}{4}$, i.e., $\frac{3}{4} \times 16 = 12$. This is an example of a within strategy.

Using Number Properties

The aim at this progression is for students to choose within and between strategies appropriately, depending on the numbers given in rate problems. They should operate on the numbers rather than relying on materials or diagrams, though folding back may be useful at times. Students should be exposed to problems involving rates created from standard measures because this has important applicability to science and technology. Through these problems, students explore derived measures such as grams per square centimetre (g/cm^2) for pressure, kilograms per cubic metre (kg/m^3) for density, and kilometres per hour (km/h) for speed. Also encourage the students to choose appropriately between mental calculation, written recording, and using a calculator. Calculators have an important role to play where the calculations are cumbersome but must be supported by conceptual understanding of what each calculation step is doing.

Suitable problems are:

"A 10 000 cm^3 (cubic centimetre) piece of pine wood has a mass of 5300 grams. What would be the mass of a 4000 cm^3 piece of the same wood?" (10 000:5300, so 1000:530, so 4000:2120)

Note: Mature pine has a density of 530 kg per m^3. m^3 is the symbol for cubic metre, which is 1 000 000 cm^3 (one million cubic centimetres).

"A bus travels 56 kilometres in 35 minutes at a constant speed along the desert road. How long does the bus take to cover the first 24 kilometres at that speed?" (56:35, so 8:5, so 24:15, or 35 is $\frac{5}{8}$ of 56, so $\frac{5}{8}$ of 24 = 15)

"A colour television uses 450 watts of electricity in the 120 minutes it is on. How many watts would it use in 70 minutes?" (450:120, so 15:4; $\frac{15}{4} = 3\frac{3}{4}$ watts per minute, $70 \times 3\frac{3}{4} = 262.5$ watts)

"Forty-eight cubic metres of water pass through the turbine of a power station in 30 seconds. How much water passes through the same turbine in 18 seconds?" ($48 \div 30 = 1.6\ m^3/s$, $18 \times 1.6 = 28.8\ m^3$)

"The vet decides how much of a drug to give dogs on the basis of body weight at a proportional rate. If a 28 kilogram dog gets 364 milligrams for each dose, how much should a 7 kilogram dog get?" ($364 \div 28 = 13$ mg/kg, $7 \times 13 = 91$ mg)

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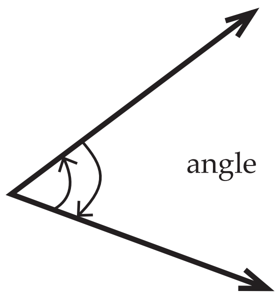
Tree-mendous Measuring

I am learning to use trigonometry to find the unknown sides and angles of right-angled triangles.

Key Mathematical Ideas

- When a shape or object is enlarged, meaning enlarged or shrunk proportionally, some features remain invariant (unchanged) while others change. Areas, volumes, and length of lines change under enlargement. However, angles and ratios of sides or edges within the shape or object stay constant.
- In algebra, $\frac{a}{b}$ means $a \div b$ as well as the variable $\frac{a}{b}$. In number, $3 \div 5$ means either "three shared among five" or "three measured by five". The share or measure is the number $\frac{3}{5}$.
- An angle represents a turn that maps two rays or line segments that have a common point onto each other, e.g.,



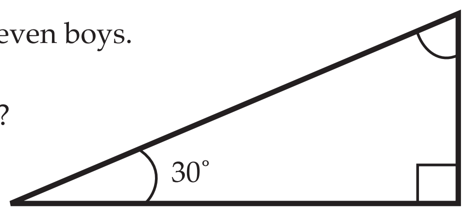
- Right-angled triangles are important in measurement because right angles are used in construction and identifying position.

Key Mathematical Knowledge

- $a \div b = \frac{a}{b}$, meaning a things shared among b people results in a share of $\frac{a}{b}$ things.
- The internal angles of a triangle add to 180º. So if one angle is 90º, the other two angles must add to 90º.

Diagnostic Snapshot

- Five pizzas are shared equally between seven boys. How much pizza does each boy get?
- What is the missing angle in this triangle?



Equipment: protractors, rulers, large protractors (enlarge a standard 180º protractor onto A3 card), metre rules, tape measures or trundle wheels, chalk, calculators (preferably scientific), paper right-angled triangles (60 mm x 104 mm x 120 mm and 80 mm x 139 mm x 160 mm)

*The students achieving success proceed to **Using Imaging**. Otherwise, they proceed to the next activity at some later time.*

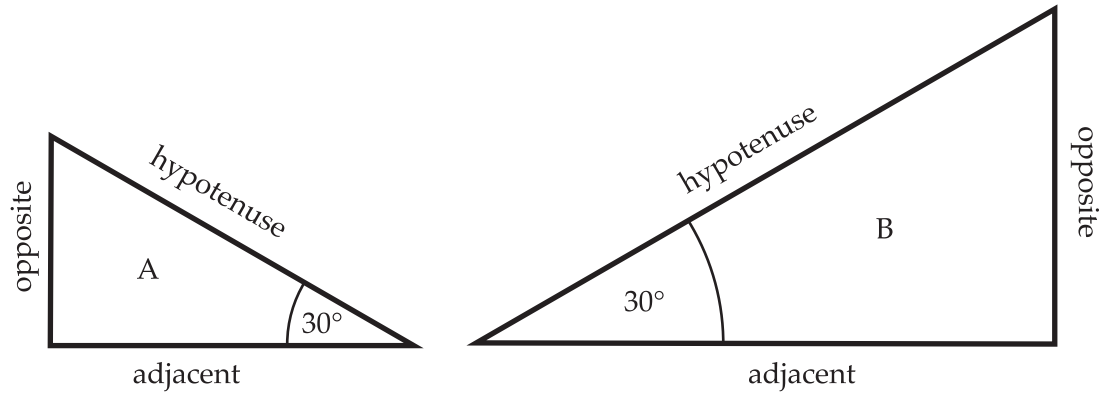
Using Materials

Give the students the two similar right-angled paper triangles. Similar means that they are enlarged or reduced copies of one another. Discuss what features of shapes helped them spot that the triangles were similar. Make a table of variant and invariant features of similar triangles, i.e., features that change and those that do not change as the triangle is enlarged.

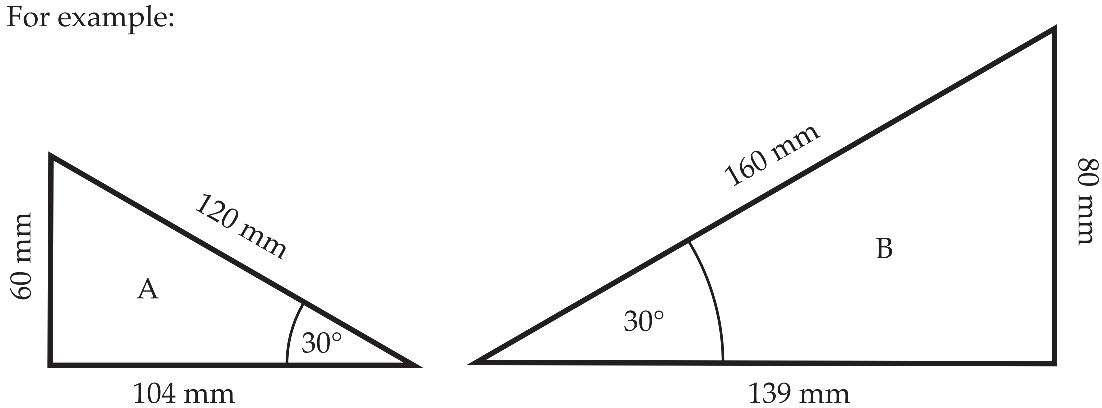
Change (variant)	Do not change (invariant)
Length of sides	Angles
Area	Shape
Perimeter	

Most students won't know that the ratios of the side lengths within and between similar triangles remain invariant under enlargement. This concept is fundamental to understanding trigonometry.

For the pair of similar triangles, first label the hypotenuse, then mark an angle and label the sides that are opposite and adjacent to that angle. Discuss the words adjacent (meaning next to), opposite (meaning across from), and hypotenuse (the side that is opposite the right angle).



Get the students to measure the side lengths of the triangles in millimetres. Stress that accuracy is important. Firstly, get the students to compare the ratios of the matching sides in the different triangles, i.e., opposite side of A : opposite side of B, adjacent side of A : adjacent side of B, and hypotenuse of A : hypotenuse of B.
For example:



The ratios 60:80, 104:139, and 120:160 contain the operator 1.33, which maps a side length from triangle A onto its matching side from triangle B. This is the scale factor. Students might like to verify that this scale factor is constant for the matching sides of a different pair of similar triangles.

Introduce the trigonometric ratios:

sine (angle) = $\frac{\text{opposite}}{\text{hypotenuse}}$ cosine (angle) = $\frac{\text{adjacent}}{\text{hypotenuse}}$ tangent (angle) = $\frac{\text{opposite}}{\text{adjacent}}$

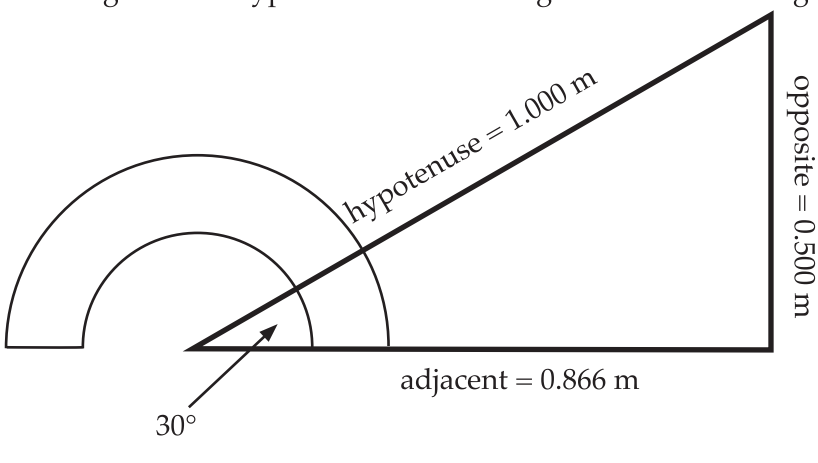
Note that these equations are algebraic, so $\frac{\text{opposite}}{\text{hypotenuse}}$ means the length of the opposite side divided by the length of the hypotenuse. Get the students to use a calculator to find the value of each trigonometric ratio for the pair of similar triangles. Record the results in a table, like this:

Triangle	Sine	Cosine	Tangent
A	0.5	0.867	0.576
B	0.5	0.869	0.576

Note there will be some measurement error, but the values should be close for each ratio [sine (30º) = 0.500, cosine (30º) = 0.866, tan (30º) = 0.577]. Tell the students that their calculator is already programmed with these numbers; this programming is activated on some calculators by entering the angle in degree mode and pressing the relevant sine, cosine, or tangent key; on other calculators, you press the relevant sine, cosine, or tangent key and then the angle in degree mode.

Have the students confirm, by checking different pairs from Material Master 7-11, that these ratios remain constant for similar right-angled triangles.

Next get the students to draw a right-angled triangle with the following specifications: "The sides meeting at the right angle are 500 millimetres (0.500 metres) and 866 millimetres (0.866 metres) long." Use the large A3 protractor on the floor surface to measure angles and draw lines on the surface using a metre ruler and chalk. Ask the students to predict the angles and length of the hypotenuse of the triangle before measuring these features.



Remind the students of the formulae for the trigonometric ratios and ask how the angles could have been predicted, given sides of 0.500 metres and 0.866 metres. Tan (θº) = $\frac{\text{opposite}}{\text{adjacent}}$, so for this triangle, the angles must have tangents of $\frac{0.500}{0.866}$ = 0.577, and $\frac{0.866}{0.500}$ = 1.732 respectively. You may like to show students how the calculator can find these angles using the inverse tangent function on the calculator (e.g., $\tan^{-1}$ (0.577) = 30º). Alternatively, use the table of trigonometric ratios (Material Master 7-12) and find the angles that match tangents of 0.577 and 1.732.

Ask the students how the length of the hypotenuse could be predicted, given an angle of 30º and an opposite side of 0.500 metres. Because sine (30º) = 0.5, for this triangle 0.5 = $\frac{0.500}{\text{hypotenuse}}$, so the hypotenuse must be 1.

Connect the measurements with the formulae for ratios introduced previously. For example, "If sine (θº) = $\frac{\text{opposite}}{\text{hypotenuse}}$ and the hypotenuse is 1, what is the length of the opposite side?" (sine θº)

Tell the students to enlarge the 30º unit triangle on the floor by a factor of three by extending the hypotenuse to 3 metres. Ask them to predict the length of the opposite and adjacent sides. They should do so by multiplying each measurement from the unit triangle by three (the scale factor). Get them to check their predictions by measuring. They should also check, using their calculators, that the trigonometric ratios for these two similar triangles have remained constant under enlargement. They should note that $\frac{\text{opposite}}{\text{hypotenuse}} = 0.500$ and $\frac{\text{adjacent}}{\text{hypotenuse}} = 0.866$ for the enlarged triangle, matching the opposite and adjacent sides of the unit triangle. The $\frac{\text{opposite}}{\text{adjacent}}$ ratio for the enlarged triangle should remain 0.577 (or very close).

Ask the students how this might be useful to solve real-life problems. You may like to get the students to use the digital learning object (http://www.tki.org.nz/r/digistore/protected/objects/?id=2326&vers=1.0), which shows how trigonometry was used historically. Tell the students that they will later use trigonometry to measure the height of the tallest trees and/or buildings in their school.

Have the students repeat this sequence for two different right-angled triangles, e.g., [90º, 45º, 45º] and [90º, 70º, 10º], firstly with an hypotenuse of 1, then using different enlargements, to establish a basis for generalisation.

Using Imaging

Use the digital learning objects Sine (http://www.tki.org.nz/r/digistore/protected/objects/?id=2329&vers=1.0), and Cosine (http://www.tki.org.nz/r/digistore/protected/objects/?id=2328&vers=1.0) to show what happens to these ratios as the angle varies between zero and ninety degrees.

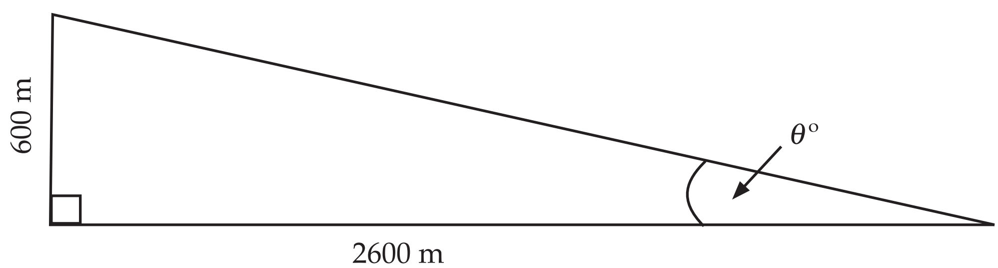
*The students achieving success proceed to **Using Number Properties**. Otherwise, they proceed to the next activity at some later time.*

Provide the students with situations in which an unknown side length or angle in a right-angled triangle must be found. Besides knowing that a given triangle is right-angled, students need only two side lengths, or another angle and one side length, to find out the other measures.

The scenarios might involve finding an unknown side or angle, such as building a skateboard ramp of given slope, the height of an aeroplane after takeoff, or the actual distance traveled by a climber, given the altitude and horizontal distance from base to a summit. Some of these problems can also be solved using Pythagoras' theorem if two side lengths are known.

For example, "An aircraft has climbed to an altitude that is 600 metres higher than where it took off. The distance on the ground it has travelled is 2600 metres. What has been the average angle of climb for the aircraft?"

A diagram of the problem might look like this:



Students need to realise that the trigonometric ratio to be used depends on the measurements available and the measurement that is required to answer the question. In this case, the opposite and adjacent sides of the triangle are known and the angle must be found. Therefore tangent must be used, i.e., $\tan(\theta°) = \frac{600}{2600} = 0.231$ so $\theta° = 13°$.

Create a scenario that involves the students finding the heights of tall trees or buildings in their local environment. This might involve working out the clearance needed when felling a tree or the height of scaffolding needed to paint a building.

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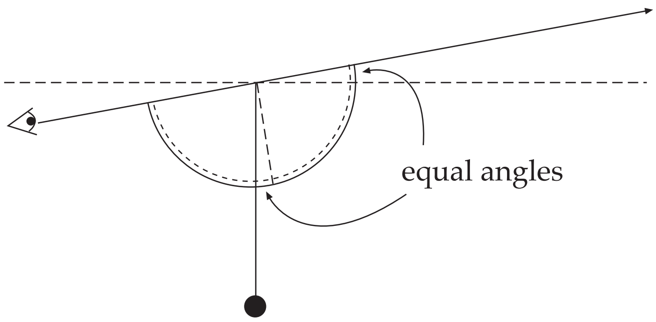
Tell the students that they can take measurements of height and angle from the ground. Introduce the clinometer as a device to measure angle of elevation. One can be made easily from a 90º protractor pasted on card, using a string and weight (two paper clips) as the plumb line.



Students may use a variety of methods. Commonly these involve scale drawing, using isosceles triangles (angles of 45º) or trigonometry. Any viable method is acceptable.

Get the students to use the learning objects at these links to apply trigonometric ratio to other problems:

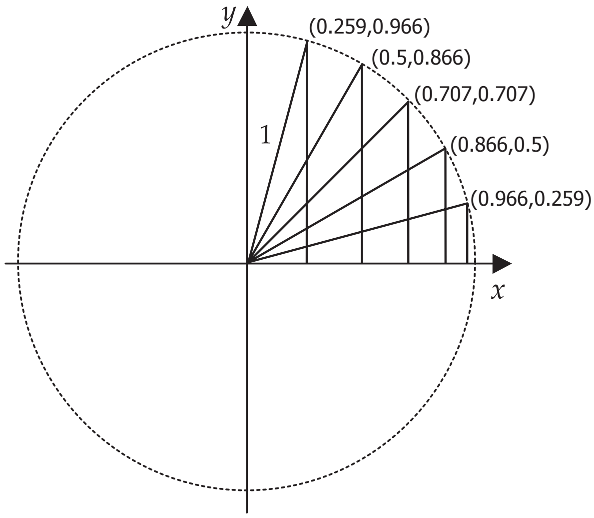
http://www.tki.org.nz/r/digistore/protected/objects/?id=2332&vers=1.0
http://www.tki.org.nz/r/digistore/protected/objects/?id=2333&vers=1.0
http://www.tki.org.nz/r/digistore/protected/objects/?id=2331&vers=1.0
http://www.tki.org.nz/r/digistore/protected/objects/?id=2334&vers=1.0
http://www.tki.org.nz/r/digistore/protected/objects/?id=2335&vers=1.0

Using Number Properties

While the use of common ratios of similar triangles is a useful way to highlight the practical applications of trigonometry, it does not lead easily to trigonometric functions. The unit circle, in which triangles with a hypotenuse of 1 are drawn, does. Introduce finding the sine and cosine of angles using co-ordinates on a number plane. Students will need fine graph paper to do this. Draw a quadrant number plane on the grid. Get the students to draw right-angled triangles within the first quadrant with hypotenuses of one unit (for convenience, 10 centimetres = one unit), as shown below (angles of 15º, 30º, 45º, 60º, 75º are suggested to show symmetry). Tell them to write coordinates for each vertex on the circumference of the circle.

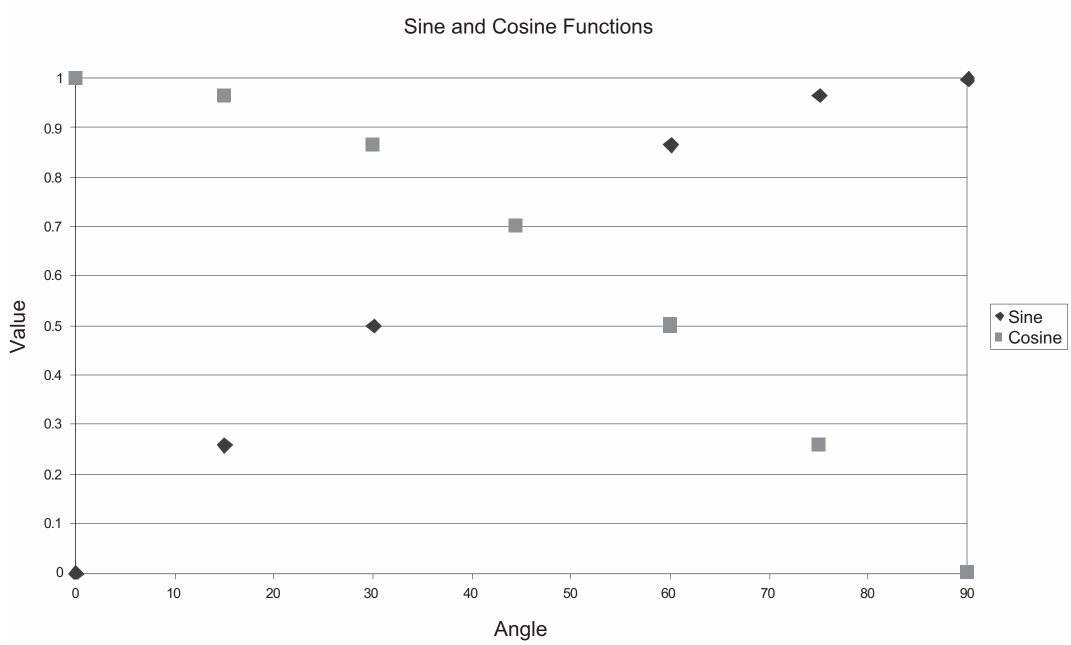
Ask the students to connect the x and y coordinates with the sine and cosine values for the angle of the triangle at the origin (0,0). They should note that the y-coordinate gives the sine value because this is the length of the side *opposite* the angle. Similarly, the x-coordinate value gives the cosine value because it is the length of the *adjacent* side.

Ask the students what the sine and cosine values will be for 0º and 90º (sine [0º] = 0 and cosine [0º] = 1; sine [90º] = 1 and cosine [90º] = 0).



Use a spreadsheet to graph sine and cosine as the angle varies in the first quadrant (goes from 0º to 90º).

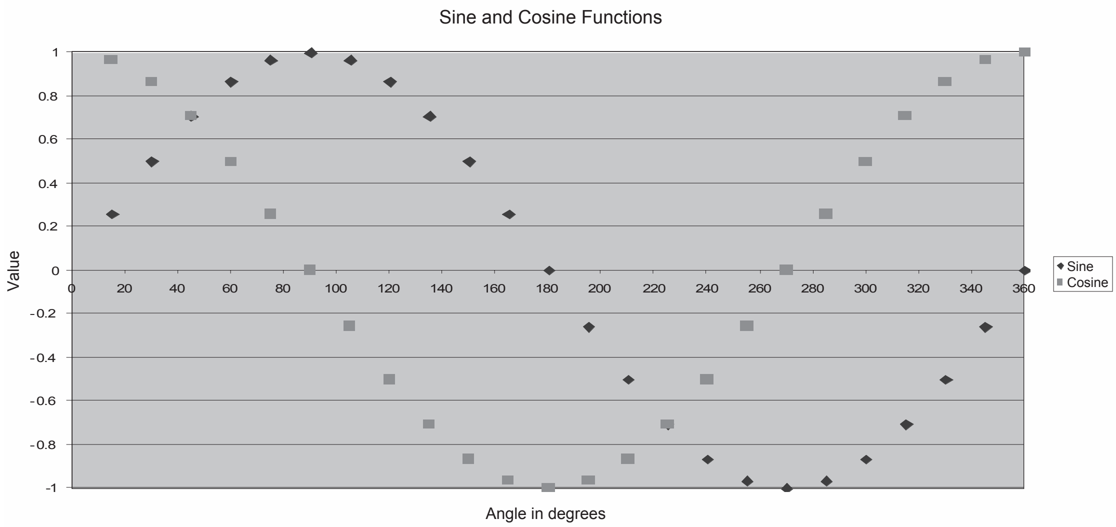
	A	B	C
1	Angle	Sine	Cosine
2	0	0	1
3	15	0.2588	0.9659
4	30	0.5	0.866
5	45	0.7071	0.7071
6	60	0.866	0.5
7	75	0.9659	0.2588
8	90	1	0



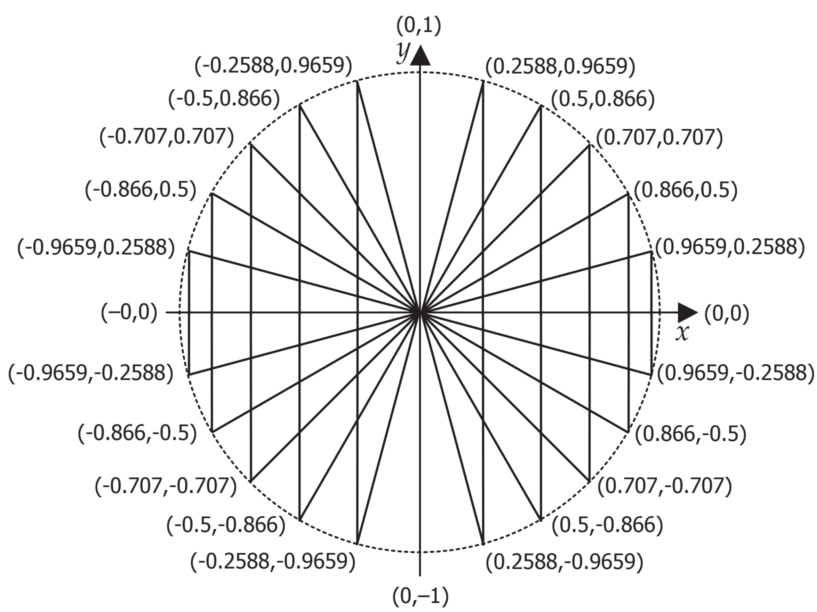
Ask the students to describe the graphs of the sine and cosine functions and to estimate values for 10º, 20º, 40º, 50º, 70º, 80º.

Now explore the tangent function in the first quadrant using the learning object at this link http://www.tki.org.nz/r/digistore/protected/objects/?id=2330&vers=1.0.

Now complete the unit circle for 15° intervals between 90° and 360° (105°, 120°, 135°, …). The students will not need to make any further measurements if they use symmetry, but they will need to attach the correct sign (+ or –) to each coordinate. They should now enter the pairs of coordinates into their spreadsheet to get graphs of the sine and cosine functions for the angles 0° to 360°.



The tangent function gives the slope of a line (in the case of the unit circle, the slope of the unit radius). Slope is "vertical height divided by horizontal distance", so the tangent value can be worked out from the sine and cosine values. The y-coordinate (sine value) gives vertical height and the x-coordinate (cosine value) gives horizontal distance, so tangent = $\frac{\text{sine}}{\text{cosine}}$.

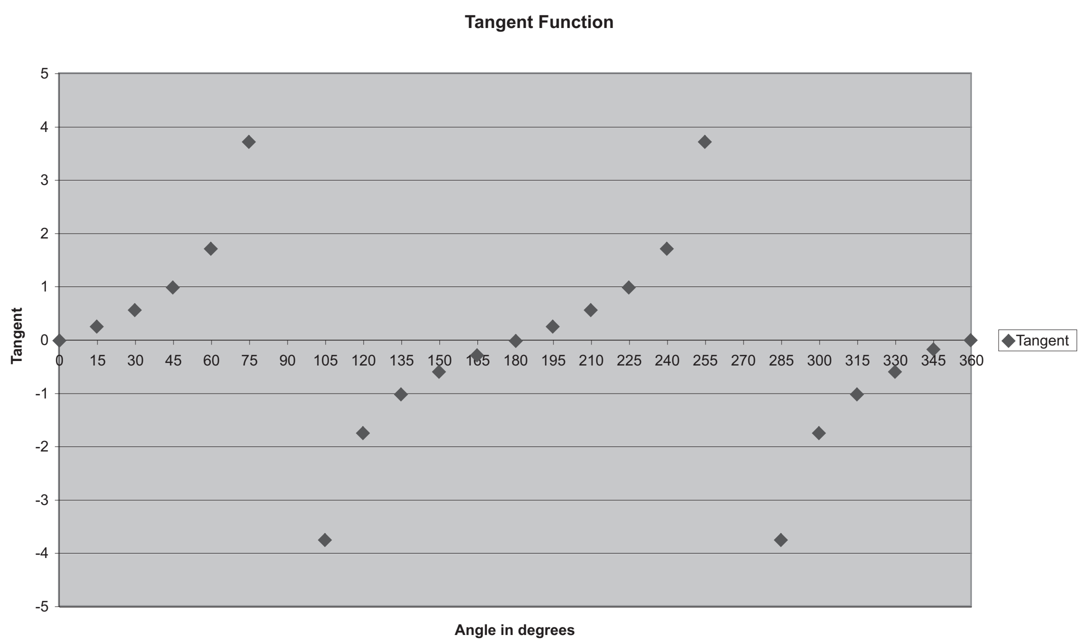


Ask the students, "For what angles is the slope of the hypotenuse equal to zero (0°, 180°, and 360°)? How do you know the slope is zero?" For 0°, 180°, and 360°, the y-coordinate (sine) is zero, so sine and cosine must be zero.

For two angles (90° and 270°), the horizontal distance, given by the *x*-coordinate or cosine, is zero. Highlight this feature and ask, "What does this mean for the tangent of these angles?" Division by zero is undefined (a calculator will show "error"), so students should consider the tangent values for angles close to 90° and 270°. For example, the tangent gets closer and closer to infinity (+) as the angle approaches 90° from below and to negative infinity (–) as the angle approaches 90° from above.

Using this knowledge, ask the students to sketch what they believe the graph of the tangent function will look like for angles in the range 0°–360°. They can confirm their graph by using a spreadsheet to calculate the tangent from sine and cosine values and getting the program to create the graph.

	A	B	C	D
1	Angle	Tangent	Opposite	Adjacent
2	0	0	0	1
3	15	0.267937	0.2588	0.9659
4	30	0.577367	0.5	0.866
5	45	1	0.707	0.707
6	60	1.732	0.866	0.5
7	75	3.732226	0.9659	0.2588
8				
9	105	-3.73223	0.9659	-0.2588
10	120	-1.732	0.866	-0.5
11	135	-1	0.707	-0.707
12	150	-0.57737	0.5	-0.866
13	165	-0.26794	0.2588	-0.9659
14	180	0	0	-1
15	195	0.267937	-0.2588	-0.9659
16	210	0.577367	-0.5	-0.866
17	225	1	-0.707	-0.707
18	240	1.732	-0.866	-0.5
19	255	3.732226	-0.9659	-0.2588
20				
21	285	-3.73223	-0.9659	0.2588
22	300	-1.732	-0.866	0.5
23	315	-1	-0.707	0.707
24	330	-0.57737	-0.5	0.866
25	345	-0.26794	-0.2588	0.9659
26	360	0	0	1



Encourage the students to consider what the graphs of the sine, cosine, and tangent functions will look like for angles greater than 360°. Make the connection with rotation of the radius of the unit circle to help students understand that the graphs are cyclic, with a period of 360° (that is, the pattern repeats every 360°).

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