Riemann's zeta function and Newton's method: Numerical experiments from a complex-dynamical viewpoint *

Tomoki Kawahira

Nagoya University, Nagoya, JAPAN

URL: http://math.nagoya-u.ac.jp/~kawahira

Abstract. We apply some root finding algorithms to characterize the zeros of Riemann's zeta. Finally we propose the *Dynamical Riemann Hypothesis* which state the Riemann Hypothesis in terms of dynamical systems.

1 Riemann's zeta and primes

For $s = \sigma + it \in \mathbb{C}$, one can easily see that the series

$$\zeta(s) \;=\; 1 + \frac{1}{2^s} + \frac{1}{3^s} \cdots \;=\; \prod_{n=1}^{\infty} \left(1 - \frac{1}{p_n^s}\right)^{-1}$$

converges if $\sigma > 1$, where p_n is the nth prime number. Indeed, $\zeta(s)$ is analytic on $\{\mathrm{Re}\, s = \sigma > 1\}$ and by analytic continuation we consider it a meromorphic function $\zeta : \mathbb{C} \to \bar{\mathbb{C}}$ with only one pole at $s = 1$, which is simple.

The Riemann Hypothesis. The most famous conjecture on Riemann's zeta function is: *ζ has non-real(non-trivial) zeros only on the critical line*

*Poster presented at Kanazawa, June 20-24, 2005. Revised on Aug 29 2006. After I presented this poster, I learned that several people were working on this subject before my experiment. Aimo Hinkkanen published a paper on some properties on relaxed Newton's method for Riemann's zeta. Dierk Schleicher and his student generated a picture of λ several years ago. Nick Sullivan generated some pictures of ν in 2002. Here I do not intend to present *new pictures*, but I suggest a possible direction of investigations of the zero-free region in the critical stripe.

$\mathrm{Re}\, s = \sigma = 1/2$ *(the Riemann Hypothesis)*. If this conjecture is affirmative, we will have a nice result on the distribution of prime numbers;

$$p_{n+1} - p_n \;=\; O(p_n^{1/2} \log p_n).$$

This is better than any known results, for example;

$$p_{n+1} - p_n \;=\; O(p_n^{0.525+\epsilon})$$

for any $\epsilon > 0$. To show the hypothesis, it is known that we only have to care the zeros on the critical stripe $\mathcal{S} = \{s \in \mathbb{C} : 0 < \mathrm{Re}\, s < 1\}$. In particular, *wider zero-free regions imply better estimates of distribution of primes.*

For example, it is known that there exists a constant $A > 0$ such that

$$\left\{ s = \sigma + it \in \mathcal{S} \;:\; \sigma \geq 1 - \frac{A}{(\log(|t|+1))^{2/3}(\log\log(|t|+1))^{1/3}} \right\}$$

is zero-free.

2 Newton's method

There are some root finding algorithms, but the most famous one would be *Newton's method*. From now on, we work with complex variable $z = x + yi$ instead of conventional s for ζ.

For a meromorphic function $f : \mathbb{C} \to \bar{\mathbb{C}}$, we define its *Newton's map* N_f by

$$N_f(z) \;=\; z - \frac{f(z)}{f'(z)},$$

which is again meromorphic. One can easily check that $f(\alpha) = 0$ iff $N_f(\alpha) = \alpha$. The idea of Newton's method is: Start with an initial value z_0 sufficiently close to α. Then the sequence $\{z_n\}$ defined by $z_{n+1} = N_f(z_n)$ converges (rapidly) to α.

More precisely, we have the following property:

If α is a simple zero of f, then $N_f(\alpha) = \alpha$ and $N_f'(\alpha) = 0$. Thus

$$N_f(z) - \alpha \;=\; O((z-\alpha)^2) \quad (z \to \alpha).$$

If α is a multiple zero, then $N_f(\alpha) = \alpha$ and $|N_f'(\alpha)| < 1$. Thus

$$|N_f(z) - \alpha| \;\leq\; C|z - \alpha| \quad (z \to \alpha)$$

for some $0 < C < 1$.

Hence the precision of z_n as an approximate value of α is exponentially or linearly increasing according to the multiplicity of α.

Newton's method as a dynamical systems. What makes this method more intriguing is the theory of iteration of holomorphic function developed by Fatou and Julia in early 1920s. For given $z_0 \in \mathbb{C}$, convergence of $z_n = N_f^n(z_0)$ (where N_f^n is nth iteration of N_f) is not guaranteed in general. To investigate the behaver of such sequence, we consider the global dynamical systems

$$\bar{\mathbb{C}} \xrightarrow{N_f} \bar{\mathbb{C}} \xrightarrow{N_f} \bar{\mathbb{C}} \xrightarrow{N_f} \cdots$$

given by iteration of Newton's map. (As we will see, we need a spacial care for poles of N_f.) For example, set $f(z) := z^3 - 1$. Then the iteration of its Newton's map gives the following picture (Figure 1):

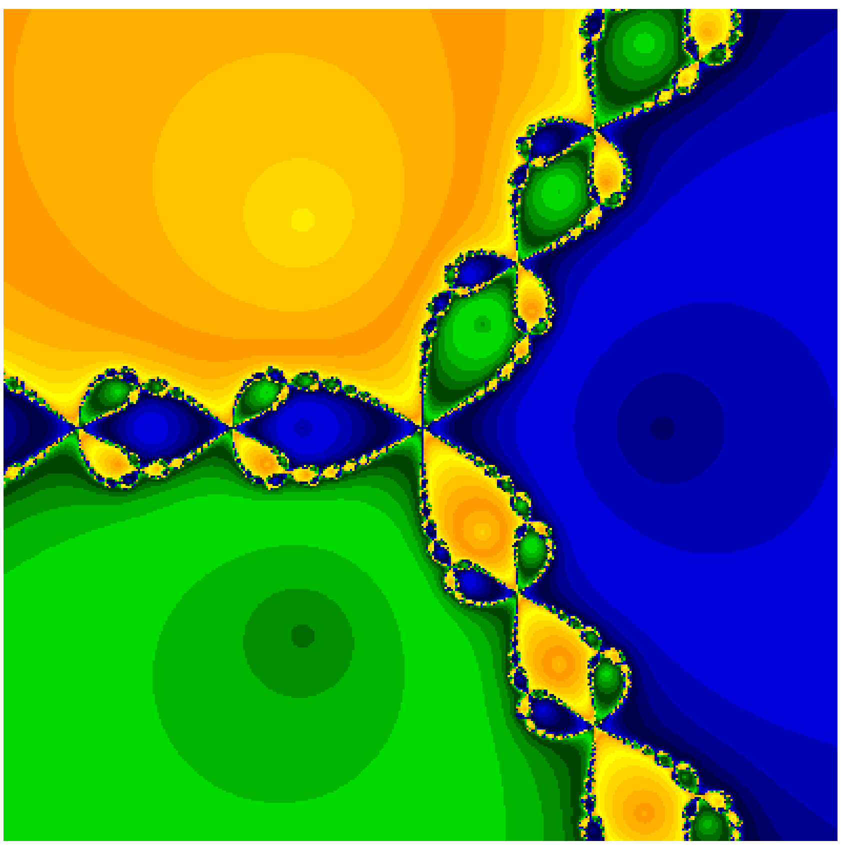

Figure 1: Dynamics of N_f for $f(z) = z^3 - 1$.

Blue, yellow, and green regions are the set of initial values z_0 such that the orbit $z_n = N_f^n(z_0)$ converges to 1, $\frac{-1+\sqrt{3}i}{2}$, and $\frac{-1-\sqrt{3}i}{2}$ respectively. Shades distinguish the number of iteration to trap the orbit in small disks around roots. The boundary of these regions has complicated structure known as *fractal*. It is the *Julia set* of N_f, where the dynamics shows chaotic behavior. In particular, orbits from the Julia set stay within the Julia set and never converge to the roots.

Newton's method for meromorphic functions. If f is a rational function, then so is N_f thus it has no essential singularity. For a meromorphic function f, its Newton's map has an essential singularity at infinity. Since $N_f(\infty)$ is indeterminate, we must stop the iteration when the orbit lands on a pole of N_f. In this particular setting, we define its *Fatou set* $F(N_f)$ by:

$$\begin{aligned} & z_0 \in F(N_f) \\ \iff & \exists U \text{ a nbd of } z_0 \text{ s.t. } \left\{N_f^n|U\right\}_{n\geq 0} \text{ is defined and a normal family} \end{aligned}$$

The *Julia set* $J(N_f)$ is the complement $\mathbb{C} - F(N_f)$.

3 Applying the method to zeta.

Now let us apply Newton's method to Riemann's zeta. For the meromorphic function $\zeta : \mathbb{C} \to \bar{\mathbb{C}}$, we set

$$\nu(z) \;:=\; N_\zeta(z) \;=\; z - \frac{\zeta(z)}{\zeta'(z)}.$$

We also apply the method to the functions

$$\eta(z) \;:=\; (z-1)\zeta(z)$$

and

$$\xi(z) \;=\; \frac{1}{2}z(1-z)\pi^{z/2}\Gamma(z/2)\zeta(z),$$

where $\xi(z)$ a classical zeta-related function with symmetry $\xi(z) = \xi(1-z)$. Since $\eta(z)$ and $\xi(z)$ are entire functions, we may expect better dynamics for

$$\mu(z) \;:=\; z - \frac{\eta(z)}{\eta'(z)} \qquad \text{and} \qquad \lambda(z) \;:=\; z - \frac{\xi(z)}{\xi'(z)}.$$

Now let us go to the gallery!

Pictures for ν. The first picture is on the dynamics of ν. The coloring indicates the number of iteration to trap the orbits in attracting fixed points:

$$0 = \text{orange} < \text{yellow} < \text{green} < \text{blue} < \text{purple} < \text{red} = \textit{maximum}.$$

Probably points colored in red are close to the Julia set.

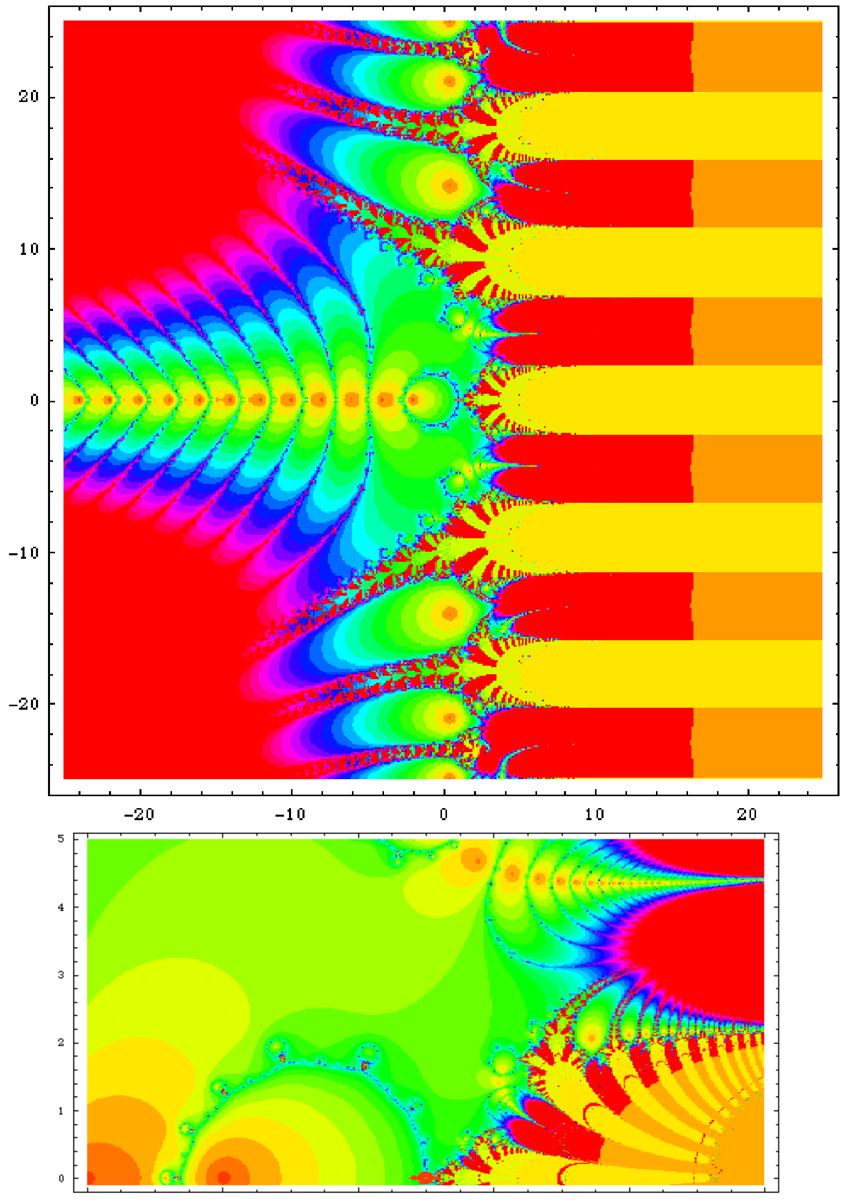

Figure 2: The orange dots are arrayed on $-2\mathbb{N}$ and the critical line. The picture in the bottom is a magnification near the origin. Probably the sequence of orange dots near $\{\operatorname{Im} z = 4.5\}$ are preimages of $-2\mathbb{N}$.

Pictures for μ Next we show the pictures of the dynamics of μ. The Julia set of μ seems much simpler.

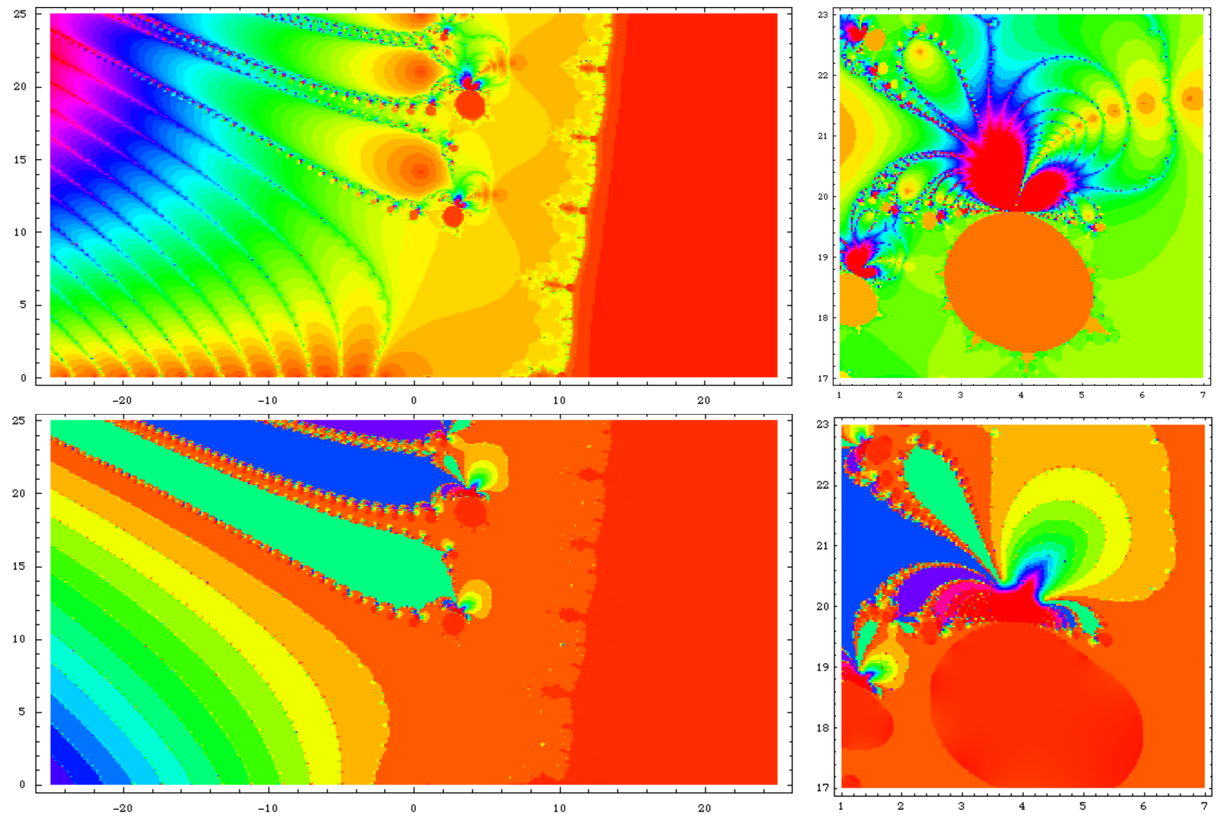

Figure 3: The Julia set of $\mu(z)$. The pictures in the second row are colored to distinguish the fixed points to converge. The pictures on the right shows the details of a prospective pole of $\mu(z)$ ("A head of chicken").

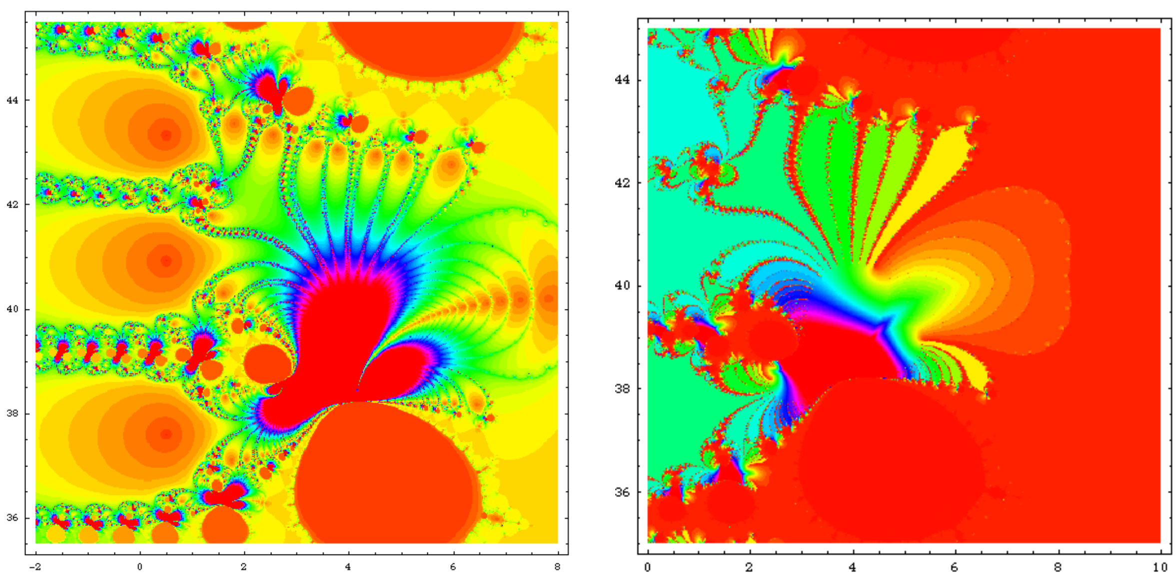

Figure 4: Head of another chicken in different colorings.

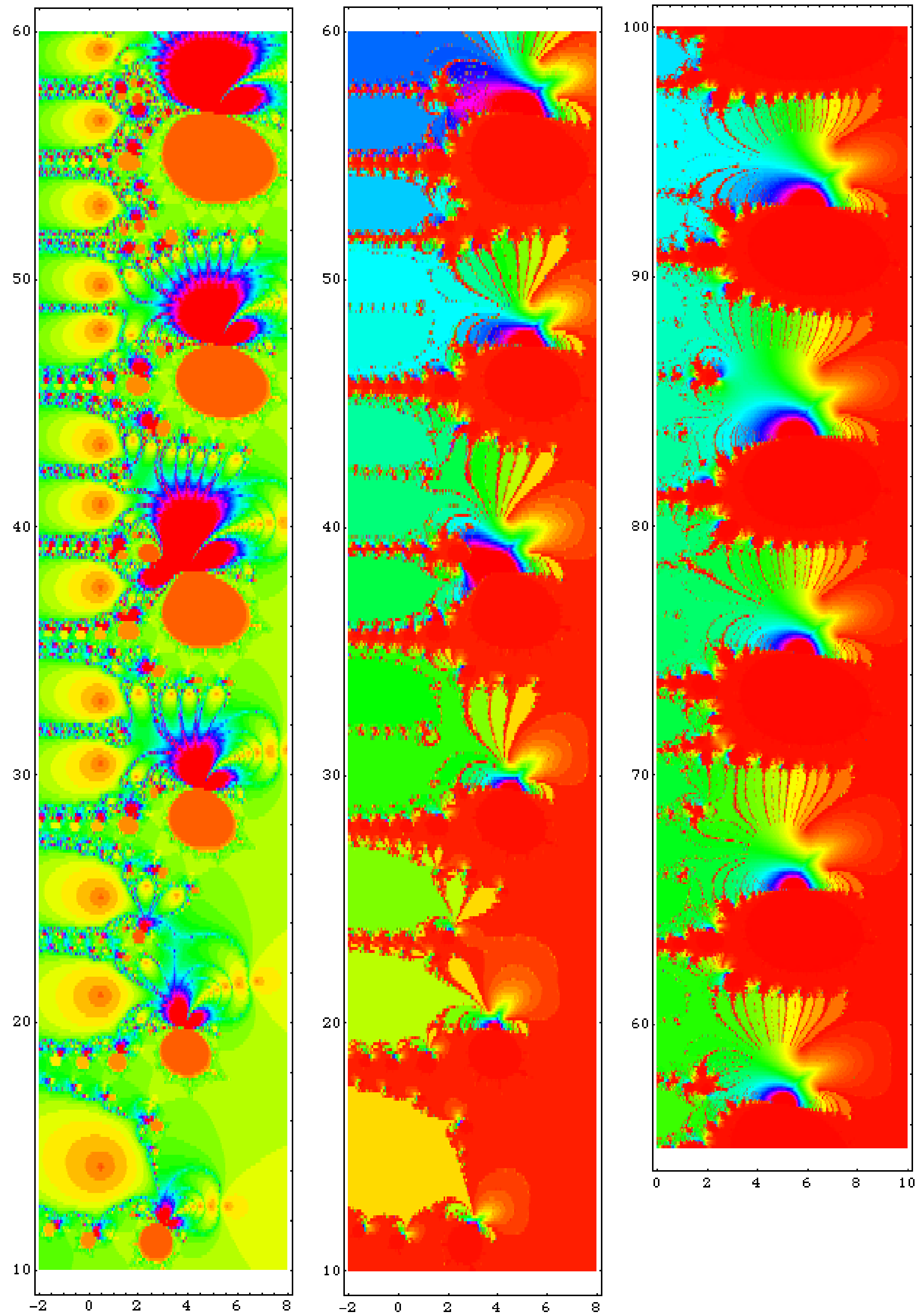

Figure 5: Chickens for $\mu(z)$. Heads appear constantly in this range, though the zeros get denser as their imaginary parts increase.

Pictures for λ. Finally we go to λ. One can easily check that the Newton's map λ has a symmetry with respect to the point $z = 1/2$. The dynamics seems the simplest, but the calculation for λ is the heaviest.

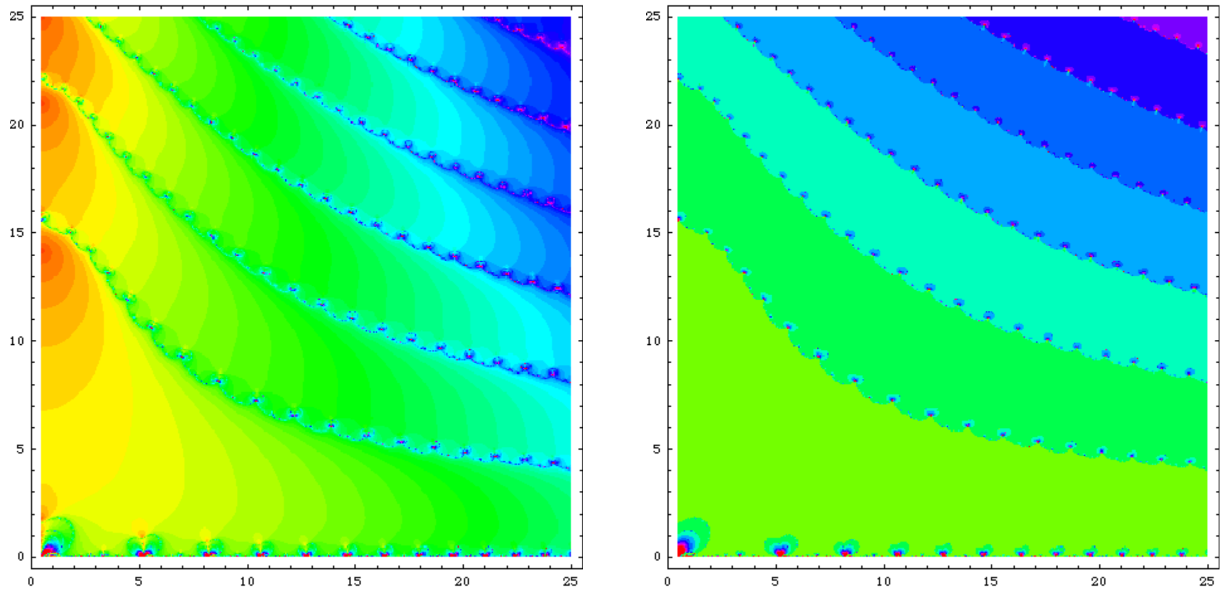

Figure 6: Julia sets for $\lambda(z)$. The dynamics seems very simple: Probably each layer has conformally the same dynamics as $z \mapsto z^2$ on the unit disk.

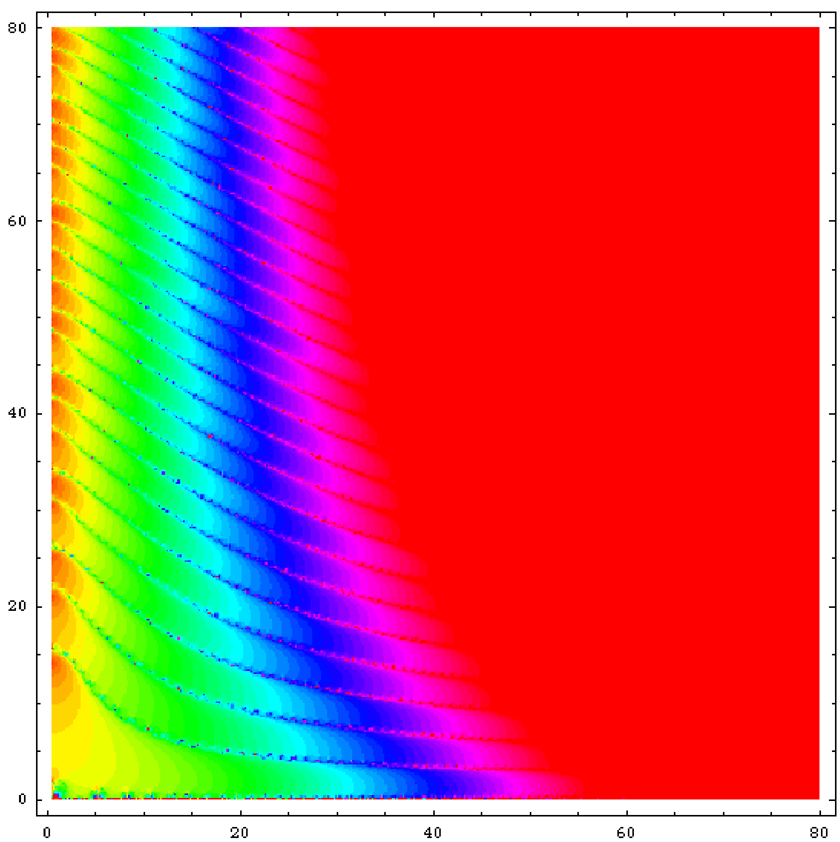

Figure 7: Julia set for $\lambda(z)$ (large scaled).

4 Further applications of the theory of dynamical systems

For these Newton's maps, we have:

$$\text{fixed-point-free region of } \nu,\ \mu, \text{ and } \lambda \implies \text{zero-free region of } \zeta.$$

For example, if $\nu(D) \cap D = \emptyset$ for some region D then ζ is zero-free on D. Hence the problem of zero-free region is translated to be rather topological.

More generally, to Riemann's zeta function $\zeta(z)$, one may apply other root finding algorithms, or functions of the form

$$\nu_h(z) = z + h(z)\zeta(z)$$

with holomorphic $h(z)$ which does not vanish on the critical stripe $\mathcal{S}$ minus the critical line. We temporarily say such functions are *generalized ν-functions*. For these functions, zeros of $\zeta(z)$ do not necessarily attract nearby points. However, $\zeta(\alpha) = 0$ still implies that α is a fixed point of ν_h.

Now our translation of the Riemann Hypothesis is:

Dynamical Riemann Hypothesis. *Any fixed points of generalized ν-functions in the critical stripe $\mathcal{S}$ are arrayed on the critical line.*

This is slightly stronger than the original.

For example, even the dynamics of

$$\nu_1(z) = z + \zeta(z)$$

is intriguing to work with. Of course, the entire function

$$\nu_{z-1}(z) = z + (z-1)\zeta(z)$$

may be easier to work with. I hope the Dynamical Riemann Hypothesis brings new perspective for the theory of zeta functions.

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5 References

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