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RICHARD COURANT AND THE FINITE ELEMENT METHOD: A FURTHER LOOK

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SUMMARIES

The finite element method has in recent years become one of the most popular and effective numerical procedures for boundary value and eigenvalue problems. In the early 1960s it was found that the method was an independent rediscovery of a simpler idea proposed in 1943 by Richard Courant. The importance of Courant's 1943 paper has been emphasized by G. Strang in 1973 and by others. This note seeks to describe briefly Courant's finite element work which led to his publication of 1943. It is shown that Courant used the finite element ideas as early as 1922 in a proof employing Dirichlet's principle.

La méthode de l'élément fini est devenue depuis les dernières années un des procédés numériques les plus populaires et les plus efficaces pour les problèmes de la valeur frontière et pour les problèmes de valeurs propres. Au début des années 60, on a constaté que cette méthode était une redécouverte indépendante d'une idée plus simple proposée par Richard Courant en 1943. En 1973 G. Strang et d'autres ont mis en relief l'importance de l'étude de Courant de 1943. L'article qui suit cherche à décrire brièvement le travail de Courant sur "l'élément fini" qui aboutit à la publication de 1943. On y montre que Courant employait déjà en 1922 l'idée de l'élément fini dans une démonstration de l'existence selon le principe de Dirichlet.

Die Methode der finiten Elemente ist in den letzten Jahren zu einem sehr populären und effektiven numerischen Verfahren zur Behandlung von Randwert- und Eigenwertaufgaben geworden. Anfang der sechziger Jahre wurde festgestellt, dass es sich dabei um eine unabhängige Wiederentdeckung einer einfacheren Idee handelt, die Richard Courant 1943 vorgeschlagen hatte. Auf die Bedeutung von Courants Arbeiten haben G. Strang (1973) und andere hingewiesen. In dieser Mitteilung wird

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Courants Beschäftigung mit finiten Elementen, die zur Veröffentlichung des Jahres 1943 führte, kurz beschrieben. Dabei wird nachgewiesen, dass Courant den Gedanken finiter Elemente bereits 1922 in Verbindung mit einem Existenzbeweis für das Dirichletsche Prinzip verwendete.

1. INTRODUCTION

The finite element method was developed in the American and European aircraft industries in the 1950s [Argyris 1960, Turner et al., 1956] and was transformed in the 1960s from a physically based procedure with a limited mathematical foundation into the present-day method resting upon variational principles [Fraeijs de Veubeke 1965, 1964, Melosh 1963]. The technique has become increasingly popular with both engineers and numerical analysts, and its applica tion has been extended far beyond the original aero-elasticity problems [Strang and Fix 1973, Williamson 1976, Zienkiewicz 1970]. It has been recognized increasingly often as a general approximation method for boundary value and eigenvalue problems, often superior to finite difference methods. The name "finite element method" was first used by Clough [Clough 1960, Zienkiewicz 1970].

The present variational form of the finite element method for boundary value problems is based upon the replacement of a governing differential equation by the equivalent extremum statement of the problem. In this form, the solution is given by the function which causes an appropriate integral functional, defined over the region for the problem, to attain that extremum. An approximate solution is obtained through the construction of a piecewise (usually polynomial) trial function continuous on a mesh of finite elements or subregions into which the region has been divided. This function contains n unspecified parameters, most often representing values of the function or its derivatives at the nodal points of the mesh. The application of the integral functional to this trial function produces a function of the n parameters, the extremum of which is found via n partial differentiations. The system of n (usually linear) equations which results is solved to fix the approximation. With proper attention to completeness of the trial function and to other considerations, convergence with mesh refinement can be expected as n approaches ∞. The procedure for eigenvalue problems is similar.

As the variational reformulation was taking place in 1960s, it was observed, probably as early as 1963 [Key 1974], that the new variational form was an independent rediscovery of a simpler method presented briefly in a [1943] paper by Richard Courant (1888-1972). While other antecedents of the variational form from outside the aircraft industry have been noted in the survey literature, for example, those of Synge [1957] and Polyá [1952], the paper by Courant [1943] is the earliest example cited [e.g., Kolata 1974, Strang 1973].

We show that this was not the only instance of Courant's use of the finite element procedure. Rather, it was the last in a series of short presentations of the idea dating from 1922, when it first appeared in an existence proof for a variation of the Riemann Mapping Theorem [Hurwitz & Courant 1922].

To indicate the essentials of the finite element method used in Courant's article of 1943 [1], consider the following passage which appears after a discussion of the Rayleigh-Ritz method [2]:

> *If the variational problems contain derivatives not higher than the first order the method of finite differences can be subordinated to the Rayleigh-Ritz method by considering in the competition only functions ϕ which are linear in the meshes of a subdivision of our net into triangles formed by diagonals of the squares of the net. For such polyhedral functions the integrals become sums expressed by the finite number of values of ϕ in the net points and the minimum conditions become our difference equations. Such an interpretation suggests a wide generalization, which provides great flexibility and seems to have considerable practical value* [Courant 1943, 10].

It is the idea of piecewise trial functions, defined over a mesh of subregions or finite elements, which is the central feature of the finite element method. While later in the article Courant applies the method to a torsion problem involving a column with doubly connected cross section, using both regular and irregular meshes of varying sizes, most of the paper is devoted to a variety of topics other than the finite element method.

2. AN EXISTENCE PROOF

Since Courant's first mention of the finite element method in 1922 occurred in the course of an existence proof, it is useful to explain something of his approach to such proofs as illustrated in [Courant 1912, 1914, 1950, Hurwitz & Courant 1922]. The goal was to show the existence of a solution to the variational problem: find the function u, selected from a prescribed class of admissible functions defined over some domain Ω, which results in a minimum value, d, for the Dirichlet integral,

$$\iint_{\Omega}\left(\left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial y}\right)^2\right)dx\,dy = D(\Phi). \tag{A}$$

Courant customarily began with a sequence of functions $\Phi_1, \Phi_2, \Phi_3, \ldots,$ such that

$$\lim_{j\to\infty} D(\Phi_j) = d; \tag{B}$$

this sequence was "smoothed" or "normed," creating another sequence which was shown either to converge to the desired solution or to provide a convergent subsequence. (That is, the original sequence wa replaced by a sequence the terms of which are harmonic in circular or rectangular subdomains of Ω.) Since only the existence of the original sequence $\{\Phi_j\}$ was necessary, and this was usually assured by the existence of a lower bound for $D(\Phi)$, the explicit specification of the functions in the sequence was not required. Courant referred to a sequence with property (B) as a minimizing sequence [*eine Minimalfolge*]. This general method of proof, based upon an integral functional (A) rather than upon an equivalent partial differential equation, is now called *Dirichlet's principle*. However, it was David Hilbert (1862-1943) who first made use of the minimizing sequence in this way [Hilbert 1904]. In so doing, he rescued Dirichlet's principle, which had fallen into disuse after K. Weierstrass, in 1870, had constructed a counterexample which invalidated its previous applications [Monna 1975, 35-65]. Moreover, Hilbert was partly responsible for Courant's interest in the Dirichlet principle, having successfully suggested it to Courant [Reid 1976, 29] as the subject of his dissertation [Courant 1910].

In 1922, the well-known *Funktionentheorie* of Hurwitz and Courant was published. Originally, Courant was only to have edited the Hurwitz two-part volume on general function theory and elliptic functions; but later he added a third part on geometric function theory [Friedrichs 1972]. In this part, Courant proved the existence of the harmonic function $u(x,y)$ required in his proof of a version of the Riemann Mapping Theorem (u is the real part of the desired mapping function $f(z) = u + iv$). The proof followed, in general outline, the arguments using Dirichlet's principle which are given above. However, because the mapping theorem required that $f(z)$ have a singularity at the origin resulting from a term of the form $1/z$, u and the corresponding admissible functions Φ needed to contain a term of the type $x/(x^2 + y^2)$. For functions containing such a singularity, $D(\phi)$ becomes infinite when the region of integration includes the origin. Consequently, Courant adjusted the class of admissible functions to allow only those of the form $\Phi = \phi - S$ where S is a given function with the prescribed singularity at the origin, and the functions ϕ are such that Φ is continuous and has piecewise continuous first derivatives. In the usual manner, Courant then introduced an unspecified minimizing $\Phi_1, \Phi_2, \Phi_3, \ldots$. Accompanying this sequence was a rather long footnote [3]:

> *The actual construction of such minimizing sequences, unimportant for the bare existence proof, presents no important difficulties. For example, if* G *is a finite domain, bounded by simple curves* C, *we imagine it covered by a net of triangles* T_i, *depending on an*

> *index* j, *so that the mesh grows finer with increasing* j. *We now consider only functions* ϕ *or* $\Phi = \phi - S$ *where the difference* $\phi - x/(x^2 + y^2)$ *is a linear function in each triangle* T_j. *For* Φ *we understand the functions corresponding to* T_j *constructed so that* $D(\Phi)$ *attains its smallest value. This condition* $D(\Phi) = min.$ *is now a problem of the minimum of a function of a finite number of variables, namely the integral interpreted in its dependence on the values of* ϕ *in the corner points of the triangular partition; this problem is certainly soluble and, moreover, as is easily seen, by means of linear equations. That the corresponding functions* Φ_j *actually form a minimizing sequence, follows from the easily provable fact that every admissible function* Φ *and its Dirichlet integral can be approximated arbitrarily accurately with the help of our construction with increasing* j [Hurwitz & Courant 1922, 338].

This sketch for constructing a minimizing sequence certainly reflects most of the essentials of the finite element method.

Of course, as Courant suggests, the specification of the minimizing sequence was not needed in the existence proof itself, but Courant had mentioned his intention of indicating "at least in principle, a method for the actual construction of the functions in question" [Hurwitz & Courant 1922, 322]. Courant was attracted to this kind of computational application of existence proof methods after the appearance, in 1908, of a paper by Walter Ritz which made the Rayleigh-Ritz method popular [Ritz 1908; Reid 1976, 114]. The latter had given a numerical approximation (without simple piecewise trial functions) of solutions to partial differential equations by using a minimizing sequence similar to one used by Hilbert [1904] in his proof based on Dirichlet's principle. Nevertheless, Courant did not suggest in *Funktionentheorie* [1922] that the idea might be anything more than an ad hoc scheme. It does not generalize easily to integrands with derivatives of order greater than one (such derivatives of piecewise linear functions vanish), and one of the changes in his next exposition of the finite element approach attempted to deal with this difficulty.

3. A NUMERICAL METHOD

In 1924, the first edition of Volume 1 of *Methoden der mathematischen Physik* was published; in it there appeared a brief description [4] of the finite element idea [Courant & Hilbert 1924, 158-159]. This time it was presented as a general numerical procedure (unconnected to an existence proof) for solving problems in one independent variable in which the minimum of integrals of the form:

$$\int_{x_0}^{x_1} F(x,y,y')\ dx = D(y) \qquad \text{(C)}$$

was sought. Here Courant proposed a minimizing sequence of piecewise linear trial functions constructed over the interval of integration which had been divided into equal subintervals [Courant & Hilbert 1924, 158-159]. He added that if derivatives higher than first order occur in the integrand, the corresponding difference quotients may be employed there.

This version of the finite element method was described in several subsequent places [Courant 1926, 1927; Courant & Hilbert 1931]. The latter two reflect a change in viewpoint [7], requiring that the sum

$$\sum_{i=1}^{n} F\left(x_i,\ y_i,\ \frac{(y_{i+1} - y_i)}{\Delta x}\right) \Delta x$$

be minimized, rather than (C), into which the piecewise linear trial function has been substituted. This method, closely related to the finite element method and now called the finite difference energy method, has found a growing number of recent applications [Felippa 1973, 1-14]. It also resembles closely, as Courant himself pointed out, the method used by L. Euler in 1744 for the derivation of the Euler differential equation [Euler 1744/1952, 1-80].

The only other published description of the finite element method by Courant occurred in his paper of 1943, already discussed, in which he returned to problems containing derivatives no higher than the first in the integrand.

4. CLOSING REMARKS

It should be emphasized that the earliest example to be found in Courant's published works of his use of finite elements was related to an existence proof in function theory which employed Dirichlet's principle; i.e., it arose from a problem in pure rather than in applied mathematics. Part of the explanation for this must lie in Courant's appreciation of the twofold usefulness of the minimizing sequence: as an important ingredient in existence proofs, and also as a mechanism for practical approximations of the limit function [Reid 1976, 114].

Despite the current importance of the finite element method, Courant, in the main, did not pursue this aspect of his work very strenuously. In his 1943 paper he did not refer to his previous work with finite elements, and many of his colleagues were unable to recall much about Courant's interest in the subject [Friedrichs 1977, Key 1974, Lewy 1977, Neugebauer 1977]. However, H. B. Keller

[6] recalls that from about 1954 Courant had at least two mathematicians at New York University, including Keller, working from time to time on finite element ideas; but "unfortunately we did not push too hard" [Keller 1979]. Nevertheless, a paper based upon piecewise linear triangular finite elements was published by Friedrichs and Keller [1966], following a similar, earlier paper by Friedrichs [1962].

By 1959, the difficulties with the Rayleigh-Ritz method, as well as with the general class of such direct methods, had been known to applied mathematicians for some time, so that finite difference methods were becoming more and more attractive. In that year, Courant invited his colleagues "to give some attention to the highly flexible Rayleigh-Ritz method and to the so-called direct variational methods which may have become unfashionable undeservedly" [1960]. Although it may not have appeared so at the time, the basis for a technique like the finite element method had been available to Courant since 1922. Later such techniques, as developed in the 1960s and 1970s, would prove to be the answer to his concern.

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NOTES

1. Although Courant's procedure required the construction of the linear system of equations for the nodal values, it did not cite the efficiencies possible in the assembly of that system. Such efficiencies account for much of the practicality of the present variational form of the finite element method.

2. The Rayleigh-Ritz method [Ritz 1908] approximates the solution of boundary value problems by replacing the governing differential equation with an equivalent statement which requires finding the extremum of an integral functional over a prescribed class of admissible functions. This extremum is approximated by a trial function which is a linear combination of n so-called "coordinate" functions with the same number of unspecified parameters as coefficients. The trial function is inserted into the functional which produces a function of the n parameters. The n conditions for the extremum of this functional yield an equal number of equations and unknowns and thus fix the approximation. With a proper choice of coordinate functions, the minimizing sequence of trial functions converges to the extremum as n increases

3. This footnote did not appear in subsequent editions of [Hurwitz & Courant 1922] with which this writer is familiar. It may have been removed by O. Neugebauer, who revised the first edition [Reid 1976, 94], although he does not recall whether he removed it or not.

4. The passage was almost certainly Courant's work rather than Hilbert's [Courant & Hilbert 1924, v-vi; Reid 1976, 97-98; Weaver 1972]. It is possible that it was written before 1922 [Reid 1976, 92-93].

5. The passage written in 1931 appears in the English edition of *Methoden der mathematischen Physik* [Courant & Hilbert 1953].

6. Keller also mentioned that the ideas of the penalty method, a scheme for treating boundary conditions other than the natural ones for a variational problem (and which Courant was the first to employ in the 1943 article), are now applied by users of the finite element method [Keller 1979].

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