Lecture Notes

Riemann Surfaces

Complex Analysis from a Differential Geometric Viewpoint

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Preface

This is my first lecture on compact Riemann surfaces and holomorphic line bundles and the very first lecture that I plan and hold by myself. It is mainly a mixture of Franz Pedit's $\mathbb{C}$-seminar [1] and Ulrich Pinkall's lectures on complex analysis [2].

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Propaganda

Though the function $z \mapsto z^2$ is not bijective, it has full rank away from zero and the inverse function theorem allows us to define a square root locally. This yields a nice analytic function which can be continued analytically to a function $z \mapsto \sqrt{z}$ defined on a slit region, say $\mathbb{C} \setminus (-\infty, 0]$. This so called branch is determined completely by the initial local function we choose to start with. Here this is basically a choice of a sign—if we would have chosen the other sign we would have ended up with another branch, namely $z \mapsto -\sqrt{z}$.

There is no reason why one of the branches should be preferred. Also the choice of the slit was completely arbitrary—one could have taken any other curve connecting zero to infinity on the Riemann sphere.

A way to resolve this problem is to change the domain of the function: The limit values of both branches on one and the other side of the negative real axis differ by sign. So they both can be glued across the negative real line to form a surface. On this surface the square root becomes a well-defined function.

Similarly, if we try to take the square root $\sqrt{p}$ of a polynomial $p(z) = \prod_{i=1}^{2n}(z - z_i)$ with pairwise distinct roots $z_i \in \mathbb{C}$, we can find a branch defined on $\mathbb{C} \setminus \{\gamma_1, \ldots, \gamma_n\}$, where γ_i are embedded curves each of which connects two of the roots of p. Again there are two branches which can be glued across the curves γ_i to form an abstract surface on which $\sqrt{p}$ becomes a well-defined function. By this we can produce compact surfaces of arbitrary topological type.

In 1913 Hermann Weyl published his "Die Idee der Riemannschen Fläche" [3] where he gave the first definition of a Riemann surface—in his eyes was the actual object of importance. A slightly newer exposition of this old approach can be found e.g. at the end of Ahlfors [4].

Though let us take yet another perspective. Consider the set $M = \{(z, w) \in \mathbb{C}^2 \mid w^2 = p(z)\}$. By the implicit function theorem it forms a complex 1-dimensional submanifold—a Riemann surface sitting in $\mathbb{C}^2 = \mathbb{R}^4$. Topology and complex structure are inherited from the ambient space and the function $\sqrt{p}$, which on M is just given by the projection to the second component, is a holomorphic function on M. Again it seems natural to regard solutions of algebraic equations, like the ones above, as functions defined on a Riemann surface. Though a famous theorem by Chow assures that all compact Riemann surfaces arise as algebraic curves, i.e. from polynomial equations, the analytic continuation picture hints at that the ambient space is of no importance at all. What has to be understood is holomorphic or meromorphic functions on Riemann surfaces.

It turns out that it is not only about functions but about holomorphic sections of holomorphic line bundles over a Riemann surface. Here one of the most famous results is the Riemann–Roch theorem. It gives us information about the dimension of the space of holomorphic sections. Here the topology of the holomorphic line bundle and the Riemann surface play a crucial role.

The main goal of this course will be to understand the Riemann–Roch theorem and its consequences from a differential geometric point of view. Therefore, we build up a rigorous framework of smooth manifolds and vector bundles and then use the elliptic theorem to draw conclusions about the topology of compact Riemann surfaces and meromorphic functions, differentials, or sections of holomorphic line bundles over Riemann surfaces. The elliptic theorem itself won't be proven here.

Content

Local Theory of Several Complex Variables

Some Foundations 1

1.1 Holomorphic Maps in Several Variables

1.1.1 Complex Differentiability

Every complex vector space has an underlying real vector space V. This allows us consider real linear maps between complex vector spaces opposed to only complex linear maps—as usual when one considers complex vector spaces as vector spaces over the field of complex numbers. In particular multiplication by the complex unit i can be considered as a distinguished real endomorphism which squares to -1—an almost complex structure.

To make things precise let us fix some notation: For two real vector spaces V and W, we set

$$\mathrm{Hom}(V;W) := \{\lambda\colon V \to W \mid \lambda \text{ linear}\}, \quad \mathrm{End}V := \mathrm{Hom}(V;V), \quad V^* := \mathrm{Hom}(V;\mathbb{R}).$$

Definition 1.1.1 (Almost Complex Structure) *Let V be a real vector space. An endomorphism $J \in \mathrm{End}V$ is called an almost complex structure if $J^2 = -I$.*

Conversely, given a real vector space V together with an almost complex structure $J \in \mathrm{End}V$, then this defines a *complex scalar* product:

$$(x+iy).v := xv + yJv \quad v \in V.$$

As such a complex vector space is equivalent to a tuple (V, J) consisting of a real vector space V and an almost complex structure $J \in \mathrm{End}V$. For convenience we usually just say that V is a complex vector space and leave the almost complex structure implicit. Clearly, $\dim V = 2\dim_{\mathbb{C}} V$.

Corollary 1.1.1 *On a vector space of odd dimension there is no almost complex structure.*

For complex vector spaces (V, J_V) and (W, J_W) we define the set $\mathrm{Hom}_+(V;W)$ of *complex linear* maps and the set $\mathrm{Hom}_-(V;W)$ of *complex antilinear* maps as follows

$$\mathrm{Hom}_\pm(V;W) := \{A \in \mathrm{Hom}(V;W) \mid AJ_V = \pm J_W A\}.$$

In particular, we have $\mathrm{End}_\pm V := \mathrm{Hom}_\pm(V;V)$.

Definition 1.1.2 (Holomorphic Map) *Let V, W be complex vector spaces and $U \subset V$. A map $f\colon U \to W$ is called complex differentiable at $p \in U$, if f is differentiable at p with complex linear differential $d_p f \in \mathrm{Hom}_+(V, W)$. If f is complex differentiable at all points $p \in U$, then f is called holomorphic.*

Exercise 1.1.1 Let f and g be holomorphic maps. Show that $f+g$ and $f \circ g$, wherever they are defined, are holomorphic. Moreover, if f and g are complex–valued, then fg and f/g are holomorphic where they are defined.

There are natural projections from the space of real linear maps to the space of complex linear and complex antilinear maps.

Theorem 1.1.2 *For each $A \in \mathrm{Hom}(V;W)$ there are is a unique $A' \in \mathrm{Hom}_+(V;W)$ and a unique $A'' \in \mathrm{Hom}_-(V;W)$ such that $A = A' + A''$. The maps A' and A'' are given by the projections*

$$A' := \tfrac{1}{2}(A - J_W A J_V), \quad A'' := \tfrac{1}{2}(A + J_W A J_V).$$

In particular, $\mathrm{Hom}(V;W) \cong \mathrm{Hom}_+(V;W) \oplus \mathrm{Hom}_-(V;W)$.

Proof. The proof is left as an exercise. □

Corollary 1.1.3 *Let $f: V \supset U \to W$ be differentiable. Then*

$$f \text{ holomorphic} \iff J(df) = (df)J \iff df = (df)' \iff (df)'' = 0.$$

Let V be a complex vector space. If we choose a complex basis $v_1, \dots, v_n$, then this defines coordinate functions $z_j = x_j + iy_j : V \to \mathbb{C}$,

$$I = \sum_{j=0}^{n} z_j v_j .$$

From this we get differentials $dx_j, dy_j \in V^*$ and $dz_j, d\bar{z}_j \in \mathrm{Hom}(V;\mathbb{C})$. Clearly, $dx_1, dy_1, \dots, dx_n, dy_n$ form a basis dual to $v_1, Jv_1 \dots, v_n, Jv_n$.

Exercise 1.1.2 The differentials $dz_1, \dots, dz_n$ resp. $d\bar{z}_1, \dots, d\bar{z}_n$ form a basis of $\mathrm{Hom}_+(V,\mathbb{C})$ resp. $\mathrm{Hom}_-(V,\mathbb{C})$.

In particular, if $f: V \to W$ is a differentiable map into another complex vector space, then df can be written in terms of dz_j and $d\bar{z}_j$:

$$df = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} dz_j + \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j ,$$

where $\frac{\partial f}{\partial z_j}$ and $\frac{\partial f}{\partial \bar{z}_j}$ are maps defined on U with values in W.

Exercise 1.1.3 (Wirtinger Derivatives) If $\frac{\partial}{\partial x_j}$ and $\frac{\partial f}{\partial y_j}$ denote differentiation with respect to the coordinates x_i, y_i, then

$$\frac{\partial f}{\partial z_j} = \frac{1}{2}\Big(\frac{\partial f}{\partial x_j} - i\frac{\partial f}{\partial y_j}\Big), \quad \frac{\partial f}{\partial \bar{z}_j} = \frac{1}{2}\Big(\frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j}\Big).$$

Thus a map $f: U \to W$ is holomorphic if and only if, for $j = 1, \dots, n$,

$$\frac{\partial f}{\partial \bar{z}_j} = \frac{1}{2}\Big(\frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j}\Big) = 0.$$

Exercise 1.1.4 Let $U \subset \mathbb{C}^m$. A map $f = (f_1, \dots, f_n): U \to \mathbb{C}^n$ is holomorphic if and only if for $i = 1, \dots, m$ the component functions $f_i: U \to \mathbb{C}$ are holomorphic.

Definition 1.1.3 (Biholomorphism) *Let V, W be complex vector spaces, $U \subset V$ and $\tilde{U} \subset W$. A map $f: U \to \tilde{U}$ is called biholomorphic if f is a holomorphic bijection with holomorphic inverse. If $f: U \to \tilde{U}$ is a biholomorphism, then U and $\tilde{U}$ are called biholomorphic.*

Notation 1.1.1 (Isomorphic Sets) If U and $\tilde{U}$ are homeomorphic, diffeomorhic resp. biholomorphic, we write $U \cong_{\mathscr{C}^0} \tilde{U}$, $U \cong_{\mathscr{C}^\infty} \tilde{U}$ resp. $U \cong_{\mathscr{O}} \tilde{U}$. If it is clear what category we are talking about we drop the index and just write $U \cong \tilde{U}$.

Exercise 1.1.5 Show that biholomorphy defines an equivalence relation.

A famous result on holomorphic functions in one variable is the Riemann mapping theorem, which we state here without proof.

Theorem 1.1.4 (Riemann Mapping Theorem) *Every non–empty simply–connected open subset $U \subsetneq \mathbb{C}$ is biholomorphic to the open unit disc $D^2 \subset \mathbb{C}$.*

Exercise 1.1.6 Consider the annuli $A_i = \{z \in \mathbb{C} \mid r_i < |z| < R_i\}$, $R_i > r_i > 0$, $i = 1, 2$. Show:

(a) $A_1 \cong_{\mathscr{C}^\infty} A_2$.
(b) $A_1 \cong_{\mathscr{O}} A_2 \Leftrightarrow R_1/r_1 = R_2/r_2$.

Exercise 1.1.7 Show that a holomorphic diffeomorphism is a biholomorphism.

It is also true that every biholomorphism is a holomorphic diffeomorphism. This follows from the fact that holomorphic functions are smooth, which we show in the next section.

1.1.2 Local Form of Complex Differentiable Functions

It turns out that the multiindex notation is neat way to deal with indexing in the higher dimensional case.

Definition 1.1.4 (Multiindex Notation) *For $\mathrm{k} \in \mathbb{Z}^n$ and $z \in \mathbb{C}^n$ we define*

$$z^{\mathrm{k}} := \prod_{j=1}^{n} z_j^{\mathrm{k}_j} .$$

Moreover, we define $\mathbf{1} := (1, \ldots, 1) \in \mathbb{Z}^n$.

For $z \in \mathbb{C}^n$ with $z_j \neq 0$, $j = 1, \ldots, n$, we then get

$$z^{-\mathbf{1}} = \prod_{j=1}^{n} z_j^{-1} = \frac{1}{\prod_{j=1}^{n} z_j} = \frac{1}{z^{\mathbf{1}}} .$$

Similarly, we write

$$\mathbf{dz} = (dz)^{\mathbf{1}} = \prod_{j=1}^{n} dz_j .$$

Definition 1.1.5 (Polydisk, n-Torus) *For $a \in \mathbb{C}^n$, $r \in (0, \infty)^n$, we define the $2n$-dimensional polydisk $D_{a,r}$ and the n-dimensional Torus with center a and radius r by*

$$D_{a,r} := \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_j - a_j| \leq r_j \forall j\} = D_{a_1,r_1} \times \cdots \times D_{a_n,r_n} ,$$
$$T_{a,r} := \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_j - a_j| = r_j \forall j\} = \partial D_{a_1,r_1} \times \cdots \times \partial D_{a_n,r_n} .$$

With this in place we can finally state the multi-dimensional version of the Cauchy–integral formula.

Theorem 1.1.5 (Cauchy–Integral Formula) *Let $U \subset \mathbb{C}^n$ be open, $D_{a,r} \subset U$ and $f : U \to \mathbb{C}$ holomorphic. Then*

$$f(z) = \tfrac{1}{(2\pi i)^n} \int_{T(a,r)} \tfrac{f(w)}{(w-z)^{\mathbf{1}}} \mathbf{dw}, \quad \forall z \in \mathring{D}_{a,r} .$$

Proof. This follows by induction on $n \in \mathbb{N}$: For $n = 1$ that's just the well-known Cauchy formula. For the induction step we just consider the function $g = f(-, z_2, \ldots, z_n)$ defined on D_{a_1,r_1}. Then, by the Cauchy–integral formula applied to g, the induction hypothesis and Fubini's theorem, we get

$$\begin{aligned} f(z_1,\ldots,z_n) &= \tfrac{1}{2\pi i}\int_{\partial D_{a_1,r_1}} \tfrac{f(w_1,z_1,\ldots,z_n)}{w_1-z_1}\,dw_1 \\ &= \tfrac{1}{2\pi i}\int_{\partial D_{a_1,r_1}} \tfrac{1}{w_1-z_1}\tfrac{1}{(2\pi i)^{n-1}}\int \tfrac{f(w_1,\ldots,w_n)}{(w_2-z_2)\cdots(w_n-z_n)}\,dw_2\cdots dw_n dw_1 \\ &= \tfrac{1}{(2\pi i)^n}\int_{T(a,r)} \tfrac{f(w)}{(w-z)^{\mathbf{1}}}\,\mathbf{dw}. \end{aligned}$$

□

Definition 1.1.6 (Multiindex Notation) *For* $\mathrm{k} \in \mathbb{N}^n$ *we define* $|\mathrm{k}| = \sum_{j=1}^n |\mathrm{k}_j|$ *and* $\mathrm{k}! = \mathrm{k}_1!\cdots\mathrm{k}_n!$. *Then*

$$f^{(\mathrm{k})}(z) := \frac{\partial^{|\mathrm{k}|}f}{\partial z^{\mathrm{k}}} := \frac{\partial^{\mathrm{k}_1+\ldots+\mathrm{k}_n}}{\partial z_1{}^{\mathrm{k}_1}\cdots\partial z_n{}^{\mathrm{k}_n}} f(z) = \frac{\partial^{\mathrm{k}_1}}{\partial z_1{}^{\mathrm{k}_1}}\cdots\frac{\partial^{\mathrm{k}_n}}{\partial z_n{}^{\mathrm{k}_n}} f(z)\,.$$

Theorem 1.1.6 *Holomorphic maps are smooth, i.e. all partial derivatives exist. In particular, all complex derivatives exist and are holomorphic. Explicitly, for* $k \in \mathbb{N}^n$ *and* $j = 1, \ldots, n$,

$$f^{(\mathrm{k})}(z) = \frac{\mathrm{k}!}{(2\pi i)^n}\int_{T(a,r)} \frac{f(w)}{(w-z)^{\mathrm{k}+\mathbf{1}}}\,\mathbf{dw}\,.$$

Proof. By the Cauchy–integral formula, we have

$$f(z) = \frac{1}{(2\pi i)^n}\int_{T(a,r)} \frac{f(w)}{(w-z)^{\mathbf{1}}}\,\mathbf{dw}\,.$$

This form shows that all partial derivatives exist and are continuous. Moreover,

$$\frac{\partial f}{\partial z_j}(z) = \frac{1}{(2\pi i)^n}\int_{T(a,r)} \frac{\partial}{\partial z_j}\frac{f(w)}{(w-z)^{\mathbf{1}}}\,\mathbf{dw} = \frac{1}{(2\pi i)^n}\int_{T(a,r)} \frac{f(w)}{(w_j-z_j)(w-z)^{\mathbf{1}}}\,\mathbf{dw}\,,$$

and holomorphicity follows from the holomorphicity of the integrand and the claim follows by induction. □

Lemma 1.1.7 (Geometric Series) *Let* $z, w \in \mathbb{C}^n$ *such that* $|z_j| < |w_j|$ *for all* $j = 1, \ldots, n$. *Then*

$$\frac{1}{(w-z)^{\mathbf{1}}} = \sum_{\mathrm{k}\in\mathbb{N}^n} \frac{z^{\mathrm{k}}}{w^{\mathrm{k}+\mathbf{1}}}\,.$$

Proof. It is known that the formula for the geometric series holds in $\mathbb{C}$: For $|q| < 1$, we have $\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$. Thus, for $|z| < |w|$,

$$\frac{1}{w-z} = \frac{1}{w}\frac{1}{1-\frac{z}{w}} = \sum_{k=0}^{\infty}\frac{z^k}{w^{k+1}}\,.$$

For the multi-variable version we make use of the absolute convergence of the geometric series. That allows us to change the order of summation: Let $z, w \in \mathbb{C}^n$ such that $|z_j| < |w_j|$ for all $j = 1, \ldots, n$. Then we have

$$\frac{1}{(w-z)^{\mathbf{1}}} = \frac{1}{w_1-z_1}\cdots\frac{1}{w_n-z_n} = \sum_{\mathrm{k}\in\mathbb{N}^n}\frac{z_1^{\mathrm{k}_1}}{w_1^{\mathrm{k}_1+1}}\cdots\frac{z_n^{\mathrm{k}_n}}{w_n^{\mathrm{k}_n+1}} = \sum_{\mathrm{k}\in\mathbb{N}^n}\frac{z^{\mathrm{k}}}{w^{\mathrm{k}+\mathbf{1}}}\,.$$

□

Theorem 1.1.8 (Power Series Expansion) *The Taylor series of a holomorphic function* $f\colon \mathring{D}_{a,r} \to \mathbb{C}$ *converges everywhere to* f*:*

$$f(z) = \sum_{\mathrm{k}\in\mathbb{N}^n} \tfrac{1}{\mathrm{k}!} f^{(\mathrm{k})}(a)(z-a)^{\mathrm{k}}.$$

Proof. As in the one-dimensional case, we start with the right-hand side and plug in the definition of $f^{(\mathrm{k})}(z)$:

$$\begin{aligned}\sum_{\mathrm{k}\in\mathbb{N}^n} \tfrac{1}{\mathrm{k}!} f^{(\mathrm{k})}(a)(z-a)^{\mathrm{k}} &= \tfrac{1}{(2\pi i)^n} \sum_{\mathrm{k}\in\mathbb{N}^n} (z-a)^{\mathrm{k}} \int_{T_{a,r}} \tfrac{f(w)}{(w-a)^{\mathrm{k}+1}} \mathbf{dw} \\ &= \tfrac{1}{(2\pi i)^n} \int_{T_{a,r}} \sum_{\mathrm{k}\in\mathbb{N}^n} \tfrac{f(w)\,(z-a)^{\mathrm{k}}}{(w-a)^{\mathrm{k}+1}} \mathbf{dw} \\ &= \tfrac{1}{(2\pi i)^n} \int_{T_{a,r}} \tfrac{f(w)}{((w-a)-(z-a))^{\mathbb{1}}} \mathbf{dw} \\ &= \tfrac{1}{(2\pi i)^n} \int_{T_{a,r}} \tfrac{f(w)}{(w-z)^{\mathbb{1}}} \mathbf{dw} \\ &= f(z)\end{aligned}$$

where we used for the last equality the Cauchy–integral theorem. □

Corollary 1.1.9 (Principle of Analytic Continuation) *Let* $U \subset \mathbb{C}^n$ *be open and connected,* $\emptyset \neq \tilde{U} \subset U$ *open and* $f : U \to \mathbb{C}$ *holomorphic. Then*

$$f|_{\tilde{U}} = 0 \implies f = 0\,.$$

Proof. Define

$$\hat{U} := \bigcap_{\mathrm{k}\in\mathbb{Z}^n} \{z \in U \mid f^{(\mathrm{k})}(z) = 0\}$$

then $\hat{U}$ is a closed subset of U and $\tilde{U} \subset \hat{U}$. Further, by the power series expansion theorem, $\hat{U}$ is open. Since U is connected we can deduce that $U = \emptyset$ or $\hat{U} = U$. Since $\hat{U} \supset \tilde{U} \neq \emptyset$ by assumption, we have $\hat{U} = U$. Thus the power series theorem implies that $f \equiv 0$. □

Remark 1.1.1 In the one dimensional case, it was even sufficient that f vanishes on a set which has an accumulation point in U. This does not generalize to higher dimensions. For example, any non-zero complex linear function is holomorphic, but vanishes on a whole hyperplane.

The power series expansion theorem and the corollary do not hold in a real $\mathscr{C}^\infty$-setting. There the situation is utterly different—there are smooth functions which are constant on certain domains. For example, $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0 & \text{for } x \le 0\,, \\ e^{-1/x} & \text{for } x > 0\,. \end{cases}$$

Theorem 1.1.10 (Open Mapping Theorem) *If* $U \subset \mathbb{C}^n$ *is connected and* $f\colon U \to \mathbb{C}$ *is a non-constant holomorphic function, then* f *is open, i.e. open set are mapped to open sets.*

Proof. Let $p \in \tilde{U} \overset{\circ}{\subset} U$, where $\tilde{U}$ is star-shaped. Then, by the principle of analytic continuation, there is $q \in \tilde{U}$ such that $f(p) \neq f(p)$. Now, if $\hat{U}$ is the intersection of $\tilde{U}$ with the affine complex line $\{p + z(q-p) \mid z \in \mathbb{C}\} \cong \mathbb{C}$, then f restricts to a non-constant holomorphic function on $\hat{U}$, which is open by the open mapping theorem in one variable. Hence $f(\hat{U})$ is open $\subset f(U)$. □

Corollary 1.1.11 *Let $U \overset{\circ}{\subset} \mathbb{C}^n$ be a domain and $f : \overline{U} \to \mathbb{C}$ holomorphic. Then $|f|$ takes its maximum on the boundary ∂U of U.*

Cauchy's integral formula and the principle of analytic continuation also yield the following theorem.

Theorem 1.1.12 (Hartogs' Extension Theorem—Polydisk Version) *Let $n \geq 2$ and $D_{a,r}, D_{a,R} \subset \mathbb{C}^n$ be polydisks such that $D_{a,r} \subset \mathring{D}_{a,R}$. Then each holomorphic map f on $\mathring{D}_{a,R} \setminus D_{a,r}$ has a unique holomorphic extension to $\mathring{D}_{a,R}$.*

This can be generalized to Hartog's theorem. We state it here without proof.

Theorem 1.1.13 (Hartogs' Extension Theorem) *Let V be a complex vector space of $\dim_{\mathbb{C}} V \geq 2$. Let $K \subset G \subset V$ such that G is open, K is compact and $G \setminus K$ is connected. Then each holomorphic map f on $G \setminus K$ has is a unique holomorphic extension to G.*

Remark 1.1.2 Hartog's theorem is wrong in complex dimension = 1—consider the function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ given by $z \mapsto 1/z$.

1.1.3 Inverse and Implicit Function Theorem for Holomorphic Maps

The inverse function theorem for holomorphic functions is an easy consequence of the inverse function theorem of smooth functions.

Theorem 1.1.14 (Inverse Function Theorem) *Let V, W be complex vector spaces and U be an open subset of V. Let $f : U \to W$ be holomorphic and $p \in U$ such that $d_p f$ is invertible. Then there are open neighborhoods $\tilde{U} \subset U$ of p and $\hat{U} \subset W$ of $f(p)$ such that $f|_{\tilde{U}} : \tilde{U} \to \hat{U}$ is a biholomorphism.*

Proof. Since holomorphic maps are smooth, there are open neighborhoods $\tilde{U} \subset U$ of p and $\hat{U} \subset W$ such that $f|_{\tilde{U}} : \tilde{U} \to \hat{U}$ is a diffeomorhism. That the inverse map is holomorphic is Exercise 1.1.7. □

For the holomorphic version of the implicit function theorem we need a little preparation.

Definition 1.1.7 (Complex Subspace) *Let (V, J) be a complex vector space. A linear subspace $\tilde{V} \subset V$ is called complex if it is J-invariant, i.e. $J\tilde{V} \subset \tilde{V}$.*

If V, W are complex vector spaces, then the complex structure on their direct sum $V \oplus W$ is given by

$$J(v, w) = (Jv, Jw)$$

for $v \in V$ and $w \in W$. The complex structure on $\mathrm{Hom}_+(V, W)$ is given as follows: For $A \in \mathrm{Hom}_+(V, W)$,

$$(JA)v = J(Av) = A(Jv).$$

Theorem 1.1.15 *Let V, W be complex vector spaces. Then $A \in \mathrm{Hom}(V, W)$ is complex linear if and only if its graph $G_A = \{(v, Av) \in V \oplus W \mid v \in V\}$ of A is a complex subspace.*

Proof. For $A \in \mathrm{Hom}_+(V,W)$ we have $J(v, A(v)) = (Jv, JA(v)) = (Jv, A(Jv)) \in G_A$. Conversely, if G_A is a complex subspace, then $(Jv, JA(v)) = J(v, A(v)) \in G_A$, thus $JA(v) = A(Jv)$. □

Theorem 1.1.16 (Implicit Function Theorem) *Let V_1, V_2 and W be complex vector spaces and let $V := V_1 \oplus V_2$. Let $U \subset V$ be open and $h\colon U \to W$ be holomorphic. If $q = (q_1, q_2) \in U$ such that $f(q) = 0$ and the restriction $d_q f\colon V_2 \to W$ is bijective, then there are open neighborhoods $U_1 \subset V_1$ of q_1 and $U_2 \subset V_2$ of q_2 and a holomorphic function $g\colon U_1 \to U_2$ such that*

$$\forall (p_1, p_2) \in U_1 \times U_2\colon\ h(p_1, p_2) = 0 \iff p_2 = g(p_1)\,.$$

Proof. Again, we know already there is such a smooth function $g\colon U_1 \to U_2$. We are left to show that g is holomorphic. Therefore it is enough to show that for all $p_1 \in U_1$, the differential $d_{p_1} g$ is complex linear. Since $d_q h(V_2) = W$ we can assume without loss of generality that $d_{(p_1, g(p_1))} h(V_2) = W$ for all $p_1 \in U_1$. From $h(p, g(p)) = 0$, we then get

$$0 = d_{(p_1, g(p_1))} h\big(v_1, d_{p_1} g(v_1)\big)\,.$$

Hence the graph of $d_{p_1} g$ is in the kernel of $d_{(p_1, g(p_1))} h$ and, since the have equal dimension, they must be equal. Since the kernel of a complex linear map is a complex linear subspace, we conclude that $d_{p_1} g$ is complex linear and thus g is holomorphic. □

Let V be a real vector space and $M \subset V$. A (global) *parametrization of M* over an open subset U of a vector space is a smooth bijective map $f\colon U \to M$ with continuous inverse, which has full rank everywhere. If such f exists, we say that M is smoothly parametrizable.

Lemma 1.1.17 *Let V be a real vector space and $M \subset V$ be smoothly parametrized by $f\colon U \to V$. Then M is locally a graph, i.e. for each $q \in M$ there are subspaces $V_1, V_2 \subset V$ such that $V_1 \oplus V_2 = V$, open subsets $U_1 \subset V_1$, $U_2 \subset V_2$ and a smooth map $g\colon U_1 \to U_2$ such that for all $p_1 \in U_1$ and $p_2 \in U_2$*

$$(p_1, p_2) \in M \iff p_2 = g(p_1)\,.$$

Proof. Without loss of generality we can assume that $q = 0$ and $f(q) = 0$. Define $V_1 := \operatorname{im} d_q f$ and choose a complementary subspace V_2, $V = V_1 \oplus V_2$. Then $f = (f_1, f_2)$. Define $\tilde\varphi\colon U \oplus V_2 \to V_1 \oplus V_2$ given by

$$(p, v_2) \mapsto (f_1(p), v_2)\,.$$

the differential $d_{(0,0)}\tilde\varphi$ is bijective. So locally there is a smooth inverse $\varphi = (\varphi_1, \varphi_2)$ defined on, say, $U_1 \times U_2$, $U_i \subset V_i$ open. Since $\tilde\varphi(p, 0) \in V_1 \times \{0\}$ and φ is bijective, we have $\tilde U := \varphi_1(U_1 \times \{0\}) \subset U$. Since φ is a homeomorphism, $\tilde U$ is open. Moreover, since $(v_1, v_2) = \tilde\varphi(\varphi(v_1, v_2)) = (f_1(\varphi_1(v_1, v_2)), \varphi_2(v_1, v_2))$, we have $f_1(\varphi_1(v_1, v_2)) = v_1$. Thus

$$f \circ \varphi(v_1, v_2) = (v_1, f_2(\varphi_1(v_1, v_2)))\,.$$

Since the inverse of f is continuous $f(\tilde U) \ni f(p)$ is open in M. After possibly shrinking U_1 and U_2 we can assume that $M \bigcap (U_1 \times U_2)$ is the graph of the map $g\colon U_1 \to U_2$ given by $v_1 \mapsto f_2(\varphi_1(v_1, 0))$. □

A flattening of M is a smooth map φ mapping an open neighborhood U of M diffeomorphically onto its image $\tilde U \subset V$ such that $M = \varphi^{-1}(\tilde U \bigcap W)$ for some m-dimensional linear subspace $W \subset V$. If such φ exists we say that M is smoothly flattenable.

Theorem 1.1.18 *Let M be a subset of a real vector space V of real dimension n. Then the following are equivalent:*

(a) Locally M is the zero set of a smooth map f of full rank into some $(n-m)$-dimensional real vector space.

(b) *Locally M looks like the graph of a smooth map g defined on an m-dimensional real subspace of V.*
(c) *Locally M is smoothly flattenable to some m-dimensional real vector space.*
(d) *Locally M is smoothly parametrizable by open sets in some m-dimensional real vector space.*

Proof. Let M be given as the zero set of a function h. Then, by the implicit function theorem, M is locally the graph of a function $g\colon V_1 \to V_2$. The function $\varphi\colon V_1 \oplus V_2 \to V_1 \oplus V_2$ given by $\varphi(v, w) = (v, w - g(w))$ is then a flattening of M. Moreover, if $\pi_2\colon V \to V_2$ denotes the projection on the second component, M is then the zero set of $\pi_2 \circ \varphi$. Thus we have seen that $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$—so they are equivalent. If M is the graph of a function $g\colon V_1 \to V_2$ then clearly $f(v_1) = (v_1, g(v_1))$ is a parametrization of M over V_1. The converse statement is Lemma 1.1.17. □

Looking through the last two proofs we can safely state that both—lemma and theorem—remain true when we change smooth to holomorphic. Thus we get a holomorphic version of theorem.

Theorem 1.1.19 *Let M be a subset of a complex vector space V of complex dimension n. Then the following are equivalent:*

(a) *Locally M is the zero set of a holomorphic map f of full rank into some $(n-m)$-dimensional complex vector space.*
(b) *Locally M looks like the graph of a smooth/holomorphic map g defined on an m-dimensional real/complex subspace of V.*
(c) *Locally M is holomorphically flattenable to some m-dimensional complex vector space.*
(d) *Locally M is holomorphically parametrizable by open sets in some m-dimensional complex vector space.*

Such flattenable spaces are called *submanifolds*. For the intrinsic geometry of M and objects living on it, as e.g. functions, the environmental vector space plays absolutely no role—in fact, in many situations it blurs the view. The next sections deal with building up a rigorous theory of abstract manifolds, which is build upon topology.

Differential Geometric Foundations

Manifolds 2

2.1 Manifolds

This section is meant to recall some basic notions of point set topology. It is written in such a way that it also may serve as a brief introduction.

2.1.1 Some Basics in Point Set Topology

Definition 2.1.1 (Topological Space) *A topological space is a pair $(X, \mathcal{O})$ consisting of a set X and a topology $\mathcal{O}$, i.e. a subset $\mathcal{O}$ of the power set $\mathcal{P}(X)$ of X such that*

(a) $\emptyset, M \in \mathcal{O}$,
(b) $\mathcal{U} \subset \mathcal{O} \Rightarrow \bigcup_{U \in \mathcal{U}} U \in \mathcal{O}$,
(c) $\mathcal{U} \subset \mathcal{O}$ *finite* $\Rightarrow \bigcap_{U \in \mathcal{U}} U \in \mathcal{O}$.

An element of $\mathcal{O}$ is called an *open* set. If it is clear what topology we are speaking about we just say that X is a topological space and speak of open sets. For convenience we then write

$$U \mathring{\subset} X \quad :\Longleftrightarrow \quad U \in \mathcal{O}_X \,.$$

A set $A \subset X$ is called a *closed* set if its complement $A^c := X \setminus A$ is open.

A broad class of topological spaces, known from analysis are metric spaces.

Example 2.1.1 (Metric Topology) Given a metric space (X, d) then for a given point $x \in X$ and a given radius $r > 0$ we can define the open ball of radius r centered at x to be

$$B_r(x) = \{y \in X \mid d(x, y) < r\} \,.$$

Let $\mathcal{B} := \{B_r(x) \mid x \in X, \;\; r > 0\}$. Then the *metric topology* is the set $\mathcal{O}_d$ of arbitrary unions of elements in $\mathcal{B}$,

$$\mathcal{O}_d := \Big\{ \bigcup_{U \in \mathcal{U}} U \mid \mathcal{U} \subset \mathcal{B} \Big\} \,.$$

Note that the open sets in the metric topology agree with the notion of open set as it is known from analysis. All the notions there that can be formulated purely in terms of open sets translate directly to general topological spaces—so we can speak of neighborhoods, limits, compactness, covers, subcovers, connectedness, connected components and so on.

Some more pathological examples of topologies are the following.

Example 2.1.2 (Trivial and Discrete Topology) Let X be a set. Then:

(a) $\mathcal{O} = \{\emptyset, X\}$ defines a topology, called the *trivial topology*.
(b) $\mathcal{O} = \mathcal{P}(X)$ defines a topology, called the *discrete topology*.

A set equipped with the discrete topology is called a *discrete space*.

Both examples are kind of extreme—each topology contains the trivial topology and is contained in the discrete topology. Actually, as sets are partially ordered by inclusion so are topologies. If

$$\mathcal{O} \subset \tilde{\mathcal{O}}\,,$$

then we say that $\mathcal{O}_X$ is *coarser* than $\tilde{\mathcal{O}}_X$ or, equivalently, $\tilde{\mathcal{O}}_X$ is *finer* than $\mathcal{O}_X$. So we can say that the trivial topology is the coarsest and the discrete topology is the finest among all topologies of X.

We are aiming at topologies which are not too coarse (Hausdorff) and not too fine (second countable).

In Example 2.1.1 the topology is generated by a set $\mathcal{B}$—such set is called a basis of topology.

Definition 2.1.2 (Basis of Topology) *Let $(X, \mathcal{O})$ be a topological space. A basis of topology is a set $\mathcal{B}$ that generates the topology, i.e. for each $U \in \mathcal{O}$ there is a subset $\mathcal{U} \subset \mathcal{B}$ such that*

$$U = \bigcup_{\tilde{U} \subset \mathcal{U}} \tilde{U}\,.$$

Definition 2.1.3 (2nd-countable Space) *A space is called 2nd-countable if it has a countable basis of topology.*

Theorem 2.1.1 (2nd-countablility of $\mathbb{R}^n$) *$\mathbb{R}^n$ equipped with its norm topology is 2nd-countable.*

Proof. Let $\mathcal{B} = \{B_r(x) \mid x \in \mathbb{Q}^n,\ \ r \in \mathbb{Q}\}$. Clearly, $\mathcal{B}$ is countable. One easily checks that $\mathcal{B}$ is a basis. □

An obvious example of a topological space which is not 2nd-countable is provided by any uncountable set equipped with the discrete topology. Though this space just has uncountably many connected components. An example of a connected space which is not 2nd-countable is the so-called long line.

Definition 2.1.4 (Subbasis) *Let $(X, \mathcal{O})$ be a topological space. A subbasis of topology is a set $\mathcal{S}$ such that the set of finite intersections*

$$\Big\{\bigcap_{\tilde{\mathcal{A}} \in \mathcal{A}} \tilde{\mathcal{A}} \mid \mathcal{A} \subset \mathcal{S}\ \text{finite}\Big\}$$

forms a basis of topology.

Taking unions of finite intersections of elements in a given set $\mathcal{S} \subset \mathcal{P}(X)$ generates a topology $\mathcal{O}$ on X, which is the coarsest topology which contains $\mathcal{S}$.

Example 2.1.3 (Order Topology) Let $(X, \leq)$ be a totally ordered set. Then the *order topology* is generated by the subbase of *open rays*

$$(a, \infty) := \{x \in X \mid a < x\}, \quad (\infty, a) := \{x \in X \mid x < a\}, \quad a \in X\,.$$

Example 2.1.4 (Long Line) There is an uncountable well-ordered set ω_1—called the *first uncountable ordinal*. The *closed long ray* is defined as the cartesian product of ω_1 with the half-open interval $[0, 1) \subset \mathbb{R}$, equipped with the order topology that arises from the lexicographical order on $\omega_1 \times [0, 1)$. The *long line* L is obtained from the closed long ray by removing the smallest element $(0, 0)$. Clearly, the long line is not 2nd-countable.

A subset of a topological space comes with a natural topology.

Example 2.1.5 (Subspace Topology) Let X be a subset of a topological space $(Y, \mathcal{O}_Y)$, then X inherits a topology $\mathcal{O}_Y$ from X. It is given as follows:

$$\mathcal{O}_X := \{U \cap X \mid U \in \mathcal{O}_Y\}\,.$$

Definition 2.1.5 (Continuous Map, Open Map, Homeomorphism) *Let $f\colon X \to Y$ be a map between topological spaces. Then we define*

$$\begin{aligned} f \ \text{continuous} \quad &:\Longleftrightarrow \quad f^{-1}(U) \overset{\circ}{\subset} X \ \text{for all} \ U \overset{\circ}{\subset} Y\,, \\ f \ \text{open} \quad &:\Longleftrightarrow \quad f(U) \overset{\circ}{\subset} Y \ \text{for all} \ U \overset{\circ}{\subset} X\,. \end{aligned}$$

If f is a continuous bijection which has a continuous inverse, then we say that f is a homeomorphism.

Exercise 2.1.1 Let X be a subset of a topological space Y. Show that the subspace topology is the coarsest topology such that the inclusion $X \hookrightarrow Y$ is continuous.

The next theorem tells us that we get 2nd-countability for free, when we are working with sets which already sit in some 2nd-countable space.

Theorem 2.1.2 *A subspace of a 2nd-countable space is 2nd-countable.*

Proof. Let $\{U_j\}_{j\in\mathbb{N}}$ be a basis of topology of X and $\tilde{X} \subset X$ Then $\{U_j \bigcap \tilde{X}\}_{j\in\mathbb{N}}$ is a basis of topology of $\tilde{X}$. ☐

Theorem 2.1.3 *A subspace of a Hausdoff space is Hausdorff.*

Proof. Let X be a subspace of a Hausdorff space Y and let $x, \tilde{x} \in X$ such that $x \neq \tilde{x}$. Then there are $U, \tilde{U} \overset{\circ}{\subset} Y$ such that $x \in U$, $\tilde{x} \in \tilde{U}$ and $U \cap \tilde{U} = \emptyset$. Then $U \cap X, U \cap X \overset{\circ}{\subset} X$ are disjoint, $x \in U \cap X$ and $\tilde{x} \in \tilde{U} \cap X$. ☐

Example 2.1.6 (Product Topology) Let $X_1, \ldots, X_n$ be topological spaces. The product topology on the cartesian product $X_1 \times \cdots \times X_n$ is defined to be the coarsest topology such that the natural projections $\pi_i\colon X_1 \times \cdots \times X_n \to X_i$, $i = 1\ldots, n$ are continuous. A subbasis is given by

$$\left\{\pi_i^{-1} U \mid U \overset{\circ}{\subset} X_i\,,\ \ 1 \leq i \leq n\right\}.$$

Theorem 2.1.4 *The product of Hausdorff spaces is Hausdorff.*

Proof. Exercise. ☐

Example 2.1.7 (Quotient Topology) Let $\hat{X}$ be a topological space and $\pi\colon \hat{X} \to X$ be surjective. Then

$$\mathbb{O}_X = \{U \subset X \mid \pi^{-1} U \in \mathbb{O}_{\hat{X}}\}$$

defines a topology on X—the *quotient topology*. It is the coarsest topology on X such that π is continuous. The map π yields an equivalence relation on $\hat{X}$ given by

$$x \sim y \quad :\Longleftrightarrow \quad \pi(x) = \pi(y)\,.$$

Conversely, the canonical projection of an equivalence relation $\sim$ is such a map π. We denote the *quotient space* by $X = \hat{X}/_\sim$.

In general, if $\hat{X}$ is 2nd-countable or Hausdorff, the quotient space might be not. If the natural projection is open, then we can say something.

Theorem 2.1.5 *If $\hat{X}$ is 2nd-countable and $\pi\colon \hat{X} \to X$ is a continuous open surjection, then X is 2nd-countable.*

Proof. Let $\hat{\mathscr{B}}$ be a countable basis of topology on $\hat{X}$. Then, since π is open, $\mathscr{B} := \{\pi(\hat{U}) \mathrel{\mathring{\subset}} X \mid \hat{U} \in \hat{\mathscr{U}}\}$. Clearly $\mathscr{B}$ is countable. To see that $\mathscr{B}$ is a basis of topology, let $U \mathrel{\mathring{\subset}} X$. Then $\hat{U} := \pi^{-1}U \mathrel{\mathring{\subset}} \hat{X}$ and there is a subset $\mathscr{A} \subset \hat{\mathscr{B}}$ such that $\hat{U} = \cup_{A\in\mathscr{A}} A$. Hence

$$U = \pi(\hat{U}) = \pi\Big(\bigcup_{A\in\mathscr{A}} A\Big) = \bigcup_{A\in\mathscr{A}} \pi(A)$$

with $\{\pi(A) \mid A \in \mathscr{A}\} \subset \mathscr{B}$. Thus $\mathscr{B}$ is a basis of topology. □

Theorem 2.1.6 *Let $\pi\colon \hat{X} \to X$ be a continuous open surjection. Then $R = \{(\hat{x},\hat{y}) \mid \pi(\hat{x}) = \pi(\hat{y})\}$ is closed in $\hat{X}\times\hat{X}$ if and only if X is Hausdorff.*

Proof. Assume R is closed. If $x, y \in X$, $x \neq y$. Then there is $(\hat{x},\hat{y}) \notin R$ such that $\pi(\hat{x}) = x$ and $\pi(\hat{y}) = y$. Since R is closed, $(\hat{X}\times\hat{X}) \setminus R$ is open and so there are open sets $\hat{U}, \hat{V} \in \hat{X}$ such that $\hat{x} \in \hat{U}$, $\hat{y} \in \hat{V}$ and $(\hat{U}\times\hat{V}) \cap R = \emptyset$. Then $U := \pi(\hat{U})$ and $V := \pi(\hat{V})$ are open with $x \in U$, $y \in V$ and $U \cap V = \emptyset$. So X is Haussdorff. Conversely, assume that X is Hausdorff. If $(\hat{x},\hat{y}) \notin R$, then $x := \pi(\hat{x}) \neq \pi(\hat{y}) =: y$ and there are disjoint open neighborhoods U of x and V of y. Then $\hat{U} := \pi^{-1}U$ and $\hat{V} := \pi^{-1}V$ are open. $\hat{U}\times\hat{V}$ is then an open neighborhood of $(\hat{x},\hat{y})$. Since U and V were disjoint, $(\hat{U}\times\hat{V}) \cap R = \emptyset$. Thus R is closed. □

Example 2.1.8 (Torus) On $\hat{X} = \mathbb{R}^n$ we define $p \sim q \;:\Longleftrightarrow\; q - p \in \mathbb{Z}^n$. If $\hat{U} \subset \hat{X}$ is open, then

$$\pi^{-1}(\hat{U}) = \bigcup_{m\in\mathbb{Z}^n} (\hat{U} + m)\,,$$

which is clearly open. Moreover, $R = \big\{(x,y) \in \hat{X}\times\hat{X} \mid x \sim y\big\} = f^{-1}(\mathbb{Z})$ for $f(x,y) = x - y$, which is closed since f is continuous. Hence $\hat{X}/_\sim$ is a 2nd-countable Hausdorff space. We also write $\mathbb{R}^n/\mathbb{Z}^n$.

The very same arguments hold if we replace $\mathbb{Z}^n$ in the equivalence relation by a *lattice*, i.e. a set $\Lambda = \{\sum_{i=1}^n m_i\omega_i \mid (m_1,\ldots,m_n) \in \mathbb{Z}^n\}$ where $\omega_1,\ldots,\omega_n \in \mathbb{R}^n$ are linearly independent. The quotient $\mathbb{R}^n/\Lambda$ is called the *real n-dimensional torus*.

Definition 2.1.6 (Covering Map) *A topological covering map is a continuous sujective map $\pi\colon \hat{X} \to X$ such that for each point $x \in X$ has a neighborhood $U \mathrel{\mathring{\subset}} X$ such that $\pi^{-1}U = \dot{\cup}_{\alpha\in A} U_\alpha$ such that for all α the restriction $\pi|_{U_\alpha}\colon U_\alpha \to U$ is a homeomorphism.*

Exercise 2.1.2 (Atlas of the Torus) Show that the canonical projection $\pi\colon \mathbb{R}^n \mapsto \mathbb{R}^n/\Lambda$ is a covering map.

Exercise 2.1.3 Consider $\hat{X} = \mathbb{R}$ and let $p \sim q \;:\Longleftrightarrow\; q - p \in \mathbb{Q}$. Show that $\hat{X}/_\sim$ is not Hausdorff.

2.1.2 Differentiable Manifolds

In this section we introduce the central objects of this lecture—complex manifolds. We start out with the basic concept of a topological manifold before we equip it with additional structures.

Definition 2.1.7 (Topological Manifold, Chart) *An m-dimensional manifold is a 2nd-countable Hausdorff space M^m which is locally homeomorphic to open sets in $\mathbb{R}^m$, i.e. for each point $p \in M$ there is $U \mathrel{\mathring{\subset}} M$ of p, $V \mathrel{\mathring{\subset}} \mathbb{R}^m$ and a homeomorphism $\varphi\colon U \to V$. Such a homeomorphism $\varphi\colon U \to V$ is called a local chart or simply a chart at p—(U,φ),* *for short. Instead of locally homeomorphic to $\mathbb{R}^m$, we also say that M is locally euclidean.*

Let us go through some examples. Many manifolds just sit in some euclidean space.

Example 2.1.9 (Sphere) The n-dimensional sphere is defined as the set $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} \mid |x|^2 = 1\}$. As a subset of euclidean space $\mathbb{S}^n$ is 2nd-countable and Hausdorff. Let $D \subset \mathbb{R}^n$ denote the open unit disk. Then, for $j = 1, \ldots, n+1$, the sets $U_j^\pm$ given by

$$U_j^\pm = \{x = (x_1, \ldots, x_{n+1}) \in \mathbb{S}^n \mid \pm x_j > 0\}$$

are open in $\mathbb{S}^n$ and $\varphi_j^\pm \colon U_j \to D$ by

$$\varphi_j^\pm(x_1, \ldots, x_{n+1}) := (x_1, \ldots, \widehat{x_j}, \ldots, x_{n+1}) := (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n+1})$$

are homeomorphisms. Since each point $p \in \mathbb{S}^n$ is contained in at least one of the open sets $U_j^\pm$, we have given for each point a neighborhood homeomorphic to $D \subset \mathbb{R}^n$.

The collection of homeomorphism $\varphi_j^\pm$ in the last example is what is called an atlas.

Definition 2.1.8 (Atlas) *Let M^m be a topological manifold. An atlas of M^m is a collection $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ of charts on M such that their domains form an open cover of M^m, $\bigcup_{\alpha \in A} U_\alpha = M$.*

Exercise 2.1.4 (Atlas of the Torus) Show that the canonical projection $\pi \colon \mathbb{R}^n \mapsto \mathbb{R}^n/\Lambda$ is a covering map. In particular, we obtain an atlas of $\mathbb{R}^n/\Lambda$ by restricting the canonical projection:

$$\mathscr{A} = \left\{(\pi|_{\hat{U}})^{-1} \colon \hat{U} \to U \mid \hat{U} \text{ such that } \pi|_{\hat{U}} \colon \hat{U} \to U \text{ homeomorphism}\right\}.$$

Show that the coordinate changes are locally translations of the form $x \mapsto x + \omega$ for some $\omega \in \Lambda$.

In some situations a manifold structure comes from a collection of compatible bijections.

Example 2.1.10 (Manifold from Atlas) Given a set M together with a collection of bijective maps $\mathscr{A} = \{\varphi_\alpha \colon U_\alpha \to V_\alpha \overset{\circ}{\subset} \mathbb{R}^m\}_{\alpha \in A}$ such that $\cup_{\alpha \in A} U_\alpha = M$ for all $\alpha, \beta \in A$ the *coordinate change*

$$\varphi_\beta \circ \varphi_\alpha^{-1} \colon \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is a homeomorphism. Then there is a unique topology on M—the *topology induces by $\mathscr{A}$*—such that for all $\alpha \in A$ the map φ_α is a homeomorphism, i.e. we have a topological space which is locally homeomorphic to $\mathbb{R}^m$. If the induced topology is 2nd-countable and Haussdorff, then M is a manifold and $\mathscr{A}$ forms an atlas for M.

Though this construction may not lead to a 2nd-countable Hausdorff topology—as illustrated by the following examples.

Remark 2.1.1 (Long Line, Line with two Origins) Note, that there are topological spaces which satisfy all but one of the three manifold properties—2nd-countability, Hausdorff property and property of being locally euclidean. As an example of a 2nd-countable Hausdorff space which is not locally euclidean, one can just consider a planar curve with self-intersections. The other examples are more difficult.

First, the *long line*, which is not 2nd-countable, but can be shown to be Hausdorff and locally euclidean.

Second, the *line with two origins*, which is obtained as a quotient—on the disjoint union of two real lines we identify all points but the origin. The line with two origins is not Hausdorff but can be shown to be 2nd-countable and locally euclidean.

In Example 2.1.10 above we have seen that if the coordinate changes—being homeomorphisms—respect topology, we can lift the topology of $\mathbb{R}^m$ up to M. Similarly, other structures can be lifted, whenever we have an atlas which respects them.

Definition 2.1.9 (Smooth Structure) *A smooth atlas on a manifold M^m is an atlas $\mathscr{A} = \{\varphi_\alpha : U_\alpha \to V_\alpha \overset{\circ}{\subset} \mathbb{R}^m\}_{\alpha \in A}$ such that for all $\alpha, \beta \in A$ the coordinate change*

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is a diffeomorphism. A maximal smooth atlas is called a smooth structure.

Definition 2.1.10 (Smooth Manifold) *A smooth manifold is a pair $(M, \mathscr{A})$ consisting of a topological manifold M and a smooth structure $\mathscr{A}$ on M.*

Remark 2.1.2 Note, that a smooth atlas determines a unique maximal atlas. Thus it is enough to specify a smooth atlas instead of a smooth structure.

Example 2.1.11 The atlases we defined above for the sphere and the torus define smooth structures.

Similarly, we define a complex structure on an even-dimensional manifold.

Definition 2.1.11 (Complex Structure) *A complex atlas on a manifold M^{2n} is an atlas $\{\varphi_\alpha : U_\alpha \to V_\alpha \overset{\circ}{\subset} \mathbb{R}^{2n} = \mathbb{C}^n\}_{\alpha \in A}$ such that for all $\alpha, \beta \in A$ the coordinate change*

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is a biholomorphism. A maximal complex atlas is called a complex differentiable or, for short, complex structure.

Definition 2.1.12 (Complex Manifold) *A complex manifold is a pair $(M, \mathscr{A})$ consisting of a topological manifold M and a complex structure $\mathscr{A}$ on M. A complex-1-dimensional manifold is also called a Riemann surface.*

Remark 2.1.3 (Differentiable Manifold) Every complex manifold is a smooth manifold. If a manifold M is smooth or complex and we want to leave it somewhat unspecified what kind of smooth structure explicitly we are dealing with we just say that M is a *differentiable manifold*.

Example 2.1.12 (Complex Torus) Let $\Lambda \subset \mathbb{R}^{2n} = \mathbb{C}^n$ be a lattice. Then the atlas consisting of restrictions of the canonical projection defines a complex structure and turns $\mathbb{C}^n/\Lambda$ into a complex manifold.

Another example is the complex projective space—due to its importance we will do it here in detail.

Example 2.1.13 (Complex Projective Space) Let $(\mathbb{C}^{n+1})^\times := \mathbb{C}^{n+1} \setminus \{0\}$. We define an equivalence relation on $(\mathbb{C}^{n+1})^\times$ as follows:

$$z \sim w \quad :\Longleftrightarrow \quad z = \lambda w, \quad \text{for } \lambda \in \mathbb{C}\,.$$

The n-dimensional complex projective space is then defined to be the quotient space $\mathbb{CP}^n = (\mathbb{C}^{n+1})^\times/_\sim$. Let $\pi : (\mathbb{C}^{n+1})^\times \to \mathbb{CP}^n$ denote the canonical projection. Since

$$\pi^{-1}(\pi(U)) = \bigcup_{\lambda \neq 0} \lambda U \overset{\circ}{\subset} (\mathbb{C}^{n+1})^\times\,,$$

the canonical projection is an open map. So we get that $\mathbb{CP}^n$ is 2nd-countable. Furthermore, if $f : (\mathbb{C}^{n+1})^\times \times$

$(\mathbb{C}^{n+1})^\times \to \mathbb{R}$ is given by $f(z,w) = \sum |z_i w_j - z_j w_i|^2$, then

$$R = \{(z,w) \in (\mathbb{C}^{n+1})^\times \times (\mathbb{C}^{n+1})^\times \mid z \sim w\} = f^{-1}\{0\}.$$

Since f is continuous, R is closed and so $\mathbb{CP}^n$ is Hausdorff. For $j = 1, \ldots, n+1$, let $U_j := \{[z] \in \mathbb{CP}^n \mid z_j \neq 0\}$,

$$\varphi_j \colon U_j \to \mathbb{C}^n, \quad [z_1, \ldots, z_{n+1}] \mapsto (z_1, \ldots, \widehat{z_j}, \ldots, z_{n+1})/z_j ,$$

where the hat means omission. We leave it as an exercise to check that $\{(U_j, \varphi_j)\}_{j=1,\ldots,n+1}$ is a complex atlas.

Once we have a smooth structure we can define smooth maps.

Definition 2.1.13 (Smooth Map) *A map $f\colon M^m \to N^n$ is called smooth, if for every two smooth charts φ of M and ψ of N such that the composition*

$$\hat{f} = \psi \circ f \circ \varphi^{-1}$$

defined, the map $\hat{f}$ is smooth as a map from some subset of $\mathbb{R}^m$ into $\mathbb{R}^n$. The set of all smooth maps from M to N will be denoted by $\mathscr{C}^\infty(M;N)$. Furthermore, $\mathscr{C}^\infty M := \mathscr{C}^\infty(M;\mathbb{R})$.

Definition 2.1.14 (Diffeomorphism) *A diffemorphism is a smooth bijection which has a smooth inverse. If there exists a diffeomorphism between two manifolds we call them diffeomorphic.*

Similary, we define holomorphic maps on a complex manifold.

Definition 2.1.15 (Holomorphic Map) *A map $f\colon M^{2m} \to N^{2n}$ is called holomorphic, if for every two complex charts φ of M and ψ of N such that the composition*

$$\hat{f} = \psi \circ f \circ \varphi^{-1}$$

defined, the map $\hat{f}$ is holomorphic as a map from some subset of $\mathbb{C}^m$ into $\mathbb{C}^n$. The set of holomorphic maps from M to N will be denoted by $\mathscr{O}(M;N)$. Furthermore, $\mathscr{O}M := \mathscr{O}(M;\mathbb{C})$.

Definition 2.1.16 (Biholomorphism) *A biholomorphism is a holomorphic bijection which has a holomorphic inverse. If there exists a biholomorphism between two complex manifolds we call them biholomorphic.*

Example 2.1.14 (Riemann Sphere) Let $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. Define $\varphi_0 \colon \widehat{\mathbb{C}} \setminus \{\infty\} \to \mathbb{C}$ and $\varphi_\infty \colon \widehat{\mathbb{C}} \setminus \{0\} \to \mathbb{C}$ by

$$\varphi_0(z) = z, \quad \varphi_\infty(z) = \begin{cases} \frac{1}{z} & \text{for } z \neq \infty \\ 0 & \text{for } z = \infty \end{cases}$$

Then $\{\varphi_0, \varphi_\infty\}$ forms an complex atlas: $\varphi_\infty \circ \varphi_0^{-1} \colon \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ is given by $z \mapsto 1/z$.

Exercise 2.1.5 Show that the biholomorphisms of the Riemann sphere $\widehat{\mathbb{C}}$ are exactly the Möbius transformations. Moreover, show that $\widehat{\mathbb{C}} \cong_{\mathscr{O}} \mathbb{CP}^1$.

Definition 2.1.17 (Local Diffeomorphism, Local Biholomorphism) *A smooth map $f\colon M \to N$ between smooth manifolds is called a local diffeomorphism if each $p \in M$ has a neighborhood $U \overset{\circ}{\subset} M$ such that the restriction $f|_U \colon U \to f(U)$ is a diffeomorphism. Similarly, a holomorhic map $f\colon M \to N$ between complex manifolds is called a local biholomorphism if each $p \in M$ has a neighborhood $U \overset{\circ}{\subset} M$ such that the restriction $f|_U \colon U \to f(U)$ is a biholomorphism.*

Exercise 2.1.6 Let $\Lambda \subset \mathbb{R}^n$ be a lattice. Show that with the atlas we defined for the torus the canonical projection is a local diffeomorphism. Show, moreover, that smooth structure on $\mathbb{R}^n/\Lambda$ is uniquely determined by the condition that the canonical projection $\mathbb{R}^n \to \mathbb{R}^n/\Lambda$ is a local diffeomorphism. The same statement is true in the complex case.

Given two manifolds we can build their cartesian product. That leads again to a manifold—as carried out by the next exercise.

Exercise 2.1.7 (Product Manifold) Show that the cartesian product $M_1 \times M_2$ of two manifolds M_1 and M_2 is a manifold. Show that, if $\mathscr{A}_1$ and $\mathscr{A}_2$ are atlases of M_1 and M_2, then

$$\{\varphi_1 \times \varphi_2 \mid \varphi_1 \in \mathscr{A}_1,\ \varphi_2 \in \mathscr{A}_2\}\,,$$

forms an atlas of $M_1 \times M_2$. Here $\varphi_1 \times \varphi_2$ denotes the cartesian product of functions,

$$(\varphi_1 \times \varphi_2)(p_1, p_2) := (\varphi_1(p_1), \varphi_2(p_2))\,.$$

Moreover, if $\mathscr{A}_1$ and $\mathscr{A}_2$ are smooth resp.complex, then so is the product atlas. Show that the natural projections $\pi_i\colon M_1 \times M_2 \to M_i$ are smooth resp. holomorphic.

Exercise 2.1.8 Show that $\mathbb{R}^n/\Lambda \cong_{\mathscr{C}^\infty} (\mathbb{S}^1)^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$.

Remark 2.1.4 The last exercise shows in particular that all n-dimensional tori $\mathbb{R}^n/\Lambda$ are diffeomorphic, but not biholomorphic.

2.1.3 Some Basics on Holomorphic Functions

Theorem 2.1.7 (Principle of Analytic Continuation) *Let M be a connected complex manifold and $f\colon M \to \mathbb{C}$ be holomorphic. If $f|_U = 0$ for some $\emptyset \neq U \mathring{\subset} M$, then $f = 0$.*

Proof. To see this consider the set $Z := \{p \in M \mid \exists U_p \mathring{\subset} M,\ p \in U_p\colon f_{U_p} = 0\}$. Clearly, Z is open and non-empty, $Z \supset U \neq \emptyset$. Moreover, if $q \in \overline{Z}$, then any open neighborhood of q intersects Z. So choose some connected complex coordinate neighborhood, (V, φ). Then there is $p \in V \cap Z$ and hence $U_p \mathring{\subset} M$ such that $f_{U_p} = 0$. Thus $f \circ \varphi^{-1}\colon \varphi(U) \to \mathbb{C}$ vanishes on some open subset. Hence $f \circ \varphi^{-1} = 0$ and so is $f_V = 0$, i.e. $p \in Z$. Hence Z is a non-empty set which is simultaneously open and closed. Since M is connected, $Z = M$. □

Theorem 2.1.8 *If M is a compact connected complex manifold and $f\colon M \to \mathbb{C}$ is holomorphic, then f is constant.*

Proof. Since $|f|$ is continuous and M is compact, there is a point $p \in M$ where $|f|$ is maximal. Expressing f in coordinates at p we obtain a holomorphic function whose modulus attains its maximum at an inner point of some open set in $\mathbb{C}^m$. Hence f is constant in some open neighborhood of p. Since M is connected, the principle of analytic continuation yields that f is then constant on all of M. □

Theorem 2.1.9 *A non-constant holomorphic function $f\colon M \to \mathbb{C}$ on a connected complex manifold is open.*

Proof. The principle of analytic continuation yields that f is nowhere locally constant. The claim then follows from the local result, since the property that a map is open is a local property. The details are left to the reader. □

Corollary 2.1.10 *A non-constant holomorphic map from a connected complex manifold to a Riemann surface is open.*

Corollary 2.1.11 *A non-constant holomorphic map from a compact connected complex manifold to a connected Riemann surface M is surjective. In particular, M is compact.*

Smooth maps are much more flexible. This is becomes particularly apparent from the existence of a *partition of unity*, which is an important construction tool and which is probably the best explanation for the necessity of 2nd-countability.

2.1.4 Partitions of Unity

A collection $\mathscr{C}$ of subsets of a manifold M is called *locally finite*, if each point $p \in M$ has a neighborhood V such that $V \cap U \neq \emptyset$ for only finitely many $U \in \mathscr{C}$. The support $\operatorname{supp} f$ of a function $f\colon M \to \mathbb{R}$ is the closure of its zero set,

$$\operatorname{supp} f := \overline{\{p \in M \mid f(p) \neq 0\}}\,.$$

Definition 2.1.18 (Partition of Unity) *A partition of unity on a smooth manifold M is a collection of smooth functions $\{\rho_\alpha\colon M \to [0,\infty)\}_{\alpha\in A}$ such that*

(a) the collection of supports $\{\operatorname{supp}\rho_\alpha\}_{\alpha\in A}$ is locally finite, and
(b) $\sum_{\alpha\in A}\rho_\alpha = 1$.

A partition of unity $\{\rho_\alpha\colon M \to [0,\infty)\}_{\alpha\in A}$ is subordinate to an open cover $\{U_\beta\}_{\beta\in B}$, if for each $\alpha \in A$ there exists $\beta \in B$ such that $\operatorname{supp}\rho_\alpha \subset U_\beta$. If $A = B$ and $\operatorname{supp}\rho_\alpha \subset U_\alpha$ for all $\alpha \in A$, we say that the partition of unity is subordinate *with the same index*.

On $\mathbb{R}^n$ there are non-negative smooth functions with compact support. By a chart these can be transferred to a manifold, where they serve as the fundamental building block for the construction of a partition of unity.

Lemma 2.1.12 *There is a smooth function $f\colon \mathbb{R}^n \to [0,1]$ such that $f(x) = 1$ for $|x| < 1$ and $f(x) = 0$ for $|x| > 2$.*

Proof. As the condition depends only on the norm of x, it is enough to show that there is such f for $n = 1$—for larger n we can then precompose the norm. The function

$$h(t) = \begin{cases} e^{-1/t} & \text{for } t > 0 \\ 0 & \text{else} \end{cases}$$

is smooth with $h(t) = 0$ for $t \leq 0$ and $h(t) > 0$ for $t > 0$. From h we define a smooth non-negative function g:

$$g(t) := \frac{h(t)}{h(t) + h(1-t)}\,.$$

Then $g(t) = 0$ for $t \leq 0$ and $g(t) = 1$ for $t \geq 1$ and so

$$f(t) := g(t+2)g(2-t)$$

has the desired properties. □

The principle to combine these localized functions to a partition of unity is quite simple. Suppose we would have a finite cover $\mathscr{C}$ and for each open set $U \in \mathscr{C}$ some function $f_U\colon M \to [0,\infty)$ with $\operatorname{supp} f_U \subset U$ and for each point $p \in M$ there is at least one such function f_U such that $f_U(p) > 0$. Then $f := \sum_{U \in \mathscr{C}} f_U > 0$ and the functions

$$\rho_U = f_U / f$$

would form partition of unity. In general we need quite some amount of topology to get into a situation similar to the one just described. The key here is 2nd-countability. Here we don't go into details and, after we introduced the needed terminology, just state the result. A proof is given in Appendix A.1.

Let $\mathscr{A}$ be an open cover of M. A refinement of $\mathscr{A}$ is an open cover $\mathscr{B}$ such that each $U \in \mathscr{B}$ is contained in some $V \in \mathscr{A}$. If each open cover of M has a locally finite refinement, then M is called *paracompact*. A *precompact subset* is a subset whose closure is compact.

Theorem 2.1.13 (Paracompactness of Manifolds) *Every locally compact 2nd-countable Hausdorff space is paracompact. In fact, given a locally compact 2nd-countable Hausdorff space X, an open cover $\mathscr{C}$ of X, and any basis $\mathscr{B}$ of the topology of X, there exists a countable, locally finite refinement of $\mathscr{C}$ consisting of elements of $\mathscr{B}$.*

To obtain then a partition of unity subordinate to a given open cover we will apply Theorem 2.1.13 to a particular adapted basis of topology.

Lemma 2.1.14 *Let M be a smooth manifold and $\mathscr{C}$ be an open cover. Then, for each $U \in \mathscr{C}$, the set*

$$\mathscr{B}_U = \left\{ f^{-1}(0, \infty) \subset M \mid f \colon M \to [0, \infty) \text{ smooth, } \operatorname{supp} f \subset U \right\}$$

forms a basis of topology of U. A basis of the topology of M is given by their union

$$\mathscr{B} = \bigcup_{U \in \mathscr{C}} \mathscr{B}_U \,.$$

Proof. Let $U \in \mathscr{C}$. To each $p \in U$ we find a coordinate chart $\varphi_p \colon U_p \to B_3(0)$ such that $U_p \overset{\circ}{\subset} U$ and $\varphi_p(p) = 0$. So we can define a function $f_p \colon M \to [0, \infty)$ as follows

$$f_p(q) = \begin{cases} f(\varphi_p(q)) & \text{for } q \in \hat{U}_p \\ 0 & \text{else} \end{cases}$$

where f is the function from Lemma 2.1.12. Then $\operatorname{supp} f_p \subset U$ and clearly f_p is smooth. In particular, $W_p := f_p^{-1}(0, \infty) \overset{\circ}{\subset} U$. Hence $U = \bigcup_{p \in U} W_p$ with $W_p \in \mathscr{B}$. Thus $\mathscr{B}_U$ is a basis of topology of U. The second claim simply follows from the fact that each $V \overset{\circ}{\subset} M$ can be written as $V = \bigcup_{U \in \mathscr{C}} U \cap V$. □

We are now ready to prove existence of the partitions of unity as stated in Warner [5].

Theorem 2.1.15 (Existence of Partitions of Unity) *Let M be a smooth manifold and $\mathscr{C}$ be an open cover of M. Then there exists a countable partition of unity $\{\rho_i\}_{i \in \mathbb{N}}$ subordintate to the cover $\mathscr{C}$ with $\operatorname{supp} \rho_i$ compact for all $i \in \mathbb{N}$. If one does not require compact supports, then there is a partition of unity $\{\rho_U\}_{U \in \mathscr{C}}$ subordinate to the cover $\mathscr{C}$ with at most countably many of the ρ_U not identically zero.*

Proof. Let $\mathscr{B}$ be the basis of topology given in Lemma 2.1.14. For each element $U \in \mathscr{B}$ in this basis there is a non-negative function $f_U \colon M \to [0, \infty)$ such that $f_U^{-1}(0, \infty) = U$. Then, by Theorem 2.1.13, there is a countable locally finite refinement $\mathscr{D}$ of $\mathscr{C}$ which consists of elements of $\mathscr{B}$. Since $\mathscr{D}$ is locally finite, we can define

$$f = \sum_{U \in \mathscr{D}} f_U \,.$$

Since $\mathscr{D}$ is a cover, we find that $f > 0$. Hence we can define

$$\rho_U := f_U / f\,.$$

By construction, $\sum_{U\in\mathscr{D}} \rho_U = 1$. Moreover, since $\mathscr{D}$ consists of the elements of $\mathscr{B} = \bigcup_{V\in\mathscr{C}} \mathscr{B}_V$, we have that $U \in \mathscr{B}_V$ for some $V \in \mathscr{C}$. Thus $\operatorname{supp}\rho_U = \operatorname{supp} f_U \subset V$. In particular, $\operatorname{supp}\rho_U$ is compact. Thus $\{\rho_U\}_{U\in\mathscr{D}}$ is a partition of unity subordinate to $\mathscr{C}$.

To obtain obtain a partition of unity with the same index, we can basically sum up the functions with support in a given set in $\mathscr{C}$: Let $\tilde{\mathscr{C}} = \{U_i\}_{i\in\mathbb{N}} \subset \mathscr{C}$ be a countable subcover. For $j \in \mathbb{N}$, we define $\mathscr{D}_j := \mathscr{D} \setminus \{U \in \mathscr{D} \mid \exists i < j : U \subset U_i\}$ and

$$\tilde{\rho}_{U_i} = \sum_{U\in\mathscr{D}_i\,:\,U\subset U_i} \rho_U\,.$$

For $U \in \mathscr{C} \setminus \tilde{\mathscr{C}}$ we set $\rho_U = 0$. Then $\{\tilde{\rho}_U\}_{U\in\mathscr{C}}$ is a partition of unity with at most countably many $\tilde{\rho}_U \neq 0$. That the support lies in U follows from the fact that, if $\mathscr{A}$ is a locally finite family of closed sets then $\overline{\bigcup_{A\in\mathscr{A}} A} = \bigcup_{A\in\mathscr{A}} A$. □

Corollary 2.1.16 (Bump Function) *Let M be a manifold, $U \overset{\circ}{\subset} M$ and $A \subset M$ be closed, $A \subset U$. Then there is a smooth function $f : M \to [0,\infty)$ such that $\operatorname{supp} f \subset U$ and $f|_A = 1$.*

2.2 Submanifolds

2.2.1 Immersions, Submersions, Embeddings

A *topological submanifold* of a manifold M is a subset $S \subset M$, which itself has the structure of a manifold. If M is a differentiable manifold, we must additionally impose some compatibility.

Definition 2.2.1 (Smooth Submanifold) *Let M be a smooth manifold of dimension m. A subset $S \subset M$ is called a smoooth submanifold of dimension s, if for each point $p \in S$ there is an adapted chart, a chart $\varphi : U \to V \overset{\circ}{\subset} \mathbb{R}^s \times \mathbb{R}^{m-s}$ such that $p \in U \overset{\circ}{\subset} M$ and*

$$U \cap M = \varphi^{-1}\big(V \cap \mathbb{R}^s \times \{0\}\big)\,.$$

The adapted charts remind us to the flattenings that we already saw in local theory. This local flattability property we also refer to as the submanifold property. Similarly, we define complex submanifolds.

Definition 2.2.2 (Complex Submanifold) *Let M be a complex manifold of complex dimension m. A subset $S \subset M$ is called a complex submanifold of dimension s, if for each point $p \in S$ there is an adapted chart, a chart $\varphi : U \to V \overset{\circ}{\subset} \mathbb{C}^s \times \mathbb{C}^{m-s}$ such that $p \in U \overset{\circ}{\subset} M$ and*

$$U \cap M = \varphi^{-1}\big(V \cap \mathbb{C}^s \times \{0\}\big)\,.$$

Let $f : M \to N$ be a differentiable map between differentiable manifolds and let $p \in M$. Then if (U,φ) is a chart at p and (V,ψ) a chart at $f(p)$, then we can write f in these coordinates and obtain a differentiable map

$$\hat{f} = \psi \circ f \circ \varphi^{-1} : \varphi(U \cap f^{-1}V) \mapsto \psi(V)\,.$$

The *rank of f at p* is then defined as the rank of $\hat{f}$ at $\varphi(p)$,

$$\operatorname{rank}_p f = \operatorname{rank}_{\varphi(p)} \hat{f} = \dim(\operatorname{im} d_{\varphi(p)} \hat{f})\,.$$

Exercise 2.2.1 Check that the definition of rank f is independent of the choice of coordinates.

We say that f is of *full rank*, if

$$\operatorname{rank} f = \min(\dim M, \dim N).$$

Definition 2.2.3 (Immersion, Submersion) *A differentiable map $f\colon M \to N$ of full rank is called*

(a) a submersion, if $\dim M \geq \dim N$,
(b) an immersion, if $\dim M \leq \dim N$.

Remark 2.2.1 (Local Diffeomorphism) A local diffeomorphism is a map of full rank between manifolds of equal dimension. In particular, it is both an immersion and a submersion.

Let $f\colon M \to N$ be a surjection. Then M is separated into preimages of points in N: For $q \in N$,

$$M_q := f^{-1}\{q\}, \quad q \in N$$

is called the *fiber of f over q*.

Remark 2.2.2 Our use of the word fiber is a slight abuse of language. The term fiber usually appears in the context of topology for fiber bundles and fibrations, which are maps which have special topological properties. Above we use it for maps between sets.

Theorem 2.2.1 (Submersion Theorem) *The fibers of a surjective submersion $f\colon M \to N$ are closed differentiable submanifolds of M of dimension* $\dim M - \dim N$.

Proof. Since f is continuous, each fiber is closed. Using charts, the submanifold property follows then just by Theorem 1.1.18 in the real and Theorem 1.1.19 in the complex case. □

Example 2.2.1 (Algebraic Curves) An *affine algebraic curve* is the zero set $\Sigma \subset \mathbb{C}^2$ of a complex polynomial $p\colon \mathbb{C}^2 \to \mathbb{C}$,

$$\Sigma = \left\{(z_1, z_2) \in \mathbb{C}^2 \mid p(z_1, z_2) = 0\right\}.$$

If p is irreducible and regular on Σ, then the submersion theorem tells us it is a 1-dimensional complex submanifold, i.e. a Riemann surface.

A homogeneous polynomial of degree d is a polynomial P such that $P(\lambda z) = \lambda^d P(z)$. Each polynomial can be homogenized by adding a dimension. In particular, if $p\colon \mathbb{C}^2 \to \mathbb{C}$ is a polynomial of degree d, then there is a homogeneous polynomial $P\colon \mathbb{C}^3 \to \mathbb{C}$ of degree d such that

$$P(z_1, z_2, 1) = p(z_1, z_2).$$

Since P is homogeneous, its zeros can be considered as a subset $\hat{\Sigma} \subset \mathbb{CP}^2$ called a *projective algebraic curve*,

$$\hat{\Sigma} = \left\{[z_1, z_2, z_3] \in \mathbb{CP}^2 \mid P(z_1, z_2, z_3) = 0\right\},$$

which is closed and, since $\mathbb{CP}^2 = \pi(\mathbb{S}^5)$ is compact, is compact as well. Furthermore, if P has full rank away from zero, then $\hat{\Sigma}$ is a compact Riemann surface which—as $\mathbb{C}^2 = \{[z_1, z_2, 1] \in \mathbb{CP}^2]\} \subset \mathbb{CP}^2$—contains Σ. So $\hat{\Sigma}$ is a *compactification* of Σ.

If $\hat{\Sigma}$ has singular points then they must lie on the subset $\{[z_1, z_2, 0] \in \mathbb{CP}^2\}$. Since P is polynomial, the rank drops only at isolated points and we have still some unique complex structure. This is basically due to

Riemann's theorem on removable singularities.

Exercise 2.2.2 (Hyperelliptic Curves) A *hyperelliptic curve* is an algebraic curve Σ of the form

$$\Sigma = \{(\mu,\lambda) \in \mathbb{C}^2 \mid \mu^2 = p(\lambda)\}, \text{ where } p(\lambda) = \prod_{j=1}^{n}(\lambda - \lambda_j), \ \lambda_1, \ldots, \lambda_n \in \mathbb{C}\,.$$

Show that Σ is non-singular if and only if $\lambda_j \neq \lambda_k$ for $j \neq k$. Moreover, show that the maps $(\mu,\lambda) \mapsto \lambda$ (away from $P_j = (0,\lambda_j)$) and $(\mu,\lambda) \mapsto \mu$ (close to $P_j = (0,\lambda_j)$) define a complex atlas for Σ.

Depending on the degree of p, we add one point P_∞ or two points $P_{\pm\infty}$ over $\lambda = \infty$ to Σ,

$$\hat{\Sigma} := \begin{cases} \Sigma \bigcup \{\infty\} & \text{if } n = 2k+1 \\ \Sigma \bigcup \{+\infty, -\infty\} & \text{if } n = 2k+2 \end{cases}$$

Show that

$$(\mu,\lambda) \mapsto \mu/\lambda^{k+1} \text{ for } (\mu,\lambda) \neq P_\infty \text{ close to } P_\infty \quad \text{resp.} \quad (\mu,\lambda) \mapsto 1/\lambda \text{ for } (\mu,\lambda) \neq P_\infty \text{ close to } P_{\pm\infty}$$

define complex compatible charts at the infinite points. Finally, show that $\hat{\Sigma}$ is compact.

Exercise 2.2.3 (Hopf fibration) Let $\mathbb{S}^{2n+1} \subset \mathbb{R}^{2n+2} = \mathbb{C}^{n+1}$ denote the unit sphere. The restriction $\pi\colon \mathbb{S}^{2n+1} \to \mathbb{CP}^n$ of the canonical projection $(\mathbb{C}^{n+1})^\times \to \mathbb{CP}^n$ to the unit sphere is a smooth submersion all of whose fibers are diffeomorphic to the unit circle $\mathbb{S}^1$—such submersion is called a fibration with fiber $\mathbb{S}^1$. The submersion π is called the *Hopf fibration*.

Remark 2.2.3 The unit circle $\mathbb{S}^1$ is Lie group, i.e. a group which is additionally a smooth manifold structure within the the group operations, multiplication and inversion, are both smooth. $\mathbb{S}^1$ acts smoothly on $\mathbb{S}^{2n+1}$ by scalar multiplication and $\mathbb{CP}^n \cong_{\mathscr{C}^\infty} \mathbb{S}^{2n+1}/\mathbb{S}^1$.

Similarly also immersions define submanifolds—at least locally.

Theorem 2.2.2 (Immersion Theorem) *Let $f\colon M \to N$ be an immersion. Then each point $p \in M$ has a neighborhood $U \overset{\circ}{\subset} M$ such that $f(U) \subset N$ is a differentiable submanifold of dimension* $\dim M$.

Proof. Again, this is an immediate consequence of Theorem 1.1.18 and Theorem 1.1.19. □

Globally the image of an immersion need not be a submanifold—e.g. there might be self-intersections. But even for injective immersions, the image can fail to be a submanifold. An additional property is needed.

Definition 2.2.4 (Topological Embedding) *A map $f\colon M \to N$ between topological spaces is called a topological embedding, if its restriction $f\colon M \to f(M)$ is a homeomorphism.*

Definition 2.2.5 (Differentiable Embedding) *A differentiable embedding is an immersion which is an topological embedding.*

Exercise 2.2.4 Show that the image of an immersion is a submanifold if and only if the immersion is an embedding.

Theorem 2.2.3 *If $f\colon M \to N$ is an injective immersion and M is compact, then f is an embedding.*

Proof. Since $f\colon M \to N$ is injective, its restriction is bijective. Since M is compact and Hausdorff, every closed set $A \subset M$ is compact and thus mapped to a compact and thus closed set $f(A) \subset f(M)$. Hence the inverse of the restriction is continuous. Hence the restriction is a homeomorphism, i.e. f is an embedding. □

Exercise 2.2.5 Let $M, \tilde{M}$ be two 1-dimensional complex submanifolds of $\mathbb{C}^2$. Show that if they intersect, then they intersect either in an isolated point or M and $\tilde{M}$ locally coincide.

Exercise 2.2.6 The Veronese map is a map $\vartheta_d\colon \mathbb{CP}^n \to \mathbb{CP}^m$, where $m = \binom{n+d}{d} - 1$, which sends $[(z_0, \dots, z_n)] \in \mathbb{CP}^n$ to the complex line spanned by monomials in $z_0, \dots, z_n$ of degree d. For $n = 1$ and $d = 3$, we have

$$\vartheta_3\colon \mathbb{CP}^1 \to \mathbb{CP}^3, \quad [(z, w)] \mapsto [(z^3, z^2 w, z w^2, w^3)]\,.$$

Show that ϑ_3 is well-defined and holomorphic and that $\vartheta_3(\mathbb{CP}^1)$ is a compact submanifold of $\mathbb{CP}^3$.

Exercise 2.2.7 (Conics) Two sets in $\mathbb{CP}^n$ are projectively equivalent, if they one can be mapped to the other by a *projective transformation*, i.e. a map $f\colon \mathbb{CP}^n \to \mathbb{CP}^n$, $[z] \mapsto [Az]$ with $A \in \mathrm{End}_+(\mathbb{C}^{n+1})$. A *conic* in $\mathbb{CP}^2$ is an algebraic curve given by a homogeneous polynomial of degree 2. Show that

(a) all non-degenerate conics in $\mathbb{CP}^2$ are projectively equivalent,
(b) each non-degenerate conic is biholomorphic to $\mathbb{CP}^1$.

Hint for (*b*): Consider the Veronese map.

2.2.2 Tangent Space of Submanifolds in $\mathbb{R}^n$

Let $M \subset \mathbb{R}^n$ be an m-dimensional smooth submanifold. By Theorem 1.1.18, M can be locally parametrized, i.e. for each $p \in M$ there is a neighborhood $V \mathring{\subset} M$, some $U \mathring{\subset} \mathbb{R}^m$ and smooth map $f\colon U \to V$ of full rank, an embedding. In particular, there is an m-dimensional vector space attached to $p \in V$, its *tangent space*

$$T_p^f M := \operatorname{im} d_{f^{-1}(p)} f\,.$$

One easily checks that $T_p M$ does not depend on the choice of the parametrization. We set $T_p M := T_p^f M$.

Theorem 2.2.4 *Let $M \subset \mathbb{R}^{2n} = \mathbb{C}^n$ be a smooth submanifold of dimension $2m$. Then M is a complex submanifold if and only if for each $p \in M$ the tangent space $T_p M$ is a complex subspace.*

Proof. If M is a complex submanifold, then locally M can be parametrized by a holomorphic map. Clearly, the image of the differential of a holomorphic map is a complex subspace. Conversely, by Theorem 1.1.18, M can locally be written as the graph of a smooth function f over some open subset U in a complex subspace. We can assume that $U \mathring{\subset} \mathbb{C}^m$ and $f\colon U \to \mathbb{C}^{n-m}$. Then a local parametrization of M is given by $\hat{f} = (\mathrm{id}_U, f)$ and, for $z \in U$, we have

$$T_{\hat{f}(z)} M = d_z \hat{f}(\mathbb{C}^m) = \left\{(X, d_z f X) \in \mathbb{C}^n \mid X \in \mathbb{C}^m\right\}.$$

Thus, by Theorem 1.1.15, $d_z f$ is complex linear, i.e. f is complex differentiable in z. Hence f is holomorphic and, by Theorem 1.1.19, M is a complex submanifold. □

In particular, if M is a complex submanifold, then each tangent space $T_p M$ inherits an almost complex structure J_p from $\mathbb{C}^n$.

Theorem 2.2.5 *Let $M \subset \mathbb{C}^n$ be a complex submanifold and $g\colon M \to \mathbb{C}^k$ be a smooth map and let $p \in M$. Then*

$$g \text{ complex differentiable in } p \iff J\,(d_p g) = (d_p g)\,J_p\,.$$

Proof. The inverse map of a complex chart of M yields a local holomorphic parametrization f of M. By definition, the map g is complex differentiable if and only if $\tilde{g} = h \circ f$ is complex differentiable. The claim then follows from

$$(J\,d_p g)\,d_x f = J\,d_x(g \circ f) = d_p g\,(d_x f\,J) = (d_p g\,J_p)\,d_x f\,, \quad p = f(x), \quad x \in U$$

and $\operatorname{im} d_x f = T_p M$. □

In particular, if $\mathscr{A} = \{(U, \varphi)\}_{\alpha \in A}$ is a smooth atlas of a submanifold $M \subset \mathbb{C}^n$, then $\mathscr{A}$ is complex if and only if, for all $(U, \varphi) \in \mathscr{A}$,

$$d_p \varphi\, J_p X = J d_p \varphi\, X \quad p \in U, \; X \in T_p M\,.$$

Now, let $M \subset \mathbb{R}^3$ be a *surface*, i.e. a real 2-dimensional smooth manifold, and N the *Gauss map* of M, i.e. a smooth map $N\colon M \to \mathbb{S}^2$ such that $N_p \perp T_p M$ for all p. Then N_p turns $T_p M$ into a complex line with almost complex structure given as follows: For $X \in T_p M$,

$$J_p X := N_p \times X\,,$$

where $\times$ denotes the cross product in $\mathbb{R}^3$. Though this almost complex structure is not coming from the surrounding space, it still makes sense to talk about holomorphic maps in the sense above. It remains the question whether M with this almost complex structure is a complex manifold.

In fact, it turns out in the end that in the case above there always exists a complex structure and, moreover, all Riemann surfaces appear as surfaces in $\mathbb{R}^3$.

In conclusion, the ambient space allowed us to talk about tangent spaces and almost complex structures on them—which turns out to be very useful when talking about holomorphic maps and complex structures. The disjoint union of the tangent spaces $T_p M$ actually inherits a smooth structure and comes with a smooth map onto M whose fibers have the structure of m-dimensional vector spaces—it forms a so-called vector bundle.

In the next chapter we are going to carry this concept over to abstract manifolds. In particular, we define the tangent bundle of a smooth manifold.

Vector Bundles 3

3.1 Smooth Vector Bundles

Loosely speaking a vector bundle of rank r is a disjoint union of r-dimensional vector spaces parametrized over some space. We want to turn this into a rigorous definition. Definitions are given for the smooth setup, but are easily translated to topological manifolds with appropriate changes. Throughout, let $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

Definition 3.1.1 (Smooth Vector Bundle) *A smooth $\mathbb{K}$-vector bundle of rank r is a triple (E, M, π) consisting of smooth manifolds E and M and a smooth map $\pi \colon E \to M$ such that*

(a) for each point $p \in M$ the fiber $E_p = \pi^{-1}\{p\}$ over p has the structure of an r-dimensional $\mathbb{K}$-vector space ,
(b) for each point $p \in M$ there is a neighborhood U and a diffeomorphism $\phi \colon \pi^{-1}U \to U \times \mathbb{K}^r$ such that

$$\pi_1 \circ \phi = \pi|_U \,,$$

where $\pi_1 \colon U \times \mathbb{R}^r \to U$ denotes the projection to the first component, and the restriction of ϕ to each fiber over U is an isomorphism of $\mathbb{K}$-vector spaces: For each $q \in U$, the restriction

$$\phi_q \colon E_q \to \{q\} \times \mathbb{K}^r \cong \mathbb{K}^r \quad \textit{isomorphism} \,.$$

E is called the total space, M the base space and π will be called the bundle projection.

Notation 3.1.1 If there is no danger of confusion, we usually just say that $\pi \colon E \to M$ or—even leaving the bundle projection implicit—that $E \to M$ is a vector bundle, sometimes that E is a vector bundle over M.

Once we have a vector bundle there are distinguished maps from M to E—so-called *sections*.

Definition 3.1.2 (Smooth Section) *Let $\pi \colon E \to M$ be a smooth vector bundle. A smooth section is a smooth map $\psi \colon M \to E$, $p \mapsto \psi_p$ such that $\pi \circ \psi = \mathrm{id}_M$. The space of smooth sections of E is denoted by ΓE.*

A section is so to say just the right-inverse of the bundle projection. The terminology also makes sense for arbitrary maps.

3.1.1 Some Examples

Example 3.1.1 (The Trivial Bundle) Given a smooth manifold M, let $\underline{\mathbb{K}}^r_M := M \times \mathbb{K}^r$ and let $\pi_1 \colon \underline{\mathbb{K}}^r_M \to M$ denote the projection to the first component. Then $(\underline{\mathbb{K}}^r_M, M, \pi_1)$ is a smooth vector bundle—*the trivial vector bundle of rank r over M*. Its sections can be canonically be identified with $\mathbb{K}^r$-valued functions,

$$\mathscr{C}^\infty(M; \mathbb{K}^r) \ni f \quad \longleftrightarrow \quad \psi = (\mathrm{id}_M, f) \in \Gamma \, \underline{\mathbb{K}}^r_M \,.$$

Exercise 3.1.1 Show that any section of a vector bundle is an embedding of the base into the total space.

Exercise 3.1.2 (Tangent Bundle of a Submanifold of $\mathbb{R}^n$) Let $M \subset \mathbb{R}^n$ be a smooth submanifold. For each $p \in M$ we then have a linear subspace $T_pM \subset \mathbb{R}^n$. The tangent bundle of the submanifold M is defined as the disjoint union

$$TM := \bigsqcup_{p\in M} T_pM \quad \subset \underline{\mathbb{R}}^n_M\,.$$

Show that TM is a smooth vector bundle of rank m.

Exercise 3.1.3 (Restricted Bundle) Let $\pi\colon E \to M$ be a smooth vector bundle of rank r. Let $S \subset M$ be a smooth submanifold and $E|_S := \pi^{-1}S$. Then $\pi_S\colon E|_S \to S$ is a smooth vector bundle of rank r.

Definition 3.1.3 (Vector Bundle Homomorphism) *Let $\pi\colon E \to M$ and $\tilde{\pi}\colon \tilde{E} \to M$ be smooth vector bundles. A vector bundle homomorphism is a smooth map $\phi\colon E \to \tilde{E}$ such that $\tilde{\pi}\circ\phi = \pi$ and for each point $p \in M$ the restriction*

$$\phi_p\colon E_p \to \tilde{E}_p \quad \text{is a homomorphism of vector spaces}\,.$$

A vector bundle isomorphism is a bijective vector bundle homomorhism. Two vector bundles E and $\tilde{E}$ are called isomorphic if there is a vector bundle isomorphism between them. In this case we write $E \cong \tilde{E}$. A vector bundle E is called trivial, if it is isomorphic to the trivial bundle.

Remark 3.1.1 The inverse of a bijective vector bundle isomorphism is automatically a bundle homomorphism.

Example 3.1.2 (Local Trivialization, Vector Bundle Atlas) Every vector bundle $E \to M$ is locally trivial: Indeed, if E is of rank r, then each point by definition has a neighborhood $U \subset M$ and a bundle isomorphism $\phi\colon E|_U \to \underline{\mathbb{R}}^r_M$. Such ϕ is called a *local trivialization* of E.

In particular, there is a collection of local trivializations $\{(E|_{U_\alpha}, \phi_\alpha)\}_{\alpha\in A}$ such that $\bigcup_{\alpha\in A} U_\alpha = M$—a so-called *vector bundle atlas*. The transition maps between two charts ϕ_α and ϕ_β are of the form

$$\phi_\beta\circ\phi_\alpha^{-1}\colon E|_{U_\alpha\cap U_\beta} \to E|_{U_\alpha\cap U_\beta}, \quad (p,v)\mapsto (p, g_{\beta\alpha}(p)v), \quad g_{\beta\alpha}\in\mathscr{C}^\infty(U_\alpha\cap U_\beta; GL(r,\mathbb{R}))\,.$$

The maps $g_{\beta\alpha}$ satisfy the *cocycle condition*:

$$g_{\alpha\gamma}g_{\gamma\beta} = g_{\beta\alpha}^{-1}, \quad \text{for all } \alpha,\beta,\gamma\in A\,.$$

Example 3.1.3 (Vector Bundles by Gluing) Let M be a smooth manifold and $\{U_\alpha\}_{\alpha\in A}$ be an open cover of M. Let $g_{\beta\alpha}\in\mathscr{C}^\infty(U_\alpha\cap U_\beta;\mathrm{GL}(\mathbb{R}^r))$ satisfy the cocycle condition. On $\hat{E} = M\times A\times\mathbb{R}^r$ we define an equivalence relation

$$(p,\alpha,v)\sim(q,\beta,w) \iff p = q \text{ and } w = g_{\beta\alpha}(p)v\,.$$

Note that the projection to the first component is well-defined on the equivalence class. Similarly, the vector space structure of $\mathbb{R}^r$ carries over to the fibers. This turns $E := \hat{E}/_\sim$ into a smooth vector bundle of rank r over M. Moreover, there is a bundle atlas whose transition maps are given by $g_{\alpha\beta}$.

Let E be a vector bundle of rank r and let $\phi\colon E|_U \to U\times\mathbb{K}^r$ be a local trivialization. Sometimes it is more convenient to think of the inverse of ϕ rather as a collection of local sections: Let $e_1,\dots,e_r\in\mathbb{K}^r$ denote the canonical basis. Then we can define $\psi_j\in\Gamma(E|_U)$,

$$\psi_{jp} := \phi^{-1}(p,e_j), \quad p\in M\,.$$

Note that for each p the vectors $\psi_{1p},\dots,\psi_{rp}$ form a basis of E_p. This motivates the following definition.

Definition 3.1.4 (Frame) *Let E be a vector bundle of rank r. A frame is a set of sections $\sigma = (\psi_1,\dots,\psi_r)$ of E such that for all $p\in M$ the vectors $\psi_{1p},\dots,\psi_{rp}$ form a basis of E_p. A local frame is a frame for a restriction $E|_U$.*

Conversely, given a smooth frame $\sigma = (\psi_1, \ldots, \psi_r)$ over U, then

$$U \times \mathbb{K}^r \ni (p, v) \mapsto \sigma_p.v := \sum_{j=1}^{r} v_j \psi_{j,p} \in E|_U$$

is a smooth bundle isomorphism from $\underline{\mathbb{K}}^r_U$ to $E|_U$. We state an immediate corollary.

Corollary 3.1.1 *A vector bundle is trivial if and only it has a global frame.*

Exercise 3.1.4 (Möbius Bundle) Let SE(2) denote the euclidean group of $\mathbb{R}^2$. Given a subgroup $G \subset \mathrm{SE}(2)$, then this defines an equilvalence relation on $\mathbb{R}^2$,

$$p \sim_G q \quad :\Longleftrightarrow \quad \exists g \in G\colon q = g.p\,.$$

This fibers the space in equivalence classes—the *orbits of* G. We denote the quotient space by $\mathbb{R}^2/G$. Now, the maps σ and τ given by

$$\sigma(x, y) = (x + 1, -y), \quad \tau(x, y) = (x + 1, y)\,.$$

generate two such discrete subgroups $G_\sigma = \{\sigma^n \mid n \in \mathbb{Z}\}$ and $G_\tau = \{\tau^n \mid n \in \mathbb{Z}\}$ of SE(2). The corresponding orbit spaces $M = \mathbb{R}^2/G_\sigma$ and $C = \mathbb{R}^2/G_\tau$ are easily proven to be smooth manifolds. Show that M and C are real vector bundles of rank 1 over the circle $\mathrm{S}^1 = \mathbb{R}/\mathbb{Z}$. Moreover, show that C is trivial while M is not. In particular, $M \ncong C$. The manifold M is called the *Möbius bundle*.

We have seen that to each submanifold of $\mathbb{R}^n$ we can assign a vector bundle consisting of tangent vectors, which is fundamental when we want to speak about the derivative of functions. For abstract manifolds we cannot rely on a surrounding vector space. So we have to clarify what a tangent vector shall be in this situation.

3.1.2 The Tangent Bundle

A tangent vector can be understood as a directional derivative, i.e. it can be applied to any smooth function and spits out a number, which basically tells us how this function infinitesimally changes if we move in the direction of the vector through a given point. Let us give a rigorous definition.

Definition 3.1.5 (Tangent Space) *Let M be a smooth manifold. The tangent space of M at $p \in M$ is defined by*

$$T_pM := \left\{X \in (\mathscr{C}^\infty M)^* \mid \exists\ \gamma\colon (-\varepsilon, \varepsilon) \to M \ \ smooth\colon Xh = (h \circ \gamma)'(0), \ \ \forall h \in \mathscr{C}^\infty M\right\}.$$

An element of T_pM is a called a tangent vector at p. Moreover, $TM := \bigsqcup_{p \in M} T_pM$ is called the tangent bundle of M.

Remark 3.1.2 Using a chart, one easily shows that T_pM is a linear subspace of $(\mathscr{C}^\infty M)^*$.

In particular, if γ is a smooth curve in M, then $\gamma'(t) \in T_{\gamma(t)}M$ is given as follows: If $h \in \mathscr{C}^\infty M$, then

$$\gamma'(t)h = (h \circ \gamma)'(t)$$

By definition, all tangent vectors appear as the derivatives of curves.

Corollary 3.1.2 (Leibniz' Law) *Let $X \in T_pM$ and $f, g \in \mathscr{C}^\infty M$. Show that*

$$X(fg) = (Xf)g(p) + f(p)(Xg)\,.$$

Our next goal is to equip TM with a canonical smooth structure which turns it into a smooth vector bundle.

Tangent Bundle of Open Subsets of $\mathbb{R}^m$: Let $U \mathring{\subset} \mathbb{R}^m$. For $g \in \mathscr{C}^\infty U$ and $(p, v) \in U \times \mathbb{R}^m$, the directional derivative of g at p in the direction v is given as follows: Let $\gamma_{p,v}(t) = p + tv$. Then

$$\partial_v\big|_p g := (g \circ \gamma_{p,v})'(0) = \tfrac{d}{dt}\big|_{t=0} g(p + tv)\,.$$

Moreover, we set $\partial_i := \partial_{e_i}$. By the definition it is clear that $\partial_v\big|_p \in T_pU$. This establishes a canonical identification between the definition of tangent spaces we gave for submanifolds of $\mathbb{R}^m$ and the abstract definition of the tangent space which is given above:

$$U \times \mathbb{R}^m \ni (p, v) \overset{1:1}{\longleftrightarrow} \partial_v\big|_p \in TU\,.$$

Note that for fixed $p \in M$, the identification is linear in v. So TU inherits a smooth vector bundle structure from $U \times \mathbb{R}^m$. We want to carry this over to abstract smooth manifolds.

Definition 3.1.6 (Differential) *Let $f\colon M \to N$ be smooth. Then the differential of f at $p \in M$ is defined to be the linear map $d_pf\colon T_pM \to T_{f(p)}N$ given as follows: For $h \in \mathscr{C}^\infty N$ and $X \in T_pM$,*

$$d_pf(X)h = X(h \circ f)\,.$$

We collect all these maps into one defined on TM: $df\colon TM \to TN$ is given by $df(X) := d_{\pi(X)}f(X)$.

Let us check for sanity what the differential looks like on open subsets of euclidean space: Let $U \mathring{\subset} \mathbb{R}^m$ and $f\colon U \to \mathbb{R}^n$ be smooth. Then, for $\partial_v|_p \in TU$ and $h \in \mathscr{C}^\infty U$,

$$df(\partial_v|_p)h = \partial_v|_p(h \circ f) = \big(\mathrm{Jac}_p(h \circ f)\big)\, v = (\mathrm{Jac}_{f(p)}h)(\mathrm{Jac}_p f)\, v = \partial_{(\mathrm{Jac}_p f)\, v}|_{f(p)}h\,,$$

where $\mathrm{Jac}_p f$ denotes the Jacobian of f at p. Hence, using the canonical identification, we get

$$U \times \mathbb{R}^m \supset T_pU \ni (p, v) \overset{df}{\longmapsto} \big(f(p), (\mathrm{Jac}_p f)\, v\big) \in T_{f(p)}\mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n\,.$$

So, in euclidean space, the differential df is just given by the usual derivative—the Jacobian of f—only that it also keeps track of the base point.

Each tangent vector comes from differentiating along a curve. On this level the differential is just mapping the curve: Let $f\colon M \to N$ be a smooth map, γ be a curve in M and $h \in \mathscr{C}^\infty N$. Then we get

$$df(\gamma')h = \gamma'(h \circ f) = ((h \circ f) \circ \gamma)' = (h \circ (f \circ \gamma))' = (f \circ \gamma)'h\,.$$

With this at hand the chain rule boils down to the associativity of the composition of maps.

Theorem 3.1.3 *Let $g\colon M \to \hat{M}$ and $f\colon \hat{M} \to \tilde{M}$ be smooth maps, then*

$$d(f \circ g) = df \circ dg\,.$$

Proof. Let $X \in TM$ and γ such that $\gamma'(0) = X$. Then

$$(df \circ dg)(\gamma') = df(dg(\gamma')) = df((g \circ \gamma)') = \big(f \circ (g \circ \gamma)\big)' = \big((f \circ g) \circ \gamma\big)' = d(f \circ g)(\gamma')\,.$$

Evaluation at zero yields $(df \circ dg)(X) = d(f \circ g)(X)$. □

Since the differential of the identity id_M at a point $p \in M$ in a smooth manifold is the identity on the tangent space id_{T_pM}, we get the following corollary.

Corollary 3.1.4 *If $f\colon M \to N$ is a diffeomorphism, $p \in M$, then $d_pf\colon T_pM \to T_{f(p)}N$ is a vector space isomorphism.*

In particular, if M is a smooth manifold and $\varphi\colon U \to V \overset{\circ}{\subset} \mathbb{R}^m$ is a smooth chart we get a bijection

$$d\varphi\colon TU \to TV = V \times \mathbb{R}^m \subset \mathbb{R}^m \times \mathbb{R}^m\,.$$

Moreover, if $\varphi_\alpha\colon U_\alpha \to V_\alpha$ and $\varphi_\beta\colon U_\beta \to V_\beta$ are two smooth charts, then, for $(x,v) \in V_\alpha \times \mathbb{R}^m$,

$$d\varphi_\beta \circ (d\varphi_\alpha)^{-1}(x,v) = d\varphi_\beta \circ d\varphi_\alpha^{-1}(x,v) = d(\varphi_\beta \circ \varphi_\alpha^{-1})(x,v) = \left((\varphi_\beta \circ \varphi_\alpha^{-1})(x), \mathrm{Jac}_x(\varphi_\beta \circ \varphi_\alpha^{-1})v\right),$$

which is smooth since φ_α and φ_β smoothly compatible charts. Thus the differentials of smoothly compatible charts are smoothly compatible bijections. This shall be our smooth charts at the end. But there is some subtlety, in which sense is TU a subset of TM?

Lemma 3.1.5 (Localization Principle) *Let $p \in U \overset{\circ}{\subset} M$, $\iota\colon U \hookrightarrow M$. Then $d_p\iota\colon T_pU \to T_pM$ is an isomorphism.*

Proof. Clearly $d_p\iota$ is surjective: If $\tilde{X} \in T_pM$, then there is a curve γ in M such that $\gamma'(0) = \tilde{X}$. Since U is open, γ restricts to U. Thus $X := \gamma'(0) \in T_pU$. Then, if $h \in \mathscr{C}^\infty M$, we have

$$d_p\iota(X)h = X(h \circ \iota) = (h \circ \iota \circ \gamma)'(0) = (h \circ \gamma)'(0) = \tilde{X}h\,.$$

It is left to see that $d_p\iota$ is injective: Let $0 \neq X \in T_pU$. Then there is a function $h \in \mathscr{C}^\infty U$ such that $\tilde{X}h \neq 0$. Then, by Corollary 2.1.16, there is a function $\rho \in \mathscr{C}^\infty M$ such that $\operatorname{supp}\rho \subset U$ and $\rho|_V = 1$ on some neighborhood $V \subset U$ of p. Thus we can extend ρh by zero on $M \setminus U$ and obtain a smooth function g on M. Then, by Leibniz' law, we get

$$d_p\iota(X)g = X(g \circ \iota) = X(g|_U) = X(\rho h) = \underbrace{(X\rho)}_{=0}\, h(p) + \underbrace{\rho(p)}_{=1}\, Xh = Xh \neq 0\,,$$

i.e. $d_p\iota(X) \neq 0$. Hence $d_p\iota$ is injective. □

Corollary 3.1.6 *The tangent spaces of an m-dimensional smooth manifold are m-dimensional.*

In particular, this shows that if $\iota\colon M \overset{\circ}{\supset} U \hookrightarrow M$ denotes the inclusion map, then $d\iota\colon TU \to TM|_U = \pi^{-1}U$ is a bijection which restricts to isomorhisms on the fibers. This understood, we see that

$$d\varphi\colon TM|_U \to V \times \mathbb{R}^m$$

is a bijection which sends fibers isomorphically to fibers. Moreover, the transition maps between any two such charts are smooth and in particular homeomorphisms. So there is an induced topology. In fact, in this case this topology leads always to a manifold structure.

Theorem 3.1.7 *Let E be a disjoint union of $\mathbb{K}$-vector spaces over a smooth manifold M. Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of M and let $\{\phi_\alpha\colon E|_{U_\alpha} \to U_\alpha \times \mathbb{K}^r\}_{\alpha\in A}$ be a collection of fiber-preserving bijections which restrict to linear maps on the fibers. If for every two $\alpha, \beta \in A$ the composition $\phi_\beta \circ \phi_\alpha\colon (U_\alpha \cap U_\beta) \times \mathbb{K}^r \to (U_\alpha \cap U_\beta) \times \mathbb{K}^r$ is a smooth vector bundle isomorphism, then E has a unique structure such that $\{\phi_\alpha\}_{\alpha\in A}$ forms a smooth bundle atlas.*

Proof. Without loss of generality one can assume that A is countable and there are smooth charts $\varphi_\alpha\colon U_\alpha \to V_\alpha$.

Since all ϕ_α shall become homeomorhisms there is no choice for the topology. We define:

$$U \mathring{\subset} E \;:\iff\; \phi_\alpha(E|_{U_\alpha} \cap U) \mathring{\subset} U_\alpha \times \mathbb{K}^r\,, \quad \forall \alpha \in A\,.$$

Since $U_\alpha \times \mathbb{K}^r \cong_{\mathscr{C}^\infty} V_\alpha \times \mathbb{K}^r$, E is locally euclidean. As a countable union of open 2nd-countable subsets E is 2nd-countable. Moreover, E is Hausdorff—two disjoint open sets are easily constructed. By construction, the ϕ_α then form a smooth atlas. □

Definition 3.1.7 (Smooth Structure of the Tangent Bundle) *The smooth structure of the tangent bundle TM of a smooth manifold M is defined by the differentials of smooth charts of M.*

Corollary 3.1.8 *If $f\colon M \to N$ is smooth, then $df\colon TM \to TN$ is smooth.*

Proof. If φ is a chart at p and ψ is a chart at $f(p)$, then $d\psi \circ df \circ (d\varphi)^{-1} = d(\psi \circ f \circ \varphi^{-1})$, which is smooth. □

Next we are interested in the local form of the sections the tangent bundle—so-called tangent vector fields.

Definition 3.1.8 (Coordinate Frame) *Let $\varphi = (x_1, \ldots, x_m)\colon U \to V \mathring{\subset} \mathbb{R}^m$ be a smooth chart of M. Then we define $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m} \in \Gamma(TU)$ as follows: For $p \in U$,*

$$\frac{\partial}{\partial x_i}\Big|_p := (d\varphi)^{-1}\big(\partial_i|_{\varphi(p)}\big)\,.$$

The local frame $\sigma_\varphi := (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m})$ of TM is called the coordinate frame of the chart φ.

Two coordinate frames are related by the differential of the coordinate change.

Theorem 3.1.9 (Transformation of Coordinate Frames) *Let (U, φ) and (V, ψ) be charts of M and let $p \in U \cap V$. Then, for all $v \in \mathbb{R}^m$,*

$$\sigma_{\varphi,p} v = \sigma_{\psi,p}.(d_{\varphi(p)}(\psi \circ \varphi^{-1})v)\,.$$

Proof. Observe that $(\varphi(p), v) = d\varphi(\sigma_{\varphi,p}.v)$. Similarly, $(\psi(p), w) = d\psi(\sigma_{\psi,p}.w)$. We have

$$d\varphi^{-1}(\varphi(p), v) = (d\psi^{-1} \circ d\psi \circ d\varphi^{-1})(\varphi(p), v) = d\psi^{-1}\big((d\psi \circ d\varphi^{-1})(\varphi(p), v)\big) = d\psi^{-1}\big(\psi(p), d_{\varphi(p)}(\psi \circ \varphi^{-1})v)\big)\,.$$

Hence $\sigma_{\varphi,p}.v = d\varphi^{-1}(\varphi(p), v) = d\psi^{-1}\big(\psi(p), d_{\varphi(p)}(\psi \circ \varphi^{-1})v)\big) = \sigma_{\psi,p}.(d_{\varphi(p)}(\psi \circ \varphi^{-1})v)$. □

Exercise 3.1.5 Let $X\colon M \to TM$ such that $\pi \circ X = \mathrm{id}_M$. Show that X is smooth if and only if one of the following equivalent conditions is fulfilled:

(a) For every smooth chart (U, φ), $X|_U = \sigma_\varphi.\alpha = \sum_i \alpha_i \frac{\partial}{\partial x_i}$ with $\alpha \in \mathscr{C}^\infty(U; \mathbb{R}^m)$.
(b) For every smooth function $h \in \mathscr{C}^\infty M$, the function $Xh\colon p \mapsto (Xh)_p$ is smooth.

3.2 Intermission—Vector Fields as Operators on Functions

For $X \in \Gamma(TM)$ and $f \in \mathscr{C}^\infty M$, the map $Xf\colon p \mapsto X_p f$ is smooth. So X can be viewed as endomorphism

$$X\colon \mathscr{C}^\infty M \to \mathscr{C}^\infty M,\; f \mapsto Xf\,.$$

Definition 3.2.1 (Lie Algebra) *A Lie algebra is a vector space $\mathfrak{g}$ together with a skew bilinear map $[.,.]\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ which satisfies the Jacobi identity,*

$$[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0.$$

Theorem 3.2.1 (Lie Algebra of Endomorphisms) *Let V be a vector space.* End$V$ *together with the commutator*

$$[.,.]\colon \mathrm{End}V\times\mathrm{End}V\to\mathrm{End}V,\quad [A,B]:=AB-BA$$

forms a Lie algebra.

Proof. That the commutator is a skew bilinear map is obvious. Further,

$$\begin{aligned}[A,[B,C]]+[B,[C,A]]+[C,[A,B]] &= A(BC-CB)-(BC-CB)A+B(CA-AC)\\ &\quad -(CA-AC)B+C(AB-BA)-(AB-BA)C,\end{aligned}$$

which is zero since each term appears twice but with opposite sign. □

Theorem 3.2.2 *For all $f,g\in\mathscr{C}^\infty M$, $X,Y\in\Gamma TM$, the following equality holds*

$$[fX,gY]=fg[X,Y]+f(Xg)Y-g(Yf)X$$

Proof. Exercise. □

Lemma 3.2.3 (Schwarz Lemma) *Let $\varphi=(x_1,\dots,x_n)$ be a coordinate chart, then*

$$\left[\tfrac{\partial}{\partial x_i},\tfrac{\partial}{\partial x_j}\right]=0$$

Proof. Exercise. □

Thus, if $X=\sum_i a_i\frac{\partial}{\partial x_i}$ and $Y=\sum_j b_j\frac{\partial}{\partial x_j}$, we get

$$[X,Y]=\sum_{i,j}\left[a_i\tfrac{\partial}{\partial x_i},b_j\tfrac{\partial}{\partial x_j}\right]=\sum_{i,j}\left(a_i\tfrac{\partial b_j}{\partial x_i}\tfrac{\partial}{\partial x_j}-b_j\tfrac{\partial a_i}{\partial x_j}\tfrac{\partial}{\partial x_i}\right)=\sum_{i,j}\left(a_j\tfrac{\partial b_i}{\partial x_j}-b_j\tfrac{\partial a_i}{\partial x_j}\right)\tfrac{\partial}{\partial x_i}.$$

Thus $[X,Y]\in\Gamma(TM)$. In particular, we get the following theorem.

Theorem 3.2.4 *The set of sections on the tangent bundle $\Gamma(TM)\subset\mathrm{End}(\mathscr{C}^\infty M)$ is a Lie subalgebra.*

3.3 Tensors and Differential Forms

3.3.1 Tensor Bundles—Digression into Linear Algebra

The cotangent bundle TM^* of a manifold M is defined to be the disjoint union of the dual spaces of the tangent spaces.

$$TM^*=\bigsqcup_{p\in M}(T_pM)^*.$$

Again there is a priori no smooth structure. Though we can assign to each basis of T_pM a basis of T_pM^*—its dual basis. Thus we can convert a smooth local frame into a local frame of TM^*—its *dual frame*.

Example 3.3.1 Let us play this through with the cotangent bundle: Let $\varphi = (x_1, \dots, x_m)$ be a smooth chart. Then the corresponding coordinate frame is given by

$$\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}.$$

The derivative of the i-th coordinate at point p is then a linear map $(dx_i)_p \colon T_pM \to T_{x_i(p)}\mathbb{R} = \mathbb{R}$ and as such an element of $(T_pM)^*$. One easily check that $(dx_i)_p(\frac{\partial}{\partial x_j}|_p) = \delta_{ij}$, where δ_{ij} is the *Kronecker delta*, i.e.

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}$$

Hence the dual frame of the coordinate frame is given by the differentials of the coordinates of the chart,

$$dx_1, \dots, dx_m.$$

A frame is basically the same as the inverse of a trivialization. If a the basis changes by some automorphism, then the dual basis changes by its adjoint. Hence the dual frames change smoothly as well. Theorem 3.1.7 then provides a smooth vector bundle structure on TM^*. In particular, a section η of TM^* is smooth if and only if has smooth coefficients when locally expressed by the $dx_1, \dots, dx_m$,

$$\eta|_U = \sum_{i=1}^m a_i\, dx_i, \quad a_i \in \mathscr{C}^\infty U.$$

Exercise 3.3.1 Let M be a smooth manifold, $f \in \mathscr{C}^\infty M$ and let $\varphi = (x_1, \dots, x_m)$ be a chart defined on $U \overset{\circ}{\subset} M$. Show that

$$df|_U = \sum_{i=1}^m \frac{\partial f}{\partial x_i}\, dx_i$$

The very same principle applies to various vector space constructions. So we can build dual bundles, direct sums and tensor products of vector bundles. We will briefly discuss the underlying linear algebra.

For the rest of this section let U, V, W and $V_1, \dots, V_k$ denote finite–dimensional $\mathbb{K}$-vector spaces, $\mathbb{K} = \mathbb{R}, \mathbb{C}$. We fix the following notation:

$$\operatorname{Mult}(V_1, \dots, V_k; W) := \{\mu \colon V_1 \times \dots \times V_k \to W \mid \mu \;\; k\text{-linear}\}, \;\; \operatorname{Mult}^k(V; W) := \operatorname{Mult}(\underbrace{V, \dots, V}_{k\text{-times}}; W),$$

$$\operatorname{Alt}^k(V; W) := \{\mu \in \operatorname{Mult}^k(V; W) \mid \mu \;\; \text{alternating}\}.$$

Moreover, we set $\operatorname{Mult}(V_1, \dots, V_k) := \operatorname{Mult}(V_1, \dots, V_k; \mathbb{K})$, $\operatorname{Mult}^k V := \operatorname{Mult}^k(V; \mathbb{K})$ and $\operatorname{Alt}^k V := \operatorname{Alt}^k(V; \mathbb{K})$.

Definition 3.3.1 (Tensor Product) *The tensor product* $\otimes \colon V_1^* \times \dots \times V_k^* \to \operatorname{Mult}(V_1, \dots, V_k)$ *is defined by*

$$\omega_1 \otimes \dots \otimes \omega_k(v_1, \dots, v_k) := \omega_1(v_1) \cdots \omega_k(v_k).$$

Remark 3.3.1 The tensor product is a k-linear map.

Exercise 3.3.2 (Basis of Multilinear Forms) Let $\{\omega^i_1, \dots, \omega^i_{n_i}\} \subset V_i^*$ be bases. Then

$$\{\omega^1_{j_1} \otimes \dots \otimes \omega^k_{j_k} \mid j_i \in \{1, \dots, n_i\} \text{ for } i = 1, \dots, k\}$$

forms a basis of $\mathrm{Mult}(V_1, \dots, V_k)$.

Theorem 3.3.1 (Universal Property of Tensor Product) *Show that for each $\mu \in \mathrm{Mult}(V_1, \dots, V_k; W)$ there is a unique linear map $\lambda\colon \mathrm{Mult}(V_1^*, \dots, V_k^*) \to W$ such that $\lambda \circ \otimes = \mu$.*

Proof. Exercise. □

This shows that, in the finite–dimensional case, any k–linear map factors over $\otimes$ to a linear map.

Definition 3.3.2 (Tensor Product of Vector Spaces) *The tensor product of $V_1, \dots, V_k$ is defined to be*

$$V_1 \otimes \dots \otimes V_k := \mathrm{Mult}(V_1^*, \dots, V_k^*)\,.$$

There are several natural isomorphisms around which should be internalized.

Exercise 3.3.3 Show that there are natural isomorphisms

$$U \otimes V \otimes W \cong (U \otimes V) \otimes W \cong U \otimes (V \otimes W)\,,$$
$$\mathrm{Mult}(V_1, \dots, V_k; W) \cong \mathrm{Hom}(V_1 \otimes \dots \otimes V_k; W) \cong V_1^* \otimes \dots \otimes V_k^* \otimes W\,.$$

In particular, $V_1^* \otimes \dots \otimes V_k^* \cong (V_1 \otimes \dots \otimes V_k)^*$. Thus we have an tensor product between arbitrary tensors.

Theorem 3.3.2 *$\otimes$ is associative.*

Exercise 3.3.4 Show that $\mathrm{Mult}(V, W)$ and $\mathrm{Hom}(V; W^*)$ are naturally isomorphic. In particular,

$$\mathrm{Hom}(V; W) = \mathrm{Mult}(V, W^*)\,.$$

Definition 3.3.3 (Almost Complex Structure) *Let E be a smooth real vector bundle. A section $J \in \Gamma\mathrm{End}E$ such that $J^2 = -I$ is called an almost complex structure.*

Remark 3.3.2 A complex bundle can be considered as a real vector bundle with almost complex structure. Let E and F be complex vector bundles. A homomorphism of complex vector bundles is a homomorphism $A\colon E \to F$ of the underlying real vector bundles such that $A \circ J = J \circ A$. If (E, J) and $(\tilde{E}, \tilde{J})$ are complex vector bundles, then the almost complex structure $\hat{J}$ of the tensor product $E \otimes \tilde{E}$ is such that the tensor product becomes complex bilinear: Let $\psi \in \Gamma E$ and $\tilde{\psi} \in \Gamma\tilde{E}$, then

$$\hat{J}(\psi \otimes \tilde{\psi}) = (J\psi) \otimes \tilde{\psi} = \psi \otimes (\tilde{J}\tilde{\psi})\,.$$

Example 3.3.2 (Tangent Bundle of Complex Manifolds) Let M be a complex manifold. Then its tangent bundle is a complex vector bundle with almost complex structure induced by its complex charts: If $\varphi\colon U \to \mathbb{C}^m$ is a complex chart at p and $X \in T_pM$, then

$$JX := d\varphi^{-1}(i\, d\varphi(X))\,.$$

Since any two complex charts are holomorphically compatible, J is independent of the choice of φ. Clearly, J is smooth. Moreover, by definition, $d\varphi \circ J = i d\varphi$.

This brings up an interesting question. If the tangent bundle of a manifold has an almost complex structure, is it induced by a complex structure of the manifold? We will answer this question later by the Newlander–Nierenberg theorem.

Definition 3.3.4 (Wedge Product) *Let* $\wedge\colon V^* \times \cdots \times V^* \to \mathrm{Alt}^k V$ *be given as follows: For* $\omega_1, \ldots, \omega_k \in V^*$, *we define*

$$\omega_1 \wedge \cdots \wedge \omega_k(v_1, \ldots, v_k) := \det\big((\omega_i(v_j))_{i,j}\big)\,.$$

Exercise 3.3.5 (Basis of Alternating Forms) Let $\{v_1, \ldots, v_n\} \subset V$ be a basis and $\{\omega_1, \ldots, \omega_n\} \subset V^*$ its dual basis. Then

$$\{\omega_{j_1} \wedge \cdots \wedge \omega_{j_k} \mid 1 \leq j_1 < \ldots < i_k \leq n\}$$

forms a basis of $\mathrm{Alt}^k V$.

Theorem 3.3.3 (Universal Property of Wedge Product) *Show that for each* $\mu \in \mathrm{Alt}^k(V;W)$ *there is a unique linear map* $\lambda\colon \mathrm{Alt}^k V^* \to W$ *such that* $\lambda \circ \wedge = \mu$.

Proof. Exercise. □

Definition 3.3.5 (Exterior Product of a Vector Spaces) *The* k*–th exterior product of a* V *is defined to be*

$$\Lambda^k V := \mathrm{Alt}^k V^*\,.$$

The exterior algebra ΛV *is then defined to be the direct sum over all possible degrees* k, $\Lambda V := \oplus_k \Lambda^k V$.

Definition 3.3.6 (Alternator) *We define a projection* $\mathrm{Alt}_k\colon \mathrm{Mult}^k V \to \mathrm{Alt}^k V$ *as follows:*

$$\mathrm{Alt}_k(\mu)(v_1, \ldots, v_k) := \tfrac{1}{k!} \sum_{\sigma \in \mathrm{S}_k} \operatorname{sgn}\sigma\, \mu(v_{\sigma_1}, \ldots, v_{\sigma_k})\,,$$

where S_k *denotes the symmetric group of* k *elements and* $\operatorname{sgn}\sigma$ *denotes the signum of the permutation* σ.

Combining the alternator with the tensor product yields the wedge product and the symmetric product.

Definition 3.3.7 (Wedge Product) *Let* $\wedge\colon \Lambda^k V \times \Lambda^\ell V \to \Lambda^{k+\ell} V$ *be given as follows:*

$$\omega \wedge \eta := \tfrac{(k+\ell)!}{k!\ell!} \mathrm{Alt}_{k+\ell}(\omega \otimes \eta)\,.$$

The wedge product formula looks quite complicated. Actually a more convenient formula can be achieved if one only sums over the set of so–called (k,ℓ)–shuffles:

$$\mathrm{Sh}(k,\ell;n) := \{\sigma \in \mathrm{S}_{k+\ell} \mid 1 \leq \sigma_1 < \ldots < \sigma_k \leq n,\ 1 \leq \sigma_{k+1} < \ldots < \sigma_{k+\ell} \leq n\}\,.$$

With this notation we get

$$\omega \wedge \eta(v_1, \ldots, v_{k+\ell}) := \sum_{\sigma \in \mathrm{Sh}(k,\ell;n)} \operatorname{sgn}\sigma\, \omega(v_{\sigma_1}, \ldots, v_{\sigma_k})\eta(v_{\sigma_{k+1}}, \ldots, v_{\sigma_{k+\ell}})\,.$$

Exercise 3.3.6 Let $\omega_1, \ldots, \omega_{k+\ell} \in V^*$. Show that

$$(\omega_1 \wedge \cdots \wedge \omega_k) \wedge (\omega_{k+1} \wedge \cdots \wedge \omega_{k+\ell}) = \omega_1 \wedge \cdots \wedge \omega_{k+\ell}\,.$$

Hint: It is enough to show equality on a basis.

The last exercise is a more general version of the Laplace expansion of the determinant and shows that the definitions of wedge products given above are consistent. In particular, it shows that the wedge product is associative.

Theorem 3.3.4 *Let $\omega \in \Lambda^k V^*$, $\eta \in \Lambda^\ell V^*$ and $\xi \in \Lambda^m V^*$. Then*

$$(\omega \wedge \eta) \wedge \xi = \omega \wedge (\eta \wedge \xi)\,.$$

Proof. Exercise. □

Theorem 3.3.5 *If $\omega \in \Lambda^k V^*$ and $\eta \in \Lambda^\ell V^*$, then*

$$\omega \wedge \eta = (-1)^{k\ell} \eta \wedge \omega\,.$$

Proof. Exercise. □

All this carries over to bundles just fiberwise. In particular, if $E \to M$ is a smooth vector bundle, then we have bundle of alternating k-forms with values in E,

$$(\Lambda^k TM^*) \otimes E\,.$$

Definition 3.3.8 (Differential Forms) *An E-valued differential form of degree k is an element of the space*

$$\Omega^k(M; E) := \Gamma((\Lambda^k TM^*) \otimes E)$$

Moreover, the space of real-valued differential forms is denoted by $\Omega^k M := \Gamma(\Lambda^k TM^*)$.

From what was said, it is clear that a differential form $\eta \in \Omega^k(M; E)$ on an m-dimensional manifold M locally is of the form

$$\eta|_U = \sum_{1 \le i_1 < \cdots < i_k \le m} \psi_{i_1 \cdots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}\,, \quad \psi_{i_1 \cdots i_k} \in \Gamma(E|_U)\,.$$

Remark 3.3.3 (Wedge Product of Differential forms) The wedge product of real-valued differential forms is defined just fiberwise. To define a wedge product for vector-valued differential forms we need an additional multiplication: Let E_i, $i = 1, 2, 3$ be vector bundles over M. Given a multiplication $\bullet \in \Gamma(\mathrm{Mult}(E_1, E_2; E_3))$, we can wedge an E_1-valued k-form with an E_2-valued ℓ-form with the very same formula to obtain an E_3-valued $(k+\ell)$-form: For $\omega \in \Omega^k(M; E_1)$ and $\eta \in \Omega^\ell(M; E_2)$,

$$\omega \wedge_\bullet \eta(X_1, \ldots, X_{k+\ell}) := \sum_{\sigma \in \mathrm{Sh}(k,\ell;n)} \operatorname{sgn} \sigma\, \omega(X_{\sigma_1}, \ldots, X_{\sigma_k}) \bullet \eta(X_{\sigma_{k+1}}, \ldots, X_{\sigma_{k+\ell}})\,.$$

Example 3.3.3 Let $M \overset{\circ}{\subset} \mathbb{R}^m$. Given two $\mathbb{R}^n$-valued forms $\omega \in \Omega^k(M;\mathbb{R}^n)$ and $\eta \in \Omega^\ell(M;\mathbb{R}^n)$,

$$\omega = \sum_{1\le i_1<\cdots<i_k\le m} \omega_{i_1\cdots i_k}\, dx_{i_1}\wedge\cdots\wedge dx_{i_k}, \quad \eta = \sum_{1\le j_1<\cdots<j_\ell\le m} \eta_{j_1\cdots j_\ell}\, dx_{j_1}\wedge\cdots\wedge dx_{j_\ell},$$

we can wedge them using the standard inner product $\langle .,.\rangle$ of $\mathbb{R}^n$:

$$\langle\omega\wedge\eta\rangle = \sum_{\substack{1\le i_1<\cdots<i_k\le m,\\ 1\le j_1<\cdots<j_\ell\le m}} \langle\omega_{i_1\cdots i_k},\eta_{j_1\cdots j_\ell}\rangle\, dx_{i_1}\wedge\cdots\wedge dx_{i_k}\wedge dx_{j_1}\wedge\cdots\wedge dx_{j_\ell}$$

Remark 3.3.4 (Properties of Wedge Product of Vector-valued Forms) The wedge product of vector valued forms depends on the given multiplications. In general, one cannot commute these forms. Though, if the forms satisfy additional properties, one gets the very same formulas: Let E_1, E_2, E_3 vector bundles. Then:

(a) $\psi_1 \bullet \psi_2 = \psi_2 \tilde{\bullet} \psi_1,\ \forall \psi_i \in \Gamma E_i, \Longrightarrow \omega_1 \wedge_\bullet \omega_2 = (-1)^{k_1k_2}\omega_2 \wedge_{\tilde{\bullet}} \omega_1,\ \forall \omega_i \in \Omega^{k_i}(M;E_i)$.

(b) $(\psi_1 \bullet \psi_2)\bar{\bullet}\psi_3 = \psi_1 \hat{\bullet}(\psi_2 \tilde{\bullet}\psi_3),\ \forall \psi_i \in \Gamma E_i \Longrightarrow (\omega_1 \wedge_\bullet \omega_2)\wedge_{\bar{\bullet}}\omega_3 = \omega_1 \wedge_{\hat{\bullet}} (\omega_2 \wedge_{\tilde{\bullet}} \omega_3),\ \forall \omega_i \in \Omega^{k_i}(M;E_i)$.

Usually the pairing will finally appear will be natural, such as the multiplication of a section by a real function or the multiplication of an endomorphism field with a section.

3.3.2 Sections along Maps and Pullback Bundles

Let $\pi\colon E \to M$ be a smooth vector bundle and $f\colon S \to M$ be a smooth map. A *section along* f is a map $\psi\colon S \to E$ such that

$$\pi\circ\psi = f\,.$$

Example 3.3.4 Let $f\colon M \to N$ be a smooth map and $X \in \Gamma(TM)$. Then $df\circ X\colon M \to TN$ and for $p \in M$ we have $df(X_p) \in T_{f(p)}N$. Hence $\pi\circ df\circ X = f$, i.e. $df\circ X$ is a section along f.

We will identify sections along maps with sections of the pullback bundle.

Definition 3.3.9 (Pullback Bundle) *Let $E \to M$ be a smooth vector vector bundle and $f\colon S \to M$ be a smooth map. Then the pullback bundle $f^*E \to S$ is defined by*

$$f^*E := \bigsqcup_{p\in S} E_{f(p)} \subset S\times E\,.$$

Exercise 3.3.7 Show that $f^*E \subset S\times E$ is a smooth submanifold.

If $\psi\colon S \to E$ is a section along f, then the corresponding sections of f^*E is given by $\hat{\psi} = (\mathrm{id}_S,\psi)$. Conversely, if $\hat{\psi} \in \Gamma(f^*E)$, then $\pi_2\circ\hat{\psi}$, where $\pi_2\colon S\times E \to E$, is a section along f:

$$\psi \overset{1:1}{\longleftrightarrow} (\mathrm{id}_S,\psi)\,.$$

We will keep this identification in mind throughout.

Definition 3.3.10 *Let $E \to M$ be a smooth vector bundle and $f\colon S \to M$ be a smooth map. If $\psi \in \Gamma E$, then $f^*\psi \in \Gamma(f^*E)$ is defined as $f^*\psi := \psi\circ f$.*

Corollary 3.3.6 *If $\psi_1,\ldots,\psi_r$ is a smooth local frame of E, then $f^*\psi_1,\ldots,f^*\psi_r$ is a smooth local frame of f^*E*

Theorem 3.3.7 *Let $f\colon M \to N$ be smooth, $X, Y \in \Gamma(TM)$ and $\tilde{X}, \tilde{Y} \in \Gamma(TN)$ such that $f^*\tilde{X} = df(X)$ and $f^*\tilde{Y} = df(Y)$. Show that*

$$df([X,Y]) = f^*[\tilde{X}, \tilde{Y}]\,.$$

Proof. Exercise. □

Example 3.3.5 (Pullback of Tautological Bundles) Let V be an n-dimensional vector space. The Grassmannian $\mathrm{Gr}_k(V)$ is the space of all k-dimensional subspaces of V. If we fix an inner product, we can identify each k-dimensional subspace with the orthogonal projection onto it. Thus,

$$\mathrm{Gr}_k(V) = \left\{P \in \mathrm{GL}(V) \mid P^2 = P,\ \ P^* = P,\ \ \operatorname{tr} P = k\right\},$$

which can be shown to be a $k(n-k)$-dimensional submanifold of $\mathrm{Sym}(V) = \{A \in \mathrm{GL}(V) \mid A^* = A\}$.

The *tautological bundle* is then defined as

$$\mathrm{Taut}(\mathrm{Gr}_k(V)) = \left\{(P, v) \in \mathrm{Gr}_k(V) \times V \mid Pv = v\right\},$$

i.e. to each k-dimensional subspace $U \subset V$ as a point in the Grassmannian we assigned U as fiber.

Now given a smooth $f\colon M \to \mathrm{Gr}_k(V)$ we obtain a rank-k bundle $f^*\mathrm{Taut}(\mathrm{Gr}_k(V)) \to M$. An interesting question is whether—up to isomorphism—every vector bundle appears as pullback of a tautological bundle.

Definition 3.3.11 (Pullback of Differential Forms) *Let $E \to M$ be a smooth vector bundle, $f\colon S \to M$ be smooth and $\omega \in \Omega^k(M;E)$. Then the pullback $f^*\omega \in \Omega^k(M; f^*E)$ of ω is defined to be $f^*\omega := (df)^*\omega$, i.e.*

$$(f^*\omega)(X_1, \ldots, X_k) = \omega(df(X_1), \ldots, df(X_k)), \quad \textit{for all}\ \ X_1, \ldots, X_k \in \Gamma(TS)\,.$$

Theorem 3.3.8 *Let $f\colon S \to M$ be smooth, $\omega \in \Omega^k M$ and $\eta \in \Omega^\ell M$. Then*

$$f^*(\omega \wedge \eta) = (f^*\omega) \wedge (f^*\eta)\,.$$

Proof. Exercise. □

3.3.3 The Newlander–Nirenberg Theorem

Definition 3.3.12 (Almost Complex Manifold) *An almost complex manifold is a pair (M, J) consisting of a smooth manifold M and an almost complex structure J on its tangent bundle TM.*

Remark 3.3.5 Any almost complex manifold is of even real dimension.

Example 3.3.2 shows that each complex manifold is an almost complex manifold.

Definition 3.3.13 (Holomorphic Maps between Almost Complex Manifolds) *A smooth map $f\colon M \to N$ between almost complex manifolds is called holomorphic, if it intertwines the almost complex structures,*

$$J \circ df = df \circ J\,.$$

Remark 3.3.6 On a complex manifold, the notion of holomorphicity coincides with the previous one.

Remark 3.3.7 Not every almost complex manifold is a complex manifold.

Theorem 3.3.9 *Let M be an almost complex manifold. An atlas consisting of holomorphic charts is a complex atlas.*

Proof. Let $J_{\mathbb{C}^m}$ denote the multiplication by i on $\mathbb{C}^m$. Let φ_α and φ_β be holomorphic charts of M, then

$$d(\varphi_\beta \circ \varphi_\alpha^{-1}) \circ J_{\mathbb{C}^m} = d\varphi_\beta \circ d\varphi_\alpha^{-1} \circ J_{\mathbb{C}^m} = d\varphi_\beta \circ J \circ d\varphi_\alpha^{-1} = J_{\mathbb{C}^m} \circ d\varphi_\beta \circ d\varphi_\alpha^{-1} = J_{\mathbb{C}^m} \circ d(\varphi_\beta \circ \varphi_\alpha^{-1}) .$$

Thus all holomorphic maps are holomorphically compatible and so define a complex structure. □

Remark 3.3.8 Thus a complex manifold is exactly the same thing as an almost complex manifold (M, J) which can be covered by holomorphic charts. If M is complex, we call J a *complex structure*.

Whether an almost complex structure is complex can be answered by looking at its Nijenhuis tensor.

Definition 3.3.14 (Nijenhuis Tensor) *Let M be a smooth manifold and $A \in \Gamma\mathrm{End}(TM)$. Then, for $X, Y \in \Gamma(TM)$, the Nijenhuis tensor $\mathcal{N}_A$ of A is given by*

$$\mathcal{N}_A(X,Y) := -A^2[X,Y] - [AX, AY] + A([AX,Y] + [X,AY]) .$$

We defined $\mathcal{N}_A \colon \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$. So, in which sense is it a tensor, i.e. a section of $\mathrm{Mult}(TM, TM; TM)$?

If E and F are vector bundles over M, then $A \in \Gamma\mathrm{Hom}(E;F)$ defines a map $\hat{A} \colon \Gamma E \to \Gamma F$ as follows:

$$(\hat{A}\psi)_p = A_p \psi_p .$$

Conversely, we have the following theorem.

Theorem 3.3.10 (Characterization of Tensors) *Let $E, F \to M$ be vector bundles and let $\hat{A} \colon \Gamma E \to \Gamma F$ be $\mathbb{R}$-linear such that*

$$\hat{A}(f\psi) = f\hat{A}(\psi), \quad \text{for all } f \in \mathscr{C}^\infty M, \psi \in \Gamma E .$$

Then there is $A \in \Gamma\mathrm{Hom}(E,F)$ such that $(\hat{A}\psi)_p = A_p(\psi_p)$ for all $\psi \in \Gamma E$, $p \in M$.

Proof. For $\tilde{\psi} \in E_p$, $p \in M$, we define $A(\tilde{\psi})$ as follows: Choose $\psi \in \Gamma E$ such that $\psi_p = \tilde{\psi}$ and set $A_p(\tilde{\psi}) = (\hat{A}\psi)_p$. We have to show that $A(\tilde{\psi})$ is well-defined, i.e $(\hat{A}\psi)_p$ depends only on ψ_p—if $\psi, \hat{\psi} \in \Gamma E$ with $\psi_p = \hat{\psi}_p$ then $(\hat{A}\psi)_p = (\hat{A}\hat{\psi})_p$. Since $\hat{A}$ is linear, it is enough to show that $(\hat{A}\psi)_p = 0$ for all $\psi \in \Gamma E$ with $\psi_p = 0$.

Choose a local frame field $(\psi_1, \ldots, \psi_r)$ of E on some neighborhood of p and a function $\rho \in \mathscr{C}^\infty M$ such that $f|_{M \setminus U} = 0$ and $f \equiv 1$ near p. In particular, $\rho\psi_1, \ldots, \rho\psi_r$ are globally defined sections. Let $\psi \in \Gamma E$ with $\psi_p = 0$. Since ψ is smooth, $\psi | U = a_1\psi_1 + \cdots + a_r\psi_r$ with $a_1, \ldots, a_r \in \mathscr{C}^\infty U$. Again $\rho a_1, \ldots, \rho a_r$ are globally defined. Then

$$\rho^2 \hat{A}\psi = \hat{A}(\rho^2\psi) = \hat{A}((\rho a_1)(\rho\psi_1) + \cdots + (\rho a_k)(\rho\psi_k)) = (\rho a_1)\hat{A}(\rho\psi_1) + \cdots + (\rho a_r)\hat{A}(\rho\psi_r)).$$

Evaluation at p yields then $(\hat{A}\psi)_p = 0$. □

Note that the sections ΓE of a smooth vector bundle form a $\mathscr{C}^\infty M$-module: If $f \in \mathscr{C}^\infty M$ and ΓE, then

$$(f\psi)_p = f(p)\psi_p .$$

Thus each tensor field $A \in \Gamma\mathrm{Hom}(E;F)$ can be identified with a unique $(\mathscr{C}^\infty M)$-linear map $\hat{A} \colon \Gamma E \to \Gamma F$. We also say that $\hat{A}$ is *tensorial*. In the following we won't distinguish between the tensor field A and its tensorial counterpart.

Corollary 3.3.11 *An $\mathbb{R}$-multilinear map $\Gamma E_1 \times \cdots \times \Gamma E_k \to \Gamma F$ is tensorial if and only if it is tensorial in each slot.*

Proof. Clearly, if a map $\hat{A} \colon \Gamma E_1 \times \cdots \times \Gamma E_k \to \Gamma F$ comes from a tensor field, then it is tensorial in each slot. Conversely, for fixed $\psi_i \in \Gamma E_i$,

$$\Gamma E_k \ni \psi \mapsto \hat{A}(\psi_1, \ldots, \psi_{k_1}, \psi) \in \Gamma F$$

is tensorial and hence $\hat{A}$ can be considered as a map $\Gamma E_1 \times \cdots \times \Gamma E_{k-1} \to \Gamma \mathrm{Hom}(E_k; F)$. By the same procedure we then get $\Gamma E_1 \times \cdots \times \Gamma E_{k-2} \to \Gamma \mathrm{Mult}(E_{k-1}, E_k; F)$ and so on—till we finally end up with $A \in \Gamma \mathrm{Mult}(E_1, \ldots, E_k; F)$. □

Now, let $A \in \Gamma \mathrm{End}(TM)$. To check that $\mathcal{N}_A$ is a tensor we have to check that it tensorial in both of its slots: Let $X, Y \in \Gamma(TM)$ and $f \in \mathscr{C}^\infty M$. Then, using the result of Exercise 3.2.2, we get

$$\begin{aligned}
\mathcal{N}_A(fX, Y) &= -A^2[fX, Y] - [AfX, AY] + A([AfX, Y] + [fX, AY]) \\
&= -A^2\big(f[X,Y] - (Yf)X\big) - \big(f[AX, AY] - ((AY)f)AX\big) + A\big(f[AX,Y] - (Yf)AX + f[X, AY] - ((AY)f)X\big) \\
&= f\mathcal{N}_A(X,Y) + ((Yf)A^2X + ((AY)f)AX - (Yf)A^2X - ((AY)f)AX) \\
&= f\mathcal{N}_A(X,Y) .
\end{aligned}$$

Tensoriality in the second slot follows then since $\mathcal{N}_a$ is anti-symmetric:

$$\mathcal{N}_A(X, fY) = -\mathcal{N}_A(fY, X) = -f\mathcal{N}_A(Y,X) = f\mathcal{N}_A(X,Y) .$$

Example 3.3.6 For $M = \mathbb{C}^m$ with $JX = iX$ we have $N_J = 0$: Since $\mathcal{N}_J$ is tensorial, we can choose $X, Y \in \Gamma T\mathbb{C}^m$ constant, then JX, JY are constant as well and all Lie brackets vanish.

This examples shows half of the following important theorem.

Theorem 3.3.12 (Newlander–Nirenberg) *An almost complex manifold (M, J) is complex if and only if its Nijenhuis tensor vanishes, $N_J = 0$.*

The other direction ($\Leftarrow$) is hard and far beyond the scope of this course. A sketch of proof and reference to the actual paper can be found in [6]. Thus we just state it here and prove instead an interesting consequence.

Theorem 3.3.13 *On a surface, i.e. a manifold of real dimension 2, every almost complex structure is complex.*

Proof. Let M be a surface with almost complex structure J. For $X \in \Gamma(TM)$, we have

$$\mathcal{N}_J(X, JX) = -J^2[X, JX] - [JX, JJX] + J([JX, JX] + [X, JJX]) = [X, JX] + [JX, X] + J([JX, JX] - [X, X]) = 0 .$$

Since $\mathcal{N}_J$ is tensorial and $X_p, JX_p \in T_pM$ form a basis whenever $X_p \neq 0$ that is sufficient. □

Example 3.3.7 Let $S \subset \mathbb{R}^3$ be a surface and $N \colon S \to \mathbb{S}^2$ such that $N_p \perp T_pS$ for all $p \in S$. Then $JX = N \times X$ defines an almost complex structure. Since S is real 2-dimensional, the Newlander–Nierenberg theorem tells us that S is a Riemann surface.

4 Differentiation in Vector Bundles

4.1 Derivative of Sections

If we think about a smooth function $f\colon M \to \mathbb{R}^r$, i.e. a section of the trivial bundle $\underline{\mathbb{R}}^n_M$, there is no question about how to its the derivative. If we instead look at a section $\psi \in \Gamma E$ of some smooth vector bundle $E \to M$ in general, we run into a problem—different points in M are mapped by ψ to different fibers. And there is a priori no canonical way to relate them at all. Actually this problem cannot be resolved without some additional structure—a *connection*—on the bundle.

4.1.1 Connections

We formalize the derivative as a map that satisfies Leibniz' law.

Definition 4.1.1 (Connection) *A connection on a vector bundle $E \to M$ is an $\mathbb{R}$-linear map $\nabla\colon \Gamma E \to \Omega^1(M;E)$, which satisfies Leibniz' law, i.e., for all $f \in \mathscr{C}^\infty M$ and $\psi \in \Gamma E$,*

$$\nabla(f\psi) = df\,\psi + f\nabla\psi\,.$$

Notation 4.1.1 If $\psi \in \Gamma E$ and $X \in \Gamma(TM)$, then one usually writes $\nabla_X\psi$ for $(\nabla\psi)(X)$.

Remark 4.1.1 A connection is sometimes called covariant derivative—we will use this terminology from time to time.

Example 4.1.1 (The Trivial Connection) Let M be a smooth manifold. A section $\psi \in \Gamma\underline{\mathbb{R}}^r_M$ is of the form $\psi = (\mathrm{id}_M, f)$, where $f\colon M \to \mathbb{R}^r$ is smooth. We define the *trivial connection* $d\colon \Gamma\underline{\mathbb{R}}^r_M \to \Omega^1(M;\underline{\mathbb{R}}^r_M)$ by the usual derivative: For $X \in TM$,

$$d_X\psi := (\pi(X), df(X))\,.$$

In particular, for $X \in \Gamma(TM)$, using the standard identification, we can write $d_X\psi = df(X)$.

Leibniz' law provides some localization in the sense that the derivative of a section ψ at a point only depends on the values of ψ in an arbitrarily small neighborhood of that point.

Theorem 4.1.1 (Localization Principle) *Let ∇ be a connection on a smooth vector bundle $E \to M$. If $U \overset{\circ}{\subset} M$ and $\psi_1, \psi_2 \in \Gamma E$ such that $\psi_1|_U = \psi_2|_U$, then $\nabla\psi_1|_U = \nabla\psi_2|_U$.*

Proof. Let $p \in U$ and $X \in T_pM$. Then there is a closed neighborhood $A \subset U$ of p and a smooth function $\rho\colon M \to \mathbb{R}$ such $\rho|_{M\setminus U} = 0$ and $\rho|_A = 1$. In particular, $\rho\psi_1 = \rho\psi_2$ on all of M. Thus

$$\nabla_X\psi_1 = d\rho(X)\psi_{1,p} + \rho(p)\nabla_X\psi_1 = \nabla_X(\rho\psi_1) = \nabla_X(\rho\psi_2) = d\rho(X)\psi_{2,p} + \rho(p)\nabla_X\psi_2 = \nabla_X\psi_2\,.$$

Since $X \in TM|_U$ was arbitrary, we conclude $\nabla\psi_1|_U = \nabla\psi_2|_U$. □

If we talk about vector bundles with connection, i.e. pairs (E, ∇), then the notion of homomorphism has to be adapted in the obvious way.

Definition 4.1.2 (Homomorphism of Vector Bundles with Connection) *Let (E,∇) and $(\tilde{E},\tilde{\nabla})$ be vector bundles with connection. A homomorphism of vector bundles with connection is a homomorphism $\phi\colon E\longrightarrow\tilde{E}$ of vector bundles such that*

$$\tilde{\nabla}\phi(\psi)=\phi(\nabla\psi)\ \text{ for all }\ \psi\in\Gamma E\,.$$

A vector bundle with connection is called trivial if it is isomorphic to the trivial bundle with the trivial connection.

Note, that a connection can be carried over from one to another bundle by an isomorphism.

Exercise 4.1.1 (Connection induced by Isomorphism) Let $E,\tilde{E}\longrightarrow M$ be vector bundles, $\phi\colon E\longrightarrow\tilde{E}$ be an isomorphism and let ∇ be a connection on E. Then there is a unique connection $\tilde{\nabla}$ on $\tilde{E}$ such that $\phi(\nabla\psi)=\tilde{\nabla}\phi(\psi)$ for all $\psi\in\Gamma E$.

Hence we can always get a locally defined connection by carrying over the trivial connection by a local trivialization. These can then be glued.

Theorem 4.1.2 *On each vector bundle there exists a connection.*

Proof. We choose local trivializations $\phi_\alpha\colon E|_{U_\alpha}\to\underline{\mathbb{R}}^r_M$ such that $M=\bigcup_{\alpha\in A}U_\alpha$. Let $\{\rho_\alpha\}_{\alpha\in A}$ denote a partition of unity subordinate to $\{U_\alpha\}_\alpha$. Now we use Exercise 4.1.1 to pull the trivial connection back to obtain a connection ∇^α on $E|_{U_\alpha}$ and define

$$\nabla\psi:=\sum_{\alpha\in A}\rho_\alpha\nabla^\alpha(\psi|_{U_\alpha})\,.$$

Clearly $\nabla\colon\Gamma E\longrightarrow\Omega^1(M;E)$. Moreover, ∇ satisfies the Leibniz law:

$$\nabla(f\psi)=\sum_{\alpha\in A}\rho_\alpha\nabla^\alpha(f\psi|_{U_\alpha})=\sum_{\alpha\in A}\rho_\alpha\big(df\psi|_{U_\alpha}+f\nabla^\alpha(\psi|_{U_\alpha})\big)=df\sum_{\alpha\in A}\rho_\alpha\psi|_{U_\alpha}+f\sum_{\alpha\in A}\rho_\alpha\nabla^\alpha(\psi|_{U_\alpha})=df\psi+f\nabla\psi\,.$$

for $f\in\mathscr{C}^\infty M$. □

The next theorem tells us that the space of connections is an affine space over the space of endomorphism-valued 1-forms.

Theorem 4.1.3 *Any two connections ∇ and $\tilde{\nabla}$ on a vector bundle $E\longrightarrow M$ differ by a 1-form $A\in\Omega^1(M;\mathrm{End}E)$:*

$$\tilde{\nabla}=\nabla+A\,.$$

Remark 4.1.2 The space of connections on a vector bundle E forms an affine space over $\Omega^1(M;\mathrm{End}E)$.

Proof. Let $\tilde{\nabla}-\nabla$, i.e. $A\colon\Gamma TM\times\Gamma E\longrightarrow\Gamma E$ is given by $(X,\psi)\mapsto\tilde{\nabla}_X\psi-\nabla_X\psi$. Clearly, A is tensorial in the first slot. Moreover, if $f\in\mathscr{C}^\infty M$ and $\psi\in\Gamma E$, then

$$A(f\psi)=\tilde{\nabla}(f\psi)-\nabla(f\psi)=df\psi+f\tilde{\nabla}\psi-df\psi-f\nabla\psi=f(\tilde{\nabla}\psi-\nabla\psi)=fA\psi\,.$$

Hence A is tensorial, i.e. $A\in\Gamma(TM^*\otimes E^*\otimes E)=\Omega^1(M;\mathrm{End}E)$. □

Example 4.1.2 (Connections on the Trivial Bundle) Consider the trivial bundle $\underline{\mathbb{R}}^r_M$. Then each connection ∇ on $\underline{\mathbb{R}}^r_M$ can be write with respect to the trivial connection d, i.e.

$$\nabla=d+A,\quad A\in\Omega^1(M;\mathrm{End}\underline{\mathbb{R}}^r_M)\,.$$

Since $\mathrm{End}\mathbb{R}^r=\mathbb{R}^{r\times r}$, we find that $A\in\Omega^1(M;\mathbb{R}^{r\times r})$, which is a matrix of 1-forms, $A=(a_{ij})_{i,j}$, $a_{ij}\in\Omega^1M$.

It is important to note that, given a vector bundle with connection, then we automatically get a connection the dual bundle. This is stated in the theorem below, whose proof—being similar to the one above—is left as exercise.

Theorem 4.1.4 (Dual Connection) *Let $E \to M$ be a vector bundle with connection ∇. Then*

$$(\nabla^*\omega)(\psi) := d(\omega(\psi)) - \omega(\nabla\psi), \quad \textit{for all } \omega \in \Gamma E^*, \ \psi \in \Gamma E\,.$$

defines a connection ∇^* on E^*—called the dual connection.

Let $\langle - | - \rangle \colon \Gamma E^* \times \Gamma E \to \underline{\mathbb{R}}_M$ denote the *dual pairing*, i.e. $\langle \omega | \psi \rangle_p = \omega_p(\psi_p)$. Then, for $\omega \in \Gamma E^*$ and $\psi \in \Gamma E$,

$$d\langle \omega|\psi\rangle = \langle \nabla^*\omega|\psi\rangle + \langle \omega|\nabla\psi\rangle\,.$$

Hence ∇ on E^* is the one and only connection which enforces the product rule for all pairs ω and ψ.

Similarly, we could define the connection for multilinear forms, i.e. for tensors. A different, less elegant, but sometimes useful way to do this uses the locality principle—we will see then that this leads finally to the same notion.

Theorem 4.1.5 (Tensor Connection) *Let (E, ∇) and $(\tilde E, \tilde\nabla)$ be vector bundles with connection over M, then there is a unique connection $\nabla^\otimes$ on $E \otimes \tilde E$ such that, for all $\psi \in \Gamma E$ and $\tilde\psi \in \Gamma\tilde E$,*

$$\nabla^\otimes(\psi \otimes \tilde\psi) = (\nabla\psi) \otimes \tilde\psi + \psi \otimes \tilde\nabla\tilde\psi\,.$$

Proof. At each point of M there is a local frame of $E \otimes \tilde E$ which consists of sections of the form $\psi_i \otimes \tilde\psi_j$. One easily checks that the above formula defines a connection: Let $\hat\psi \in \Gamma(E \otimes \tilde E)$ and $X \in T_pM$, then there is a neighborhood $U \mathrel{\mathring{\subset}} M$ of p such that $\hat\psi|_U = \sum a_{ij}\psi_i \otimes \tilde\psi_j$ for some $\psi_i \in \Gamma E$, $\psi \in \Gamma\tilde E$ and $a_{ij} \in \mathscr{C}^\infty M$. If there is some connection $\nabla^\otimes$, it must satisfy

$$\nabla^\otimes_X\hat\psi := (\nabla^\otimes|_U)_X\hat\psi|_U = \sum(\nabla^\otimes|_U)_X(a_{ij}\psi_i \otimes \tilde\psi_j)|_U = \sum \nabla^\otimes_X(a_{ij}\psi_i \otimes \tilde\psi_j) = \sum\big(da_{ij}(X)\,\psi_i \otimes \tilde\psi_j + a_{ij}\nabla^\otimes_X(\psi_i \otimes \tilde\psi_j)\big)\,.$$

This shows already uniqueness. For existence we define $\nabla^\otimes$ by the right-hand side. To verify that this defines a connection is an easy exercise. □

Remark 4.1.3 The previous proof shows that a connection is uniquely determined by its values on frames.

So once we have bundles with connection we get a connection on their tensor product. Which connection is meant becomes usually clear from the context and we just write ∇.

Example 4.1.3 (Derivative of Tensors) In fact, a more useful formula for the derivative of tensor fields is the following: If $\omega \in \Gamma E^*$ and $\tilde\psi \in \Gamma\tilde E$, then $\nabla(\tilde\psi \otimes \omega) = \nabla\tilde\psi \otimes \omega + \tilde\psi \otimes \nabla\omega$, i.e. for $\psi \in \Gamma E$, we get

$$\big(\nabla(\tilde\psi \otimes \omega)\big)(\psi) = \nabla\tilde\psi \otimes \omega(\psi) + \tilde\psi \otimes (\nabla\omega)(\psi) = \nabla\tilde\psi \otimes \omega(\psi) + \tilde\psi \otimes d(\omega(\psi)) - \tilde\psi \otimes \omega(\nabla\psi) = \nabla\big(\tilde\psi \otimes \omega(\psi)\big) - \tilde\psi \otimes \omega(\nabla\psi)\,.$$

Since each $A \in \Gamma\mathrm{Hom}(E, \tilde E)$ is a sum of such fields, we get

$$(\nabla A)\psi = \nabla\big(A(\psi)\big) - A(\nabla\psi)\,.$$

Note that $\mathrm{Mult}(E_1, \ldots, E_k; F) = \mathrm{Hom}(E_1 \otimes \cdots \otimes E_k; F)$, i.e. if $A \in \Gamma\mathrm{Mult}(E_1, \ldots, E_k; F)$, then

$$A(\psi_1, \ldots, \psi_k) = \hat A(\psi_1 \otimes \cdots \otimes \psi_k)$$

for a unique $\hat{A} \in \Gamma\mathrm{Hom}(E_1 \otimes \cdots \otimes E_k; F)$. Hence

$$(\nabla A)(\psi_1, \dots, \psi_k) = \nabla(\hat{A}(\psi_1 \otimes \cdots \otimes \psi_k)) - \hat{A}(\nabla(\psi_1 \otimes \cdots \otimes \psi_k)) = \nabla(A(\psi_1, \dots, \psi_k)) - \sum_i A(\psi_1, \dots, \nabla\psi_i, \dots, \psi_k)).$$

Since since the alternating forms are a subspace of the multilinear forms, $\mathrm{Alt}^k(E;F)$ forms a subbundle of $\mathrm{Mult}^k(E;F)$. Moreover, we have a vector bundle homomorphism

$$\mathrm{Alt}_k : \mathrm{Mult}^k(E;F) \to \mathrm{Alt}^k(E;F).$$

The next theorem tells us that it is compatible with the induced connection. It is very easy to prove using the following lemma.

Lemma 4.1.6 *Let $E \to M$ be a vector bundle with connection ∇. Then for each $p \in M$ there exists a local frame $\psi_1, \dots, \psi_r \in \Gamma E$ at p such that*

$$\nabla\psi_1|_{T_pM} = \cdots = \nabla\psi_r|_{T_pM} = 0.$$

Proof. Exercise. □

Theorem 4.1.7 *Let $E, F \to M$ be vector bundles with connection and $\omega \in \mathrm{Mult}^k(E;F)$. Then*

$$\nabla(\mathrm{Alt}_k(\omega)) = \mathrm{Alt}_k(\nabla\omega).$$

In particular, if $\omega \in \Gamma\mathrm{Alt}^k(E;F)$, then $\nabla\omega \in \Omega^1(M;\mathrm{Alt}^k(E;F))$.

Proof. Let $\omega \in \Gamma\mathrm{Mult}^k(E;F)$, $\psi_1, \dots, \psi_k \in \Gamma E$ and $X \in T_pM$. By the last lemma, we can without loss of generality assume that $\nabla_X\psi_i = 0$ for all $i = 1, \dots, k$. Then

$$\begin{aligned}(\nabla_X \mathrm{Alt}_k\omega)(\psi_1, \dots, \psi_k) &= \tfrac{1}{k!}\sum_{\sigma\in S_k} \mathrm{sgn}\sigma\, \nabla_X(\omega(\psi_{\sigma_1}, \dots, \psi_{\sigma_k})) \\ &= \tfrac{1}{k!}\sum_{\sigma\in S_k} \mathrm{sgn}\sigma\, (\nabla_X\omega)(\psi_{\sigma_1}, \dots, \psi_{\sigma_k}) = (\mathrm{Alt}_k(\nabla_X\omega))(\psi_1, \dots, \psi_k).\end{aligned}$$

Since Alt_k is a projection, $\nabla\omega = \nabla(\mathrm{Alt}_k(\omega)) = \mathrm{Alt}_k(\nabla\omega)$. Hence $\nabla\omega \in \Omega^1(M;\mathrm{Alt}^k(E;F))$. □

If we have a splitting of a vector bundle E with connection into a sum of two subbundles $E = E_1 \oplus E_2$, then any connection splits in four parts.

Exercise 4.1.2 Let $E = E_1 \oplus E_2$ be a vector bundle with connection ∇, then there are connections ∇^i on E_i, $A \in \Omega^1(M;\mathrm{Hom}(E_1;E_2))$ and $B \in \Omega^1(M;\mathrm{Hom}(E_2;E_1))$ such that

$$\nabla = \begin{pmatrix}\nabla^1 & B \\ A & \nabla^2\end{pmatrix}.$$

4.1.2 Parallel Transport

The analogue of a constant function in a vector bundle is a parallel section.

Definition 4.1.3 (Parallel Section) *A section ψ of a vector bundle E with connection ∇ is called parallel if $\nabla\psi = 0$.*

Example 4.1.4 Parallel sections of the trivial bundle $\underline{\mathbb{R}}^r_M$ correspond to constant functions $f \in \mathscr{C}^\infty(M, \mathbb{R}^r)$.

Example 4.1.5 The identity $I \in \Gamma\mathrm{End}E$ is always parallel, $\nabla I = 0$: If $\psi \in \Gamma E$, then

$$(\nabla I)\psi = \nabla(I\psi) - I\nabla\psi = \nabla\psi - \nabla\psi = 0\,.$$

Example 4.1.6 A vector bundle homomorphism $A\colon E \to \tilde{E}$ restricts in each point to a linear map $A_p\colon E_p \to \tilde{E}_p$, i.e. $A_p \in \mathrm{Hom}(E, \tilde{E})$. Thus A can be considered as a section of $\mathrm{Hom}(E, \tilde{E})$. If E and $\tilde{E}$ come with connections, then saying A is an isomorphism of vector bundles with connection is equivalent to

$$\nabla(A\psi) = A(\nabla\psi) \iff \nabla A = 0\,.$$

Definition 4.1.4 (Parallel Frame) *Let $E \to M$ be a vector bundle with connection ∇. A parallel frame is a frame $\sigma = (\psi_1, \ldots, \psi_r)$ of E such that $\nabla\psi_1 = \cdots = \nabla\psi_r = 0$.*

Theorem 4.1.8 *Let $E \to M$ be a vector bundle of rank r with connection ∇ and σ be a parallel frame. Then*

$$\nabla(\sigma.v) = \sigma.(dv), \quad \forall\ v \in \mathscr{C}^\infty(M; \mathbb{R}^r)\,.$$

Proof. Exercise. □

As an immediate consequence we obtain the following theorem.

Theorem 4.1.9 *A vector bundle with connection is trivial if and only if there exists a global parallel frame.*

Lemma 4.1.10 *Let $E \to M$ be a vector bundle of rank r with connection ∇ and σ and $\tilde{\sigma}$ be parallel frames. Then locally σ and $\tilde{\sigma}$ differ by a constant matrix $A \in \mathbb{R}^{r\times r}$.*

Proof. Since σ and $\tilde{\sigma}$ are parallel we have $\nabla(\sigma.v) = \sigma(dv)$ and $\nabla(\tilde{\sigma}.v) = \tilde{\sigma}.(dv)$. Moreover, since they both are frames, there is a $\mathrm{GL}(\mathbb{R}^r)$-valued map A such that $\tilde{\sigma} = \sigma.A$. Thus

$$\sigma.(Adv) = \tilde{\sigma}.(dv) = \nabla(\tilde{\sigma}.v) = \nabla(\sigma.(Av)) = \sigma.d(Av) = \sigma.((dA)v + Adv).$$

Hence $dA = 0$, i.e. A is locally constant. □

Theorem 4.1.11 (Triviality of Vector Bundles over Intervals) *Every vector bundle with connection over an interval $I \subset \mathbb{R}$ is trivial.*

Proof. Let $I \subset \mathbb{R}$ be an interval and $E \to I$ be a smooth rank r vector bundle with connection ∇. At first we show that, at each point $t_0 \in I$, there exists a local parallel frame, i.e. a frame consisting of parallel sections: To see this, we choose some local frame σ defined on a neighborhood $U \overset{\circ}{\subset} I$ of t_0. This yields a trivialization $\underline{\mathbb{R}}^r_M \ni (t, v) \mapsto \sigma_t.v \in E_t$. The connection induced from ∇ on the trivial bundle by this trivialization is of the form $d - A$, where $A \in \Omega^1(U; \mathbb{R}^{r\times r})$. Thus we have, for a $v\colon U \to \mathbb{R}^r$,

$$0 = \nabla_{\frac{\partial}{\partial t}}(\sigma_t.v_t) = \sigma_t.\big(\tfrac{\partial}{\partial t}v - A(\tfrac{\partial}{\partial t}v_t)\big) \iff \tfrac{\partial}{\partial t}v = A(\tfrac{\partial}{\partial t})v_t\,,$$

which is an ordinary linear differential equation with smooth coefficients $A(\frac{\partial}{\partial t})$ and thus can be solved on all of U. Solving the initial value problem on a whole basis we obtain a parallel frame σ on U. Now, using the local existence, it is not hard to see that set $V \subset I$ to which σ can be extended is open and closed: We have

$$V = \{t \in I \mid \exists \textit{ frame } \tilde{\sigma} \textit{ defined on } \tilde{U} \overset{\circ}{\subset} I: \ t \in \tilde{U},\ U \subset \tilde{U},\ \tilde{\sigma}|_U = \sigma\}.$$

Clearly, V is open. To see that V is closed take a limit point t_1 of V. Then there exists a local frame $\hat{\sigma}$ define on an open neighborhood $\hat{V} \subset I$ of t_1. Then $V \cap W \neq \emptyset$ and there is a from $\tilde{\sigma}$ defined on $\tilde{V} \overset{\circ}{\subset} I$ with $\tilde{V} \supset U$ and $t \in \tilde{V}$. We can assume that $\hat{V} \cap \tilde{V}$ is connected. Then, by the previous lemma, $\hat{\sigma}.A = \tilde{\sigma}$ on $\hat{V} \cap \tilde{V}$ for some $A \in \mathbb{R}^{r \times r}$ and we can define a new frame τ defined on $\hat{V} \cup \tilde{V} \overset{\circ}{\subset} I$ as follows:

$$\tau|_{\tilde{V}} = \tilde{\sigma}, \quad \tau|_{\hat{V}} = \hat{\sigma}.A\,.$$

Hence V is closed. Since I is connected and $V \supset U \neq \emptyset$ we conclude that $V = I$. □

Theorem 4.1.12 (Pullback Connection) *Let $E \to M$ be a vector bundle with connection ∇ and $f\colon S \to M$ be a smooth map. Then there is a unique connection $f^*\nabla$ on f^*E such that*

$$(f^*\nabla)(f^*\psi) = f^*(\nabla\psi) \quad \textit{for all } \psi \in \Gamma E$$

Proof. The proof is basically the same as for the existence and uniqueness to the tensor connection. □

As an immediate consequence of Theorem 4.1.11 we get the following theorem.

Corollary 4.1.13 *Let $E \to M$ be a vector bundle with connection and $\gamma\colon I \to M$ be a smooth curve defined on some interval $I \subset \mathbb{R}$. Then there exists a parallel frame along γ.*

Definition 4.1.5 (Parallel Transport) *Let $E \to M$ be a vector bundle with connection ∇ and $\gamma\colon [0,1] \to M$ be a smooth curve. The parallel transport $P_\gamma^\nabla\colon T_{\gamma(0)}E \to T_{\gamma(1)}E$ along γ is given by*

$$P_\gamma^\nabla(\psi_0) = \psi(1),$$

where $\psi \in \Gamma(\gamma^*E)$ is parallel such that $\psi(0) = \psi_0$.

Exercise 4.1.3 Show that P_γ^∇ is a linear isomorphism.

Exercise 4.1.4 Let $\gamma\colon \mathbb{R} \to M$ be a smooth curve and let $P_t\colon E_{\gamma(0)} \to E_{\gamma(t)}$ denote the parallel transport along its restriction $\gamma|_{[0,t]}$. Show that

$$(\gamma^*\nabla)_{\frac{\partial}{\partial t}\big|_{t=0}} \psi = \tfrac{d}{dt}\big|_{t=0} P_t^{-1}\big(\psi(t)\big)\,,$$

for all $\psi \in \Gamma(\gamma^*E)$.

4.1.3 Affine Connections

An *affine connection* is a name to denote a connection on the tangent bundle. What is special about is that they come with a particular tensor.

Definition 4.1.6 (Torsion Tensor) *The torsion tensor $T^\nabla\colon TM \times TM \to TM$ of an affine connection ∇ is defined as follows: For $X, Y \in \Gamma TM$,*

$$T^\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]\,.$$

Remark 4.1.4 T^∇ is skew.

We still need to check that T^∇ is a tensor: Since T^∇ is skew, it suffices to check tensoriality in only one argument. For $f \in C^\infty M$ and $X, Y \in \Gamma(TM)$, we have

$$T^\nabla(fX, Y) = f\nabla_X Y - (Yf)X - f\nabla_Y X - f[X,Y] + (Yf)X = fT^\nabla(X,Y).$$

Note that $TM^* \otimes \mathrm{End}(TM) = \mathrm{Mult}^2(TM;TM)$. In particular, any two affine connections differ by some section $A \in \Gamma\mathrm{Mult}^2(TM;TM)$.

Lemma 4.1.14 *Let ∇ be an affine connection on M and $\tilde{\nabla} = \nabla + A$ for some $A \in \Gamma\mathrm{Mult}^2(TM;TM)$. Then,*

$$T^{\tilde{\nabla}} = T^\nabla + 2\mathrm{Alt}_2(A).$$

Proof. For $X, Y \in \Gamma TM$, we have

$$T^{\tilde{\nabla}}(X,Y) = \nabla_X Y + A(X,Y) - \nabla_Y X - A(Y,X) - [X,Y] = T^\nabla(X,Y) + A(X,Y) - A(Y,X) = T^\nabla + 2\mathrm{Alt}_2(A)(X,Y).$$

□

Definition 4.1.7 (Torsion-free) *An affine connection ∇ is called torsion-free, if $T^\nabla = 0$.*

Theorem 4.1.15 *On each tangent bundle there is a torsion-free connection.*

Proof. Let ∇ be some affine connection and define $A(X,Y) = -\frac{1}{2}T^\nabla(X,Y)$. Then $\tilde{\nabla} := \nabla + A$ has torsion

$$T^{\tilde{\nabla}} = T^\nabla + 2\mathrm{Alt}_2(A) = T^\nabla - T^\nabla = 0.$$

Here the last equality used that T^∇ is skew. □

Example 4.1.7 Take a look at our formula for the Lie-bracket expressed in local coordinates. We see that the trivial connection ∇ on $T\mathbb{R}^n$ is torsion-free, $T^\nabla = 0$.

The bilinear forms Bil^2 on a vector space splits into a direct sum of two subspaces—the skew bilinear forms Alt^2 and the symmetric bilinear forms Sym^2. Similarly, we get a splitting for the bundle $\mathrm{Bil}^2(TM;TM)$ of TM-valued bilinear forms defined on TM. The following theorem is a direct consequence of Lemma 4.1.14.

Corollary 4.1.16 *The torsion-free affine connections on M form an affine space over $\Gamma\mathrm{Sym}^2(TM;TM)$.*

Proof. The difference of any two torsion-free affine connections is a skew bilinear form, which vanishes if and only if the connections are equal. □

4.2 Derivative of Differential Forms

4.2.1 Exterior Derivative

We have seen how to differentiate tensors. For skew-symmetric tensors, i.e. differential forms, there is a another important notion of derivative—it only needs a connection on E.

Definition 4.2.1 (Exterior Derivative) *Let $E \to M$ be a smooth vector bundle with connection. Then the exterior derivative $d^\nabla : \Omega^k(M;E) \to \Omega^{k+1}(M;E)$ is defined as follows: Let $\omega \in \Omega^k(M;E)$ and $X_0, \dots, X_k \in \Gamma(TM)$, then*

$$d^\nabla \omega(X_0, \dots, X_k) := \sum_{i=0}^{k} (-1)^i \nabla_{X_i} \left(\omega(X_0, \dots, \widehat{X_i}, \dots, X_k)\right) + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_k),$$

where hat means that omission, i.e. $\omega(X_0, \dots, \widehat{X_i}, \dots, X_k) = \omega(X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_k)$.

From the definition it is not clear that $d^\nabla \omega$ is tensorial. One can directly check this. Since each manifold has a torsion-free connection. It also follows from the next theorem which is interesting itself as it relates the exterior derivative to the covariant derivative $\nabla \omega$ of a form $\omega \in \Omega^k(M;E)$ when considered as a section of $\mathrm{Mult}^{k+1}(TM;E)$.

Theorem 4.2.1 *Let $E \to M$ be a vector bundle with connection. Then each choice of a torsion-free affine connection on M induces a connection ∇ on $\mathrm{Mult}^k(TM;E) \supset \mathrm{Alt}^k(TM;E)$ and we have*

$$d^\nabla \omega = (k+1)\,\mathrm{Alt}_{k+1}(\nabla \omega) = \nabla \wedge \omega\,.$$

Proof. Let $X_0, \dots, X_k \in \Gamma TM$. Then

$$(k+1)\,\mathrm{Alt}_{k+1}(\nabla\omega)(X_0, \dots, X_k) = \sum_{\sigma \in \mathrm{Sh}(1,k;n)} \mathrm{sgn}\sigma \left(\nabla_{X_{\sigma_0}} \omega\right)(X_{\sigma_1}, \dots, X_{\sigma_k}) = \sum_{i=0}^{k} (-1)^i \left(\nabla_{X_i} \omega\right)(X_1, \dots, \widehat{X_i}, \dots, X_k)$$
$$= \sum_{i=0}^{k} (-1)^i \left(\nabla_{X_i} \omega(X_1, \dots, \widehat{X_i}, \dots, X_k)\right) - \sum_{i \neq j} (-1)^i \omega(X_1, \dots, \nabla_{X_i} X_j, \dots, \widehat{X_i}, \dots, X_k)\,.$$

Now,

$$\sum_{i \neq j} (-1)^i \omega(X_1, \dots, \nabla_{X_i} X_j, \dots, \widehat{X_i}, \dots, X_k)$$
$$= \sum_{i<j}^{k} (-1)^{i+j+1} \omega(\nabla_{X_i} X_j, X_1, \dots, \widehat{X_j}, \dots, \widehat{X_i}, \dots, X_k) + \sum_{i>j}^{k} (-1)^{i+j} \omega(\nabla_{X_i} X_j, X_1, \dots, \widehat{X_j}, \dots, \widehat{X_i}, \dots, X_k)$$
$$= -\sum_{i<j}^{k} (-1)^{i+j} \omega(\underbrace{\nabla_{X_i} X_j - \nabla_{X_j} X_i}_{=[X_i, X_j]}, X_1, \dots, \widehat{X_j}, \dots, \widehat{X_i}, \dots, X_k),$$

as claimed. □

Corollary 4.2.2 *Let $E \to M$ be a vector bundle with connection ∇ and $\tilde{\nabla} = \nabla + A$ with $A \in \Omega^1(M; \mathrm{End}E)$. Then*

$$d^{\tilde{\nabla}} = d^\nabla + A \wedge\,.$$

Remark 4.2.1 (Tautological 1-Form) On the tangent bundle TM we have the tautological 1-form $\omega \in \Omega^1(M;TM)$ given by $\omega(X) = IX = X$. Given connection on TM, we can compute its exterior derivative

$$d^\nabla \omega(X,Y) = \nabla_X \omega(Y) - \nabla_Y \omega(X) - \omega([X,Y]) = \nabla_X Y - \nabla_Y X - [X,Y] = T^\nabla(X,Y)\,.$$

Another important property of the exterior derivative is the following product rule for wedge products.

Theorem 4.2.3 (Product Rule) *Let E_1, E_2 and E_3 be vector bundles with connection, let $\bullet \in \mathrm{Mult}^2(E_1, E_2; E_3)$ be parallel and let $\omega \in \Omega^k(M; E_1)$ and $\eta \in \Omega^\ell(M; E_2)$. Then*

$$d^\nabla(\omega \wedge_\bullet \eta) = d^\nabla\omega \wedge_\bullet \eta + (-1)^k \omega \wedge_\bullet d^\nabla \eta\,.$$

Proof. Again, we choose a torsion-free connection on the tangent bundle. Then, since Alt_{k+l} is parallel,

$$\begin{aligned} d^\nabla(\omega \wedge_\bullet \eta) &= (k+\ell+1)\mathrm{Alt}_{k+\ell+1}\big(\nabla(\omega \wedge_\bullet \eta)\big) \\ &= (k+\ell+1)\mathrm{Alt}_{k+\ell+1}\big(\nabla \tfrac{(k+\ell)!}{k!\ell!}\mathrm{Alt}_{k+\ell}(\omega \otimes_\bullet \eta)\big) \\ &= (k+\ell+1)\mathrm{Alt}_{k+\ell+1}\big(\tfrac{(k+\ell)!}{k!\ell!}\mathrm{Alt}_{k+\ell}(\nabla(\omega \otimes_\bullet \eta))\big)\,. \end{aligned}$$

Now, if $X \in TM$, then

$$\nabla_X(\omega \otimes_\bullet \eta) = (\nabla_X \omega) \otimes_\bullet \eta + \omega \otimes_\bullet (\nabla_X \eta)\,.$$

If we choose a chart, we can write the involved 1-forms with respect to the corresponding coframe $dx_1, \dots, dx_m$,

$$\nabla\omega = \sum_{i=1}^m dx_i \otimes \omega_i'\,, \quad \text{with } \omega_i' \in \Omega^k(M; E_1)\,, \qquad \nabla\eta = \sum_{i=1}^m dx_i \otimes \eta_i'\,, \quad \text{with } \eta_i' \in \Omega^\ell(M; E_2)\,.$$

So we get

$$\tfrac{(k+\ell)!}{k!\ell!}\mathrm{Alt}_{k+\ell}\big((\nabla\omega) \otimes \bullet\eta + \omega \otimes \bullet(\nabla\eta)\big) = \sum_{i=1}^m dx_i \otimes \tfrac{(k+\ell)!}{k!\ell!}\mathrm{Alt}_{k+\ell}\big(\omega_i' \otimes \bullet\eta + \omega \otimes \bullet\eta_i'\big) = \sum_{i=1}^m dx_i \otimes \big(\omega_i' \wedge \bullet\eta + \omega \wedge \bullet\eta_i'\big)\,.$$

Hence,

$$\begin{aligned} d^\nabla(\omega \wedge_\bullet \eta) &= \sum_{i=1}^m (k+\ell+1)\mathrm{Alt}_{k+\ell+1}\big(dx_i \otimes \big(\omega_i' \wedge \bullet\eta + \omega \wedge \bullet\eta_i'\big)\big) \\ &= \sum_{i=1}^m dx_i \wedge \big(\omega_i' \wedge \bullet\eta + \omega \wedge \bullet\eta_i'\big) \\ &= \Big(\sum_{i=1}^m dx_i \wedge \omega_i'\Big) \wedge \bullet\eta + (-1)^k\omega \wedge \bullet\Big(\sum_{i=1}^m dx_i \wedge \eta_i'\Big) \end{aligned}$$

Moreover,

$$d^\nabla\omega = (k+1)\mathrm{Alt}_{k+1}(\nabla\omega) = (k+1)\mathrm{Alt}_{k+1}\Big(\sum_{i=1}^m dx_i \otimes \omega_i'\Big) = \sum_{i=1}^m dx_i \wedge \omega_i'$$

and, similarly, $d^\nabla\eta = \sum_{i=1}^m dx_i \wedge \eta_i'$. Plugging this into the formula for $d^\nabla(\omega \wedge \bullet\eta)$ derive above then completes the proof. □

Remark 4.2.2 Most pairings we usually work with are parallel. For an arbitrary pairing one would need to include its derivative. The formula then becomes

$$d^\nabla(\omega \wedge_\bullet \eta) = d^\nabla\omega \wedge_\bullet \eta + (-1)^k\omega \wedge (d^\nabla \bullet) \wedge \eta + (-1)^k \omega \wedge_\bullet d^\nabla\eta\,.$$

As a first application we will show that for real valued forms the second exterior derivative always vanishes. In this context the exterior derivative always means the exterior derivative with respect to the trivial connection—unless explicitly stated differently. It will be denoted just by d.

Theorem 4.2.4 *Let $\omega \in \Omega^k(M;\mathbb{R}^r)$, then $d^2\omega = 0$.*

Proof. If $f \in \mathscr{C}^\infty(M;\mathbb{R}^r)$, $X, Y \in \Gamma TM$, then $d^2 f(X,Y) = X(Yf) - Y(Xf) - df[X,Y] = 0$. It is enough to check the claim for k-forms of the form $\omega = f_0\, df_1 \wedge \cdots \wedge df_k$, where $f_0 \in \mathscr{C}^\infty(m;\mathbb{R}^r)$ and $f_1, \ldots, f_k \in \mathscr{C}^\infty M$. This is because every k-form is locally a sum of such and—as a connection only depends on a the values in a small neighborhood—it is enough to check the equality locally. Then the product rule yields

$$d\omega = df_0 \wedge df_1 \wedge \cdots \wedge df_k$$

and thus—again by the product rule—we conclude $d^2\omega = 0$. □

Definition 4.2.2 (Exact and Closed Forms) *Let (E,∇) be a smooth vector bundle with connection over M. The space $Z^k(M;E)$ of closed k-forms and the space $B^k(M;E)$ of exact k-forms are defined as follows:*

$$B^k(M;E) = \operatorname{im}\left(d^\nabla : \Omega^{k-1}(M;E) \to \Omega^k(M;E)\right), \quad Z^k(M;E) = \ker\left(d^\nabla : \Omega^k(M;E) \to \Omega^{k+1}(M;E)\right).$$

We set $B^kM = B^k(M;\underline{\mathbb{R}}_M)$ and $Z^kM = Z^k(M;\underline{\mathbb{R}}_M)$.

Corollary 4.2.5 $B^kM \subset Z^kM$

Exercise 4.2.1 Let $M = \mathbb{R}^3$. Determine which of the following forms are closed and which are exact

(a) $\omega = yz\,dx + xz\,dy + xy\,dz\,,$
(b) $\omega = x\,dx + x^2y^2\,dy + yz\,dz\,,$
(c) $\omega = 2xy^2\,dx \wedge dy + z\,dy \wedge dz\,.$

If ω is exact, write down the potential form θ explicitly.

Definition 4.2.3 (De-Rahm Cohomology)

$$H^kM := Z^kM/B^kM\,.$$

As a second application we show that pullback and exterior derivative commute.

Theorem 4.2.6 (Naturality of Pullback) *Let $E \to M$ be a vector bundle with connection ∇, let $f\colon S \to M$ be smooth and let $\omega \in \Omega^k(M;E)$. Then*

$$d^{f^*\nabla} f^*\omega = f^*(d^\nabla\omega)\,.$$

Proof. It is enough to check the equality for k-forms of the form $\omega = \psi\, dg_1 \wedge \cdots \wedge dg_k$, where $\psi \in \Gamma E$ and $g_1, \ldots, g_k \in \mathscr{C}^\infty M$. Now, first, the equation holds for the derivative of real functions: For $g \in \mathscr{C}^\infty M$, we have

$$f^*dg = dg \circ df = d(g \circ f) = d(f^*g)\,.$$

In particular, we get $d(f^*dg) = d^2(f^*g) = 0$ and thus

$$\begin{aligned} d^{f^*\nabla} f^*\omega &= d^{f^*\nabla}(f^*\psi\, f^*(dg_1 \wedge \cdots \wedge dg_k)) = d^{f^*\nabla}(f^*\psi\,(f^*dg_1 \wedge \cdots \wedge f^*dg_k)) = (d^{f^*\nabla} f^*\psi) \wedge (f^*dg_1 \wedge \cdots \wedge f^*dg_k) \\ &= ((f^*\nabla)(f^*\psi)) \wedge f^*(dg_1 \wedge \cdots \wedge dg_k) = f^*(\nabla\psi) \wedge f^*(dg_1 \wedge \cdots \wedge dg_k) = f^*(\nabla\psi \wedge dg_1 \wedge \cdots \wedge dg_k) \end{aligned}$$

Hence we have $f^*(d^\nabla\omega) = f^*(d^\nabla\psi) \wedge dg_1 \wedge \cdots \wedge dg_k) = f^*(\nabla\psi \wedge dg_1 \wedge \cdots \wedge dg_k) = d^{f^*\nabla} f^*\omega$. □

4.2.2 Curvature—the 2nd Exterior Derivative

In general the second exterior derivative $(d^\nabla)^2$ does not vanish: Let $\omega \in \Omega^k(M;E)$ and $f\colon \mathscr{C}^\infty M$, then

$$(d^\nabla)^2(f\omega) = d^\nabla(df\,\psi + f\,d^\nabla\omega) = (d^2f)\wedge\omega - df\wedge\,d^\nabla\omega + df\wedge d^\nabla\omega + f\,d^\nabla d^\nabla\omega = f\,(d^\nabla)^2\omega\,.$$

So $(d^\nabla)^2$ is actually tensorial and as such a section of $\Omega^2(M;\mathrm{End}E)$. This tensor plays an extraordinary role.

Definition 4.2.4 (Curvature) *Let $E \to M$ be a vector bundle. The curvature $F^\nabla \in \Omega^2(M;\mathrm{End}E)$ of a connection ∇ is given by*

$$F^\nabla\psi = d^\nabla d^\nabla\psi\,,\quad \psi\in\Gamma E\,.$$

Theorem 4.2.7 (2nd Exterior Derivative) *Let E be a vector bundle with connection and $\omega\in\Omega^2(M;E)$. Then*

$$(d^\nabla)^2\omega = F^\nabla\wedge\omega\,.$$

Proof. Again we can assume without loss of generality that $\omega = \psi df_1\wedge\cdots df_k$ for $\psi\in\Gamma E$ and $f_i\in\mathscr{C}^\infty M$. Then we get

$$(d^\nabla)^2\omega = d^\nabla(d^\nabla\psi\wedge df_1\wedge\cdots df_k) = (d^\nabla)^2\psi\wedge df_1\wedge\cdots df_k = F^\nabla\psi\wedge df_1\wedge\cdots df_k = F^\nabla\wedge\psi df_1\wedge\cdots df_k\,.$$

Hence we get $(d^\nabla)^2\omega = F^\nabla\wedge\omega$. □

Theorem 4.2.8 (2nd Bianchi Identty) *For each vector bundle E with connection ∇ one has*

$$d^\nabla F^\nabla = 0\,.$$

Proof. Let $\omega\in\Omega^k(M;E)$. Then the product rule yields

$$(d^\nabla F^\nabla)\wedge\omega = d^\nabla(F^\nabla\wedge\omega) - F^\nabla\wedge d^\nabla\omega = d^\nabla((d^\nabla)^2\omega) - (d^\nabla)^2(d^\nabla\omega) = ((d^\nabla)^3 - (d^\nabla)^3)\omega = 0\,.$$

□

Theorem 4.2.9 *Le (E,∇) and $(\tilde E,\tilde\nabla)$ be vector bundles with connection and $\phi\colon E\to\tilde E$ be an isomorphism of vector bundles with connection. Then*

$$F^{\tilde\nabla} = \phi\,F^\nabla\phi^{-1}\,.$$

Proof. We have already seen that ϕ can be considered as a parallel section of $\mathrm{Hom}(E;\tilde E)$, i.e.

$$\phi\in\Omega^0(M;\mathrm{Hom}(E,\tilde E)),\quad d^{\hat\nabla}\phi = \hat\nabla\phi = 0\,,$$

where $\hat\nabla$ is the induced connection on $\mathrm{Hom}(E,\tilde E)$. In particular, we can wedge it with E-valued k-forms from the right: Let $\omega\in\Omega^k(M;E)$ The product rule then yields

$$d^{\tilde\nabla}(\phi\wedge\omega) = (d^{\hat\nabla}\phi)\wedge\omega + \phi\wedge d^\nabla\omega = \phi\wedge d^\nabla\omega\,.$$

Hence

$$F^{\tilde\nabla}\phi\wedge\omega = (d^{\tilde\nabla})^2\phi\wedge\omega = \phi\wedge(d^\nabla)^2\omega = \phi\,F^\nabla\wedge\omega\,.$$

Since ω was arbitrary, the claim follows. □

Theorem 4.2.10 *On a trivial bundle always has curvature zero.*

Proof. Theorem 4.2.4 says the trivial bundle has curvature zero. Theorem 4.2.9 says any trivial bundle has curvature zero. □

Exercise 4.2.2 If $E \to M$ is a vector bundle with connection ∇, $f\colon S \to M$ is smooth, then $F^{f^*\nabla} = f^*F^\nabla$.

Theorem 4.2.11 (Change of Curvature from Change of Connection) *Let $E \to M$ be a vector bundle with connection ∇ and let $\tilde\nabla = \nabla + A$. Then*

$$F^{\tilde\nabla} = F^\nabla + d^\nabla A + A \wedge A\,.$$

Proof. We know that $d^{\tilde\nabla} = d^\nabla + A\wedge$. Let $\omega \in \Omega^k(M;E)$. Then

$$F^{\tilde\nabla} \wedge \omega = (d^{\tilde\nabla})^2\omega = (d^\nabla + A\wedge)(d^\nabla\omega + A \wedge \omega) = d^\nabla\omega + A \wedge \omega)$$
$$= (d^\nabla)^2\omega + d^\nabla(A \wedge \omega) + A \wedge d^\nabla\omega + A \wedge A \wedge \omega = F^\nabla \wedge \omega + (d^\nabla A) \wedge \omega + A \wedge A \wedge \omega$$

□

Remark 4.2.3 (Local Expression of Exterior Derivative and Curvature) Let $\sigma = (\psi_1, \ldots, \psi_r)$ be a local frame of a rank r vector bundle E with connection ∇. On the support U of σ each $\omega \in \Omega^k(M;E)$ can be written in terms of σ and $\alpha \in \Omega^k(U;\mathbb{R}^r)$,

$$\omega_U = \sigma.\alpha = \sum_{i=1}^r \alpha_i\,\psi_i\,.$$

This actually defines a local trivialization and thus a connection $\tilde\nabla$ on the trivial bundle—which then can be written with respect to the trivial connection, $\tilde\nabla = d + A$. Thus, on U,

$$d^\nabla\omega = d^\nabla(\sigma.\alpha) = \sigma.(d^{\tilde\nabla}\alpha) = \sigma.(d\alpha + A \wedge \alpha)\,.$$

In particular, since the curvature of the trivial connection is zero, we get

$$F^\nabla\sigma = \sigma.(dA + A \wedge A)\,.$$

4.2.3 Fundamental Theorem of Flat Bundles

A flat vector bundle is a bundle which locally looks like the trivial bundle.

Definition 4.2.5 (Flat Vector Bundle) *A vector bundle with connection is called flat if it is locally isomorphic to the trivial bundle with trivial connection.*

We immediately get the following equivalent description.

Corollary 4.2.12 *A vector bundle is flat if and only if at each point there exist local parallel frames.*

A flat bundle, being locally isomorphic to the trivial bundle, has vanishing curvature tensor. If we are able to show that, whenever the curvature tensor vanishes, there is a local parallel frame, then we would have shown the following theorem.

Theorem 4.2.13 (Fundamental Theorem of Flat Vector Bundles) *(E, ∇) is flat if and only if $F^\nabla = 0$.*

Remark 4.2.4 If $F^\nabla = 0$ we call ∇ a flat connection.

We say that a set $U \mathring{\subset} M$ is a *star-region* with respect to p if it is the image of a diffeomorphism $f\colon V \to U$, where $V \mathring{\subset} \mathbb{R}^m$ is star-shaped with respect to 0 and $f(0) = p$. In particular, each star-region comes with a particular radial vector field ν—the push-forward of $\frac{\partial}{\partial r} \in \Gamma(TV)$ given by $\frac{\partial}{\partial r}\big|_x g = \frac{d}{dt}\big|_{t=0} g((1+t)x)$,

$$\nu_q = (f_* \tfrac{\partial}{\partial r})_q = df\left(\tfrac{\partial}{\partial r}\big|_{f^{-1}(q)}\right).$$

If (E, ∇) is a vector bundle over M and $U \mathring{\subset} M$ is a star region with radial field ν, then $\psi \in \Gamma(E|_U)$ is called *radially parallel*, if $\nabla_\nu \psi = 0$.

Lemma 4.2.14 *On every star-region there exists a radially parallel frame.*

Proof. If we choose a basis of E_p then each basis vector can be parallel transported along radial lines in U. The result is a radially parallel frame σ. That the sections are smooth is basically the smooth dependence on parameters for solutions of ordinary differential equations: We have $U = f(V)$, where V is star-shaped. Consider $\tilde{f}\colon [0,1] \times V \to U$ given by $\tilde{f}(t, x) = f(tx)$. Then $\tilde{\sigma} = \tilde{f}^*\sigma$ solves

$$\tilde{f}^* \nabla_{\frac{\partial}{\partial t}} \tilde{\sigma} = f^*(\nabla\sigma)(\tfrac{\partial}{\partial t}) = \nabla_{d\tilde{f}(\frac{\partial}{\partial t})} \sigma = \nabla_{t\nu}\sigma = 0,$$

i.e. $\tilde{\sigma}$ solves an initial value problem in $\tilde{f}^*E$ with smooth coefficients and smooth initial data and, as such, is smooth. In particular, since $f(x) = \tilde{f}(1, x)$, we have that $\sigma = (f^{-1})^*\tilde{\sigma}$ is smooth. □

Theorem 4.2.15 *If $F^\nabla = 0$, a radially parallel frame is parallel.*

Proof. Clearly $\nabla\psi\big|_{T_pM} = 0$. On $U \setminus \{p\}$ we have polar coordinates $r, \alpha_1, \ldots, \alpha_{m-1}$. We have

$$0 = F^\nabla(\tfrac{\partial}{\partial r}, \tfrac{\partial}{\partial \alpha_i})\psi = d^\nabla \nabla\psi(\tfrac{\partial}{\partial r}, \tfrac{\partial}{\partial \alpha_i}) = \nabla_{\frac{\partial}{\partial r}} \nabla_{\frac{\partial}{\partial \alpha_i}} \psi - \nabla_{\frac{\partial}{\partial \alpha_i}} \underbrace{\nabla_{\frac{\partial}{\partial r}} \psi}_{=0} - \underbrace{\nabla_{[\frac{\partial}{\partial \alpha_i}, \frac{\partial}{\partial r}]}}_{=0} = \nabla_{\frac{\partial}{\partial r}} \nabla_{\frac{\partial}{\partial \alpha_i}} \psi\,.$$

Since $\nabla_{\frac{\partial}{\partial \alpha_i}} \psi$ is radially parallel and vanishes at p, we get $\nabla_{\frac{\partial}{\partial \alpha_i}} \psi = 0$. Hence $\nabla\psi = 0$ on all of U. □

Remark 4.2.5 (Maurer–Cartan Lemma) If we express the connection with respect to a local frame we can write ∇ as $d + A$. Then the fundamental theorem of flat vector bundles becomes the well-known *Maurer–Cartan lemma*:

$$\exists g \in \mathrm{GL}(\mathbb{R}^r)\colon g^{-1} dg = A \iff dA + A \wedge A = 0\,.$$

Exercise 4.2.3 Let $M \subset \mathbb{R}^2$ be open. On $E = M \times \mathbb{R}^2$ we define two connections ∇ and $\tilde{\nabla}$ as follows:

$$\nabla = d + \begin{pmatrix} 0 & -x\,dy \\ x\,dy & 0 \end{pmatrix}, \quad \tilde{\nabla} = d + \begin{pmatrix} 0 & -x\,dx \\ x\,dx & 0 \end{pmatrix}.$$

Show that (E, ∇) is not trivial. Further construct an explicit isomorphism between $(E, \tilde{\nabla})$ and the trivial bundle (E, d).

Kähler Manifolds 5

5.1 Additional Structures and Compatibility

In many applications the vector bundle carries additional structures—such as almost complex or hermitian structures—and the connection is required to be compatible with them, i.e. basically the structure behave under differentiation as they were constant.

5.1.1 Euclidean Vector Bundles

Important structures are the so-called euclidean structures which provide an inner product per fiber so one can can measure lengths and angles.

Definition 5.1.1 (Fiber Metric) *A fiber metric $\langle .,. \rangle$ on a vector bundle E is a section $\langle .,. \rangle$ of the bundle* $\mathrm{Sym}(E)$ *of symmetric bilinear forms such that $\langle .,. \rangle_p$ is positive-definite for all $p \in M$. A $(E, \langle .,. \rangle)$ consisting of a vector bundle $E \to M$ and a fiber metric $\langle .,. \rangle$ on E is called a euclidean vector bundle.*

Theorem 5.1.1 *Every vector bundle admits a fiber metric.*

Proof. Let E be a vector bundle. Choose a bundle atlas $\{\phi\colon E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^r\}_{\alpha\in A}$ and a partition of unity $\{\rho_\alpha\}_{\alpha\in A}$ subordinate to $\{U_\alpha\}_{\alpha\in A}$ with the same index. On $E|_{U_\alpha}$ a metric $\langle .,. \rangle_\alpha$ is given by $\langle \psi_1, \psi_2 \rangle_\alpha = \langle \phi(\psi_1), \phi(\psi_2) \rangle_{\mathbb{R}^r}$. Then one can check that $\langle .,. \rangle = \sum_{\alpha\in A} \rho_\alpha \langle .,. \rangle_\alpha$ defines a fiber metric on E. □

Definition 5.1.2 (Metric Connection) *Let $(E, \langle .,. \rangle)$ be a euclidean vector bundle over M. A connection ∇ on E is called metric, if $\nabla \langle .,. \rangle = 0$.*

Theorem 5.1.2 *Each euclidean vector bundle admits a metric connection.*

Proof. Let $(E, \langle .,. \rangle)$ be a euclidean vector bundle and $\tilde\nabla$ be a connection on E. Then $\tilde\nabla \langle .,. \rangle \in \Omega^1(M; \mathrm{Sym}^2(TM))$ which defines $A \in \Omega^1(M; \mathrm{End}E)$ through $\tilde\nabla \langle .,. \rangle = \langle A.,. \rangle$. Clearly, $A^* = A$. Set $\nabla := \tilde\nabla + \frac{1}{2}A$. Then, for $\psi, \varphi \in \Gamma E$,

$$\begin{aligned}(\nabla\langle .,. \rangle)(\psi,\varphi) &= d\langle \psi,\varphi\rangle - \langle \tilde\nabla\psi,\varphi\rangle - \langle \psi,\tilde\nabla\varphi\rangle = d\langle \psi,\varphi\rangle - \langle(\nabla + \tfrac{1}{2}A)\psi,\varphi\rangle - \langle \psi,(\tilde\nabla + \tfrac{1}{2}A)\varphi\rangle \\ &= d\langle \psi,\varphi\rangle - \langle \tilde\nabla\psi,\varphi\rangle - \langle \psi,\tilde\nabla\varphi\rangle - \tfrac{1}{2}\langle A\psi,\varphi\rangle - \tfrac{1}{2}\langle \psi, A\varphi\rangle = \tilde\nabla\langle .,. \rangle(\psi,\varphi) - \langle A\psi,\varphi\rangle = 0\,,\end{aligned}$$

i.e. ∇ is metric. □

Again the difference of two metric connections has a particular type.

Theorem 5.1.3 *Let $(E, \langle .,. \rangle)$ be a euclidean vector bundle and ∇ and $\tilde\nabla$ be two metric connections. Then*

$$\tilde\nabla = \nabla + A, \quad A^* = -A\,.$$

Proof. Let $A = \tilde{\nabla} - \nabla$. Then, for $\psi, \varphi \in \Gamma E$,

$$\begin{aligned}\langle A\psi, \varphi\rangle &= \langle \tilde{\nabla}\psi, \varphi\rangle - \langle \nabla\psi, \varphi\rangle = d\langle \psi, \varphi\rangle - \langle \psi, \tilde{\nabla}\varphi\rangle - d\langle \psi, \varphi\rangle + \langle \psi, \nabla\varphi\rangle \\ &= \langle \psi, \nabla\varphi\rangle - \langle \psi, \tilde{\nabla}\varphi\rangle = -\langle \psi, A\varphi\rangle\,.\end{aligned}$$

Hence $A^* = -A$. □

Also the curvature of a metric connection is skew-adjoint.

Theorem 5.1.4 *Let $(E, \langle .,. \rangle)$ be a euclidean vector bundle and ∇ be a metric connection. Then*

$$(F^\nabla)^* = -F^\nabla\,.$$

Proof. Let $\psi, \varphi \in \Gamma E$. Since $\langle .,. \rangle$ is parallel, the product rule yields $d\langle \psi, \varphi\rangle = \langle d^\nabla\psi, \varphi\rangle + \langle \psi, d^\nabla\varphi\rangle$ and thus

$$0 = d^2\langle \psi, \varphi\rangle = \langle F^\nabla\psi, \varphi\rangle - \langle d^\nabla\psi \wedge d^\nabla\varphi\rangle + \langle d^\nabla\psi \wedge d^\nabla\varphi\rangle + \langle \psi, F^\nabla\varphi\rangle \iff \langle \psi, F^\nabla\varphi\rangle = -\langle F^\nabla\psi, \varphi\rangle\,.$$

Hence, $(F^\nabla)^* = -F^\nabla$. □

Exercise 5.1.1 Let $E_1, E_2 \to M$ be euclidean vector bundles and $E = E_1 \oplus_\perp E_2$. Show that a connection ∇ on E is metric if and only if

$$\nabla = \begin{pmatrix} \nabla^1 & -A^* \\ A & \nabla^2 \end{pmatrix},$$

where ∇^1 and ∇^2 metric connections and $A \in \Omega^1(M; \mathrm{Hom}(E_1; E_2))$.

5.1.2 Hermitian Vector Bundles

Now let us look at complex vector bundles.

Definition 5.1.3 (Complex Connection) *Let (E, J) be a complex vector bundle. A connection ∇ on E is called complex if $\nabla J = 0$.*

Remark 5.1.1 A complex connection is complex linear.

Theorem 5.1.5 *Every complex vector bundle has a complex connection.*

Proof. Let (E, J) be a complex vector bundle and let ∇ be some connection on E. Differentiating the equation $J^2 = -I$ yields $\nabla J \in \Omega^1(M; \mathrm{End}_- E)$. Set $\tilde{\nabla} = \nabla - \frac{1}{2}J\nabla J$. Then, for $\psi \in \Gamma E$,

$$\tilde{\nabla}(J\psi) = \nabla(J\psi) - \tfrac{1}{2}J(\nabla J)(J\psi) = J\nabla\psi + (\nabla J)\psi - \tfrac{1}{2}(J\nabla J)(J\psi) = J\nabla\psi + \tfrac{1}{2}(\nabla J)\psi = J\big(\nabla\psi - \tfrac{1}{2}(J\nabla J)(\psi)\big) = J\tilde{\nabla}\psi\,,$$

i.e. $\tilde{\nabla}$ is complex. □

It is then natural to ask how two complex connections differ.

Recall that, given an almost complex structure J, the endomorphisms split in complex linear and complex antilinear ones

$$\mathrm{End}E = \mathrm{End}_+ E \oplus \mathrm{End}_- E\,.$$

Proposition 5.1.6 *Any two complex connections on a complex vector bundle E differ by a $\mathrm{End}_+ E$-valued 1-form.*

Proof. If we have two connections which commute with J, so does their difference. □

Similarly, we get the following theorem.

Theorem 5.1.7 *The curvature of a complex connection is complex linear.*

In the presence of an almost complex structure not all fiber metrics are equally well. We want to single out those fiber metrics which turn J into an orthogonal endomorphism.

Definition 5.1.4 (Hermitian Fiber Metric) *Let E be a complex vector bundle. A fiber metric $\langle .,. \rangle$ is called hermitian if J is orthogonal with respect to it, i.e. $\langle J\psi, J\varphi \rangle = \langle \psi, \varphi \rangle$ for all $\psi, \varphi \in \Gamma E$.*

Remark 5.1.2 If $\langle .,. \rangle$ is a hermitian fiber metric on a complex vector bundle E, then J is skew, i.e. $J^* = -J$: Let $\psi, \varphi \in \Gamma E$, then

$$\langle J\psi, \varphi \rangle = -\langle J\psi, J^2\varphi \rangle = -\langle \psi, J\varphi \rangle \,.$$

Remark 5.1.3 (Associated Complex-Valued Hermitian Fiber Metric) Usually the term hermitian is used to denote a complex valued metric, whereas here a hermitian metric denotes a real valued bilinear form. Though both notions are equivalent: If $\langle .,. \rangle_{\mathbb{C}}$ denotes a complex valued hermitian form, then its real part is a hermitian form in the sense above. Conversely, given a real valued hermitian form $\langle .,. \rangle$, then there is a corresponding *complex valued hermitian form*:

$$\langle \psi, \varphi \rangle_{\mathbb{C}} = \langle \psi, \varphi \rangle + i \langle J\psi, \varphi \rangle \,. \quad \text{for all } \psi, \tilde{\psi} \in \Gamma E \,.$$

Definition 5.1.5 (Hermitian Vector Bundle) *A hermitian vector bundle is a pair $(E, \langle .,. \rangle)$ consisting of a complex vector bundle E and a hermitian fiber metric $\langle .,. \rangle$.*

Definition 5.1.6 (Unitary Connection) *Let E be a hermitian vector bundle. A connection is called unitary if it is both complex and metric. A hermitian vector bundle with unitary connection is called a unitary vector bundle.*

Remark 5.1.4 Form the results above we conclude immediately that any two unitary connections differ by a skew-adjoint complex linear endomorhism-valued 1-form.

We have seen above, that every vector bundle has a Riemannian metric, but not every bundle admits all of the structures—e.g. there exists no complex structure J on $T\mathbb{S}^4$. Though given a complex vector bundle it has always a hermitian structure.

Theorem 5.1.8 *Every complex vector bundle has a hermitian metric.*

Proof. Choose some fiber metric $\langle .,. \rangle_\sim$ on E and for $\psi, \varphi \in \Gamma E$ we set $\langle \psi, \varphi \rangle := \frac{1}{2}(\langle \psi, \varphi \rangle_\sim + \langle J\psi, J\varphi \rangle_\sim)$. Then, since $J^2 = -1$,

$$\langle J\psi, J\varphi \rangle = \tfrac{1}{2}(\langle J\psi, J\varphi \rangle_\sim + \langle J(J\psi), J(J\varphi) \rangle_\sim) = \tfrac{1}{2}(\langle J\psi, J\varphi \rangle_\sim + \langle \psi, \varphi \rangle_\sim) = \langle \psi, \varphi \rangle \,.$$

Hence $\langle .,. \rangle$ is hermitian. □

Theorem 5.1.9 *Every hermitian vector bundle has a unitary connection.*

Proof. Let $E \to M$ be a hermitian vector bundle. We already know that there is a metric connection, say ∇. Then $\tilde{\nabla} = \nabla - \frac{1}{2} J \nabla J$ is complex. Since J is skew-adjoint so is ∇J. To see this, let $\psi, \varphi \in \Gamma E$. Then

$$\begin{aligned}\langle (\nabla J)\psi, \varphi \rangle &= \langle \nabla(J\psi) - J\nabla\psi, \varphi \rangle = \langle \nabla(J\psi), \varphi \rangle - \langle J\nabla\psi, \varphi \rangle = \langle \nabla(J\psi), \varphi \rangle + \langle \nabla\psi, J\varphi \rangle \\ &= d\langle J\psi, \varphi \rangle - \langle J\psi, \nabla\varphi \rangle + d\langle \psi, J\varphi \rangle - \langle \psi, \nabla(J\varphi) \rangle = \langle \psi, J\nabla\varphi \rangle - \langle \psi, \nabla(J\varphi) \rangle = -\langle \psi, (\nabla J)\varphi \rangle\end{aligned}$$

Moreover, differentiating $J^2 = -I$, we get $(\nabla J)J = -J(\nabla J)$. Thus

$$\langle J(\nabla J)\psi, \varphi \rangle = -\langle (\nabla J)\psi, J\varphi \rangle = \langle \psi, (\nabla J)J\varphi \rangle = -\langle \psi, J(\nabla J)\varphi \rangle \,.$$

Hence $\tilde{\nabla}$ is still metric. □

Exercise 5.1.2 The standard hermitian metric $\langle .,. \rangle$ on the trivial bundle $\underline{\mathbb{C}}^{n+1}_{\mathbb{CP}^n}$ induces a hermitian metric on the tautological line bundle $L = \mathrm{Taut}(\mathbb{CP}^n)$. Let $\pi_L \colon \underline{\mathbb{C}}^{n+1}_{\mathbb{CP}^n} \to L$ denote the orthogonal projection in the fiber. Show that $\nabla \colon \Gamma(L) \to \Omega^1(\mathbb{CP}^n; L)$ given by

$$\nabla\psi = \pi_L(d\psi)\,,$$

defines a unitary connection on L.

Exercise 5.1.3 Let E be a vector bundle with connection ∇ and let P^∇ denote the parallel transport along a curve γ. Show:

(a) P^∇ is complex linear if and only if ∇ is complex.
(b) P^∇ is an isometry if and only if ∇ is metric.

5.2 Kähler Manifolds

5.2.1 Riemannian and Kähler Manifolds

Vaguely, a Kähler manifold is a manifold which has a complex and a metric structure on its tangent bundle which play together nicely. Let us make this precise. We start with the metric structure.

Definition 5.2.1 (Riemannian Manifold) *A Riemannian manifold is a pair $(M, \langle .,. \rangle)$ consisting of manifold M and a Riemannian metric $\langle .,. \rangle$ on M, i.e. $\langle .,. \rangle$ is a fiber metric on TM. A 2-dimensional Riemannian manifold is called a Riemannian surface.*

Theorem 5.2.1 (Koszul Formula) *Let $(M, \langle .,. \rangle)$ be a Riemannian manifold. A metric and torsion-free connection ∇ satisfies the following formula: Let $X, Y, Z \in \Gamma(TM)$, then*

$$2\langle \nabla_X Y, Z \rangle = \begin{array}{lll} +\ X\langle Y, Z\rangle & +\ Y\langle X, Z\rangle & -\ Z\langle X, Y\rangle \\ -\langle X, [Y, Z]\rangle & -\langle Y, [X, Z]\rangle & +\langle Z, [X, Y]\rangle\,. \end{array}$$

Proof. Let ∇ denote the Levi-Civita connection. Then

$$\begin{aligned} X\langle Y,Z\rangle + Y\langle X,Z\rangle - Z\langle X,Y\rangle &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle + \langle \nabla_Y X, Z\rangle + \langle X, \nabla_Y Z\rangle - \langle \nabla_Z X, Y\rangle - \langle X, \nabla_Z Y\rangle \\ &= \langle \nabla_X Y + \nabla_Y X, Z\rangle + \langle Y, \nabla_X Z - \nabla_Z X\rangle + \langle X, \nabla_Y Z - \nabla_Z Y\rangle \\ &= \langle 2\nabla_X Y - [X,Y], Z\rangle + \langle Y, [X,Z]\rangle + \langle X, [Y,Z]\rangle \end{aligned}$$

Shifting all Lie bracket terms over to the left yields the desired formula. □

The following theorem is sometimes called the *fundamental theorem of Riemannian geometry*.

Theorem 5.2.2 (Levi-Civita Connection) *Let $(M, \langle .,.\rangle)$ be a Riemannian manifold. Then there is a unique metric torsion-free affine connection ∇. The connection ∇ is called the Levi-Civita connection of $(M, \langle .,.\rangle)$.*

Proof. Uniqueness follows from the Koszul formula. For the existence we define ∇ by the right-hand side of the Koszul formula. It is left to show that ∇ really defines a connection which is metric and torsion-free. We leave this as an exercise. □

Theorem 5.2.3 (Change of Levi-Civita from Change of Riemannian Metric) *Let M be a smooth manifold and let $\langle .,.\rangle$ and $\langle .,.\rangle_\sim$ be a two Riemannian metrics on M. Then there is a unique positive-definite self-adjoint $B \in \Gamma\mathrm{End}(TM)$ such that $\langle .,.\rangle_\sim = \langle B.,.\rangle$ and the corresponding Levi-Civita connections ∇ and $\tilde\nabla$ are related as follows:*

$$2B(\tilde\nabla - \nabla) = (\nabla B) \odot I - \mathrm{grad}^\nabla B\,,$$

where I denotes the identity, $\odot$ the symmetric product and $(grad)^\nabla$ the gradient with respect to the metric $\langle .,.\rangle$, i.e.

$$\big((\nabla B)\odot I\big)(X,Y) = (\nabla_X B)Y + (\nabla_Y B)X\,, \quad \big\langle (\mathrm{grad}^\nabla B)(X,Y), Z\big\rangle := \big\langle (\nabla_Z B)X, Y\big\rangle\,, \quad (X,Y,Y \in T_pM)\,.$$

Proof. The claim follows by a straight-forward computation using the Koszul formula. We leave it as exercise. □

Definition 5.2.2 (Almost Hermitian Manifold) *An almost complex manifold (M, J) together with a hermitian Riemannian metric is called an almost hermitian manifold.*

Remark 5.2.1 Almost hermitian manifold are sometimes also called unitary manifolds.

Definition 5.2.3 (Kähler Manifold) *A Kähler manifold is an almost hermitian manifold whose Levi–Civita connection is complex.*

Theorem 5.2.4 *An almost complex manifold is complex $\Longleftrightarrow$ there is a complex torsion-free affine connection.*

Proof. Let (M, J) be complex, then using an atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$, we locally find complex torsion-free connections ∇_α. These can then be glued together using a partition of unity. Conversely, given a complex torsion-free connection, we have

$$\begin{aligned} N_J(X,Y) &= [X,Y] - [JX,JY] + J([JX,Y] + [X,JY]) \\ &= \nabla_X Y - \nabla_Y X - J\nabla_{JX} Y + J\nabla_{JY} X + J(\nabla_{JX} Y - J\nabla_Y(X) + J\nabla_X Y - \nabla_{JY} X) \\ &= 0\,. \end{aligned}$$

By Theorem 3.3.12, (M, J) is complex. □

As a direct consequence of Theorem 5.2.4 we get the following.

Corollary 5.2.5 *A Kähler manifold is complex.*

Definition 5.2.4 (Kähler Form) *Let $(M, \langle .,. \rangle)$ be an almost hermitian manifold. The associated Kähler form $\omega \in \Omega^2 M$ is given by*

$$\omega(X, Y) = \langle JX, Y \rangle .$$

Theorem 5.2.6 *Let M be an almost hermitian manifold with associated Kähler form ω. If M is complex, then*

$$d\omega(X,Y,Z) = \langle (\nabla_X J)Y, Z\rangle + \langle (\nabla_Y J)Z, X\rangle + \langle (\nabla_Z)X, Y\rangle ,$$
$$2\langle (\nabla_X J)Y, Z\rangle = d\omega(X,Y,Z) - d\omega(X, JY, JZ) .$$

In particular, M is Kähler if and only if $d\omega = 0$.

Proof. Since M is complex one can assume that X, Y, JY, Z, JZ are commuting vector fields. The proof is straight-forward and left as an exercise. □

Remark 5.2.2 A symplectic manifold is a manifold together with a symplectic form, i.e. a closed non-degenerate 2-form $\sigma \in \Omega^2 M$. The previous theorem shows that every Kähler manifold is symplectic.

Theorem 5.2.7 *Let V be a vector space of dimension $2n < \infty$ and $\sigma \colon V \times V \to \mathbb{R}$ be a skew-symmetric non-degenerate bilinear form. Then there is a basis with respect to which σ is represented by the diagonal block matrix*

$$\operatorname{diag}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) .$$

In particular, for $\omega = dx_1 \wedge dy_1 + \cdots + dx_m \wedge dy_m$, we get

$$\omega \wedge \cdots \wedge \omega = m! dx_1 \wedge dy_1 \wedge \cdots \wedge dx_m \wedge dy_m \neq 0 .$$

Theorem 5.2.8 *If M is a compact Kähler manifold with Kähler form ω, then ω is not exact.*

Proof. Assume that $\omega = d\alpha$ for some $\alpha \in \Omega^1 M$. Then $m! d\mathrm{vol}_M = \omega \wedge \omega \wedge \cdots \wedge \omega = d(\alpha \wedge \omega \wedge \cdots \wedge \omega)$. Hence

$$0 < \int_M \omega \wedge \omega \cdots \wedge \omega = \int_M d(\alpha \wedge \omega \wedge \cdots \wedge \omega) = 0 ,$$

which is a contradiction. □

Corollary 5.2.9 *If M is a Kähler manifold, then $H^2 M \neq \{0\}$.*

Example 5.2.1 The unit circle $\mathbb{S}^1 \subset \mathbb{C}$ acts isometrically on $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$. Thus the complex projective space $\mathbb{CP}^n = \mathbb{S}^{2n+1}/_{\mathbb{S}^1}$ comes with an induced metric—its *Fubini–Study* metric—turning the canonical projection

$\pi\colon \mathbb{S}^{2n+1} \to \mathbb{CP}^n$ into Riemannian submersion, i.e. for each point $p \in \mathbb{S}^{2n+1}$ the restriction

$$d_p\pi\colon (\ker d_p\pi)^\perp \to T_{\pi(p)}\mathbb{CP}^n$$

is an isometry of vector spaces (see Appendix A.2). Moreover $\mathbb{CP}^n$ is complex. The Kähler form of ω of $\mathbb{C}^{n+1}$ and the Kähler form ω_{FS} of the Fubini–Study metric are related as follows: If $\iota\colon \mathbb{S}^{2n+1} \hookrightarrow \mathbb{C}^{n+1}$ denotes the inclusion, then

$$\pi^*\omega_{FS} = \iota^*\omega.$$

In particular, $\pi^* d\omega = 0$. Hence the Fubini–Study metric turns $\mathbb{CP}^n$ into a Kähler manifold.

Exercise 5.2.1 Let ω_{FS} denote the Fubini-Study Kähler form of $\mathbb{CP}^n$. Show that the tautological complex line bundle over $\mathbb{CP}^n$ has curvature $F^\nabla = -\omega_{FS}J$.

Exercise 5.2.2 Show that a complex submanifold of a Kähler manifold is Kähler.

Theorem 5.2.10 (A Version of Kodaira Embedding Theorem) *Every compact Kähler manifold admits a complex (not necessarily isometric) embedding in some* $\mathbb{CP}^n$.

Proof. The proof is way to difficult to do it in the scope of this lecture, so we will skip it. It may be seen in a lecture about algebraic geometry, as the result is one of the highlights of that lecture. □

The manifold $M := (\mathbb{C}^2 \setminus \{0\})/_{\{z\mapsto 2z\}} \cong \mathbb{S}^3 \times \mathbb{S}^1$ is a complex manifold—called a *Hopf manifold*.

The Künneth formula (1923) states that if $[\alpha_1], \dots, [\alpha_n]$ is a basis of $H^*M = H^0M \oplus \cdots H^mM$ and $[\tilde\alpha_1], \dots, [\tilde\alpha_{\tilde n}]$ is a basis of $H^\bullet\tilde M$. Then $\beta_{ij} = \alpha_i \wedge \tilde\alpha_j$ form a basis of $H^\bullet M \otimes H^\bullet \tilde M \cong H^\bullet(M \times \tilde M)$.

Corollary 5.2.11 *The Hopf manifold $M \cong \mathbb{S}^3 \times \mathbb{S}^1$ is not Kähler and therefore cannot be realized in any* $\mathbb{CP}^n$.

Proof. By the Künneth formula, $H^2(\mathbb{S}^3 \times \mathbb{S}^1) = \{0\}$. □

5.2.2 Riemann Surfaces and Oriented Riemannian Surfaces

On the ordered bases of an n-dimensional vector space we have an equivalence relation the corresponding coordinate change has a positive determinant:

For each n–dimensional real vector space V we have an equivalence relation on the set of ordered bases $\mathcal{B}_V := \{(v_1, \dots, v_n) \in V^n \mid v_1, \dots, v_n \text{ basis}\}$ given as follows: Any two elements $B_1 = (v_1, \dots, v_n)$ and $B_2 = (w_1, \dots, w_n)$ of $\mathcal{B}$ are related by a linear *coordinate change*, i.e. there is a unique isomorphism $A_{B_1B_2} \in \operatorname{End}V$ such that $A_{B_1B_2}v_i = w_i$ for $i = 1, \dots, n$. Then we set

$$B_1 \sim B_2 \quad :\Longleftrightarrow \quad \det(A_{B_1B_2}) > 0\,.$$

Definition 5.2.5 (Vector Space Orientation) *An element in $[B] \in \mathcal{B}_V/_\sim$ is called an orientation of V.*

Complex vector spaces come with a canonical orientation. To see this we need a bit complex linear algebra.

Lemma 5.2.12 *Let $A \in \mathbb{C}^{n\times n}$ and $A_\mathbb{R} \in \mathbb{R}^{2n\times 2n}$ denote the corresponding real matrix. Then* $\det A_\mathbb{R} = |\det_\mathbb{C} A|^2$.

Proof. Write $A = B + iC$ with $B, C \in \mathbb{R}^n$. Then, with respect to the basis $e_1, \dots, e_n, ie_1, \dots, ie_n$, the matrix $A_\mathbb{R}$ is of the form

$$A_\mathbb{R} = \begin{pmatrix} B & -C \\ C & B \end{pmatrix} .$$

No apply elementary operations:

$$|\det_\mathbb{C} A|^2 = \det_\mathbb{C} \bar{A}^t \det_\mathbb{C} A = \det_\mathbb{C} \begin{pmatrix} B - iC & 0 \\ C & B + iC \end{pmatrix} = \det_\mathbb{C} \begin{pmatrix} B & -C + iB \\ C & B + iC \end{pmatrix} = \det_\mathbb{C} \begin{pmatrix} B & -C \\ C & B \end{pmatrix} = \det_\mathbb{R} A_\mathbb{R} .$$

□

Corollary 5.2.13 *Let V be a complex vector spaces and $A \in \mathrm{End}_+ V$. Then* $\det A \geq 0$.

Proof. Each $A \in \mathrm{End}_+ V$ is represented by a complex matrix. □

Corollary 5.2.14 (Canonical Orientation of Complex Vector Spaces) *If $(v_1, \dots, v_n)$ and $(w_1, \dots, w_n)$ are complex bases of a complex vector space V, then the corresponding real bases are of the same orientation:*

$$(v_1, Jv_1, \dots, v_n, Jv_n) \sim (w_1, Jw_1, \dots, w_n, Jw_n) .$$

This orientation we call the canonical orientation of the complex vector space V.

Proof. The coordinate change $A \in \mathrm{End} V$ is complex linear, i.e. $A \in \mathrm{End}_+ V$. So $\det A > 0$. □

Equivalently an orientation is given by an equivalence class of determinants, i.e. non-vanishing n-forms, where we consider two such determinants as equivalent if they differ by a positive scalar. This can be easily carried over to manifolds.

Let M be a smooth manifold of dimension m. A nowhere-vanishing $\omega \in \Omega^m M$ is called a *volume form*. On the define an equivalence relation on the space of volume forms as follows:

$$\tilde{\omega} \sim \omega \quad :\Longleftrightarrow \quad \exists f \in \mathscr{C}^\infty M, f > 0 : \quad \tilde{\omega} = f\omega .$$

Definition 5.2.6 (Manifold Orientation) *A manifold is called orientable if it has a volume form. An equivalence class of volume forms is called an orientation. An oriented manifold is a pair $(M, [\omega])$ consisting of a manifold and an orientation on it.*

From what was said above the following statement is obvious.

Corollary 5.2.15 *Every almost complex manifold has a canonical orientation.*

For surfaces we have the following statement.

Theorem 5.2.16 *Each oriented Riemannian surface* $(M, \langle .,. \rangle)$ *has a unique almost complex structure* J *such that* $\langle .,. \rangle$ *is hermitian and the canonical orientation of* (M, J) *coincides with the given one.*

Proof. Define J to be the rotation by $\pi/2$ in the positive sense. □

Theorem 5.2.17 *Every oriented Riemannian surface is Kähler.*

Proof. Since J is skew-adjoint, so is ∇J. Since we are on a surface, the skew-adjoint endomorphisms are spanned by J and so $\nabla J = \lambda J$ for some real valued function λ. Differentiating $J^2 = -1$, we find

$$0 = (\nabla J)J + J(\nabla J) = (\lambda J)J + J(\lambda J) = -2\lambda \,.$$

Hence $\nabla J = 0$. □

Corollary 5.2.18 *The Nijenhuis tensor of an oriented Riemannian surface vanishes.*

So each oriented Riemannian surface is a Riemann surface. Conversely, given a Riemann surface. This determines a Riemannian surface up to conformal equivlence.

Definition 5.2.7 (Conformal Equivalence) *Two Riemannian metrics* g *and* $\tilde{g}$ *on a manifold* M *are called conformally equivalent, if there is* $u \in \mathscr{C}^\infty M$ *such that*

$$\tilde{g} = e^{2u} g \,.$$

An equivalence class of conformal metrics is called a conformal structure.

Theorem 5.2.19 *On a Riemann surface* (M, J) *there is a unique conformal structure and orientation such that for all* $0 \neq X \in TM$ *the vectors* X, JX *have the same length and form an orthogonal positively oriented basis.*

Proof. If X, JX are positively oriented, then the orientation is the canonical orientation. If $\langle .,. \rangle$ is such that $|X| = |JX|$, then $\langle .,. \rangle$ is hermitian—such metric always exists. The conformal structure is unique, since the space of hermitian sesquilinear forms on a complex line is of real dimension 1. □

Integration 6

6.1 Integration on Manifolds

6.1.1 Oriented Charts

Let $(M,[\omega])$ be an oriented manifold. Then, given a chart $\varphi\colon U \to \mathbb{R}^m$, the pullback $\varphi^*\mathrm{det}_{\mathbb{R}^m}$ of the determinant

$$\mathrm{det}_{\mathbb{R}^m} \in \Omega^m\mathbb{R}^m$$

yields a volume form on U.

Definition 6.1.1 (Oriented Charts) *A chart (U,φ) of an oriented manifold $(M,[\omega])$ is called positively oriented, if the orientation given by $\varphi^*\mathrm{det}_{\mathbb{R}^m}$ coincides with the orientation induced by ω, i.e. $[\varphi^*\mathrm{det}_{\mathbb{R}^m}] = [\omega|_U]$. If $[\varphi^*\mathrm{det}_{\mathbb{R}^m}] \neq [\omega|_U]$, the chart φ is called negatively oriented.*

If a chart φ is negatively oriented, then we can change it to a positively oriented chart by postcomposing a reflection, as e.g. $(x_1, x_2, \ldots, x_m) \mapsto (-x_1, x_2, \ldots, x_m)$.

Corollary 6.1.1 *Every oriented manifold has an oriented atlas, i.e. an atlas consisting of positively oriented charts.*

The Jacobian of the coordinate change of any two equally oriented charts φ and $\tilde{\varphi}$ has a positive determinant: Let $e_1, \ldots, e_m$ denote the standard basis of $\mathbb{R}^m$. Then, with $\varphi^*\mathrm{det}_{\mathbb{R}^m} = \lambda\omega$ and $\tilde{\varphi}^*\mathrm{det}_{\mathbb{R}^m} = \tilde{\lambda}\omega$, we get

$$\begin{aligned}\mathrm{det}_{\mathbb{R}^m}\big(\mathrm{Jac}(\tilde{\varphi}\circ\varphi^{-1})\big) &= \mathrm{det}_{\mathbb{R}^m}\big(d\tilde{\varphi}(d\varphi^{-1}(e_1)), \ldots, d\tilde{\varphi}(d\varphi^{-1}(e_m))\big) = (\tilde{\varphi}^*\mathrm{det}_{\mathbb{R}^m})\big(d\varphi^{-1}(e_1), \ldots, d\varphi^{-1}(e_m)\big)\\ &= (\tilde{\lambda}\omega)\big(d\varphi^{-1}(e_1), \ldots, d\varphi^{-1}(e_m)\big) = (\tilde{\lambda}/\lambda)\circ\varphi^{-1}\mathrm{det}_{\mathbb{R}^m}(e_1, \ldots, e_m) = (\tilde{\lambda}/\lambda)\circ\varphi^{-1} > 0\,.\end{aligned}$$

Note, that we can speak of equally oriented charts independently of a given orientation.

Theorem 6.1.2 *An atlas of pairwise equally oriented charts defines an orientation.*

Proof. Given an atlas $\{(U_\alpha, \varphi_\alpha)\}_\alpha$, we obtain locally defined volume forms $\varphi_\alpha^*\mathrm{det}_{\mathbb{R}^m} \in \Omega^m U_\alpha$ which can be extended using a subordinate partition of unity to smooth forms on $\omega_\alpha \in \Omega^m M$. The sum $\omega = \sum_\alpha \omega_\alpha$ then defines a volume form on M and thus an orientation $[\omega]$. One easily checks that the charts φ_α are positively oriented with respect to $[\omega]$. □

6.1.2 Integration of Differential Forms

Let $E \to M$ be a vector bundle. The *support of a section* $\psi \in \Gamma E$ is the closure of the set of points in M which are mapped by ψ to a non-zero vector,

$$\operatorname{supp}\psi = \overline{\{p \in M \mid \psi_p \neq 0\}}\,.$$

Definition 6.1.2 (Forms with Compact Support)

$$\Omega_0^k M := \{\omega \in \Omega^k M \mid \operatorname{supp}\omega \ \textit{compact}\}\,.$$

We first define the integral of compactly supported forms on $\mathbb{R}^m$: Let $\omega \in \Omega_0^m \mathbb{R}^m$. Then $\omega = f \det_{\mathbb{R}^m}$ for some compactly supported smooth function $f \in \mathscr{C}^\infty \mathbb{R}^m$ and we define

$$\int_{\mathbb{R}^m} \omega := \int_{\mathbb{R}^m} f \, d\mu_m\,,$$

where μ_m denotes the Lebesgue measure on $\mathbb{R}^m$.

Lemma 6.1.3 *Let $\omega, \tilde\omega \in \Omega_0^m \mathbb{R}^m$ and $\varphi\colon \operatorname{supp}\tilde\omega \to \operatorname{supp}\omega$ be a diffeomorphism such that $\tilde\omega = \varphi^*\omega$ and $\det_{\mathbb{R}^m}(\mathrm{Jac}(\varphi)) > 0$, then*

$$\int_{\mathbb{R}^m} \omega = \int_{\mathbb{R}^m} \tilde\omega\,.$$

Proof. Let $\omega = f\det_{\mathbb{R}^m}$ and $\tilde\omega = \tilde f\det_{\mathbb{R}^m}$. Let $e_1, \dots, e_m$ denote the standard basis of $\mathbb{R}^m$, then

$$(\varphi^*\det_{\mathbb{R}^m})(e_1,\dots,e_m) = \det_{\mathbb{R}^m}\big(d\varphi(e_1),\dots,d\varphi(e_m)\big) = \det_{\mathbb{R}^m}(\mathrm{Jac}(\varphi))\,.$$

Since $\mathrm{Alt}^m\mathbb{R}^m$ is one dimensional, we conclude that

$$\varphi^*\det_{\mathbb{R}^m} = \det_{\mathbb{R}^m}\big(\mathrm{Jac}(\varphi)\big)\det_{\mathbb{R}^m}\,.$$

Hence, with $\det_{\mathbb{R}^m}(\mathrm{Jac}(\varphi)) > 0$, we get

$$\tilde f\det_{\mathbb{R}^m} = \tilde\omega = \varphi^*\omega = (f\circ\varphi)\,\varphi^*\det_{\mathbb{R}^m} = (f\circ\varphi)\det_{\mathbb{R}^m}\big(\mathrm{Jac}(\varphi)\big)\det_{\mathbb{R}^m} = (f\circ\varphi)\big|\det_{\mathbb{R}^m}\big(\mathrm{Jac}(\varphi)\big)\big|\det_{\mathbb{R}^m}\,.$$

So that

$$\int_{\mathbb{R}^m}\tilde\omega = \int_{\mathbb{R}^m}\tilde f\,d\mu_m = \int_{\mathbb{R}^m}(f\circ\varphi)\big|\det_{\mathbb{R}^m}\big(\mathrm{Jac}(\varphi)\big)\big|\,d\mu_m = \int_{\mathbb{R}^m} f\,d\mu_m\,,$$

by the usual transformation formula of the Lebesgue integral. □

Now we can define the integral for compactly supported m-forms on an oriented manifold whose support is contained in a chart region: Let (U,φ) be a positively oriented chart on M and $\omega \in \Omega_0^m M$ such that $\operatorname{supp}\omega \subset U$, then

$$\int_M \omega = \int_{\mathbb{R}^m} (\varphi^{-1})^*\omega\,.$$

By the last lemma, this integral is well-defined—the Jacobian of the coordinate change between two positively oriented charts has a positive determinant.

Definition 6.1.3 (Integral of Compactly Supported Forms on a Manifold) *Let M be an oriented manifold and $\omega \in \Omega_0^m M$, then we define*

$$\int_M \omega := \sum_{\alpha\in A}\int_M \rho_\alpha\omega\,,$$

where $\{(U_\alpha,\varphi_\alpha)\}_{\alpha\in A}$ is an oriented atlas and $\{\rho_\alpha\}_{\alpha\in A}$ is a partition of unity subordinate to it.

Theorem 6.1.4 *The definition of the integral is independent of the choices.*

Proof. Without loss of generality we can assume that the atlas contains all possible charts, so the independence of the atlas is clear. Now, let $\{\rho_\alpha\}_{\alpha\in A}$ and $\{\tilde\rho_\alpha\}_{\alpha\in A}$ be two partitions of unity subordinate to the atlas, then

$$\sum_{\alpha\in A}\int_M \rho_\alpha\omega = \sum_{\alpha\in A}\int_M (\sum_{\beta\in A}\tilde\rho_\beta)\rho_\alpha\omega = \sum_{\alpha,\beta\in A}\int_M \tilde\rho_\beta\rho_\alpha\omega = \sum_{\beta\in A}\int_M (\sum_{\alpha\in A}\rho_\alpha)\tilde\rho_\beta\omega = \sum_{\beta\in A}\int_M \tilde\rho_\beta\omega\,.$$

Thus the definition is independent of the choice of the partition of unity as well. □

Let $(\tilde M,[\tilde\omega])$ and $(M,[\omega])$ be oriented manifolds. A diffeomorphism $f\colon M\to\tilde M$ is called *orientation-preserving*, if $[f^*\tilde\omega]=[\omega]$. From Lemma 6.1.3 we immediately get a transformation formula for the integral of differential forms. The proof is left as an exercise.

Theorem 6.1.5 (Transformation Formula for Forms) *If $\eta\in\Omega_0^m\tilde M$ and $f\colon M\to\tilde M$ is an orientation preserving diffeomorphism between m-dimensional manifold, then*

$$\int_M f^*\eta = \int_{\tilde M}\eta\,.$$

6.2 Stokes' Theorem

Stokes' Theorem can be regarded as the manifold version of the fundamental theorem of calculus—it relates the integral of the exterior derivative $d\omega$ of an $(m-1)$-form on an m-dimensional manifold M to the integral of ω over the boundary ∂M. But what do mean by boundary?

6.2.1 Manifolds with Boundary

A manifolds with boundary are manifolds modeled over *euclidean half-space* H^m,

$$\mathrm{H}^m := \{(x_1,\ldots,x_m)\in\mathbb{R}^m \mid x_1\le 0\}\,.$$

Definition 6.2.1 (Manifold with Boundary) *A smooth m-dimensional manifold with boundary is a 2nd-countable Hausdorff space M which is locally homeomorphic to open sets in H^m together with a maximal smooth atlas $\{\varphi_\alpha\colon U_\alpha\to V_\alpha \overset{\circ}{\subset} \mathrm{H}^m\}_{\alpha\in A}$.*

Remark 6.2.1 Note that in this situation the coordinate changes are between open subsets of H^m. Here smooth means in the sense of Milnor: A map f defined on an arbitrary subset $A\subset\mathbb{R}^m$ is called smooth if it has a smooth extension to an open neighborhood of A.

The boundary $\partial\mathrm{H}^m$ of H^m is defined as follows:

$$\partial\mathrm{H}^m = \{0\}\times\mathbb{R}^{m-1}\,.$$

The boundary of M simply consists of all its points which are mapped to $\partial\mathrm{H}^m$.

Definition 6.2.2 (Boundary) *Let M be a manifold with boundary, then the boundary ∂M is defined by*

$$\partial M := \{p\in M \mid \varphi(p)\in\partial\mathrm{H}^m \ \text{for some smooth chart}\ \ \varphi\}\,.$$

Exercise 6.2.1 Check that ∂M is well-defined, i.e. if $\varphi(p) \in \partial \mathrm{H}^m$ for some smooth chart φ, then $\varphi(p) \in \partial \mathrm{H}^m$ for all smooth charts φ.

Remark 6.2.2 If M is an m-dimensional manifold with boundary, then

(a) $\mathring{M} := M \setminus \partial M$ is an m-dimensional manifold,
(b) ∂M is an $(m-1)$-dimensional submanifold of M

—both without boundary.

Definition 6.2.3 (Outward-Pointing Vectors) *Let M be a manifold with boundary and $X \in TM|_{\partial M}$. Then X is called outwoard-pointing, if there is a smooth chart $\varphi = (x_1, \ldots, x_m)$ such that $dx_1(X) > 0$.*

Remark 6.2.3 Again it should be checked that the definition does not depend on the chart.

Lemma 6.2.1 (Existence of Outward-Pointing Fields) *Each manifold with boundary has an outward-pointing field, i.e. a smooth vector field $v \in \Gamma TM$ such that for all $p \in \partial M$ the vector v_p is outward-pointing.*

Proof. This is again a partition of unity argument. The boundary ∂M can be covered by smooth charts $\{(U_i, \varphi_i)\}_{i \in I}$ and on each U_i we have a smooth vector field given by $v_{i,p} := d\varphi_i^{-1}(\varphi_i(p), e_1)$. Then there is a partition of unity $\{\rho_{\mathring{M}}\} \bigcup \{\rho_i\}_{i \in I}$ subordinate to $\mathring{M} \bigcup \{U_i\}_{i \in I}$. Then define $v = \sum_{i \in I} \rho_i v_i$. Clearly, v is smooth. That v is outward-pointing follows since convex combinations of outward-pointing vectors are outward-pointing. □

Given a k-form $\eta \in \Omega^k M$. For a given vector field $X \in \Gamma(TM)$ we define a $(k-1)$-form $X \lrcorner \eta \in \Omega^{k-1} M$ as follows: For $X_1, \ldots, X_{k-1} \in T_p M$,

$$X \lrcorner \eta\, (X_1, \ldots, X_{k-1}) := \eta(X_p, X_1, \ldots, X_{k-1}).$$

Now, if $(M, [\omega])$ is an oriented manifold with boundary, $\iota_{\partial M} \colon \partial M \hookrightarrow M$ denotes the inclusion and $v \in \Gamma(TM)$ is an outward-pointing field. Then $\iota_{\partial M}{}^*(v \lrcorner \omega)$ is a volume form.

Definition 6.2.4 (Induced Orientation) *The orientation of the boundary of an oriented manifold with boundary $(M, [\omega])$ is defined to be $[\iota_{\partial M}{}^*(v \lrcorner \omega)]$, where v is an outward-pointing field.*

6.2.2 Stokes' Theorem

So we are finally ready to state and proof Stokes' theorem.

Theorem 6.2.2 (Stokes' Theorem) *Let M be an oriented manifold with boundary ∂M and $\omega \in \Omega_0^{m-1} M$. Then*

$$\int_M d\omega = \int_{\partial M} \omega\,.$$

Proof. The integral is defined via a partition of unity. All summands are integrals over H^m. Without loss of generality we can assume that $M = H^m$. The form ω is then of the form

$$\omega = \sum_{i=1}^{m} (-1)^{i-1} \omega_i\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_m\,.$$

Thus $\omega|_{\partial M} = \omega_1\, dx_2 \wedge \cdots \wedge dx_m$ and

$$\int_{\partial M} \omega = \int_{\{0\}\times\mathbb{R}^{m-1}} \omega_1\, d\mu_{m-1} = \int_{\mathbb{R}^{m-1}} \omega_1(0,.)\, d\mu_{m-1}\,.$$

On the other hand we have

$$d\omega = \sum_{i=1}^{m}(-1)^{i-1} d\omega_i \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_m = \Big(\sum_{i=1}^{m} \frac{\partial \omega_i}{\partial x_i}\Big)\, dx_1 \wedge \cdots \wedge dx_m = \Big(\sum_{i=1}^{m} \frac{\partial \omega_i}{\partial x_i}\Big) \det\nolimits_{\mathbb{R}^m}$$

and, by Fubini's theorem and the fundamental theorem of calculus,

$$\int_M d\omega = \sum_{i=1}^{m} \int_{H^m} \frac{\partial \omega_i}{\partial x_i}\, d\mu_m = \int_{\mathbb{R}^{m-1}} \underbrace{\int_{-\infty}^{0} \frac{\partial \omega_1}{\partial x_1}\, d\mu_1}_{=\omega_1(0,.)\text{ (comp. supp.)}}\, d\mu_{m-1} + \sum_{i=2}^{m} \int_{\mathbb{R}^{m-1}} \underbrace{\int_{-\infty}^{\infty} \frac{\partial \omega_i}{\partial x_i}\, d\mu_1}_{=0\text{ (comp. supp.)}}\, d\mu_{m-1} = \int_{\mathbb{R}^{m-1}} \omega_1(0,.)\, d\mu_{m-1}\,,$$

i.e. both integrals are equal. □

6.2.3 Integration of Functions on Riemannian Manifolds

So far we can only integrate top-dimensional forms on oriented manifolds. If the manifold is Riemannian, we can also integrate functions. This is basically because each oriented Riemannian manifold comes with a canonical volume form.

Theorem 6.2.3 (Riemannian Volume Form) *Let M be an oriented m-dimensional Riemannian manifold. Then there is a unique m-form dvol_M such that*

$$\mathrm{dvol}_M(X_1, \cdots, X_m) = 1\,,$$

on each positive oriented orthonormal basis $X_1, \cdots, X_m \in T_pM$. The form dvol_M is called the Riemannian volume form.

With this volume form at hand we can integrate functions just turning them into an m-form.

Definition 6.2.5 (Integral of Functions) *Let M be an oriented Riemannian manifold and $f \in \mathscr{C}^\infty M$. Then*

$$\int_M f = \int_M f\, \mathrm{dvol}_M$$

Holomorphic Line Bundles over Compact Riemann Surfaces

Complex Line Bundles 7

A *complex line bundle* is a complex vector bundle, whose fibers are complex 1-dimensional vector spaces—complex lines. The goal of this chapter is to classify complex line bundles over an oriented compact surface. It turns out that each line bundle L comes with a *degree* $\deg L \in \mathbb{Z}$ which is the only topological invariant: If $L \to M$ and $\tilde{L} \to M$ are complex line bundles, then

$$L \cong \tilde{L} \iff \deg L = \deg \tilde{L}$$

7.1 Complex Line Bundles over Surfaces

7.1.1 The Degree

Let L be a complex line bundle over a compact oriented surface M. We know already that there are complex connections on L and any two of such differ by a 1-form taking values in End_+L—which is spanned by the parallel endomorphism fields I and J and thus is trivial as complex vector bundle as well as complex vector bundle with connection. Explicitly,

$$\underline{\mathbb{C}}_M \ni \alpha + i\beta \longleftrightarrow \alpha I + \beta J \in \mathrm{End}_+ L\,.$$

In particular, the curvature of a complex complex connection ∇ on L is commuting with J and can thus be regarded as a complex valued 2-form

$$F^\nabla \in \Omega^2(M;\mathbb{C})\,.$$

Now, if $\tilde{\nabla}$ is another complex connection then $\tilde{\nabla} = \nabla + \eta$ for some $\eta \in \Omega^1(M;\mathbb{C})$ then the usual change of curvature formula yields

$$F^{\tilde{\nabla}} = F^\nabla + d\eta + \underbrace{\eta \wedge \eta}_{=0} = F^\nabla + d\eta\,.$$

Hence Stokes' theorem yields

$$\int_M F^{\tilde{\nabla}} = \int_M F^\nabla + d\eta = \int_M F^\nabla\,.$$

Thus the integral of curvature is independent of the particular choice of the complex connection.

Definition 7.1.1 (Degree) *The degree of a complex line bundle L over a compact oriented surface M is defined as*

$$\deg L := \tfrac{i}{2\pi}\int_M F^\nabla\,,$$

where ∇ is some complex connection on L.

Later the Poincaré–Hopf index theorem will reveal that the degree is actually an integer. For now we show that the degree is a real number.

Theorem 7.1.1 *Let $L \to M$ be a unitary line bundle. Then*

$$F^\nabla = -2\pi i\omega, \quad \omega \in \Omega^2 M\,.$$

Proof. F^∇ is both complex linear and skew-adjoint. Thus its real part of F^∇ must vanish. □

Since we are free to choose a hermitian metric and a unitary connection, when we compute the degree, we obtain the following corollary.

Corollary 7.1.2 *The degree of a complex line bundle is real.*

Recall, two complex line bundles $L \to M$ and $\tilde{L} \to M$ are isomorphic if there exists an isomorphism of complex vector bundles between them. This mean that there is a nowhere-vanishing section of $\mathrm{End}_+(L, \tilde{L})$. A complex line bundle is trivial if it is isomorphic to the trivial complex line bundle $\underline{\mathbb{C}}_M$.

Exercise 7.1.1 If $L_1 \cong L_2$ and $\tilde{L}_1 \cong \tilde{L}_2$, then $L_1 \otimes L_2 \cong \tilde{L}_1 \otimes \tilde{L}_2$.

Thus the complex tensor product $\otimes$ descends to the space $\mathscr{L}$ of isomorphism classes of complex line bundles. Hence $(\mathscr{L}, \otimes)$ is a abelian group—the inverse element of a bundle L is given by its dual bundle $L^{-1} := L^*$. The neutral element is given by the trivial bundle $\mathbf{1} = \underline{\mathbb{C}}_M$.

Theorem 7.1.3 *If (L, ∇) and $(\tilde{L}, \tilde{\nabla})$ are complex line bundles with complex connections and $\hat{\nabla}$ is the corresponding tensor connection, then*

$$F^{\hat{\nabla}} = F^\nabla + F^{\tilde{\nabla}}.$$

Proof. This is a local computation. Without loss of generality let $\psi \in \Gamma L$ and $\tilde{\psi} \in \Gamma \tilde{L}$ be non-vanishing sections. Then $\psi \otimes \tilde{\psi}$ is a non-vanishing section and, since the tensor connection is defined such that the tensor product is parallel, we get

$$\begin{aligned} F^{\hat{\nabla}}(\psi \otimes \tilde{\psi}) &= (d^{\hat{\nabla}})^2(\psi \otimes \tilde{\psi}) = d^{\hat{\nabla}}\big((d^\nabla \psi) \otimes \tilde{\psi} + \psi \otimes (d^{\tilde{\nabla}}\tilde{\psi})\big) \\ &= ((d^\nabla)^2\psi) \otimes \tilde{\psi} - d^\nabla \psi \wedge^\otimes d^{\tilde{\nabla}}\tilde{\psi} + d^\nabla \psi \wedge^\otimes d^{\tilde{\nabla}}\tilde{\psi} + \psi \otimes ((d^{\tilde{\nabla}})^2 \tilde{\psi}) \\ &= (F^\nabla \psi) \otimes \tilde{\psi} + \psi \otimes (F^{\tilde{\nabla}}\tilde{\psi}). \end{aligned}$$

Since the tensor product is complex bilinear, we get $F^{\hat{\nabla}} = F^\nabla + F^{\tilde{\nabla}}$. □

Corollary 7.1.4 $F^{\nabla^*} = -F^\nabla$.

Clearly, we have $\deg \underline{\mathbb{C}}_M = 0$.

Corollary 7.1.5 *The map* $\deg\colon \mathscr{L} \to (\mathbb{R}, +)$ *is a group homomorphism.*

We are going now to show that the degree is an integer which counts the zeros of a section counted with multiplicity—if the zeros are isolated.

7.1.2 Transversality

Let $f\colon M^m \to N^n$ be a smooth map. A point $p \in M$ is called a *regular point* of f, if the differential $d_p f$ has full rank. A point which is not regular is called a *critical point*. Let $C_f \subset M$ denote the set of critical points of f.

Sard's lemma then states that the *set of critical values* $f(C_f)$ of a smooth function $f\colon M^m \to N^n$ has measure zero. With some effort this can be used to show one of the main results in differential topology—that, generically, a smooth map intersects a submanifold transversely.

Definition 7.1.2 (Transverse Intersection) *Let N be a smooth manifold, $M \subset N$ be a submanifold and $f\colon S \to N$ be a smooth map. We say that f intersects M transversely, if*

$$d_p f(T_p S) + T_{f(p)}M = T_{f(p)}N \quad \text{for all } p \in f^{-1}M\,.$$

We say that two submanifolds intersect transversely, if the inclusion map of one intersects the other transversely.

Remark 7.1.1 (Transverse Sections) Since a section of a vector bundle is always an embedding we can speak of transverse sections.

The transversality theorem then states that transversality is a generic property—any smooth map can be perturbed by an arbitrary small amount to a transverse map.

Theorem 7.1.6 (Transversality Theorem) *Let N and S be a smooth manifold and $M \subset N$ be a submanifold. Then the set of transverse maps $f\colon S \to N$ is dense in the $\mathscr{C}^\infty$-topology.*

The proof of this theorem is far beyond the scope of this course. Instead let us look at some consequences.

Corollary 7.1.7 *Let N be a smooth manifold, $M \subset N$ be a submanifold and $f\colon S \to N$ be a transverse map. Let $n = \dim N$, $m = \dim M$ and $s = \dim S$. Then $f^{-1}M \subset S$ is a closed $(s+m-n)$-dimensional submanifold.*

Proof. Without loss of generality, $M = \mathbb{R}^m \times \{0\} \subset \mathbb{R}^n = N$. Let $f_2 = \pi_2 \circ f$, where $\pi_2\colon \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^{n-m}$. Then $f^{-1}M = f_2^{-1}\{0\}$. Transversality yields that $\operatorname{rank}_p f_2 = n - m$ for all $p \in f_2^{-1}\{0\}$. Hence, by the submersion theorem, $f^{-1}M = f_2^{-1}\{0\}$ is a submanifold of dimension $s - (n-m) = s + m - n$. □

Let us look at what this means for the zero set of a generic section of a vector bundle.

Corollary 7.1.8 *If $\psi \in \Gamma E$ is a smooth section, then any section φ can be perturbed to a section transverse to the zero section.*

If $E \to M$ is a smooth rank r vector bundle, then the zeros of a section $\psi \in \Gamma E$ are exactly the intersection points of ψ and the *zero section* $p \mapsto 0 \in E_p$. The image of the zero section is an m-dimensional submanifold, where m is the dimension of M. The total space E has dimension $m + r$. Thus

Corollary 7.1.9 *Let E be a smooth rank r vector bundle over an m-dimensional manifold M and let $\psi \in \Gamma E$ be transverse to the zero section, then the zero set $Z_\psi \subset M$ of ψ is a smooth submanifold of dimension $m - r$.*

Proof. The dimension of the total space is $n = m + r$, the dimension of the image of the zero section is m. Hence the zero set of a generic section is a submanifold of dimension $m + m - (m + r) = m - r$. □

Remark 7.1.2 In particular, a generic section of a complex line bundle over a compact surface has finitely many isolated zeros.

7.1.3 Poincaré–Hopf Index Theorem

Let L be a complex line bundle over a compact oriented surface M. If $\psi \in \Gamma L$ be a section with isolated zeros, then each zero comes with a particular sign:

Given a smooth function $z\colon M \to \mathbb{C}$ and a point $p \in M$, one can count how often z winds around zero when one runs around p by integration of $\frac{1}{2\pi i} d \log z = \frac{1}{2\pi i}\frac{dz}{z}$ over the boundary of a small enough disk D containing p. Here small enough just means that z has no zeros in $D \setminus \{p\}$.

A similar formula allows us to define the index of a direction field.

Definition 7.1.3 (Direction Field) *A direction field of L is a smooth section ψ of L defined away from a discrete set S such that $\psi_p \neq 0$ for all $p \in M \setminus S$. We call S the set of isolated singularities of ψ.*

Definition 7.1.4 (Index) *Let ψ be a direction field of L with isolated singularities S. The index of ψ at $p \in M$ is defined as follows:*

$$\mathrm{ind}_p\psi := \frac{1}{2\pi i}\int_{\partial D} d\log z\,,$$

where $D \ni p$ is a disk such that $D \cap S = \{p\}$ and $\psi = z\varphi$ for some $\varphi \in \Gamma L$ which has no zeros in D.

Remark 7.1.3 The definition is independent of both the choice of φ and the choice of D.

Remark 7.1.4 Clearly, $\mathrm{ind}_p\psi = 0$ for $p \in M \setminus S$.

Remark 7.1.5 If ψ is transverse to the zero section, then its zeros are isolated, the function z has only simple zeros and the index of ψ at $p \in M$ agrees with the sign of the determinant of $d_p z$. In particular, ψ is a direction field with isolated singularities.

Now, we are ready to formulate the Poincaré–Hopf index theorem. For a smooth section with isolated zeros it basically states that the number of zeros of section, when counted with multiplicity, is prescribed by the degree of the complex line bundle.

Theorem 7.1.10 (Poincaré–Hopf Index Theorem) *Let ∇ be a complex connection on L and $\psi \in \Gamma L$ be a direction field. Then*

$$i\int_M F^\nabla = 2\pi \sum_{p\in M} \mathrm{ind}_p\psi\,.$$

Remark 7.1.6 Since the singularities of ψ are isolated and M is compact, there are only finitely many singularities and the sum on the right-hand side is a sum over finitely many points.

Corollary 7.1.11 *The degree of a complex line bundle is an integer.*

Remark 7.1.7 Note that the right-hand side—and thus the degree of the bundle—does not depend on J or ∇ but only depends only on the oriented rank 2 bundle. As such it is purely topological. A change of orientation results in a change of sign.

The Poincaré–Hopf index theorem will follow as a corollary from Theorem 7.1.13, which is based on the following observation.

Lemma 7.1.12 *Let ∇ be a complex connection on L, $\psi \in \Gamma L$ nowhere-vanishing and $\nabla\psi = \eta\psi$, then*

$$F^\nabla = d\eta\,.$$

The form η will be called the logarithmic derivative of ψ.

Proof. If $\nabla\psi = \eta\psi$, then $F^\nabla\psi = d^\nabla\nabla\psi = d^\nabla(\eta\psi) = (d\eta)\psi - \eta\wedge\nabla\psi = (d\eta)\psi - \underbrace{\eta\wedge\eta}_{=0}\,\psi = (d\eta)\psi$. □

Theorem 7.1.13 *Let (L, ∇) be a complex line bundle with connection over a compact oriented surface M with boundary, let ψ be a direction field of L with isolated singularities $S \subset \mathring{M}$ and let η denote the logarithmic derivative of ψ. Then*

$$\int_{\partial M} \eta = \int_M F^\nabla + 2\pi i \sum_{p \in M} \mathrm{ind}_p \psi\,.$$

Proof. Since M is compact, ψ has only finitely many zeros, say $p_1, \ldots, p_n$. Choose disks $D_i \subset$ M and sections φ_i such that $p_i \in D_i$ and $p_j \notin D_j$ for $j \neq i$, and such that the restrictions $\varphi_i|_{D_i}$ have no zeros. Furthermore, define $M_0 := M \setminus (\bigcup_{i=1}^n \mathring{U}_i)$. Then, with $\nabla\psi = \eta\psi$ and $\nabla\varphi_i = \eta_i\varphi_i$,

$$\int_M F^\nabla = \int_{M_0} F^\nabla + \sum_{i=1}^n \int_{U_i} F^\nabla = \int_{M_0} d\eta + \sum_{i=1}^n \int_{U_i} d\eta_i = \int_{\partial M_0} \eta + \sum_{i=1}^n \int_{\partial U_i} \eta_i = \int_{\partial M} \eta + \sum_{i=1}^n \int_{\partial U_i} \eta_i - \eta\,.$$

since $\partial M_0 = \partial M - \sum_{i=1}^n \partial U_i$—the boundary of M_0 contains the boundary of $\bigcup_{i=1}^n U_i$ but with opposite induced orientation. Now, on U_i we can write $\psi = z_i\varphi_i$. Thus

$$\eta\psi = \nabla\psi = \nabla(z_i\varphi_i) = (dz_i)\varphi_i + z_i\nabla\varphi_i = (d\log z_i)z_i\varphi + z_i\eta_i\varphi = (d\log z_i + \eta_i)\psi$$

Hence $\eta_i - \eta = -d\log z_i$. Thus

$$\int_{\partial M} \eta = \int_M F^\nabla - \sum_{i=1}^n \int_{\partial U_i} \eta_i - \eta = \int_M F^\nabla + \sum_{i=1}^n \int_{\partial U_i} d\log z_i = \int_M F^\nabla + 2\pi \sum_{i=1}^n \mathrm{ind}_{p_i}\psi\,.$$

as was claimed. □

7.1.4 Euler Characteristic

A nice application of the Poincaré–Hopf index theorem is the Theorem of Gauss–Bonnet which relates the *Euler characteristic* $\chi(M) := \deg(TM)$ of a compact Riemannian surface M to its Gaussian curvature K given by

$$R^\nabla = -iK d\mathrm{vol}_M\,.$$

Here R^∇ denotes the *Riemannian curvature* i.e. the curvature of the Levi-Civita connection and $d\mathrm{vol}_M$ the Riemannian volume form.

Theorem 7.1.14 (Gauss–Bonnet Theorem) *Let M be a compact oriented Riemannian surface, then*

$$\int_M K\, d\mathrm{vol}_M = 2\pi\chi(M)$$

Remark 7.1.8 From this we can easily deduce the Hairy Ball Theorem: Every smooth vector field on a sphere must have a zero.

Another consequence of the transversality theorem is that every manifold has a *height function*, i.e. a function $h \in \mathscr{C}^\infty M$ such that dh vanishes only at isolated points.

One way manufacture a height function for an immersed surface $f\colon M \to \mathbb{R}^3$ with Gauss map N is the following: Choose some regular value of $a \in \mathbb{S}^2$ of N and define

$$h = \langle a, f\rangle\,.$$

Consider the corresponding gradient field $\operatorname{grad} h \in \Gamma(TM)$, i.e. $\langle \operatorname{grad} h, . \rangle = dh$. The critical points split into minima and maxima, where $\operatorname{ind}(\operatorname{grad} h) = 1$, and saddles, where $\operatorname{ind}(\operatorname{grad} h) = -1$.

Corollary 7.1.15 *The Euler characteristic can be expressed by*

$$\chi(M) = \#maxima - \#saddles + \#minima\,.$$

It is more common to define the Euler characteristic in terms of a cell decomposition or a triangulation.

A *k-cell* is a subset $C \subset M$ which is diffeomorhic to a convex polytope in $P \subset \mathbb{R}^k$. Set $\dim C := k$. If P is a simplex, i.e. the convex hull of $k+1$ points in general position, the we call C a *k-simplex*.

Definition 7.1.5 (Cell Decomposition) *A cell decomposition of a compact smooth manifold M is a finite collection of cells $\mathfrak{D} = \{C_\alpha\}_{\alpha \in A}$ such that $M = \bigcup_{\alpha \in A} C_\alpha$ and for all $\alpha, \beta \in A$ there exists $\gamma \in A$ such that $C_\alpha \bigcap C_\beta = C_\gamma$. A simplicial decomposition is a cell decomposition which consists of simplices. A simplicial decomposition of a surface is also called a triangulation.*

A triangulation consists of cells of dimension $k = 0, 1, 2$. In this context we call the 0-cells *vertices*, the 1-cells *edges* and the 2-cells *triangles*. Given a triangulation $\mathfrak{D}$ of a compact oriented surface, one can construct a vector field $X \in \Gamma(TM)$ with

$$\begin{cases} \text{a source at each face center,} \\ \text{a sink at each vertex,} \\ \text{a saddle at each edge center.} \end{cases}$$

This then yields

$$\chi(M) = \#vertices - \#edges + \#faces\,.$$

7.2 Classification of Complex Line Bundles

7.2.1 Complex Line Bundles over Surfaces

We have seen that $\deg\colon \mathscr{L} \to \mathbb{Z}$ is a group homomorphism. We will see now that it is actually an isomorphism. Thus the complex line bundles over a surface are classified by their degree.

To show surjectivity of deg we need to come up with a line bundle L for an arbitrarily given degree $d \in \mathbb{Z}$.

Given an integer $d \in \mathbb{Z}$ and a point p on a Riemann surface M we can define a complex line bundle $\operatorname{sky}(p, d)$ over M as follows: Let (U, z) be a complex chart at p such that $z(p) = 0$. Then $U_0 = U$ and $U_1 = M \setminus \{p\}$ and is an open cover of M. On $U_0 \bigcap U_1 = U \setminus \{p\}$ we define $g_{01}\colon U_0 \bigcap U_1 \to \mathbb{C}^\times \cong \mathrm{GL}(1, \mathbb{C})$ by

$$g_{01}(q) = (z(q))^d\,.$$

This defines a cocycle and thus a complex line bundle $\operatorname{sky}(p, d)$ over M (compare Example 3.1.3).

Definition 7.2.1 (Skyscraper Bundle) *The bundle $\operatorname{sky}(p, d)$ is called the skyscraper bundle of degree d at p.*

More concretely, $\operatorname{sky}(p, d)$ is the quotient of the disjoint union $(U_0 \times \mathbb{C}) \bigsqcup (U_1 \times \mathbb{C})$ with respect to the equivalence relation $\sim$ determined by

$$(0, q_1, v) \sim (1, q_2, w) \iff q_1 = q_2 \;\text{ and }\; v = g_{01}(q_2)w = (z(q_2))^d v\,.$$

In other words we glued the trivial bundles $\underline{\mathbb{C}}_{U_0}$ and $\underline{\mathbb{C}}_{U_1}$ over $U \setminus \{p\}$ by g_{01}.

In particular, we have an inclusion $\underline{\mathbb{C}}_{U_1} \hookrightarrow \mathrm{sky}(p,d)$. Under this inclusion each non-zero constant sections of $\underline{\mathbb{C}}_{U_1}$ is mapped to a nowhere-vanishing smooth section of $\mathrm{sky}(p,d)$ defined away from p:

$$\xi_q := \begin{cases} [(0,q,(z(q))^d)] & \text{for } q \in U_0\,, \\ [(1,q,1)] & \text{for } q \in U_1\,. \end{cases}$$

The section $\xi \in \Gamma\mathrm{sky}(p,d)$ is called the *famous section*. In particular, $\xi_q \neq 0$ for $q \neq p$. By construction, for small enough $\varepsilon > 0$,

$$\mathrm{ind}_p \xi = \tfrac{1}{2\pi i} \int_{|z|=\varepsilon} d\log(z^d) = d\,.$$

The Poincaré–Hopf index theorem applied to the famous section then yields the following.

Corollary 7.2.1 $\deg \mathrm{sky}(p,d) = d$.

In particular, since every oriented surface has a complex structure, we constructed for every degree d a complex line bundle of that degree.

Corollary 7.2.2 $\deg\colon \mathcal{L} \mapsto \mathbb{Z}$ *is surjective.*

It is left to show injectivity. Since deg is a homomorphism, it is enough to show that every line bundle of degree zero is trivial or, equivalently, there is a nowhere vanishing section. This needs some preparation.

Lemma 7.2.3 *Let M be a connected m-dimensioanal manifold, $U \overset{\circ}{\subset} M$ and $p \in M$. Then there is a diffeomorphism $f\colon M \to M$ such that $f(U) \subset U$ and $f(p) \in U$.*

Proof. Since M is connected, there is a smooth $\gamma\colon [0,1] \to M$ with $\gamma(0) = p$ and $\gamma(1) =: q \in U$. By the transversality theorem, we can assume that γ is embedded—if not, γ can be slightly perturbed to have transverse self-intersections. These can be resolved to get a collection of embedded curves. If we delete the loops we obtain an embedded curve from p to q. By the *"tubular neighborhood theorem"*, we then get a diffeomorphism $g\colon [-\varepsilon, 1+\varepsilon] \times D^{m-1} \to V \subset M$ such that $g(0,0) = p$ and $g(1,0) = q$.

Furthermore we can construct a diffeomorphism $\tilde{f}\colon [-\varepsilon,1+\varepsilon]\times D^{m-1} \to [-\varepsilon,1+\varepsilon]\times D^{m-1}$ such that $\tilde{f}(x) = x$ on some neighborhood of the boundary of $[-\varepsilon,1+\varepsilon]\times D^{m-1}$, but $\tilde{f}(0,0) = (1,0)$. Then $g \circ \tilde{f} \circ g^{-1}$ defined on V extends by the identity to a smooth map defined on M and does the trick. □

Proposition 7.2.4 *If M is a connected smooth m-dimensional manifold and $p_1,\ldots,p_n \in M$, then there is a $U \subset M$ diffeomorphic to D^m such that $p_1,\ldots,p_n \in U$.*

Proof. We will proof this by induction on $n \in \mathbb{N}$. For $n = 1$ the statement is clear. By the induction hypothesis $p_1,\ldots,p_{m-1}$ are already contained in some $\tilde{U}$. Use the above Lemma to find a diffeomorphism $f\colon M \to M$ with $f(\tilde{U}) \subset \tilde{U}$ and $f(p_n) \in \tilde{U}$. If we define $U := f^{-1}\tilde{U}$, then $p_1,\ldots,p_n \in U$. □

Proposition 7.2.5 $\deg\colon \mathcal{L} \to \mathbb{Z}$ *is injective.*

Proof. Let L be a complex line bundle of degree zero over an m-dimensional manifold. We need to show that there is a nowhere-vanishing section. By the transversality theorem we can choose a section $\psi \in \Gamma L$ with isolated zeros $p_1,\ldots,p_n$. Then, by the Poincaré–Hopf index theorem, we have

$$\sum_{i=1}^{n} \mathrm{ind}_{p_i}\psi = \deg L = 0\,.$$

By the proposition above, there is a U diffeomorphic to D^m such that $p_1, \ldots, p_n \in U$. In particular this can also be done such that this is true for some $U_1 \subset U$. Since $L|_U$ is trivial, there is a nowhere-vanishing section $\varphi \in \Gamma(L|_U)$. In particular, $\psi|_U = g\varphi$ for some $g \in C^\infty(U, \mathbb{C})$. Since all zeros are contained in $U_1 \subset U$ with winding numbers summing up to zero, we get

$$\int_\gamma d \log g = 0$$

for all closed curves γ in $U \setminus U_1$. So there exists $\alpha \in C^\infty(U \setminus U_1; \mathbb{C})$ such that $g = e^\alpha$.

Now choose some $\rho \in C^\infty M$ such that $\rho|_{U_1} = 0$ and $\rho|_{M\setminus U} = 1$, and define a smooth section $\tilde{\psi} \in \Gamma L$ as follows:

$$\tilde{\psi} = \begin{cases} e^{\rho\alpha}\varphi & \text{on } U \\ \psi & \text{on } M \setminus U \end{cases}$$

By construction, $\tilde{\psi}$ has no zeros. □

The previous results can be compactly summarized by the following theorem.

Theorem 7.2.6 *The map* $\deg\colon \mathscr{L} \to \mathbb{Z}$ *is an isomorphism of abelian groups.*

7.2.2 Complex Line Bundles with Connection

Now, let us look at the isomorphism classes of complex bundles with connection. Since any two isomorphic bundles with connection are in particular isomorphic as complex bundles, they must have the same degree. So we are free to just look at different connections ∇ and $\tilde{\nabla}$ on a single complex line bundle $L \to M$. Thus

$$\tilde{\nabla} = \nabla - 2\pi i\, \eta\,, \quad \eta \in \Omega^1(M; \mathbb{C})\,.$$

In this situation an isomorphism from (L, ∇) to $(L, \tilde{\nabla})$ just becomes a function $g\colon M \to \mathbb{C}^\times$ such that

$$\tilde{\nabla}(g\psi) = g\nabla\psi\,, \quad \text{for all } \psi \in \Gamma L\,.$$

Hence

$$g\nabla\psi = \tilde{\nabla}(g\psi) = \nabla(g\psi) - 2\pi i\,\eta(g\psi) = g(\nabla\psi + d\log g\,\psi - 2\pi i\,\eta\,\psi) \iff d\log g = 2\pi i\,\eta\,.$$

In particular, η is an *integral* 1-form, i.e.

$$\int_\gamma \eta \in \mathbb{Z}\,, \quad \text{for all closed curves } \gamma\colon \mathbb{S}^1 \to M\,.$$

Conversely, any such 1-form defines a $\mathbb{C}^\times$-valued function. Therefore we choose a point $o \in M$ and define the function $g\colon M \to \mathbb{C}^\times$ as follows:

$$g(p) = \exp\Big(2\pi i \int_o^p \eta\Big)\,.$$

Here the integral is taken along some path from o to p. Clearly, $dg = 2\pi i\eta g$. By the computations above we see then that g defines an isomorphism from (L, ∇) to $(L, \tilde{\nabla})$.

We still have to show that g is well-defined. To see this note that paths can be reversed and concatenated.

Let $\gamma\colon [0, 1] \to M$ be a path, then the *reversed path* $\gamma^{-1}\colon [0, 1] \to M$ is given by

$$\gamma^{-1}(t) := \gamma(1 - t)\,.$$

Clearly, we have

$$\int_{\gamma^{-1}} \xi = -\int_\gamma \xi\,, \quad \text{for all } \xi \in \Omega^1(M,\mathbb{C})\,.$$

Moreover, given two paths $\gamma\colon [0,1] \to M$ and $\tilde\gamma\colon [0,1] \to M$ such that $\gamma(1) = \tilde\gamma(0)$, we can build their *concatenation* $\tilde\gamma * \gamma\colon [0,1] \to M$ as follows:

$$(\tilde\gamma * \gamma)(t) := \begin{cases} \gamma(2t) & \text{for } t \in [0,\tfrac12]\,,\\ \tilde\gamma(2t-1) & \text{for } t \in [\tfrac12,1]\,.\end{cases}$$

Clearly,

$$\int_{\tilde\gamma*\gamma} \xi = \int_{\tilde\gamma} \xi + \int_\gamma \xi\,, \quad \text{for all } \xi \in \Omega^1(M,\mathbb{C})\,.$$

In general the concatenation is only piecewise smooth. We are talking here about curves up to reparametrization, i.e. one is free to precompose the curve by diffeomorphisms. If one is worried about that one can precompose γ by a smooth map $r\colon [a,b] \to [a,b]$ such that $\gamma(r(x)) = \gamma(b)$ for all $x \in (b-\varepsilon, b]$. Similarly, we can reparametrize $\tilde\gamma$ such that $\tilde\gamma(x)$ is constant for x close to c.

Now, if γ and $\tilde\gamma$ are two paths from o to p, then $\tilde\gamma * \gamma^{-1}$ is a closed path in M. Thus

$$\int_{\tilde\gamma} \eta - \int_\gamma \eta = \int_{\tilde\gamma*\gamma^{-1}} \eta \in \mathbb{Z}\,.$$

Hence

$$\exp\Big(2\pi i \int_{\tilde\gamma} \eta\Big) = \exp\Big(2\pi i \int_\gamma \eta\Big)\,,$$

i.e. g is well-defined. Thus we have shown the following theorem.

Theorem 7.2.7 *Two complex line bundles with connection (L,∇) and $(L,\tilde\nabla)$ are isomorphic if and only if*

$$\tilde\nabla = \nabla - 2\pi i\,\eta \;\; \textit{for some integral} \;\; \eta \in Z^1(M;\mathbb{C})\,.$$

In particular, isomorphic line bundles have the same curvature, which can be translated to a statement about parallel transport along boundaries by the following Gauss-Bonnet type theorem.

Theorem 7.2.8 *Let (L,∇) be a complex line bundle with connection. Then, if $\gamma\colon [0,1] \to M$ parametrizes the boundary of a region $U \subset M$, $\gamma(0) = p = \gamma(1)$ and $P^\nabla_\gamma\colon L_p \to L_p$ denotes the parallel transport along γ, then*

$$P^\nabla_\gamma = \exp\Big(-\int_U F^\nabla\Big)\,.$$

Proof. Without loss of generality we can assume that $U = M$ and γ parametrizes its boundary. Let $\varphi \in \Gamma\gamma^*E$ be a non-vanishing parallel field, so

$$P^\nabla_\gamma \varphi_0 = \varphi_1\,.$$

Now, let $\psi \in \Gamma E$ be a transverse section. We can assume that ψ has no zeros on the boundary. Then, on ∂M, we can write $\gamma^*\psi = g\varphi$ for some function $g\colon [0,1] \to \mathbb{C}$ and get

$$g(0)\varphi_0 = \psi_{\gamma(0)} = \psi_{\gamma(1)} = g(1)\varphi_1 = g(1)P^\nabla_\gamma\varphi_0\,.$$

Hence we have $P^\nabla_\gamma = g(0)/g(1)$. On the other hand, if we look at the logarithmic derivative η of ψ, we get

$$\gamma^*\eta\,\gamma^*\psi = \gamma^*(\eta\psi) = \gamma^*(\nabla\psi) = \gamma^*\nabla\gamma^*\psi = \gamma^*(\nabla(g\varphi)) = dg\,\varphi = (d\log g)\,g\varphi = (d\log g)\,\gamma^*\psi\,.$$

Hence we have

$$\gamma^*\eta = d\log g$$

and thus, by Theorem 7.1.13,

$$\int_0^1 d\log g = \int_0^1 \gamma^*\eta = \int_{\partial M}\eta = \int_M F^\nabla + 2\pi i \sum_{p\in M} \mathrm{ind}_p\psi \equiv \int_M F^\nabla \mod 2\pi\mathbb{Z}\,.$$

Hence $P_\gamma^\nabla = g(0)/g(1) = \exp(-\int_U F^\nabla)$. □

Note that the parallel transport along a closed path is given by a non-zero complex number, which is independent of where we started on the path.

Lemma 7.2.9 *Let (L,∇) and $(\tilde{L},\tilde{\nabla})$ be a complex vector bundles with connection and γ be a closed path, then*

$$P_\gamma^{\nabla^\otimes} = P_\gamma^\nabla P_\gamma^{\tilde{\nabla}}\,.$$

Proof. This easily follows from the fact that the tensor product of two parallel sections is parallel. □

Corollary 7.2.10 *Let (L,∇) be a complex vector bundle with connection and γ be a closed path, then*

$$P_\gamma^{\nabla^*} = (P_\gamma^\nabla)^{-1}\,.$$

From this we can easily deduce the following theorem.

Theorem 7.2.11 *Let (L,∇) and $(\tilde{L},\tilde{\nabla})$ be complex line bundles with connection over a compact oriented surface. Then*

$$(L,\nabla)\cong(\tilde{L},\tilde{\nabla}) \iff P_\gamma^\nabla = P_\gamma^{\tilde{\nabla}} \text{ for all } \gamma\colon \mathbb{S}^1\to M\,.$$

Proof. Clearly, if two complex line bundles are isomorphic, then the parallel transports coincide. Conversely, let (L,∇) and $(\tilde{L},\tilde{\nabla})$ be two complex line bundles with connection. Then $\mathrm{Hom}_+(L;\tilde{L})$ has an induced complex connection $\hat{\nabla}$. We want to show that $\mathrm{Hom}_+(L;\tilde{L})$ has a global parallel section. Since $\mathrm{Hom}_+(L;\tilde{L}) = \tilde{L}\otimes L^*$. The parallel transport along any closed path γ is trivial,

$$P_\gamma^{\hat{\nabla}} = P_\gamma^{\tilde{\nabla}} P_\gamma^{\nabla^*} = P_\gamma^{\tilde{\nabla}}(P_\gamma^\nabla)^{-1} = P_\gamma^\nabla(P_\gamma^\nabla)^{-1} = 1\,.$$

Moreover, clearly $P_{\gamma^{-1}}^{\hat{\nabla}} = (P_\gamma^{\hat{\nabla}})^{-1}$. Hence the parallel transport from a point $o\in M$ to another point $p\in M$ is independent of the path we take. Hence we can fix some non-zero complex linear map $\phi_o\colon L_o\to\tilde{L}_o$ and define as $\phi_p = P_\gamma^{\nabla^\otimes}\phi_o$ for some path from 0 to p. So we constructed a global non-zero parallel section ϕ. □

So, if we put this together, a complex line bundle with connection (L,∇) is—up to isomorphism—determined by its parallel transport along loops γ. If the loop is the boundary of a domain then the parallel transport is determined by the curvature of the bundle. By theorem 7.2.7 we see that this freedom is controlled by some $\omega\in Z^1(M;\mathbb{C})$.

Actually each $\omega\in\Omega^2M$ with $\int_M\omega\in 2\pi\mathbb{Z}$ can be realized as curvature of a hermintian line bundle.

Theorem 7.2.12 *For each $\omega\in\Omega^2(M;\mathbb{R})$ with $\int_M\omega\in\mathbb{Z}$ there is a hermitian line bundle L with connection ∇ such that $F^\nabla = -2\pi i\omega$. Moreover, all hermitian line bundles of curvature $-2\pi i\omega$ can be parametrized over $Z^1M/Z^1_{int}M$, where $Z^1_{int}M$ denotes the set of real-valued integral 1-forms.*

Proof. Let $\omega \in \Omega^2 M$ such that $d := \int_M \omega \in \mathbb{Z}$. Then there is a hermitian line bundle of degree d. Moreover, there exists a unitary connection $\tilde{\nabla}$. Any other unitary connection ∇ is of the form $\nabla = \tilde{\nabla} + i\alpha$ for $\alpha \in \Omega^1 M$. Its curvature is then given by

$$F^\nabla = F^{\tilde{\nabla}} + i d\alpha - \underbrace{\alpha \wedge \alpha}_{=0} = F^{\tilde{\nabla}} + i\, d\alpha\,.$$

So if we can solve $d\alpha = iF^{\tilde{\nabla}} - 2\pi\omega$ we are done. As we will see (Theorem 8.2.15) this equation has a solution whenever $\int_M F^{\tilde{\nabla}} - 2\pi\omega = 0$, which is true by construction. Moreover, once we have a unitary connection ∇ such that $F^\nabla = -2\pi i\omega$. Then $\nabla + 2\pi i\beta$ with β closed is also a solution and, by the above considerations adding an integral β leads to isomorphic bundle, so the space of isomorphism classes can be parametrized by the quotient $Z^1 M/Z^1_{int} M$. □

The previous theorem is—almost verbatim—true for hermitian line bundles over an arbitrary compact manifold M. It is the starting point for geometric quantization.

In contrast the obstruction here is not a single integer but an integer for any closed oriented surfaces inside M, i.e. $\omega \in \Omega^2 M$ must be integral,

$$\int_S \omega \in \mathbb{Z}\,, \quad \text{for all closed surfaces} \quad S \subset M\,.$$

Theorem 7.2.13 (Weil's Theorem) *For every integral $\omega \in \Omega^2 M$ there is a hermitian line bundle L with connection ∇ such that $F^\nabla = -2\pi i\,\omega$ and all hermitian line bundles of curvature $-2\pi i\,\omega$ can be parametrized over $Z^1 M/Z^1_{int} M$.*

Remark 7.2.1 The curvature F^∇ of a complex line bundle L is closed and thus determines an element of the second de Rahm cohomology: $\frac{i}{2\pi} F^\nabla \in H^2 M$ is called the *first Chern class* of L.

$\bar\partial$-Operators and Elliptic Problems | 8

A $\bar\partial$-operator is a certain first-order elliptic differential operator on a complex vector bundle over an almost complex manifold. Any holomorphic vector bundle comes with a $\bar\partial$-operator. In this chapter we describe the relation between holomorphic vector bundles and complex vector bundles over complex manifolds equipped with $\bar\partial$-operator. It turns out that an almost complex surface is always a complex curve and that any $\bar\partial$-operator on a complex vector bundle over an almost complex surface always uniquely turns it into a holomorphic vector bundle.

8.1 $\bar\partial$-Operators

8.1.1 The Canonical Bundle

Let $E \to M$ be a complex vector bundle over an almost complex $2m$-dimensional manifold (M, J). The complex-valued linear forms on M split into complex linear and complex antilinear forms $\mathrm{Hom}_+(TM, E)$ and $\mathrm{Hom}_-(TM; E)$. Similarly, the differential forms split into forms with a certain bidegree.

Definition 8.1.1 (Bigraded Alternating Forms) *For $p, q \in \mathbb{N}$, we set*

$$\mathrm{Alt}^{p,q}(TM; E) := \left\{\eta \in \mathrm{Alt}^{p+q}(M; E) \mid \eta(zX_1, \dots, zX_{p+q}) = z^p \bar z^q \eta(X_1, \dots, X_{p+q}),\ \ \forall z \in \mathbb{C}\right\}.$$

The space of differential form of bidegree (p, q) is given by

$$\Omega^{p,q}(M; E) := \Gamma\mathrm{Alt}^{p,q}(M; E), \quad \Omega^{p,q}M = \Omega^{p,q}(M; \mathbb{C}).$$

Definition 8.1.2 (Canonical Bundle K, Anticanonical Bundle $\bar K$)

$$K := \mathrm{Alt}^{m,0}(TM; \mathbb{C}), \quad \bar K := \mathrm{Alt}^{0,m}(TM; \mathbb{C}).$$

Moreover, we set $KE := \mathrm{Alt}^{m,0}(TM, E)$ and $\bar K E := \mathrm{Alt}^{0,m}(TM, E)$.

Theorem 8.1.1 (Type Argument) *Let E and $\tilde E$ be complex vector bundles over an almost complex manifold M and $\bullet \in \Gamma\mathrm{Mult}(E, \tilde E; \mathbb{C})$ be complex bilinear. Let*

$$\omega \in \Gamma(KE), \quad \tilde\omega \in \Gamma(K\tilde E), \quad \eta \in \Gamma(\bar K E), \quad \tilde\eta \in \Gamma(\bar K \tilde E).$$

Then

$$\omega \wedge\bullet\, \tilde\omega = 0, \quad \eta \wedge\bullet\, \tilde\eta = 0.$$

Furthermore, if $\bullet$ is non-degenerate, then

$$\omega \wedge\bullet\, \tilde\eta = 0 \iff \omega = 0 \text{ or } \tilde\eta = 0.$$

Proof. Let $\omega_1, \dots, \omega_m$ be a basis of $\mathrm{Hom}_+(T_pM; \mathbb{C})$. Then there are for some $\psi \in E_p$ and $\tilde\psi \in \tilde E_p$ such that

$$\omega_p = \omega_1 \wedge \dots \wedge \omega_m\, \psi, \quad \tilde\omega_p = \omega_1 \wedge \dots \wedge \omega_m\, \tilde\psi.$$

So we get

$$(\omega \wedge\bullet \tilde\omega)_p = (\omega_1 \wedge \cdots \wedge \omega_m \psi) \wedge \bullet(\omega_1 \wedge \cdots \wedge \omega_m \tilde\psi) = \underbrace{(\omega_1 \wedge \omega_1)}_{=0} \wedge \cdots \wedge \underbrace{(\omega_m \wedge \omega_m)}_{=0} (\psi \bullet \tilde\psi) = 0\,.$$

Similarly, we can write η and $\tilde\eta$ with respect to $\bar\omega_1 \ldots, \bar\omega_m$. The same calculation show that $\eta \wedge \tilde\eta = 0$.

If $X_1, \ldots, X_m$ denotes the dual basis of $\omega_1, \ldots, X_m$, then $\omega_i \wedge \bar\omega_i(X_j, JX_j) = 0$ and

$$\omega_i \wedge \bar\omega_i(X_i, JX_i) = \omega_i(X_i)\bar\omega_i(JX_i) - \omega_i(JX_i)\bar\omega_i(X_i) = -i\omega_i(X_i)\bar\omega_i(X_i) - i\omega_i(X_i)\bar\omega_i(X_i) = -2i$$

Hence, if

$$\tilde\eta_p = \bar\omega_1 \wedge \cdots \wedge \bar\omega_m \tilde\varphi\,.$$

for $\tilde\varphi \in E_p$, then $\omega \wedge\bullet \tilde\eta = \pm\omega_1 \wedge \bar\omega_1 \wedge \cdots \wedge \omega_m \wedge \bar\omega_m (\psi \bullet \tilde\varphi)$ and so

$$(\omega \wedge\bullet \tilde\eta)_p(X_1, JX_1, \ldots, X_m, JX_m) = \pm(-2i)^m \psi \bullet \tilde\varphi\,.$$

This yields the additional claim. □

Remark 8.1.1 In general, one has $\mathrm{Alt}^m(TM;\mathbb{C}) = \bigoplus_{p+q=m} \mathrm{Alt}^{p,q}(TM;\mathbb{C})$.

8.1.2 $\bar\partial$-Operators

Let M be an almost complex manifold. The trivial connection d on $\Gamma\underline{\mathbb{C}}_M = \mathscr{C}^\infty(M;\mathbb{C})$ splits into a complex linear and complex antilinear part: For $f \in \mathscr{C}^\infty(M;\mathbb{C})$,

$$df = d'f + d''f\,, \quad d'f = (df)' \in \Omega^{1,0}M\,, \quad d''f = (df)'' \in \Omega^{0,1}M\,.$$

More concretely,

$$d'f = \tfrac{1}{2}(df - i df \circ J)\,, \quad d''f = \tfrac{1}{2}(df + i df \circ J)\,.$$

Clearly,

$$f\colon M \to \mathbb{C} \text{ holomorphic} \iff df \in \Omega^{1,0}M \iff d''f = 0 \iff f \in \ker d''\,.$$

Definition 8.1.3 ($\bar\partial$-Operator) *Let E be a complex vector bundle over an almost complex manifold M. A $\bar\partial$-operator is a linear operator $\bar\partial\colon \Gamma E \to \Omega^{0,1}(M;E)$ such that, for all $\psi \in \Gamma E$ and $f \in \mathscr{C}^\infty(M;\mathbb{C})$,*

$$\bar\partial(f\psi) = (d''f)\psi + f\bar\partial\psi\,.$$

Remark 8.1.2 The operator d'' is a $\bar\partial$-operator and referred to as *the canonical $\bar\partial$-operator of the trivial bundle*.

Remark 8.1.3 Clearly, as for connections, the product rule assures that the operator is local in the sense that, if $X \in T_pM$ and $\psi, \tilde\psi \in \Gamma E$ are such that $\psi|_U = \tilde\psi|_U$ for some neighborhood U of p, then $\bar\partial_X\psi = \bar\partial_X\tilde\psi$.

Definition 8.1.4 (Holomorphic Section) *Let E be a complex vector bundle over an almost complex manifold M with $\bar\partial$-operator. A section $\psi \in \Gamma E$ is called holomorphic, if $\bar\partial\psi = 0$. The space of holomorphic sections is denoted by*

$$H^0E = \ker\bar\partial\,.$$

Moreover, we define $h^0E := \dim_{\mathbb{C}} H^0E$.

Theorem 8.1.2 *Let $\bar{\partial}$ and $\tilde{\bar{\partial}}$ be $\bar{\partial}$-operators on a complex vector bundle E over an almost complex manifold M. Then*

$$\tilde{\bar{\partial}} = \bar{\partial} + \xi, \quad \xi \in \Omega^{0,1}(M; \mathrm{End}_+ E).$$

Proof. Let $\xi := \tilde{\bar{\partial}} - \bar{\partial}$ and let $f \in \mathscr{C}^\infty(M;\mathbb{C})$, $X \in \Gamma M$ and $\psi \in \Gamma E$. Then clearly, $\xi_{fX} = \bar{f}\omega_X$. Moreover, by the product rule, we get

$$\xi(f\psi) = \tilde{\bar{\partial}}(f\psi) - \bar{\partial}(f\psi) = (\bar{\partial} f)\psi + f\tilde{\bar{\partial}}\psi - (\bar{\partial} f)\psi - f\bar{\partial}\psi = f\tilde{\bar{\partial}}\psi - f\bar{\partial}\psi = f\xi\psi\,.$$

Hence ξ is tensorial—complex antilinear in X and complex linear in ψ—i.e. $\xi \in \Omega^{0,1}(M;\mathrm{End}_+ E)$. □

Example 8.1.1 ($\bar{\partial}$-Operator from Complex Connection) Let (E, J) be a complex vector bundle over an almost complex manifold M. If ∇ is a complex connection, then

$$\nabla = \nabla' + \nabla'',$$

where

$$\nabla' \colon \Gamma E \to \Omega^{1,0}(M;E), \quad \nabla'' \colon \Gamma E \to \Omega^{0,1}(M;E).$$

Both, ∇' and ∇'' are given by the usual splitting $TM^* \otimes E = \mathrm{Hom}_+(TM;\mathbb{C}) \oplus \mathrm{Hom}_-(M;\mathbb{C}) \otimes E$. Concretely,

$$\nabla' = \tfrac{1}{2}(\nabla + J * \nabla), \quad \nabla'' = \tfrac{1}{2}(\nabla - J * \nabla).$$

Clearly, ∇'' is a $\bar{\partial}$-operator. In particular, since every complex vector bundle has a complex connection, there always exists a $\bar{\partial}$-operator.

Theorem 8.1.3 *Let $E \to M$ be a complex vector bundle over an almost complex manifold. Given a $\bar{\partial}$-operator $\bar{\partial}$ on E, then there is a complex connection ∇ on E such that $\nabla'' = \bar{\partial}$.*

Proof. Any choice of a complex connection $\tilde{\nabla}$ defines a $\bar{\partial}$-operator $\tilde{\bar{\partial}}$. Then $\bar{\partial} = \tilde{\bar{\partial}} + \xi$ with $\xi \in \Omega^{0,1}(M;\mathrm{End}_+ E)$ and $\nabla = \tilde{\nabla} + \xi$ is a complex connection with $\nabla'' = \bar{\partial}$. □

Theorem 8.1.4 *Let ∇ be a complex torsion-free connection on an almost complex manifold M, $X, Y \in \Gamma(TM)$. Then*

$$\nabla''_X Y = \tfrac{1}{2}\big([X,Y] + J[JX,Y]\big).$$

Proof. $\nabla''_X Y = \frac{1}{2}\big(\nabla_X Y + J\nabla_{JX} Y\big) = \frac{1}{2}\big(\nabla_Y X + [X,Y] + \underbrace{J\nabla_Y JX}_{=-\nabla_Y X} + J[JX,Y])\big) = \frac{1}{2}\big([X,Y] + J[JX,Y]\big)$. □

Example 8.1.2 (Canonical $\bar{\partial}$-Operator on the Canonical Bundle of an Almost Complex Surface) On an almost complex surface M, we can identify $\bar{K}K$ and $\mathrm{Alt}^2(TM;\mathbb{C})$ as follows:

$$\bar{K}K \ni \beta \longleftrightarrow 2\mathrm{Alt}_2(\beta) \in \mathrm{Alt}^2(TM;\mathbb{C}).$$

Moreover, $\Gamma K \subset \Omega^1(M;\mathbb{C})$ and $\Omega^{0,1}(M;K) = \Gamma(\bar{K}K)$. The exterior derivative hence restricts to a map

$$\bar{\partial} \colon \Gamma K \to \Omega^2(M;\mathbb{C}) = \Omega^{0,1}(M;K), \quad \bar{\partial}\omega = d\omega\,.$$

Actually, $\bar\partial$ is a $\bar\partial$-operator on K. To see this let $\omega \in \Gamma K$ and $f \in \mathscr{C}^\infty(M;\mathbb{C})$. Then

$$\bar\partial(f\omega) = d(f\omega) = df \wedge \omega + f d\omega = (\bar\partial f) \wedge \omega + f d\omega = (\bar\partial f)\omega + f\bar\partial\omega\,.$$

Here the third equality uses the type argument—Theorem 8.1.1.

Remark 8.1.4 In case M is a complex curve, K is a holomorphic line bundle. A complex coordinate z yields a holomorphic frame dz. Then $\omega = g dz$ for some $\mathbb{C}$-valued function g and

$$\bar\partial\omega = d(g\,dz) = dg \wedge dz = (g_z\,dz + g_{\bar z}\,d\bar z) \wedge dz = g_{\bar z}\,d\bar z \wedge dz\,,$$

where $g_{\bar z}$ is the usual Wirtinger derivative. Hence ω holomorphic if and only if f holomorphic, Thus the canonical $\bar\partial$-operator of K coincides with the $\bar\partial$-operator of the holomorphic line bundle.

Corollary 8.1.5 *Let $\eta \in \Gamma K$. Then:*

$$\eta \text{ holomorphic} \iff \eta \text{ closed}\,.$$

As for connections one can show that given vector bundles with $\bar\partial$-operators, then there are unique induced $\bar\partial$-operators on the dual and tensor bundles such that the product rule holds. We leave the proofs as an exercise.

Theorem 8.1.6 (Dual Holomorphic Structure) *Given a $\bar\partial$-operator $\bar\partial$ on a complex vector bundle E over an almost complex manifold, then there is a unique $\bar\partial$-operator $\hat{\bar\partial}$ on E^* such that, for all $\varphi \in \Gamma E^*$ and $\psi \in \Gamma E$,*

$$\bar\partial\langle\varphi|\psi\rangle = \langle\hat{\bar\partial}\varphi|\psi\rangle + \langle\varphi|\bar\partial\psi\rangle\,.$$

Here $\langle.|.\rangle$ denotes the natural complex bilinear pairing of E^* and E.

Theorem 8.1.7 (Tensor Holomorphic Structure) *Given $\bar\partial$-operators $\bar\partial$ and $\tilde{\bar\partial}$ on the complex vector bundles E and $\tilde E$ over an almost complex manifold, then there is a unique $\bar\partial$-operator $\hat{\bar\partial}$ on $E \otimes \tilde E$ such that, for all $\varphi \in \Gamma E, \psi \in \Gamma\tilde E$,*

$$\hat{\bar\partial}(\varphi \otimes \psi) = (\bar\partial\varphi) \otimes \psi + \varphi \otimes (\tilde{\bar\partial}\psi)\,.$$

As usual, unless explicitly stated differently, the $\bar\partial$-operator on dual bundles or tensor bundles are the induced ones. This understood, we do not explicitly distinguish the corresponding operators but just write $\bar\partial$.

Exercise 8.1.1 (Canonical $\bar\partial$-operator on K^*) Let K be the canonical bundle of an almost complex surface M. Show that $TM = K^*$. As the dual bundle of K the tangent bundle has a canonical $\bar\partial$-operator $\bar\partial$. Show that for any torsion-free connection ∇,

$$\bar\partial = \nabla''\,.$$

8.1.3 Holomorphic Vector Bundles

Definition 8.1.5 (Holomorphic Vector Bundle) *A holomorphic vector bundle of rank r is a complex vector bundle $\pi\colon E \longrightarrow M$ of complex rank r where E and M are complex manifolds, $\pi\colon E \longrightarrow M$ is holomorphic and at each point $p \in M$ there is a biholomorphic trivialization $\phi\colon E|_U \to U \times \mathbb{C}^r$, $U \ni p$.*

Every holomorphic vector bundle comes with a canonical $\bar\partial$-operator.

Example 8.1.3 ($\bar\partial$-Operator of a Holomorphic Vector Bundle) Let $E \longrightarrow M$ be a holomorphic vector bundle.

Then we define a $\bar\partial$-operator $\bar\partial$ as follows: If σ is a holomorphic frame and $\psi \in \Gamma E$, then $\psi = \sigma.\xi$ for some $\mathbb{C}^r$-valued function ξ. Define

$$\bar\partial(\psi) := \sigma.(\bar\partial\xi)\,.$$

Since any two holomorphic frames are related by a $\mathrm{GL}(\mathbb{C}^r)$-valued holomorphic map, $\bar\partial$ is well-defined. Moreover, one easily checks that $\bar\partial$, as defined above, satisfies

$$\bar\partial(f\psi) = (\bar\partial f)\,\psi + f(\bar\partial\psi)\,.$$

for all $\psi \in \Gamma E$ and $f \in \mathscr{C}^\infty(M;\mathbb{C})$.

Corollary 8.1.8 *If E be a holomorphic vector bundle, then*

$$H^0 E = \{\psi \in \Gamma E \mid \psi\colon M \to E \quad \text{holomorphic}\}\,.$$

Example 8.1.4 (Trivial Holomorphic Vector Bundle) Let M be a compact connected complex manifold, then $E = \underline{\mathbb{C}}_M$ is a holomorphic vector bundle. Its holomorphic sections are holomorphic functions. Thus, if M is compact and connected, $H^0 E$ consists only of constant functions and $h^0 E = 1$.

Example 8.1.5 (Tangent Bundle of a Complex Manifold) If M is a complex manifold, then TM is a holomorphic vector bundle—the transition maps between coordinate frames of holomorphic charts are holomorphic.

Theorem 8.1.9 *Let E be a complex vector bundle over a complex manifold M. If there is a vector bundle atlas $\{\phi_\alpha\colon E|_{U_\alpha} \to U_\alpha \times \mathbb{C}^r\}_\alpha$ such that the corresponding cocycle $g_{\beta\alpha}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(\mathbb{C}^r)$ consists only of holomorphic maps, then this turns E into a holomorphic vector bundle.*

Proof. The bundle charts—combined with charts of M—yield charts of E which by the holomorphicity of the coycle are holomorphically compatible. Holomorphicity of the projection follows from construction. □

Example 8.1.6 (Skyscraper Bundle) Let M be a Riemann surface, $p \in M$ and $d \in \mathbb{Z}$. Then $\mathrm{sky}(p, d)$ is a holomorphic line bundle—the transition map used to construct it is obviously holomorphic.

Example 8.1.7 (Tautological Line Bundle) The tautological line bundle $\mathrm{Taut}(\mathbb{CP}^n)$ over $\mathbb{CP}^n$ is a holomorphic line bundle. The open set $U_i = \{[z] \in \mathbb{CP}^n \mid z_i \neq 0\}$ form a cover of $\mathbb{CP}^n$. On each U_i we have a smooth non-vanishing section

$$\psi_{i,[z]} = (z_0, \ldots, z_n)/z_i\,.$$

This defines a smooth vector bundle atlas $\phi_i\colon \mathrm{Taut}(\mathbb{CP}^n)|_{U_i} \to U_i \times \mathbb{C}$, $\mathrm{Taut}(\mathbb{CP}^n)|_{U_i} \ni \psi_{i,[z]}w \mapsto ([z], w) \in U_i \times \mathbb{C}$. The corresponding cocycle is $g_{ji}\colon U_i \cap U_i \to \mathbb{C}^*$ given by $g_{ji}([z]) = z_j/z_i$.

Theorem 8.1.10 *A holomorphic line bundle E of negative degree over a compact Riemann surface has no non-zero holomorphic sections, $h^0 E = 0$*

Proof. Follows from Poincaré–Hopf and the fact that holomorphic sections have non-negative index. □

8.1.4 Connections on Holomorphic Vector Bundles

We want to characterize those complex bundles with connection which admit a complex structure so that they become a holomorphic vector bundle. We follow the discussion in [7].

Let $E \to M$ be a complex vector bundle over a complex manifold. Then each complex connection ∇ on E splits into a holomorphic ∇' and antiholomorphic structure ∇'', $\nabla = \nabla' + \nabla''$. Similarly, the exterior derivative splits: $d^\nabla = d^{\nabla'} + d^{\nabla''}$, where

$$d^{\nabla'} : \Omega^{p,q}(M;E) \to \Omega^{p+1,q}(M;E), \quad d^{\nabla''} : \Omega^{p,q}(M;E) \to \Omega^{p,q+1}(M;E).$$

and decomposition of the product rule with respect to the *bidegree* (p,q) yields, for $\psi \in \Gamma E$ and $\omega \colon \Omega^{p,q}M$,

$$d^{\nabla'}(\psi\omega) = \nabla'\psi \wedge \omega + \psi\, d'\omega\,, \quad d^{\nabla''}(\psi\omega) = \nabla''\psi \wedge \omega + \psi\, d''\omega\,.$$

The curvature then spits into 3-parts,

$$F^\nabla = (d^{\nabla'} + d^{\nabla''})(d^{\nabla'} + d^{\nabla''}) = (d^{\nabla'})^2 + (d^{\nabla'}d^{\nabla''} + d^{\nabla''}d^{\nabla'}) + (d^{\nabla''})^2\,,$$

where

$$(d^{\nabla'})^2 \in \Omega^{2,0}(M;\mathrm{End}_+E)\,, \quad (d^{\nabla''})^2 \in \Omega^{0,2}(M;\mathrm{End}_+E)\,,$$
$$d^{\nabla'}d^{\nabla''} + d^{\nabla''}d^{\nabla'} \in \Omega^{1,1}(M;\mathrm{End}_+E)\,.$$

Proposition 8.1.11 *Let E be a holomorphic vector bundle. If ∇ is a complex connection such that $\nabla'' = \bar{\partial}$, then*

$$(d^{\nabla''})^2 = 0\,.$$

Proof. Let σ be a holomorphic frame. Then $d^{\nabla''}\sigma = \nabla''\sigma = \bar{\partial}\sigma = 0$ and thus $(d^{\nabla''})^2$ vanishes on a frame. □

Conversely, one as the following proposition—a proof can be found in [7].

Proposition 8.1.12 *Let E be a complex vector bundle over a complex manifold. If ∇ is a connection on E such that $(d^{\nabla''})^2 = 0$, then there is a unique complex structure on E which turns E into a holomorphic vector bundle with holomorphic structure $\bar{\partial} = \nabla''$.*

Now, if we have just a complex vector bundle $E \to M$ over a complex manifold with holomorphic structure $\bar{\partial}$, then we may ask, whether there is a complex structure on E which turns it into a holomorphic vector bundle such that $H^0E = \ker\bar{\partial}$. The above theorem tells us that we can do so, whenever we have a complex connection ∇ with $\nabla'' = \bar{\partial}$ and $(d^{\nabla''})^2 = 0$. In general, it is not easy to come up with such a connection.

The problem boils down to the existence of local flat connections: Suppose we have given a some complex connection ∇ and a frame σ of E. Then we have

$$\nabla\sigma = \sigma.(\alpha + \beta)\,, \quad \alpha \in \Omega^{1,0}(M;\mathbb{C}^{r\times r}), \ \ \beta \in \Omega^{0,1}(M;\mathbb{C}^{r\times r})\,.$$

In particular, we have

$$F^\nabla\sigma = \sigma.\big(d(\alpha+\beta) + (\alpha+\beta)\wedge(\alpha+\beta)\big) = \sigma.\big(d\alpha + d\beta + \alpha\wedge\alpha + \beta\wedge\beta + [\alpha\wedge\beta]\big)\,,$$

where $[.,.]$ denotes the Lie bracket of $\mathbb{C}^{r\times r}$, i.e. $[\alpha\wedge\beta](X,Y) = [\alpha(X),\beta(Y)] - [\alpha(Y),\beta(X)]$. Thus we get

$$(F^\nabla)^{2,0}\sigma = \sigma.(d'\alpha + \alpha\wedge\alpha)\,, \quad (F^\nabla)^{0,2}\sigma = \sigma.(d''\beta + \beta\wedge\beta)\,,$$
$$(F^\nabla)^{1,1}\sigma = \sigma.\big(d''\alpha + d'\beta + [\alpha\wedge\beta]\big)\,.$$

So, in order to apply the proposition above, one needs, first, that M is complex and, second, to solve a nonlinear differential equation

$$d''\beta + \beta \wedge \beta = 0$$

If M is an almost complex surface, the situation becomes a bit better. Here $\mathrm{Alt}^{2,0}M$ and $\mathrm{Alt}^{0,2}M$ are trivial and we are left with just one component,

$$F^{\nabla}\sigma = \sigma.(d\alpha + d\beta + [\alpha \wedge \beta])\,,$$

Hence

$$F^{\nabla} = 0 \iff d\alpha + [\alpha \wedge \beta] = -d\beta \iff \bar{\partial}\alpha + [\alpha \wedge \beta] = -d\beta\,,$$

where we identified $\Gamma(\bar{K}K)$ and $\Omega^2(M;\mathbb{C})$ as usual. Prescribing $\bar{\partial}$ on E is the same as prescribing β. The existence of a flat ∇ with prescribed $bar\partial$-operator is thus related to the solvability of an inhomogeneous equation involving $\bar{\partial} + [.,\beta]$—which itself is a $\bar{\partial}$-operator on $\underline{\mathbb{C}}^{r\times r}_M$. This problem is an elliptic problem—a so called $\bar{\partial}$-problem. This problem is not asy to solve by elementary tools. In particular, since we cannot assume the existence of complex charts, one cannot even apply the local complex theory.

Soon we will see that—at least for line bundles—for each $\bar{\partial}$-operator there always exist locally a corresponding flat connection (Theorem 8.2.8) which then, by the fundamental theorem of flat bundles, assures the existence of local holomorphic frames (Theorem 8.2.9). As a consequence we obtain the following theorem.

Theorem 8.1.13 *A complex line bundle E with $\bar{\partial}$-operator over an almost complex surface can be turned uniquely into a holomorphic line bundle such that its induced $\bar{\partial}$-operator equals the given one.*

8.2 Elliptic Problems

Already twice we came to the point that got stuck with a first-order differential equation, which we called elliptic—once in Weil's theorem 7.2.13 as we tried to classify the hermitian line bundles with connection, and once when we tried to show that a complex line bundle with $\bar{\partial}$-operator over an almost complex surface is the same thing as a holomorphic line bundle. Now we are going to explain what elliptic means and when such an equation can be solved. Finally we look at certain problems.

8.2.1 Fundamental Theorem of Elliptic Theory

Since we only need to solve linear first-order equations we restrict our attention to first-order operators, which keeps the setup quite simple and transparent.

Definition 8.2.1 (Linear first-order differential operator) *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Let E and $\tilde{E}$ be $\mathbb{K}$-vector bundles over a smooth manifold M. Then a $\mathbb{K}$-linear map $D\colon \Gamma E \to \Gamma\tilde{E}$ is called linear first-order differential operator, if there is a $A \in \Gamma\mathrm{Hom}(TM^*;\mathrm{Hom}(E;\tilde{E}))$ such that, for all $f \in \mathcal{C}^\infty M$ and $\psi \in \Gamma E$,*

$$D(f\psi) = A_{df}\psi + fD\psi\,.$$

Example 8.2.1 ($\bar{\partial}$-Operators) A $\bar{\partial}$-operator on a vector bundle $E \to M$ is a linear first-order differential operator—the field $A \in \Gamma\mathrm{Hom}(TM^*;\mathrm{Hom}(E;E))$ is the projection to the complex antilinear part,

$$A_{df} = \bar{\partial}f = d''f\,.$$

The field A is actually uniquely determined by D. To see this, let $\varphi_1, \dots, \varphi_r$ be local frames of E and let $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}$ be a coordinate frame of M all defined on a common neighborhood U of p. Then, for $\psi = \sum_i y_i \varphi_i$, we get

$$D\psi = \sum_i \big(A_{dy_i}\varphi_i + y_i D\varphi_i\big) = \sum_i \big(\sum_\ell \tfrac{\partial y_i}{\partial x_\ell} A_{dx_\ell}\varphi_i + y_i D\varphi_i\big)\,.$$

For given j, k there are functions $y_{jk,i}$ such that $y_{jk,i}(p) = 0$ and $\frac{\partial y_{jk,i}}{\partial x_\ell} = \delta_{ik}\delta_{j\ell}$. Thus

$$D\Big(\sum_i y_{jk,i}\varphi_i\Big)\Big|_p = \sum_i \sum_\ell \tfrac{\partial y_{jk,i}}{\partial x_\ell} A_{dx_\ell}\varphi_i\Big|_p = A_{dx_j}\varphi_k\Big|_p\,.$$

Hence A is uniquely determined.

Definition 8.2.2 (Symbol of First-Order Differential Operator) *The uniquely determined field A in the definition of a linear first-order differential operator D is called the symbol of D.*

Let E be a real vector bundle over a compact oriented m-dimensional manifold M. Then there is a natural non-degenerate pairing between $\Omega^m(M; E^*)$ and ΓE: For $\omega \in \Omega^m(M; E^*)$ and $\psi \in \Gamma E$, let

$$\langle\!\langle \omega | \psi \rangle\!\rangle = \int_M \langle \omega | \psi \rangle\,.$$

Definition 8.2.3 (Formal Adjoint) *Given non-degenerate pairings between the possibly infinite-dimensional vector spaces $\tilde{V}$ and V and between $\tilde{W}$ and W. Then, if $B\colon V \to W$ is a linear map, a linear map $\tilde{B}\colon \tilde{W} \to \tilde{V}$ is called an adjoint of B if, for all $v \in V$ and $\tilde{w} \in \tilde{W}$,*

$$\langle\!\langle \tilde{w} | Bv \rangle\!\rangle = \langle\!\langle \tilde{B}\tilde{w} | v \rangle\!\rangle\,.$$

If $\tilde{B}$ exists, it is unique and we write $\tilde{B} = B^*$.

The following theorem can be proven using basically integration by parts.

Theorem 8.2.1 *If E and $\tilde{E}$ are vector bundles over a compact oriented manifold M and $D\colon \Gamma E \to \Gamma\tilde{E}$ is a linear first-order differential operator, then D has an adjoint*

$$D^*\colon \Omega^m(M; \tilde{E}^*) \to \Omega^m(M; E^*)$$

and D^* is again a linear first-order differential operator.

Though in practice D^* can usually be written down explicitly.

Definition 8.2.4 (Elliptic Operator) *Let $D\colon \Gamma E \to \Gamma\tilde{E}$ be a linear first-order differential operator with symbol A.*

$$D \text{ elliptic} \quad :\Longleftrightarrow \quad \forall 0 \neq \omega \in TM^*\colon A_\omega\colon E_{\pi(\omega)} \to \tilde{E}_{\pi(\omega)} \text{ isomorphism of vector spaces.}$$

Remark 8.2.1 Note that, as M is a real manifold, ω here always refers to real-valued covectors.

Example 8.2.2 ($\bar{\partial}$-Operators) The symbol of a $\bar{\partial}$-operator is given by

$$A_\omega = \omega'' = \tfrac{1}{2}(\omega + i\,\omega \circ J), \qquad \omega \in TM^*\,.$$

Clearly, $A_\omega = 0$ only if $\omega = 0$. Hence $\bar{\partial}$ is elliptic.

Now we are ready to formulate the elliptic theorem in a form adapted to our needs.

Theorem 8.2.2 (Fundamental Theorem of Elliptic Theory) *Let E and $\tilde{E}$ be vector bundles over a compact oriented manifold M and let $D\colon \Gamma E \to \Gamma\tilde{E}$ be an elliptic linear first-order differential operator. Then:*

(i) $\dim\ker D < \infty$.
(ii) $D^*\colon \Omega^m(M;\tilde{E}^*) \to \Omega^m(M;E^*)$ *is elliptic and* $\operatorname{im} D = (\ker D^*)^\perp$. *In particular,* $(\operatorname{im} D)^\perp = \ker D^*$.

This theorem is well-known to experts. Though proofs are somehow rare to find—especially for the specific version given above. One quite excellent reference is [8]. Another reference is [5]—there the theorem is contained as an exercise at the end of the book.

Corollary 8.2.3 *For a holomorphic vector bundle E over a compact complex manifold, the space of holomorphic sections H^0E is finite-dimensional.*

Definition 8.2.5 (Genus of a Compact Riemann Surface) *The genus g of a compact Riemann surface is defined as the complex dimension of the space of holomorphic sections of the canonical bundle K,*

$$g := \dim_{\mathbb{C}} H^0 K\,.$$

8.2.2 $\bar\partial$-Problems

Let M denote a compact almost complex surface. Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Two $\mathbb{K}$-vector bundles $E, \tilde{E} \to M$ are called *paired vector bundle* if there is a non-degenerate tensorial pairing $\langle .|.\rangle_{\mathbb{K}}\colon \Gamma\tilde{E}\times\Gamma E \to \Omega^2(M;\mathbb{K})$. In particular,

$$\langle\!\langle .,.\rangle\!\rangle_{\mathbb{K}}\colon \Gamma E\times\Gamma\tilde{E} \to \mathbb{K}, \quad \langle\!\langle\tilde\psi|\psi\rangle\!\rangle_{\mathbb{K}} = \int_M \langle\tilde\psi|\psi\rangle_{\mathbb{K}}\,.$$

is a non-degenerate as well. Usually we drop the subindex.

Remark 8.2.2 If $E, \tilde{E} \to M$ are paired bundles, there is a natural isomorphism $\tilde{E} \cong \Lambda^2 TM^*\otimes E^*$.

Theorem 8.2.4 *Let $E \to M$ be a complex vector bundle over an almost complex surface. Then there is a natural isomorphism $\Phi\colon \bar{K}KE \to \Lambda^2 TM^*\otimes E$ given by*

$$\bar{K}KE \ni \eta \mapsto \eta\circ\wedge \in \Lambda^2 TM^*\otimes E\,.$$

Moreover, if ∇ is a complex connection on E and $\nabla'' = \bar\partial$, then $\Phi\circ\bar\partial = d^\nabla$.

Proof. To see that Φ is an isomorphism we explicitly write down the inverse isomorphism. Therefore, given $\sigma \in \Lambda^2(M;E)$, we define $\hat\sigma \in \bar{K}(KE)$ as follows: For $X, Y \in T_pM$,

$$\hat\sigma_X(Y) := \tfrac{1}{2}\big(\sigma(X,Y) + J\sigma(JX,Y)\big)\,.$$

We leave it as an exercise to verify that $\hat\sigma$ has the correct types and that $\Phi(\hat\sigma) = \sigma$. That $\Phi\circ\bar\partial = d^\nabla$ follows from local considerations by an easy type argument: For $\psi \in \Gamma E$ and $\omega \in \Gamma K$, we have

$$d^\nabla\eta = d^\nabla(\psi\omega) = \nabla\psi\wedge\omega + \psi d\omega = \nabla''\psi\wedge\omega + \psi d\omega = \Phi((\bar\partial\psi)\omega) + \Phi(\psi\bar\partial\omega) = \Phi(\bar\partial(\psi\omega)) = \Phi(\bar\partial\eta)\,.$$

Since every $\eta \in \Gamma(KE)$ can be locally written as such a product we are done. □

Let $E \to M$ be a vector bundle with $\bar\partial$-operator over an almost complex surface. Then, under the isomorphism above, E^* is naturally paired with $\bar{K}KE$. Similarly, $\bar{K}E^*$ is naturally paired with KE. We have the following theorem.

Theorem 8.2.5 *For $\varphi \in \Gamma E^*$ and $\eta \in \Gamma KE$*

$$\langle\langle \varphi | \bar\partial \eta \rangle\rangle = -\langle\langle \bar\partial \varphi | \eta \rangle\rangle .$$

Proof. Let ∇ be a complex connection on E such that $\nabla'' = \bar\partial$. Then we have

$$\langle\langle \varphi | \bar\partial \eta \rangle\rangle = \int_M \langle \varphi | \Phi(\bar\partial \eta)\rangle = \int_M \langle \varphi | d^\nabla \eta\rangle = \int_M d\langle \varphi | \eta\rangle - \int_M \langle \nabla\varphi \wedge \eta\rangle = -\int_M \langle \nabla\varphi \wedge \eta\rangle = -\int_M \langle \bar\partial\varphi \wedge \eta\rangle = -\langle\langle \bar\partial \varphi | \eta \rangle\rangle ,$$

where again a type argument was used. □

As an immediate consequence of part (ii) of the fundamental theorem we obtain the following theorems. The details of the proofs are left as an exercise.

Theorem 8.2.6 *Let $\omega \in \Omega^2(M; E)$, then*

$$\exists \eta \in \Gamma KE : \omega = d^\nabla \eta \iff \forall \varphi \in H^0 E^* : \langle\langle \varphi | \omega \rangle\rangle = 0 .$$

Theorem 8.2.7 *Let $\eta \in \Gamma \bar K E$, then*

$$\exists \psi \in \Gamma E : \eta = \bar\partial \psi \iff \forall \varphi \in H^0 K E^* : \langle\langle \varphi | \eta \rangle\rangle = 0 .$$

Theorem 8.2.8 *Let L be a complex line bundle with $\bar\partial$-operator. Then, locally, there is a flat complex connection ∇ such that $\nabla'' = \bar\partial$.*

Proof. Let $\tilde\nabla$ be some complex connection such that $\tilde\nabla'' = \bar\partial$. If $\eta \in \Gamma K$ and $\nabla = \tilde\nabla - \eta$ then also ∇''. The curvature of ∇ is given by

$$F^\nabla = F^{\tilde\nabla} - d\eta .$$

Thus ∇ is flat if and only if $d\eta = F^{\tilde\nabla}$, which is locally solvable by Theorem 8.2.6. The details are left as exercise. □

Theorem 8.2.9 *Let $E \to M$ be a complex line bundle with $\bar\partial$-operator over a compact almost complex surface. Then for each $p \in M$ there exists a local holomorphic frame at p.*

Proof. Locally there is a flat connection ∇ such that $\nabla'' = \bar\partial$. By the fundamental theorem of flat bundles, there is a local parallel frame $\psi \in \Gamma L$. In particular, $\bar\partial\psi = \nabla''\psi = 0$. □

In particular, this proves that almost complex surface are complex curves—without using the Newlander–Nirenberg Theorem.

Theorem 8.2.10 *Every almost complex surface is complex.*

Proof. The canonical bundle has a local holomorphic frames. In particular, at each point there is locally a closed nowhere-vanishing $\eta \in \Gamma K$. A complex chart is defined by $z = \int \eta$. □

Furthermore this ensures the local solvability of $\bar\partial$-problems by gluing a complex chart neighborhood to $\mathbb{CP}^1$. The details are left as exercise.

Theorem 8.2.11 *For every $\eta \in \Gamma \bar K E$, the $\bar\partial$-problem $\bar\partial \psi = \eta$ has always local solutions.*

8.2.3 Hodge-Decomposition of 1-Forms

Definition 8.2.6 (Hodge–$*$–Operator) *For $\omega \in \Omega^1(M;E)$,*

$$*\omega = -\omega \circ J\,.$$

Note that the Hodge-star operator is a natural almost complex structure on the—a priori real— bundle TM^*. As such TM^* becomes a complex line bundle which is naturally isomorphic to the canonical bundle K via the isomorphism

$$TM^* \ni \alpha = \operatorname{Re}\eta \longleftrightarrow \eta = \alpha + i * \alpha \in K\,.$$

One easily checks that the isomorphism is well-defined and complex linear.

Definition 8.2.7 (Harmonic 1-Forms) *A harmonic 1-form is an element of the following space*

$$\operatorname{Harm} M := \{\alpha \in \Omega^1 M \mid d\alpha = 0 = d * \alpha\} = \ker d \cap \ker d *\,.$$

Corollary 8.2.12 *Under the above identification, we have*

$$\operatorname{Harm} M \cong H^0 K\,.$$

Remark 8.2.3 As such the harmonic forms form a g-dimensional complex subspace. In particular, there is an underlying $2g$-dimensional real space.

Similarly, we can identify TM^* with $\bar{K}$. Thus, if $\eta \in \Gamma\bar{K}$ and $\xi \in \Gamma K$, then we there are $\alpha, \beta \in \Omega^1 M$ such that

$$\eta = \alpha - i * \alpha, \quad \xi = \beta + i * \beta\,.$$

If we express now the pairing of $\bar{K}$ and K in terms of α and β we get

$$\int_M \eta \wedge \xi = \int_M (\alpha - i * \alpha) \wedge (\beta + i * \beta) = \int_M (\alpha \wedge \beta + *\alpha \wedge *\beta) + i \int_M (\alpha \wedge *\beta - *\alpha \wedge \beta) = 2 \int_M \alpha \wedge \beta + 2i \int_M \alpha \wedge *\beta\,.$$

Its imaginary part yields an inner product $\langle\!\langle .,. \rangle\!\rangle \colon \Omega^1 M \times \Omega^1 M \to \mathbb{R}$,

$$\langle\!\langle \alpha, \beta \rangle\!\rangle = \int_M \alpha \wedge *\beta\,.$$

Definition 8.2.8 *Let $\alpha \in \Omega^1 M$. Then*

$$\alpha \text{ coclosed } :\Longleftrightarrow \; d * \alpha = 0 \; \Longleftrightarrow \; \alpha \in \ker d*\,,$$
$$\alpha \text{ coexact } :\Longleftrightarrow \; \exists\, f \in \mathscr{C}^\infty M : \alpha = *df \; \Longleftrightarrow \; \alpha \in \operatorname{im} * d\,.$$

Remark 8.2.4 Note that coexact implies coclosed.

Using Stokes theorem we get for $f \in \mathscr{C}^\infty M$ and $\beta \in \Omega^1 M$,

$$\langle\!\langle df, \beta \rangle\!\rangle = -\int_M f\, d * \beta\,, \quad \langle\!\langle *df, \beta \rangle\!\rangle = -\int_M f\, d\beta\,.$$

Thus $\ker d \perp \operatorname{im} * d$ and $\ker *d \perp \operatorname{im} d$. We can say even more.

Theorem 8.2.13 (Hodge-Decomposition)

$$\Omega^1 M = \operatorname{im} d \;\; \oplus_\perp \;\; \operatorname{im} * d \;\; \oplus_\perp \;\; \operatorname{Harm} M \,.$$

Proof. Let $\alpha \in \Omega^1 M$. We have to show that we can uniquely decompose α into an orthogonal sum of an exact and coexact and a harmonic form. Therefore we choose a orhonormal basis $h_1, \ldots, h_{2g} \in \operatorname{Harm} M$ and define $\tilde\alpha = \alpha - h$, where $h := \sum_i \langle\langle \alpha, h_i \rangle\rangle h_i$. Then $\tilde\alpha \perp \operatorname{Harm} M$. Let $\eta := \tilde\alpha - i * \tilde\alpha \in \Gamma \bar K$ and $\xi = \beta + i * \beta \in H^0 K$. Then

$$\int_M \eta \wedge \xi = -2\langle\langle \tilde\alpha, *\beta \rangle\rangle + 2i \langle\langle \tilde\alpha, \beta \rangle\rangle = 0 \,.$$

By Theorem 8.2.7 we hence get that $\eta = 2\bar\partial f$ for some function $f = u + iv \in \mathscr{C}^\infty(M;\mathbb{C})$ and

$$\tilde\alpha = 2 \operatorname{Re}(\bar\partial f) = \operatorname{Re}\big((d - i * d)(u + iv)\big) = du + *dv \,.$$

Thus we have $\alpha = \tilde\alpha + h = du + *dv + h$, as desired. □

Corollary 8.2.14

$$\operatorname{Harm} M \cong H^1 M \,.$$

In combination with Theorem 8.2.6 we get almost immediately the following important theorem.

Theorem 8.2.15 *Let M be a compact almost complex surface and $\omega \in \Omega^2 M$. Then*

$$\exists\, f \in \mathscr{C}^\infty M : \;\; d * df = \omega \quad \Longleftrightarrow \quad \int_M \omega = 0 \,.$$

The previous theorem basically tells us when a *Poisson problem* can be solved: Given a metric on M, we can define the *Laplace operator* $\Delta \colon \mathscr{C}^\infty M \to \mathscr{C}^\infty M$ through

$$(\Delta f)\, d\mathrm{vol}_M := d * df \,,$$

where $d\mathrm{vol}_M$ denotes the Riemannian volume form. Poisson's equation is then $\Delta f = g$.

Corollary 8.2.16 *Let M be a compact oriented Riemannian surface and $g \in \mathscr{C}^\infty M$. Then*

$$\exists\, f \in \mathscr{C}^\infty M : \;\; \Delta f = g \quad \Longleftrightarrow \quad \int_M g = 0 \,.$$

Proof of theorem. Clearly, if $\omega = d * df$, then $\int_M \omega = 0$ by Stokes' theorem. Conversely, let $\int_M \omega = 0$. Then, since $\Omega^2 M \subset \Omega^2(M;\mathbb{C})$, Theorem 8.2.6 yields an $\eta \in \Gamma K$ such that $d\eta = \omega$. Since $\eta \in \Gamma K$, there is $\alpha \in \Omega^1 M$ such that $\eta = \alpha + i * \alpha$. Hodge-decomposition yields $\alpha = *df + dg + h$, where $f, g \in \mathscr{C}^\infty M$ and $h \in \operatorname{Harm} M$. Taking the real part of $d\eta = \omega$ yields $d\alpha = \omega$. Thus $\omega = d\alpha = d * df$. □

Holomorphic Line Bundles 9

In this section we classify the holomorphic line bundles over a compact Riemann surface. Furthermore we prove the Mittag–Leffler Theorem, which yields conditions on the existence or non-existence of meromorphic sections in terms of their their residues. In particular, we obtain existence theorems for meromorphic differentials—so called Abelian differentials. This at hand we are able to prove the Riemann–Roch theorem which is then applied to certain problems.

9.1 Classification of Holomorphic Line Bundles

Throughout let M denote a compact Riemann surface of genus g.

9.1.1 First Homology of Riemann Surfaces

We have seen that the first cohomology of M is isomorphic to Harm M and thus forms a $2g$-dimensional real vector space.

Definition 9.1.1 (First Homology Vector Space) *The first homology vector space is given by*

$$H_1(M;\mathbb{R}) := (H^1 M)^*$$

Remark 9.1.1 In general, on every m-dimensional compact manifold all cohomolgy groups are finite-dimensional. The above definition can be modified in the obvious way to define the k-th homology vector space. Though, for our purposes, it is enough to define the first homology. This has the advantage that a 1-cycle in M (in the sense of algebraic topology) can simply be considered as a map from a compact oriented 1-dimensional manifold—a finite disjoint union of oriented circles—into M. This allows us to avoid the definition of the boundary operator on chains. This does not work for k-cycles in general—for $k > 2$ a k-cycle is the image of a manifold but a so-called pseudomanifold.

Definition 9.1.2 (1-Cycle) *A* 1*-cycle is a smooth map* $\gamma\colon C \to M$*, where* C *is* 1*-dimensional, compact and oriented.*

Definition 9.1.3 (Homologuous Cycles) *Two* 1*-cycles* γ *and* $\tilde{\gamma}$ *are called homologuous, if*

$$\int_\gamma \alpha = \int_{\tilde{\gamma}} \alpha\,, \quad \textit{for all} \quad \alpha \in Z^1 M\,.$$

A 1*-cycle is called null-homologuous, if it is homologuous to a constant cycle.*

Remark 9.1.2 Two 1-cycles $\gamma_0, \gamma_1\colon C \to M$ are homotopic, if there exists a smooth map $H\colon [0,1]\times C \to M$ such that $H(0,.) = \gamma_0$ and $H(1,.) = \gamma_1$—a *homotopy*. By Stokes' theorem homotopic cycles are homologuous.

To be homologuous defines an equivalence relation on the space of 1-cycles in M. We write $\gamma \sim \tilde{\gamma}$. The set of homology classes $[\gamma]$ forms an abelian group: If $\gamma\colon C \to M$ and $\tilde{\gamma}\colon \tilde{C} \to M$ are 1-cycles, then $\gamma + \tilde{\gamma}\colon C \sqcup \tilde{C} \to M$ is given by

$$(\gamma + \tilde{\gamma})|_C = \gamma, \quad (\gamma + \tilde{\gamma})|_{\tilde{C}} = \tilde{\gamma}\,.$$

The inverse of $\gamma\colon C \to M$, denoted by $-\gamma$, is simply obtained by switching the orientation of C.

Definition 9.1.4 (First Homology) *The first homology group of M*

$$H_1(M;\mathbb{Z}) := \{[\gamma] \mid \gamma \text{ 1-cycle in } M\}.$$

Theorem 9.1.1 *Every homology class $[\gamma] \in H_1(M;\mathbb{Z})$ has a representative, which consists of finitely many non-intersecting embedded curves.*

Proof. By the transversality theorem, we obtain a representative which consists of finitely many curves that intersect transversely. The transverse intersections can be eliminated by cutting and gluing. □

Each homology class $[\gamma] \in H_1(M;\mathbb{Z})$ can be considered as an element of $H_1(M;\mathbb{R})$: For $[\alpha] \in H^1M$,

$$\langle\!\langle [\gamma] | [\alpha] \rangle\!\rangle = \int_\gamma \alpha.$$

By definition, the inclusion $H_1(M;\mathbb{Z}) \hookrightarrow H_1(M;\mathbb{R})$ is injective. Moreover, we have the following theorem.

Theorem 9.1.2 *Let $\alpha \in \Omega^1 M$. Then*

$$\alpha \text{ exact} \iff \int_\gamma \alpha = 0 \text{ for all closed curves } \gamma\colon S^1 \to M.$$

Proof. One direction follows immediately from Stokes' theorem. For the other one, suppose that the integral of α vanishes along any closed path. Assume without loss of generality that M is connected. Fix $p_0 \in M$ and define $f(p) = \int_{p_0}^p \alpha$, where the integral is taken along an arbitrary path from p_0 to p. Since any two paths from p_0 to p differ by a closed curve, f is well-defined. Clearly, $df = \alpha$. □

Corollary 9.1.3

$$H_1(M;\mathbb{Z})^\perp = \{0\}.$$

Since $(H^1M)^* = \operatorname{Harm} M$, we immediately get the following theorem.

Theorem 9.1.4 (Poincaré Dual) *If $[\gamma] \in H_1(M;Z)$, then there is a unique $\alpha_\gamma \in \operatorname{Harm} M$ such that*

$$\int_\gamma \beta = \int_M \alpha_\gamma \wedge \beta \quad \text{for all } \beta \in Z^1M.$$

Though it is worth to look at an explicit construction: Let $\gamma\colon C \to M$ be a embedding, C oriented. Then, by the tubular neighborhood theorem, there is an embedding $f\colon [-\varepsilon,\varepsilon] \times C \to M$ such that $f(0,.) = \gamma$. Without loss of generality we can assume that f is also oriented. On the image of f there is a real-valued function h which takes values in $[-\varepsilon,\varepsilon]$. Now, let $g\colon \mathbb{R} \to \mathbb{R}$ smooth such that $g(x) = 0$ for $x < -\varepsilon/2$ and $g(x) = 1$ for $x > \varepsilon/2$. Then $d(g \circ h)$ has support in the image of f and hence extends to a form $\tilde{\alpha}_\gamma \in \Omega^1 M$. Clearly, $\tilde{\alpha}_\gamma$ is closed. For $\beta \in Z^1M$, we get

$$\begin{aligned}\int_M \tilde{\alpha}_\gamma \wedge \beta &= \int_{f([-\varepsilon,\varepsilon]\times C)} \tilde{\alpha}_\gamma \wedge \beta = \int_{f([-\varepsilon,\varepsilon]\times C)} d(g\circ h) \wedge \beta = \int_{f([-\varepsilon,\varepsilon]\times C)} d((g\circ h)\beta) \\ &= \int_{\partial f([-\varepsilon,\varepsilon]\times C)} (g\circ h)\beta = \int_{\partial f(\{\varepsilon\}\times C)} \beta = \int_{f(\{0\}\times C)} \beta = \int_\gamma \beta.\end{aligned}$$

Hence $\tilde{\alpha}_\gamma = \alpha_\gamma + d\tilde{g}$.

Note that this yields a skew bilinear product on 1-cycles: If $[\gamma], [\tilde{\gamma}] \in H_1(M;\mathbb{Z})$, then

$$[\gamma] \bullet [\tilde{\gamma}] := \langle\!\langle \alpha_\gamma | \alpha_{\tilde{\gamma}} \rangle\!\rangle .$$

Given two transversely embedded 1-cycles γ and $\tilde{\gamma}$ and $p \in \operatorname{im}\gamma \cap \operatorname{im}\tilde{\gamma}$. Then we set

$$s_p(\gamma, \tilde{\gamma}) = \operatorname{sign} \sigma(\gamma'(t), \tilde{\gamma}'(\tilde{t})) ,$$

where $\gamma(t) = p = \tilde{\gamma}(\tilde{t})$ and $\sigma \in \Omega^2 M$ represents the canonical orientation of M.

Theorem 9.1.5 *The product $[\gamma] \bullet [\tilde{\gamma}]$ of two transversely embedded 1-cycles γ and $\tilde{\gamma}$ equals to the number of intersections counted with sign,*

$$[\gamma] \bullet [\tilde{\gamma}] = \sum_{p \in \gamma \cap \tilde{\gamma}} s_p(\gamma, \tilde{\gamma}) .$$

In particular, $\bullet\colon H_1(M;\mathbb{Z}) \times H_1(M;\mathbb{Z}) \to \mathbb{Z}$—called the intersection product.

Proof. Let $\gamma, \tilde{\gamma}$ be two transversely embedded 1-cycles and $\tilde{\alpha}_\gamma$ and $\tilde{\alpha}_{\tilde{\gamma}}$ be the corresponding closed representatives coming from the explicit construction above. Then, since transverse intersections are isolated and $\tilde{\gamma}$ is compact, there are only finitely many intersections. Given a tubular neighborhood $\tilde{f}\colon (-\varepsilon, \varepsilon) \times \tilde{C} \to M$ of $\tilde{\gamma}$ this splits γ into finitely many components—curves γ_i in the image of $\tilde{f}$ starting and ending at the boundary of $\tilde{f}((-\varepsilon, \varepsilon) \times \tilde{C})$. By choosing ε small enough, we can assume that each component γ_i contains exactly one intersection point p_i. Then

$$[\gamma] \bullet [\tilde{\gamma}] = \int_\gamma \tilde{\alpha}_{\tilde{\gamma}} = \sum_i \int_{\gamma_i} d(g \circ \tilde{h}) = \sum_i s_{p_i}(\gamma, \tilde{\gamma}) .$$

Here we used in the last equality that $s_{p_i}(\gamma, \tilde{\gamma}) = \pm 1$, if γ_i starts in $\tilde{f}(\{\mp\varepsilon\} \times \tilde{C})$ and ends in $\tilde{f}(\{\pm\varepsilon\} \times \tilde{C})$. □

Theorem 9.1.6 *$H_1(M;\mathbb{Z}) \subset H_1(M;\mathbb{R})$ forms a $2g$-dimensional lattice, i.e. a discrete additive subgroup. In particular, $H_1(M;\mathbb{Z}) \cong \mathbb{Z}^{2g}$.*

Proof. Clearly, $H_1(M;\mathbb{Z})$ forms a subgroup. It is left to show that it is discrete. Therefore it is enough to show that there is a neighborhood of zero which contains no other element than the zero cycle. Since $H^1(M;\mathbb{Z})^\perp = 0$, we can choose a cycles $\gamma_1, \ldots, \gamma_{2g}$ such that the corresponding forms $\alpha_1, \ldots, \alpha_{2g} \in H^1 M$ form a basis. Now we define an inner product as follows

$$\|\xi\|^2 = \sum_i \langle\!\langle \alpha_i | \xi \rangle\!\rangle^2 .$$

Now, let $[\gamma] \in H_1(M;\mathbb{Z})$. Then

$$\|[\gamma]\|^2 = \sum_i \langle\!\langle \alpha_i | [\gamma] \rangle\!\rangle^2 = \sum_i ([\gamma_i] \bullet [\gamma])^2 \in \mathbb{Z} .$$

Hence there is a neighborhood of zero which contain no element of $H_1(M;\mathbb{Z}) \setminus \{0\}$. Since on a finite-dimensional vector space, all inner products are equivalent and we are done. □

Definition 9.1.5 (Canonical Basis of Homology) *A canonical basis of homology is a basis $a_1, \ldots, a_g, b_1, \ldots, b_g \in H_1(M;\mathbb{Z})$ such that*

$$a_i \bullet a_j = b_i \bullet b_j = 0 , \quad a_i \bullet b_j = \delta_{ij} .$$

So a canonical basis is a basis with respect to which the intersection form is represented by the matrix

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

To show that there is always a canonical basis of homology we need some preparation.

Lemma 9.1.7 *Let $\gamma\colon \mathbb{S}^1 \to M$ be an embedding such that $M \setminus \gamma$ is connected. Then there is an embedding $\tilde{\gamma}\colon \mathbb{S}^1 \to M$ such that γ and $\tilde{\gamma}$ have a single positive intersection point. In particular, $[\gamma] \bullet [\tilde{\gamma}] = 1$ and $[\gamma] \neq 0$.*

Proof. Since $M \setminus \gamma$ is connected, there is a curve in $M \setminus \gamma$ from one to the other side of γ. Using the transversality theorem we can resolve self-intersections to get an embedded curve. Connecting then both ends passing through γ yields an embedded closed curve $\tilde{\gamma}$ in M. With the right choice of orientation we can achieve that the intersection is positive, $[\gamma] \bullet [\tilde{\gamma}] = 1$. Since $\bullet$ is non-degenerate $[\gamma] \neq 0$. □

In particular, an embedded cycle is zero if and only if it splits the surface into two parts.

Theorem 9.1.8 *Let $\gamma\colon \mathbb{S}^1 \to M$ be an embedding. Then*

$$[\gamma] = 0 \iff \exists \text{ surface } S \subset M \text{ with boundary } \partial S = \gamma\,.$$

Proof. If $\gamma = \partial S$, then $\int_\gamma \alpha = \int_{\partial S} \alpha = \int_S d\alpha = 0$ for all $\alpha \in \operatorname{Harm} M$. Hence $[\gamma] = 0$. Conversely, we can assume that M is connected. If γ is not a boundary, then $M \setminus \gamma$ is connected. Hence $[\gamma] \neq 0$. □

Corollary 9.1.9 *Let M be connected and $\gamma\colon \mathbb{S}^1 \to M$ be an embedding, $[\gamma] \neq 0$. Then there is an embedding $\tilde{\gamma}\colon \mathbb{S}^1 \to M$ such that γ and $\tilde{\gamma}$ have a single positive intersection point. In particular, $[\gamma] \bullet [\tilde{\gamma}] = 1$.*

Lemma 9.1.10 *Let $a, b\colon \mathbb{S}^1 \to M$ be transverse embeddings which intersect in a single point $p \in M$ and $[c] \in H_1(M;\mathbb{Z})$ such that $[a] \bullet [c] = [b] \bullet [c] = 0$. Then there are tubular neighborhoods A of a and B of b such that $b^{-1}(M \setminus A)$ and $a^{-1}(M \setminus B)$ are connected and an embedded representative γ of $[c]$ with image in $M \setminus (A \cup B)$. In particular, γ does not intersect a nor b.*

Proof. Without loss of generality we can choose an embedded representative γ of $[c]$ which intersects a and b transversely. Choose a tubular neighborhoods A of a and B of b such that $A \cap B$ is a small cube around p and $b^{-1}(M \setminus A)$ is connected. Since $[a] \bullet [c] = 0$, there are as many positive as negative intersection points between a and γ. Order them to pairs (p_j, q_j) and let a_j denote the segment of a joining p_j to q_j. Then we can add loops $a_j a_j^{-1}$ to γ and, after cutting and gluing, homotope γ out of A. The resulting homotoped γ may have self-intersections. Again, by cutting and gluing, we can assure by a small perturbation that γ is embedded in $M \setminus A$ intersecting b transversely. In particular, the intersections of γ and b lie outside A. Since $[b] \bullet [\gamma] = 0$, the positive and negative intersection points again come in pairs. Since $b^{-1}(M \setminus A)$ is connected we can proceed as before and join them by segments of b contained in $M \setminus A$, use them to resolve the intersections with b and then push γ out of $A \cup B$. Again we can repair possible self-intersections and, after a small perturbation, obtain a representative γ embedded in $M \setminus (A \cup B)$. □

Theorem 9.1.11 (Existence of Canonical Basis) *Each compact Riemann surface has a canonical basis of homology which consists of transversely embedded closed curves any two of which intersect in at most a single point.*

Proof. Without loss of generality M is connected. Suppose $a_1, \ldots, a_n, b_1, \ldots, b_n \colon \mathbb{S}^1 \to M$ are transverse embeddings such that

$$a_i \bullet b_j = \delta_{ij}\,, \quad a_i \bullet a_j = 0 = b_i \bullet b_j\,,$$

where a_i intersects b_i in a single point. Then, if there is some $[\gamma] \in H_1(M;\mathbb{Z})$ with $[\gamma] \bullet [a_i] = 0 = [\gamma] \bullet [b_i]$ for $i = 1, \ldots, n$, we can successively apply the previous lemma to obtain an embedded representative which does not intersect any of the $a_1, \ldots, a_n, b_1, \ldots, b_n$. Pick a non-trivial component $a_{n+1} \colon \mathbb{S}^1 \to M$ of γ. Since a_{n+1} is non-trivial, there is $\tilde{\gamma} \colon \mathbb{S}^1 \to M$ such that $[a_{n+1}] \bullet [\tilde{\gamma}] = 1$. Then

$$\hat{\gamma} := \tilde{\gamma} - \sum_{j=1}^{n} \big([\tilde{\gamma}] \bullet [b_j] a_j - [\tilde{\gamma}] \bullet [a_j] b_j\big)$$

satisfies $[a_{n+1}] \bullet [\hat{\gamma}] = [a_{n+1}] \bullet [\tilde{\gamma}] = 1$. For $i \le n$ we get

$$[a_i] \bullet [\hat{\gamma}] = [a_i] \bullet [\tilde{\gamma}] - \sum_{j=1}^{n} \big([\tilde{\gamma}] \bullet [b_j][a_i] \bullet [a_j] - [\tilde{\gamma}] \bullet [a_j][a_i] \bullet [b_j]\big) = [a_i] \bullet [\tilde{\gamma}] + \sum_{j=1}^{n} [\tilde{\gamma}] \bullet [a_j] \delta_{ij} = 0\,,$$

$$[b_i] \bullet [\hat{\gamma}] = [b_i] \bullet [\tilde{\gamma}] - \sum_{j=1}^{n} \big([\tilde{\gamma}] \bullet [b_j][b_i] \bullet [a_j] - [\tilde{\gamma}] \bullet [a_j][b_i] \bullet [b_j]\big) = [b_i] \bullet [\tilde{\gamma}] + \sum_{j=1}^{n} [\tilde{\gamma}] \bullet [b_j] \delta_{ij} = 0\,.$$

Hence, again by the previous lemma, we can find an embedded representative of $\hat{\gamma}$ which does not intersect any of the $a_1, \ldots, a_n, b_1, \ldots, b_n$. Keeping $\hat{\gamma}$ fixed in a small tubular neighborhood of a_{n+1}, we can assure that $\hat{\gamma}$ has a component b_{n+1} that intersects a_{n+1} transversely in a single point, $[a_{n+1}] \bullet [b_{n+1}]$. Clearly, $[b_{n+1}] \bullet [b_{n+1}] = 0$. Starting from $n = 0$ we repeat this process. Since $H_1(M;\mathbb{Z}) \cong \mathbb{Z}^{2g}$ it stops for $n = g$ and we are left with $2g$ embedded curves $a_1, \ldots, a_g, b_1, \ldots, b_g$, which form a canonical basis of homology. The additional statement on transversality can be established by the usual perturbation arguments. □

Connecting the intersection points p_i of a_i and b_i by an embedded curve to a fixed $p_0 \in M$ we can change the a_i and b_i so that they all start and end at p_0. Cutting M open along these curves we obtain a $4g$-gon—the fundamental polygon $\mathcal{F}$ with boundary

$$\partial \mathcal{F} = \sum_i (a_i + b_i + a_i^{-1} + b_i^{-1})\,.$$

9.1.2 The Picard Group

Let M be a compact Riemann surface. Then

$$\operatorname{Pic} M := \{\textit{isomorphism classes of holomorphic line bundles } (L, \bar{\partial}) \textit{ over } M\}\,.$$

Theorem 9.1.12 *The tensor product defines a multiplication* $\operatorname{Pic} M \times \operatorname{Pic} M \to \operatorname{Pic} M$*, which turns* $\operatorname{Pic} M$ *into an abelian group with identity element given by the trivial holomorphic line bundle* $\underline{\mathbb{C}}_M$ *and inverse given by* $L^{-1} = L^*$*.*

Proof. Exercise. □

Definition 9.1.6 (Picard Group) *The group* $(\operatorname{Pic} M, \otimes)$ *is called the Picard group.*

Let $p \in M$ and $d \in \mathbb{Z}$. The skyscraper bundle $\operatorname{sky}(p, d)$ is then a holomorphic line bundle of degree d. Hence the following theorem is an immediate consequence of Theorem 7.2.6.

Theorem 9.1.13 *The degree* $\deg\colon \operatorname{Pic} M \to \mathbb{Z}$ *is a surjective group homomorphism.*

Definition 9.1.7 (Jacobi Variety) *For* $d \in \mathbb{Z}$ *we define*

$$\operatorname{Pic}_d M := \deg^{-1}\{d\}\,.$$

The group $\operatorname{Jac} M := \operatorname{Pic}_0 M$ *is then called the Jacobian variety.*

We have $\operatorname{Pic}_{d_1} M \otimes \operatorname{Pic}_{d_2} = \operatorname{Pic}_{d_1+d_2} M$ and $\operatorname{Jac} M \subset \operatorname{Pic} M$. Thus $\operatorname{Jac} M$ acts on $\operatorname{Pic} M$.

Exercise 9.1.1 The action $\operatorname{Jac} M \times \operatorname{Pic} M \to \operatorname{Pic} M$ is free and transitive.

If we fix $L_0 \in \operatorname{Pic}_d M$, we obtain a bijection

$$\operatorname{Jac} M \to \operatorname{Pic}_d M, \quad L \mapsto L \otimes L_0\,.$$

This allows us to identify $\operatorname{Pic}_d M$ with $\operatorname{Jac} M$ so that, in order to put more structure on $\operatorname{Pic}_d M$, we can focus solely on $\operatorname{Jac} M$.

As we show now we can assign to each holomorphic line bundle of degree zero a unique connection.

Recall that every holomorphic line bundle $(L, \bar\partial)$ has a hermitian metric. The next theorem tells us that the choice of such a metric fixes a connection on L.

Theorem 9.1.14 *Let* $(L, \bar\partial)$ *be a holomorphic line bundle over* M *with hermitian metric* $\langle .,. \rangle$*. Then there is a unique unitary connection* ∇ *such that* $\nabla'' = \bar\partial$*.*

Proof. There exists some unitary connection $\tilde\nabla$. Then $\bar\partial = \tilde\nabla'' - i\xi$ for $\xi \in \Gamma\bar K$. Hence there is $\alpha \in \Omega^1 M$ such that $2\xi = \alpha - i * \alpha = 2\alpha''$. Then $\nabla = \tilde\nabla - i\alpha$ is unitary and

$$\nabla'' = \tilde\nabla'' - i\alpha'' = \tilde\nabla'' - i\xi = \bar\partial\,,$$

as desired. □

Moreover, we have the following result.

Theorem 9.1.15 *Let* $(L, \bar\partial)$ *be a holomorphic line bundle of degree* d *and* $\sigma \in \Omega^2 M$ *such that* $\int_M \sigma = 2\pi i d$*. Then there is a hermitian metric on* L *(unique up to constant scale) such that the corresponding unitary connection* ∇ *with* $\nabla'' = \bar\partial$ *has curvature* $F^\nabla = -i\sigma$*.*

Proof. Let $\langle .,. \rangle$ be some hermitian metric and its associated connection ∇. If we change the metric to $e^{2u}\langle .,. \rangle$ for some $u \in \mathscr{C}^\infty M$, we obtain a new associated connection $\tilde\nabla$. Since J is parallel with respect to both ∇ and $\tilde\nabla$ and $\nabla'' = \bar\partial = \tilde\nabla''$, we have

$$\tilde\nabla = \nabla + \eta, \quad \eta \in \Gamma K\,.$$

Furthermore, since $\tilde\nabla$ is metric with respect to $e^{2u}\langle .,. \rangle$, we get for $\psi \in \Gamma L$

$$2e^{2u}\langle \tilde\nabla\psi, \psi \rangle = d(e^{2u}|\psi|^2) = 2e^{2u}(\langle \psi, \nabla\psi \rangle + |\psi|^2 du) = 2e^{2u}(\langle \tilde\nabla\psi, \psi \rangle + |\psi|^2(du - \operatorname{Re}\eta)\,,$$

which implies that $du = \operatorname{Re}\eta$ and hence $\eta = 2\partial u$. Thus we get the following relation of curvatures:

$$F^{\tilde\nabla} = F^\nabla + 2d\bar\partial u = F^\nabla + id * du\,.$$

Thus $F^{\tilde{\nabla}} = -i\sigma$ if and only if $d * du = iF^{\nabla} - \sigma$, which by Theorem 8.2.15 is solvable, since $\int_M iF^{\nabla} - \sigma = 0$. The solution is unique up to an additive constant. □

The following theorem is now an immediate consequence.

Theorem 9.1.16 *Each holomorphic line bundle $(L, \bar\partial)$ with $\deg L = 0$ has a unique flat connection ∇ such that $\nabla'' = \bar\partial$, which is unitary in the sense that there exists a parallel hermitian metric.*

In summary, we have identified $\operatorname{Jac} M$ with the space of flat unitary connections. Hence, using Weil's theorem, $\operatorname{Jac} M$ may be parametrized by $Z^1 M / Z^1_{int} M \cong \operatorname{Harm} M / H_1(M;\mathbb{Z}) \cong \mathbb{R}^{2g}/\mathbb{Z}^{2g}$—a $2g$-dimensional torus.

Remark 9.1.3 Let us make this identification more precise using the trivial unitary bundle $\underline{\mathbb{C}}_M$. One easily checks that the associated connection is the trivial connection d. The corresponding point in $\operatorname{Jac} M$ is then

$$(\underline{\mathbb{C}}_M, d'') .$$

Any other flat unitary line bundle class is then obtained by changing the trivial connection $\nabla = d - 2\pi i\alpha$, where $\alpha \in \operatorname{Harm} M$. The corresponding holomorphic structure is obtained by extracting the $\bar K$-part,

$$\nabla'' = d'' - 2\pi i \alpha'' .$$

Since α is harmonic, α'' is closed. So we get an affine map to the holomorphic bundles of degree zero

$$\ker\big(d \colon \Gamma \bar K \to \Omega^2(M;\mathbb{C})\big) \ni \xi \quad \mapsto \quad \big(\underline{\mathbb{C}}_M, d'' - 2\pi i \xi\big) \in \operatorname{Jac} M .$$

Note that $\ker\big(d \colon \Gamma \bar K \to \Omega^2(M;\mathbb{C})\big) \cong \Gamma \bar K / \operatorname{im} \bar\partial$. The non-degenerate pairing $\langle\!\langle .|. \rangle\!\rangle \colon \Gamma \bar K \times \Gamma K \to \mathbb{C}$ descends to a non-degenerate pairing $\langle\!\langle .|. \rangle\!\rangle \colon \Gamma \bar K / \operatorname{im} \bar\partial \times H^0 K \to \mathbb{C}$ giving us an isomorphism

$$(H^0 K)^* \cong \Gamma \bar K / \operatorname{im} \bar\partial \cong \ker\big(d \colon \Gamma \bar K \to \Omega^2(M;\mathbb{C})\big) .$$

Moreover, each $\gamma \in H_1(M;\mathbb{C})$, can be considered as an element of $(H^0 K)^*$ as usual, $\eta \mapsto \int_\gamma \eta$. In terms of the harmonic form α_γ corresponding to γ, $\eta \mapsto \int_\gamma \eta$ is then represented as follows: If $\eta = \beta + i * \beta$, then

$$\int_\gamma \eta = \int_\gamma \beta + i \int_\gamma * \beta = \int_M \alpha_\gamma \wedge \beta + i \int_M \alpha_\gamma \wedge * \beta = \int_M \alpha''_\gamma \wedge \eta .$$

In particular, $\alpha_\gamma \in Z^1_{int}(M;\mathbb{R})$. Thus we have shown that

$$\operatorname{Jac} M \cong (H^0 K)^* / H_1(M;\mathbb{Z}) .$$

Corollary 9.1.17 *If M is diffeomorphic to $\mathbb{S}^2$, then $\deg \colon \operatorname{Pic} M \to \mathbb{Z}$ is an isomorphism. In particular, up to isomorphism, for each degree there is a unique holomorphic line bundle.*

Proof. If M is diffeomorphic to $\mathbb{S}^2$, then $H^0 K = \{0\}$. □

9.2 Meromorphic Sections

Throughout this section L denotes a holomorphic line bundle over a connected compact Riemann surface M of genus g.

9.2.1 Mittag–Leffler Theorem

A *meromorphic section* of a holomorphic bundle is a section which, with respect to a holomorphic frame, is represented by a meromorhic function. We are interested in what configurations of poles appear as the poles of a meromorphic section.

Definition 9.2.1 (Section with Poles) *Let $p_1, \ldots, p_n \in M$ be pairwise distinct. Then $\psi \in \Gamma L|_{M\setminus\{p_1,\ldots,p_n\}}$ is called a section with poles $p_1, \ldots, p_n$, if there is $\eta \in \Gamma \bar{K} L$ such that*

$$\bar{\partial}\psi = \eta\,,$$

and for each $j \in \{1, \ldots, n\}$ and every metric ψ either smoothly extends to p_j or

$$|\psi_p| \longrightarrow \infty \text{ for } p \rightarrow p_j\,.$$

Remark 9.2.1 Note ψ is a meromorphic section of L if and only if ψ is a section with poles such that $\bar{\partial}\psi$. The limit condition in the definition excludes the possibility of essential singularities.

Theorem 9.2.1 *If ψ is a section of L with poles $p_1, \ldots, p_n$, then for each $j \in \{1, \ldots, n\}$ there is a neighborhood U_j of p_j, a meromorphic section ψ of $U|_{U_j}$ with a single pole at p_j and a smooth $\varphi_j \in \Gamma(L|_{U_j})$ such that*

$$\psi|_{U_j} = \psi_j + \varphi_j\,.$$

Proof. Let $\eta := \bar{\partial}\psi$. By Theorem 8.2.11, there is a neighborhood U_j of p_j and $\varphi_j \in \Gamma L|_{U_j}$ such that

$$\bar{\partial}\varphi_j = \eta\,.$$

Then $\psi_j := \psi_{U_j\setminus\{p_j\}} - \varphi_j$ satisfies $\bar{\partial}\psi_j = 0$ and is either smoothly extendable or satisfies $|\psi_{j,p}| \rightarrow \infty$ for $p \rightarrow p_j$, i.e. ψ_j is meromorphic. □

Definition 9.2.2 (Residue) *Let ω be section of K with poles. Then the residue of ω at $p \in M$ is defined by*

$$2\pi i \operatorname{Res}_p \omega := \lim_{r\rightarrow 0} \int_{|z|=r} \omega\,,$$

where z is a complex chart at p.

Exercise 9.2.1 Show that the residue is well-defined.

Theorem 9.2.2 (Mittag–Leffler Theorem) *Let ψ be a section of L with poles $p_1, \ldots p_n$. Then there is a meromorphic section $\tilde{\psi}$ of L such that $\tilde{\psi} \equiv \psi \mod \Gamma L$ if and only if*

$$\sum_{j=1}^{n} \operatorname{Res}_{p_j} \langle \xi | \psi \rangle = 0\,, \text{ for all } \xi \in H^0 K L^{-1}\,.$$

Proof. By Theorem 9.2.1 we can assume without loss of generality that there are pairwise disjoint open disks $U_1, \ldots, U_n$ around $p_1, \ldots, p_n$ such that $\bar{\partial}\psi|_{U_j\setminus\{p_j\}} = 0$. Define $M_0 := M \setminus (\cup_j U_j)$, $\eta \in \Gamma \bar{K} L$ and let $\eta|_{M\setminus\{p_1,\ldots,p_n\}} = \bar{\partial}\psi|_{M\setminus\{p_1,\ldots,p_n\}}$. In particular, $\eta|_{U_j} = 0$. Now suppose that there is $\varphi \in \Gamma L$ such that $\psi - \varphi$ is meromorphic. Then $\eta - \bar{\partial}\varphi = \bar{\partial}(\psi - \varphi) = 0$ and the $\bar{\partial}$-problem $\bar{\partial}\psi = \eta$ has a solution. Conversely, if $\bar{\partial}\varphi = \eta$ for some $\varphi \in \Gamma L$, then $\psi - \varphi$ is meromorphic. Hence

$$\exists\, \varphi \in \Gamma L \;\; \textit{such that} \;\; \psi - \varphi \;\; \textit{is meromorphic} \quad \Longleftrightarrow \quad \exists\, \varphi \in \Gamma L \;\; \textit{such that} \;\; \bar{\partial}\varphi = \eta\,,$$

which, by Theorem 8.2.7, is the case if and only if, for all $\xi \in H^0 K L^{-1}$,

$$0 = \langle\!\langle \eta | \xi \rangle\!\rangle = \int_M \langle \eta \wedge \xi \rangle = \int_{M_0} \langle \eta \wedge \xi \rangle = \int_{M_0} \langle \bar{\partial}\psi \wedge \xi \rangle = -\int_{\partial M_0} \langle \psi \wedge \xi \rangle = \sum_{j=1}^{n} \int_{\partial U_j} \langle \psi \wedge \xi \rangle = 2\pi i \sum_{j=1}^{n} \operatorname{Res}_{p_j} \langle \xi | \psi \rangle\,,$$

where the last equality uses that ψ is holomorphic in $U_j \setminus \{p_j\}$. □

Corollary 9.2.3 *The residues of a meromorphic differential $\eta \in \Gamma K$ on a compact Riemann surface sum to zero.*

9.2.2 Abelian Differentials and Riemann Bilinear Identity

An *Abelian differential* is a meromorhic section of the canonical bundle K. They divide into three classes.

Definition 9.2.3 (Abelian Differentials) *An Abelian differential is called*

- *of the first kind, if it has no poles,*
- *of the second kind, if it has no residues,*
- *of the third kind, if it has residues.*

Remark 9.2.2 Away from singularities Abelian differentials are holomorphic and hence closed. In particular, an Abelian differential of the first kind is nothing else then a holomorphic differential.

Let $a_1, \ldots, a_g, b_1, \ldots, b_g$ be a canonical basis of homology of M. The Riemann bilinear identity relates the integral of the wedge product of two closed complex-valued 1-forms ω and η to their a- and b-periods.

Theorem 9.2.4 (Riemann Bilinear Identity) *Let $\omega, \eta \in \Omega^1(M;\mathbb{C})$ be closed. Then*

$$\int_M \omega \wedge \eta = \sum_{i=1}^{g} \Big(\big(\int_{a_i} \omega\big)\big(\int_{b_i} \eta\big) - \big(\int_{a_i} \eta\big)\big(\int_{b_i} \omega\big) \Big)\,.$$

Proof. We can assume that the a- and b cycles are all embedded starting and ending at the same point. If we cut M along the cycles we obtain a $4g$-gon with boundary $\partial\mathcal{F} = \sum_i (a_i + b_i + a_i^{-1} + b_i^{-1})$. Any closed curve in $\mathring{\mathcal{F}}$ corresponds to a closed curve in M which, since it does not intersect any a- or b-cycle, is null-homologous. In particular, any closed 1-form on $\mathcal{F}$ is exact. In particular, if $p_0 \in \mathring{\mathcal{F}}$, then

$$f(p) = \int_{p_0}^{p} \omega\,,$$

where the integration is taken along some path γ_p from p_0 to p in $\mathcal{F}$, yields a potential of ω, i.e. $df = \omega$. In particular we get

$$\int_M \omega \wedge \eta = \int_{\mathcal{F}} df \wedge \eta = \int_{\mathcal{F}} d(f \wedge \eta) = \int_{\partial\mathcal{F}} f\eta = \sum_{i=1}^{g} \Big(\int_{a_i} f\eta + \int_{b_i} f\eta + \int_{a_i^{-1}} f\eta + \int_{b_i^{-1}} f\eta \Big)\,.$$

By construction, if $p \in a_i \subset \mathcal{F}$ and $\tilde{p} \in a_i^{-1} \subset \mathcal{F}$ correspond the same point in M, then the path γ_p^{-1} concatenated with $\gamma_{\tilde{p}}$ closes in M and is homologous to b_i. Hence

$$f(\tilde{p}) = f(p) + \int_{b_i} \omega\,.$$

Similarly, for $p \in b_i \subset \mathcal{F}$ and $\tilde{p} \in b_i^{-1} \subset \mathcal{F}$ which correspond to the same point in M we get

$$f(\tilde{p}) = f(p) - \int_{a_i} \omega\,.$$

Hence we get

$$\int_M \omega \wedge \eta = \sum_{i=1}^{g}\Big(\int_{a_i} f\eta + \int_{b_i} f\eta - \int_{a_i}(f + \int_{b_i}\omega)\eta - \int_{b_i}(f - \int_{a_i}\omega)\eta\Big) = \sum_{i=1}^{g}\Big(-\int_{a_i}(\int_{b_i}\omega)\eta + \int_{b_i}(\int_{a_i}\omega)\eta\Big),$$

which yields the desired formula after pulling the constants $\int_{b_i}\omega$ and $\int_{a_i}\omega$ out of the integrals. ☐

Corollary 9.2.5 *If $\eta = \alpha + i * \alpha$ is a holomorphic differential, then*

$$\int_M \alpha \wedge *\alpha = \sum_{i=1}^{g} \operatorname{Im}\Big(\overline{(\int_{a_i}\eta)}(\int_{b_i}\eta)\Big).$$

Proof. Exercise. ☐

Corollary 9.2.6 *Let $\eta \in H^0K$. Then $\eta = 0$ if and only if all its a-periods vanish.*

Corollary 9.2.7 *Let $\eta \in H^0K$. Then $\eta = 0$ if and only if all its a- and b-periods are real.*

Remark 9.2.3 Note that, by Corollary 9.2.6, $a_1, \ldots, a_g$ form a complex basis of $(H^0K)^*$.

Definition 9.2.4 (Canonical Basis of Holomorphic Differentials) *The basis $\omega_1, \ldots, \omega_g \in H^0K$ dual to $a_1, \ldots, a_g \in (H^0K)^*$ is called canonical.*

Definition 9.2.5 *An Abelian differential η is callled normalized, if $\int_{a_i}\eta = 0$ for all $i = 1, \ldots, g$.*

Since $H^0\underline{\mathbb{C}}_M$ consists only of constant. The Mittag–Leffler theorem tells us that the only condition on the poles is that the residues have to sum up to zero. This yields the following theorem.

Theorem 9.2.8 *Let (U, z) be a complex chart at $p \in M$ and (V, w) be a complex chart at $q \in M$. Then:*

(a) For each $1 \leq n \in \mathbb{N}$ there is a unique normalized Abelian differential of the second kind ω_p^n with a single pole at p such that

$$\omega_p^n|_U = \big(\tfrac{1}{z^{n+1}} + \mathcal{O}(1)\big)\,dz\,.$$

(b) There is a unique normalized Abelian differential of the third kind ω_{pq} with exactly two poles at p and q such that

$$\omega_{pq}|_U = \big(\tfrac{1}{z} + \mathcal{O}(1)\big)\,dz\,, \quad \omega_{pq}|_V = \big(-\tfrac{1}{w} + \mathcal{O}(1)\big)\,dw\,.$$

Proof. Existence of Abelian differentials with these poles is assured by the Mittag–Leffler Theorem. They are unique since the difference of two normalized Abelian differentials with the same poles is a holomorphic differential with vanishing a-periods. □

Remark 9.2.4 Note that ω_p^n integrates to a meromorphic function $f\colon \mathcal{F} \to \mathbb{C}$ with a pole at p of the form

$$f|_U = -\tfrac{1}{z^n} + \mathcal{O}(1)\,.$$

If γ is an embedded path from p to q in $\mathcal{F}$, then ω_{pq} integrates to a holomorphic function $\varphi\colon \mathcal{F} \setminus \gamma \to \mathbb{C}$ such that

$$\varphi|_U = \log z + \mathcal{O}(1)\,, \quad \varphi|_V = -\log w + \mathcal{O}(1)\,.$$

In particular, $g = e^{\varphi}$ becomes a holomorphic function defined on $\mathcal{F}$ which has a single simple zero at p and a single simple pole at q:

$$g|_U = z(c + \mathcal{O}(z))\,, \quad g|_V = \tfrac{1}{w}(d + \mathcal{O}(w))\,,$$

where c, d are non-zero constants.

Theorem 9.2.9 *Let ω_{pq} be a normalized Abelian differential of the third kind with only poles at p and q such that*

$$\mathrm{Res}_p \omega_{pq} = 1\,, \quad \mathrm{Res}_q \omega_{pq} = -1$$

and $\omega_1, \ldots, \omega_g$ be the canonical basis of holomorphic differentials. Then

$$\int_{b_i} \omega_{pq} = 2\pi i \int_q^p \omega_i\,, \quad i = 1, \ldots, g\,.$$

Proof. On $\mathcal{F}$ there is f_i such that $df_i = \omega_i$. Since $f_i \omega_{pq}$ is holomorphic on $\mathcal{F} \setminus \{p, q\}$ we have

$$\int_{\partial\mathcal{F}} f_i \omega_{pq} = \int_{|z|=\varepsilon} f_i \omega_{pq} + \int_{|w|=\varepsilon} f_i \omega_{pq} = 2\pi i f_i(p) - 2\pi i f_i(q) = 2\pi i \int_q^p df_i = 2\pi i \int_q^p \omega_i\,.$$

On the other hand, as in the proof of the Riemann bilinear identity one shows that

$$\int_{\partial\mathcal{F}} f_i \omega_{pq} = \sum_j \Big(\int_{a_j} \omega_i \int_{b_j} \omega_{pq} - \int_{b_j} \omega_i \int_{a_j} \omega_{pq} \Big)\,.$$

Hence, using $\int_{a_j} \omega_i = \delta_{ij}$ and $\int_{a_j} \omega_{pq} = 0$, we find that $\int_{b_i} \omega_{pq} = 2\pi i \int_q^p \omega_i$. □

9.2.3 Divisors and the Abel–Jacobi Map

Let M be a compact Riemann surface. A *divisor* is a map $D\colon M \to \mathbb{Z}$ with discrete $\operatorname{supp} D = \{p_1, \ldots, p_n\} \subset M$. If we identify a point $p \in M$ with the divisor which assigns the value 1 to p and vanishes else, we can write

$$D = \sum_{p \in M} D(p)\, p = \sum_{i=1}^n d_i\, p_i\,.$$

Clearly, with addition of functions the divisors form an Abelian group, which we denote by $\operatorname{Div} M$.

Example 9.2.1 (Divisor of Direction Fields) Let ψ be a direction field. Then the index of ψ is a divisor:

$$(\psi) := \operatorname{ind}\psi \in \operatorname{Div} M\,.$$

Definition 9.2.6 (Divisor Class Group) *A divisor is called a principal divisor if it is the divisor of a meromorphic function. Two divisors are called equivalent if and only if their difference is principal. The corresponding quotient group is called the divisor class group* $\operatorname{Cl} M = \operatorname{Div} M/_\sim$.

To each divisor we can assign a holomorphic line bundle as follows: To each divisor D we assign the holomorphic line bundle

$$[D] := \bigotimes_{p\in M} \operatorname{sky}(p, D(p))\,.$$

Clearly, $[D] \cong [\tilde{D}]$ if and only if $D \sim \tilde{D}$. Hence the map $D \mapsto [D]$ defines an injective group homomorphism

$$\operatorname{Cl} M \hookrightarrow \operatorname{Pic} M\,.$$

We will see soon that this is actually an isomorphism.

Remark 9.2.5 (Degree of Divisors) Since $\operatorname{Cl} M \subset \operatorname{Pic} M$, each divisor has a degree. To compute it, note that each skyscraper bundle comes with a meromorphic section—its famous section. The product of these famous sections is then a meromorphic section of S_D. In particular, if $D = \sum_{i=1}^n d_i p_i$, then it follows from the Poincaré–Hopf index theorem that

$$\deg D := \deg\,[D] = \sum_{i=1}^n d_i\,.$$

In particular, we have divisors of a certain degree $\operatorname{Div}_d M = \operatorname{Div} M \cap \operatorname{Pic}_d M$.

The restriction of the map S to the divisors of degree zero is called the *Abel–Jacobi* map.

Definition 9.2.7 (Abel–Jacobi Map) *The map* $\mathscr{A}\colon \operatorname{Div}_0 M \to \operatorname{Jac} M$ *given by* $D \mapsto [D]$ *is called Abel–Jacobi map.*

For $p, q \in M$ we have $p - q \in \operatorname{Div}_0 M$. Hence, if we fix $q \in M$, we obtain a map $A\colon M \to \operatorname{Jac} M$:

$$A(p) = \mathscr{A}(p-q) = [p-q]\,.$$

We want to show that A is holomorphic. Therefore we need to get out which element in $(H^0K)^*$ belongs to $p - q$. To do so we explicitly construct an isomorphism from $A(p)$ to the trivial bundle.

Let $a_1, \ldots, a_g, b_1, \ldots, b_g$ be a canonical basis of homology and $\omega_1, \ldots, \omega_g \in H^0K$ be the corresponding canonical basis of holomorphic differentials.

Note that the famous section ψ of $L = [p-q]$ is a holomorphic section with a single simple zero at p and a single simple pole at q. Let ω_{pq} be a normalized Abelian differential of the third kind with $\operatorname{res}_p \omega_{pq} = 1$ and $\operatorname{res}_q \omega_{pq} = -1$. Since ω_{pq} is normalized, it has no a-periods.

Recall, to each cycle we can assign the Poincaré dual, i.e. to each a_i we have a harmonic form α_i and to each b_i a harmonic form β_i such that for all $\eta \in H^0K$

$$\int_{a_j} \eta = \int_M \alpha_i'' \wedge \eta\,, \qquad \int_{b_j} \eta = \int_M \beta_i'' \wedge \eta\,.$$

Define

$$\omega = \omega_{pq} + \sum_{i=1}^{g} c_i\, \alpha_i'' \,, \quad c_i = \int_{b_i} \omega_{pq} \,.$$

Then ω has no periods we get a function $f = \exp\big(-\int \omega\big)$ with a single pole at p and a single zero at q.

Hence $\varphi = f\psi$ extends to a smooth section of L and defines an isomorphism with the trivial bundle $\underline{\mathbb{C}}_M$,

$$\mathscr{C}^\infty(M;\mathbb{C}) \ni z \mapsto \Phi(z) := z\varphi \in \Gamma L \,.$$

Under this isomorphism the $\bar\partial$-operator of L takes the form $d'' - 2\pi i\xi$ with $\xi \in \Gamma\bar K$ closed. To determine ξ we can use that ψ is meromorphic. We have

$$\bar\partial\Phi(z) = \bar\partial(zf\psi) = \big(d''(zf)\big)\psi = (d''z + d''\log f)\varphi = \Phi(d''z + d''\log f)\,.$$

Since $d\log f = -\omega_{pq} - \sum_{i=1}^{g} c_i\,\alpha_i''$ and $\omega_{pq}'' = 0$, we get $2\pi i\xi = -d''\log f = \sum_{i=1}^{g} c_i\,\alpha_i''$. With Theorem 9.2.9, we thus obtain the following coordinate expression for $\mathscr{A}$:

$$A(p) = \tfrac{1}{2\pi i}(c_1,\ldots,c_n) = \int_q^p (\omega_1,\ldots,\omega_g)\,.$$

which can be identified with just the integral operator $\int_p^q$ considered as an element of $(H^0K)^*$.

Theorem 9.2.10 *Let $D = \sum_i (p_i - q_i)$, then*

$$\mathscr{A}(D) = \sum_i \int_{q_i}^{p_i} : H^0K \to \mathbb{C}$$

9.2.4 Existence of Non-Trivial Meromorphic Sections and the Riemann–Roch Theorem

The Riemann–Roch theorem then relates the dimensions of H^0L and H^0KL^{-1} to the degree of L and the genus g of M.

Theorem 9.2.11 (Riemann–Roch Theorem)

$$h^0L - h^0KL^{-1} = \deg L + 1 - g\,.$$

If we subtract $\deg L$ on both sides of the equality we are left with an integer on the right-hand side that only depends on the genus of the surface so does the left-hand side. We give it a name:

$$n_L := h^0L - h^0KL^{-1} - \deg L\,.$$

Since $n_{\underline{\mathbb{C}}_M} = h^0\underline{\mathbb{C}}_M - h^0K - \deg\underline{\mathbb{C}}_M = 1-g$, the Riemann–Roch theorem just becomes $n_L = n_{\underline{\mathbb{C}}_M}$.

We will show n_L is invariant under tensoring in skyscraper and thus divisor bundles. The Riemann–Roch Theorem then follows directly from the existence of a meromorphic sections.

Let $p \in M$. The skyscraper bundle $[p] = \mathrm{sky}(p,1)$ then comes with its famous section, which is a holomorphic section with a single simple zero at p. Every other holomorphic bundle wiht this property is holomorphically equivalent: If S is another such holomorphic line bundle, then the quotient $[p]S^{-1}$ has a section with a removable singularity, i.e. it is holomorphically trivial.

Tensoring in a skyscraper bundle changes the dimension of the space of holomophic sections by at most 1: Let $\varphi \in H^0[p]$ denote the famous section. Then we have an injection $H^0L \hookrightarrow H^0[p]L$ given by

$$H^0L \ni \psi \mapsto \varphi \otimes \psi \in H^0[p]L\,.$$

The image of this map equals $\ker(\tilde{\psi} \mapsto \tilde{\psi}_p)$. In particular, we get the following corollary.

Corollary 9.2.12 $h^0[p]L - h^0L \in \{0, 1\}$

This previous theorem says that the difference $h^0[p]L - h^0L$ is boolean. Moreover, it proves half of the following theorem—namely, $(b) \Leftrightarrow (c)$.

Theorem 9.2.13 *The following are equivalent:*

(a) There is a meromorphic section ψ of L with a single simple pole at p.
(b) There is $\tilde{\psi} \in H^0[p]L$ with $\tilde{\psi}_p \neq 0$.
(c) $h^0[p]L > h^0L$.

Proof. It is left to show $(a) \Leftrightarrow (b)$. Let ψ be a meromorphic section of L with a single simple pole at p and $0 \neq \varphi \in H^0[p]$. Then $\varphi \otimes \psi \in H^0[p]L$ with $(\varphi \otimes \psi)_p \neq 0$. Conversely, if there is $\tilde{\psi} \in H^0[p]L$ with $\tilde{\psi}_p \neq 0$, then $\psi = \tilde{\psi}/\varphi$ is meromorphic with single simple pole at p. □

Moreover, the Mittag–Leffler Theorem yields the following lemma.

Lemma 9.2.14 *There is a holomorphic section ψ of L with $\psi_p \neq 0$ if and only if there is no meromorphic section η of KL^{-1} with a single simple pole at p.*

Proof. "$\Rightarrow$": If there would be such ψ and η then $\langle \eta | \psi \rangle$ would have a single simple pole at p, which contradicts the fact that the residues of a meromorphic 1-form sum up to zero. "$\Leftarrow$": Suppose all $\psi \in H^0L$ would vanish at p. Let ξ be a smooth section of KL^{-1} with a single simple pole at p. Then $\mathrm{Res}_p \langle \xi | \psi \rangle = 0$ for all $\psi \in H^0L$ and, by the Mittag–Leffler theorem, there is a meromorphic $\eta \equiv \xi \mod \Gamma KL^{-1}$. □

This together yields the following theorem.

Theorem 9.2.15 *The following are equivalent:*

(a) $h^0L[p] > h^0L$.
(b) There is $\tilde{\psi} \in H^0L[p]$ with $\tilde{\psi}_p \neq 0$.
(c) There is a no meromorphic section of $KL^{-1}[-p]$ with a single pole at p.
(d) not($h^0KL^{-1} > h^0KL^{-1}[-p]$).

Boolean algebra-wise, the equivalence of (a) and (d) yields in particular that

$$h^0L[p] - h^0L = 1 - \left(h^0KL^{-1} - h^0KL^{-1}[-p]\right).$$

Thus

$$h^0L[p] - h^0KL^{-1}[-p] - \deg L[p] = h^0L[p] - h^0KL^{-1}[-p] - \deg L - 1 = h^0L - h^0KL^{-1} - \deg L = n_L\,.$$

Hence we have shown that n_L is invariant under tensoring with a divisor.

Lemma 9.2.16 *Let $D \in \mathrm{Div} M$, then*

$$n_L = n_{L[D]}\,.$$

Proof. The above computation shows that $n_L = n_{L[p]}$. Hence $n_L = n_{L[D]}$. □

Theorem 9.2.17 *Every holomorphic line bundle has a non-trivial meromorphic section.*

Proof. For large n the bundle $\tilde{L} = L[np]$ satisfies $\deg K\tilde{L}^{-1} < 0$. Thus $h^0 K\tilde{L}^{-1} = 0$ and so we have

$$n_L = n_{\tilde{L}} = h^0\tilde{L} - \deg \tilde{L}\,.$$

Hence $h^0\tilde{L} > 0$ for n large enough. Hence there is $\psi \in H^0\tilde{L}$. Then, if $\varphi \in H^0[np]$ denotes the famous section, the section ψ/φ is a meromorphic section of L. □

Given the existence of a non-trivial meromorphic section and that n_L is invariant under tensoring in point bundles, the proof of the Riemann–Roch Theorem becomes a triviality:

Proof of the Riemann–Roch Theorem. Let ψ be a meromorphic section of L and $D = (\psi)$, then $L[-D]$ is trivial and hence $n_L = n_{L[-D]} = n_{\underline{\mathbb{C}}_M}$. □

9.3 Some Applications of the Riemann–Roch Theorem

Through out this section, let M be a compact Riemann surface of genus g.

9.3.1 The Riemann–Hurwitz Formula

We can use the Riemann–Roch theorem to compute the degree of the canonical bundle K of M.

Theorem 9.3.1 $\deg K = 2g - 2$.

Proof. By the Riemann–Roch theorem, $1 - g = n_K = h^0K - h^0\underline{\mathbb{C}}_M - \deg K = g - 1 - \deg K$. □

Corollary 9.3.2 $\chi(M) = 2 - 2g$.

The Riemann–Hurwitz formula relates the branch order of a holomorphic map to degree and genus:

Definition 9.3.1 (Degree of a Holomorphic Map) *Let $\tilde{M}$ and M be compact Riemann surfaces and $f\colon \tilde{M} \to M$ be holomorphic. The degree of f is given by*

$$\deg f := \#sheets = \frac{1}{\mathrm{vol}\, M}\int_{\tilde{M}} f^*\omega_M\,,$$

where ω_M denotes the volume form of M.

Remark 9.3.1 In fact, for each smooth map $f\colon \tilde{M} \to \hat{M}$ between equidimensional connected compact manifolds, $\dim \tilde{M} = m = \dim \hat{M}$, one can show that there is $\deg f \in \mathbb{Z}$ such that

$$\int_{\tilde{M}} f^*\omega = \deg f \int_{\hat{M}} \omega$$

for all $\omega \in \Omega^m \hat{M}$. Also in this situation the degree is a sum over sheets, but counted with sign—as explained e.g. in [9]. A proof of the degree formula can be found in [10].

The following lemma is then an easy exercise.

Lemma 9.3.3 *Let $\tilde{M}, M$ be compact Riemann surfaces, $f\colon \tilde{M} \to M$ be holomorphic and $L \to M$ be a complex line bundle. Then*

$$\deg f^*L = \deg f \cdot \deg L$$

Now, let $\tilde{M}$ and M be compact Riemann surfaces and $f\colon \tilde{M} \to M$ be holomorphic. Then we can consider df as a section of $\mathrm{Hom}(T\tilde{M}; f^*TM)$. Since f is holomorphic, we have $Jdf = df \circ J$. Hence $df \in \Gamma(\tilde{K} f^*TM)$, where $\tilde{K}$ denotes the canonical bundle of $\tilde{M}$. So df is a section of a complex line bundle. Moreover, df has isolated zeros. In this context the index of df is called the branch order.

Definition 9.3.2 (Branch Order) *Let $\tilde{M}, M$ be compact Riemann surfaces and $f\colon \tilde{M} \to M$ be holomorphic. Then the branch order of f at $p \in \tilde{M}$ is defined to be $b_p(f) := \mathrm{ind}_p(df)$. The branch order of f is then defined as*

$$b(f) = \sum_{p \in \tilde{M}} b_p(f)\,.$$

Example 9.3.1 If $f(z) = z^n$ we have $df = nz^{n-1}dz$ and thus $b_0 f = n-1$. Recall that every holomorphic function locally looks like this.

In particular, by the Poincaré–Hopf index theorem the branch order of f equals the degree of $\tilde{K} f^*TM$, i.e.

$$b(f) = \sum_{p \in \tilde{M}} \mathrm{ind}_p(df) = \deg(\tilde{K} f^*TM) = \deg \tilde{K} + \deg f^*TM = \deg \tilde{K} - \deg f \deg K = 2(\tilde{g} - 1 - \deg f(g-1))\,.$$

Here g and $\tilde{g}$ denote the genus of M and $\tilde{M}$, respectively. The last two equalities follows from Lemma 9.3.3 and Theorem 9.3.1. This proves the Riemann–Hurwitz formula.

Theorem 9.3.4 (Riemann–Hurwitz Formula) *Let $\tilde{M}$ and M be compact Riemann surfaces of genus $\tilde{g}$ and g, respectively. If $f\colon \tilde{M} \to M$ is holomorphic, then*

$$b(f) = 2\big(\tilde{g} - 1 - \deg f(g-1)\big)\,.$$

E.g. this enables us to compute the genus of hyperelliptic curves.

Exercise 9.3.1 Show that the differentials $\omega_j = \frac{z^{j-1}}{w}dz$, $j = 1, \ldots, g$, form a basis of holomorphic differentials of the hyperelliptic Riemann surface given by $w^2 = \prod_{i=1}^n (z - z_i)$, $z_i \neq z_j$ for $i \neq j$, $n = 2g+1$ or $n = 2g+2$.

9.3.2 The Abel Theorem

By the Poincaré–Hopf index theorem any principle divisor is of degree zero. Abel's theorem answers precisely the question when a given degree zero divisor is principal. This is based on the simple observation that a holomorphic line bundle of degree zero is trivial if and only if there exists a non-trivial holomorphic section.

Theorem 9.3.5 (Abel's Theorem) *Let $D \in \mathrm{Div}_0 M$. Then*

$$D \text{ principal} \iff \mathscr{A}(D) = 0\,.$$

Proof. Let $L = [D]$ and φ its famous section. Clearly, φ is meromorphic with $(\varphi) = D$. Thus D is principal if and only if there is a meromorphic function f such that

$$D = (f) \iff \psi = \varphi/f \in H^0L \iff H^0L \neq \{0\} \iff L \text{ trivial}\,,$$

i.e. $\mathscr{A}(D) = 0$. □

The existence of non-trivial meromorphic sections implies that the Abel–Jacobi map is surjective group homomorphism. With Abel's theorem we get that $\mathscr{A}$ descends to a group isomorphism from the divisor class group to the Jacobi variety.

Corollary 9.3.6 $\mathscr{A}\colon \mathrm{Cl}\, M \to \mathrm{Jac}\, M$ *is an isomorphism of Abelian groups.*

9.3.3 The Jacobi Inversion Theorem

The Jacobi inversion theorem is a refinement of Corollary 9.3.6 on the particular form of the divisor. It says that each element of $\mathrm{Jac}\, M$ can be represented by a divisor of the form $D = \sum_{i=1}^{g}(p_i - p_0) = D_0 - gp_0$, $0 \leq D_0 \in \mathrm{Div}_g M$. One way to show this uses the following fact (cf. [11], page 237).

Proposition 9.3.7 *Let $f\colon \tilde{M} \to \hat{M}$ be a holomorphic map between equidimensional compact connected complex manifolds.If there is exists a point $p \in \tilde{M}$ such that $d_p f$ is bijective, then f is surjective.*

To apply the theorem to the problem one identifies the positive divisors of degree g on a compact Riemann surface M with unordered g-tuples, i.e. with points in the g-th *symmetric product* $M^{(g)}$—which is defined as the quotient of the g-th cartesian product M^g by the action of the symmetric group S_g: For $\sigma \in S_g$,

$$(p_1, \ldots, p_g) \sim (z_{\sigma_1}, \ldots, z_{\sigma_g})\,.$$

For a Riemann surface M the n-th symmetric product $M^{(n)}$ has a natural complex structure. This is basically due to the fundamental theorem of algebra—every unordered n-tuple of points in the complex plane can be identified with the zero set of a polynomial of degree n, which is unique up to a multiplicative constant and as such can be considered as a point in the n-dimensional complex projective space.

Clearly, if M is connected and compact, so is $M^{(g)}$. Hence the Jacobi inversion theorem in principle follows from the following lemma.

Lemma 9.3.8 *Generically the map $A\colon M^g \to (H^0K)^*$ given by $A(p_1, \ldots, p_g) = \sum_{i=1}^{g} \int_{p_0}^{p_i}$ is regular.*

Proof. If $\omega_1, \ldots \omega_g \in H^0K$ is a basis of holomorphic differentials, then its dual basis gives coordinates on $(H^0K)^*$. With respect to these coordinates, we have

$$A(p_1, \ldots, p_g) = \sum_i \int_{p_0}^{p_i} (\omega_1, \ldots, \omega_g) \in \mathbb{C}^g\,.$$

In particular, if z_i are coordinate charts at p_i, its Jacobi matrix is of the form

$$\begin{pmatrix} \frac{\omega_1}{dz_1} & \cdots & \frac{\omega_1}{dz_g} \\ \vdots & & \vdots \\ \frac{\omega_g}{dz_1} & \cdots & \frac{\omega_g}{dz_g} \end{pmatrix}.$$

Now we can choose $p_1 \in M$ such that $\frac{\omega_1}{dz_1}|_{p_1} \neq 0$ and subtract multiples of ω_1 from $\omega_2, \ldots, \omega_g$ such that $\frac{\omega_i}{dz_1}|_{p_1} = 0$ for $i > 1$. Then choose p_2 such that $\frac{\omega_2}{dz_2}|_{p_2} \neq 0$ and subtract multiples of ω_2 from $\omega_3, \ldots, \omega_g$ to annihilate the entries below $\frac{\omega_2}{dz_2}|_{p_2}$. Proceeding like that the Jacobian of $\mathcal{A}$ becomes an upper triangular matrix with non-zero entries on the diagonal and we are done. □

Another more direct way to prove the Jacobi inversion theorem is based on the Riemann–Roch theorem.

Theorem 9.3.9 (Jacobi Inversion Theorem) $\forall\, \mu \in \operatorname{Jac} M, p_0 \in M \;\; \exists p_1, \ldots, p_g \in M \colon \mathcal{A}\big(\sum_{i=1}^{g}(p_i - p_0)\big) = \mu$.

Proof. By Lemma 9.3.8 there are $(\tilde{p}_1, \ldots, \tilde{p}_g)$ such that $d_{(\tilde{p}_1,\ldots,\tilde{p}_g)}A$ is bijective. Thus A is locally invertible, i.e. there are open neighborhoods U_i of $\tilde{p}_i$ and V of $\tilde{\mu} = A(\tilde{p}_1, \ldots, \tilde{p}_g)$ such that $A \colon U_1 \times \cdots U_g \to V$ is invertible. In particular, given $\mu \in (H^0K)^*$ there is $n \in \mathbb{N}$ such that $\tilde{\mu} - \frac{1}{N}\mu \in V$. Hence there are $\hat{p}_1, \ldots, \hat{p}_g$ such that $\tilde{\mu} - \frac{1}{N}\mu = A(\hat{p}_1, \ldots, \hat{p}_g)$. Thus, with $\tilde{D} = \sum_{i=1}^{g} \tilde{p}_i$ and $\hat{D} = \sum_{i=1}^{g} \hat{p}_i$, we get

$$\mathcal{A}(\hat{D} - gp_0) = \mathcal{A}(\tilde{D} - gp_0) - \tfrac{1}{N}\mu \iff \mu = N\big(\mathcal{A}(\hat{D} - gp_0) - \mathcal{A}(\tilde{D} - gp_0)\big) + \mathcal{A}\big(N(\hat{D} - \tilde{D})\big).$$

It is left to show that $N(\hat{D} - \tilde{D}) \sim D - gp_0$ for some $D \geq 0$ with $\deg D = g$. Therefore consider the bundle $L = [N(\hat{D} - \tilde{D}) + gp_0]$. Then the Riemann–Roch theorem yields

$$h^0L \geq h^0L - h^0KL^{-1} = \deg L + 1 - g = 1.$$

So if $0 \neq \psi H^0 L$ and φ denotes the famous section of L we get a meromorphic function $f = \psi/\varphi$ with

$$(f) = D - N(\hat{D} - \tilde{D}) - gp_0 \iff N(\hat{D} - \tilde{D}) + (f) = D - gp_0,$$

where $D = (\psi) = \sum_{i=1}^{g} p_i$. Hence $\mu = \mathcal{A}(N(\hat{D} - \tilde{D})) = \mathcal{A}(D - gp_0) = \mathcal{A}\big(\sum_{i=1}^{g}(p_1 - p_0)\big)$. □

9.3.4 Branched Coverings of Riemann Sphere

Let $L \to M$ be a holomorphic line bundle of degree d over a compact Riemann surface of genus g. Then, if $d = g + 1$, the Riemann–Roch theorem yields that

$$h^0L \geq h^0L - h^0KL^{-1} = \deg L + (1 - g) = 2.$$

Thus there are two linearly independent sections $\psi_1, \psi_2 \in H^0L$ which by $f = \psi_2/\psi_1$ define a non-constant moromorphic function, i.e. a holomorphic map $f \colon M \to \mathbb{CP}^1$. Since f is non-constant, f is surjective—a branched covering. Surjectivity also yields that f is of degree ≥ 1. On the other hand, counted with multiplicity, ψ_1 has $d + 1$ zeros. Thus f has at most $d + 1$ poles and so $\deg f \leq f$. Since on every compact Riemann surface there exists holomorphic line bundles of arbitrary degree we have shown the following theorem.

Theorem 9.3.10 *Let M be a compact Riemann surface of genus g. Then there exists a holomorphic map $f \colon M \to \mathbb{CP}^1$ whose degree satisfies* $1 \leq \deg f \leq g + 1$.

In particular, since a holomorphic map f of degree one is a biholomorphism we get the following corollary.

Corollary 9.3.11 *Any Riemann surface of genus zero is biholomorphic to* $\mathbb{CP}^1$.

Proof. Choose some holomorphic line bundle $L \to M$ of degree one. Then $h^0L = 2$. If $\psi_1, \psi_2 \in H^0L$ are linearly independent, then $f := \psi_2/\psi_1$ is a meromorphic function, i.e. a holomorphic map $M \to \mathbb{CP}^1$. A holomorphic section of L has exactly one simple zero and so the preimage of $\infty \in \mathbb{CP}^1$ is a single point. In particular, $\deg f = 1$ and thus f is a biholomorphism. □

Corollary 9.3.12 *There is a holomorphic line bundle* $L \to M$ *such that* $\deg L = 1$ *and* $h^0L = 2$ *if and only if* M *is of genus zero.*

9.3.5 The Abel–Jacobi Map on Surfaces of Genus $g > 0$

From Corollary 9.3.12 we can draw the following conclusion on the zeros of holomorphic differentials.

Theorem 9.3.13 *On a Riemann surface of genus* $g > 0$ *there is no point at which all holomorphic differentials vanish simultaneously.*

Proof. Suppose that all holomorphic differentials vanish at a point $p \in M$. Then $h^0K[-p] = g$ and

$$h^0[p] - g = h^0[p] - h^0K[-p] = 1 + 1 - g = 2 - g\,.$$

Hence $h^0[p] = 2$ and so M is of genus zero—a contradiction. □

Corollary 9.3.14 *Let* M *be of genus* $g > 0$, $p_0 \in M$. *Then* $A\colon M \to \operatorname{Jac} M$, $A(p) := \mathcal{A}(p - p_0)$ *is an embedding.*

Proof. From the last theorem we get that A is an immersion, which is injective by Abel's theorem. Since M is compact, A is an embedding (Theorem 2.2.3). □

Thus each Riemann surface of genus larger than zero is holomorphically embedded into its Jacobian.

Moreover, for surfaces of genus $g = 1$ the Jacobi inversion theorem yields surjectivity of the Abel–Jacobi map. Thus we get the following theorem.

Theorem 9.3.15 *Every Riemann surface of genus one is biholomorphic to its Jacobi variety.*

In particular, each Riemann surface of genus one is a conformally equivalent to a flat torus $\mathbb{C}/\Lambda$. Clearly, if $\Lambda' = c\Lambda$ for $c \in \mathbb{C}^\times$, then $\mathbb{C}/\Lambda \cong \mathbb{C}/\Lambda'$. Hence we can assume that

$$\Lambda = \{m_1 + m_2\tau \mid m_1, m_2 \in \mathbb{Z}\}\,.$$

where $\operatorname{Im}\tau > 0$. Moreover, one can show that two such tori are isometric if and only if $\Lambda' = \Lambda$. In this case the generators $(1, \tau')$ and $(1, \tau)$ are then related by some $M \in \mathrm{SL}(2, \mathbb{Z})$. The corresponding action of $\mathrm{Sl}(2, \mathbb{Z})$ on τ is generated by the following two transformations:

$$\tau \mapsto \tau + 1\,, \quad \tau \mapsto -1/\tau\,.$$

Hence the Riemann surfaces of genus one can be identified with equivalence classes of points in the upper halfplane H^2. Each class has a representative τ with $|\tau| \geq 1$, $-\frac{1}{2} \leq \operatorname{Re}\tau \leq \frac{1}{2}$ and $\operatorname{Im}\tau \geq 0$. If the point does not lie on the boundary the representative is unique. The quotient $\mathrm{H}^2/\mathrm{SL}(2, \mathbb{Z})$ is called the moduli space.

9.4 Holomorphic Maps into Complex Projective Space

Let M be a complex manifold. There is a natural one-to-one correspondence between maps $M \to \mathbb{CP}^n$ and complex line subbundles of $\underline{\mathbb{C}}^{n+1}_M$:

$$\mathscr{C}^\infty(M;\mathbb{CP}^n) \ni f \mapsto f^*\mathrm{Taut}(\mathbb{CP}^n) \subset \underline{\mathbb{C}}^{n+1}_M .$$

Conversely, given a complex line subbundle $L \subset \underline{\mathbb{C}}^{n+1}_M$ we can define $f_L\colon M \to \mathbb{CP}^n$ by $f(p) = \pi(L_p) \in \mathbb{CP}^n$.

Remark 9.4.1 Similarly, higher rank subbundles correspond to maps into Grassmannians.

Recall that the tautological line bundle $\mathrm{Taut}(\mathbb{CP}^n)$ is a holomorphic line bundle.

Proposition 9.4.1 *A map $f\colon M \to \mathbb{CP}^n$ is holomorphic if and only if $f^*\mathrm{Taut}(\mathbb{CP}^n)$ is a holomorphic subbundle.*

Proof. If f is holomorphic, then any holomorphic frame of $\mathrm{Taut}(\mathbb{CP}^n)$ pulls back to a holomorphic frame of $f^*\mathrm{Taut}(\mathbb{CP}^n)$. Conversely, if $f^*\mathrm{Taut}(\mathbb{CP}^n) \subset \underline{\mathbb{C}}^{n+1}_M$ is a holomorphic subbundle, then at each p we find a holomorphic frame ψ of $f^*\mathrm{Taut}(\mathbb{CP}^n)$, i.e. a holomorphic map $\psi\colon M \to (\mathbb{C}^{n+1})^\times$ such that $\pi \circ \psi = f$. In particular, f is holomorphic. □

Definition 9.4.1 (Linearly Full) *A map $f\colon M \to \mathbb{CP}^n$ is called linearly full, if its image $f(M)$ is not contained in any projective hyperplane.*

Corollary 9.4.2 *Let $L := f^*\mathrm{Taut}(\mathbb{CP}^n)$, where $f\colon M \to \mathbb{CP}^n$ is holomorphic and linearly full. Then $h^0L^* \geq n+1$.*

Proof. Each element of $(\mathbb{C}^{n+1})^*$ restricts to a section ψ of $\mathrm{Taut}(\mathbb{CP}^n)^*$. Clearly, ψ is holomorphic. Since f is holomorphic, $f^*\psi \in H^0L^*$. This defines a linear map $(\mathbb{C}^{n+1})^* \to H^0L^*$. Since f is linearly full, this map is injective. Hence $h^0L^* \geq n+1$. □

9.4.1 Linear Systems and Some Bounds on their Dimension

Now let L be a complex line bundle over a complex manifold M. We want to construct a map from M into complex projective space. Therefore we start with a basepoint-free linear system.

Definition 9.4.2 (Basepoint-Free Linear System) *A basepoint-free linear system is a finite-dimensional complex subspace $V \subset \Gamma L$ such that*

(a) For each $p \in M$ there is $\psi \in V$ such that $\psi_p \neq 0$.
(b) For all $\psi, \varphi \in V$ the function $\psi/\varphi\colon M \setminus \varphi^{-1}\{0\} \to \mathbb{C}$ is holomorphic.

Such a basepoint-free linear system determines a unique $\bar\partial$-operator.

Theorem 9.4.3 *If $V \subset \Gamma L$ is a basepoint-free linear system, then there is a unique $\bar\partial$-operator such that $V \subset H^0L$.*

Proof. For each $p \in M$ there is $\psi \in V$ such that $\psi_p \neq 0$. Hence for each $\varphi \in \Gamma L$ there are $\psi_i \in V$ and $z_i \in \mathscr{C}^\infty M$ such that $\varphi = \sum_{i=1}^m z_i\,\psi_i$. Then

$$\bar\partial\varphi := \sum_{i=1}^m (d'' z_i)\,\psi_i .$$

defines a $\bar{\partial}$-operator. That $\bar{\partial}\varphi$ does not depend on the choice of z_i and ψ_i follows since any two sections in V differ by a holomorphic function. Clearly $V \subset H^0 L$ and $\bar{\partial}$ is uniquely determined by this condition. □

To each point $\xi \in L_p^*$ we can assign a linear functional $\delta_\xi \in (\Gamma L)^*$ as follows: For $\psi \in \Gamma L$,

$$\delta_\xi(\psi) := \xi(\psi_p)\,.$$

This yields a complex linear map $\delta\colon L^* \to (\Gamma L)^*$. Given a basepoint-free linear system $V \subset H^0 L$, the restriction $\delta\colon L^* \to V^*$ is smooth and descends to a smooth map $f_V : M \to \mathbb{P}(V^*)$:

$$f_V(p) = \delta_{L_p^*}\,.$$

The map f_V is called the *Kodaira correspondence*.

Theorem 9.4.4 *If $V \subset H^0 L$ is a basepoint-free linear system of complex dimension $m > 0$, then the Kodaira correspondence $f_V : M \to \mathbb{P}(V^*) \cong \mathbb{CP}^{m-1}$ is a linearly full holomorphic map.*

Proof. Exercise. □

Let us fix the notation for the degree and the dimension of holomorphic sections: For a holomorphic line bundle $L \to M$ over a compact Riemann surface of genus g, we write

$$d = \deg L\,, \quad n = h^0 L\,.$$

The Riemann–Roch theorem relates d to n. We are interested in bounds on n in terms of d. Let us collect the most obvious relations in the following proposition.

Proposition 9.4.5 *We have $n \geq d + 1 - g$. Moreover:*

(a) If $d < 0$, then $n = 0$.
(b) If $d = 0$, then $n \in \{0, 1\}$.
(c) If $d > 2g - 2$, then $n = d + 1 - g$.
(d) If $d = 2g - 2$, then $n \in \{d + 1 - g, d + 2 - g\}$.

Proof. (a) is the statement of Theorem 8.1.10 followed from the Poincaré–Hopf index theorem and the fact that holomorphic sections only have zeros of positive index. In particular, a holomorphic section of a degree zero bundle has no zeros at all. So there are only two cases—either L is non-trivial ($n = 0$) or L is trivial ($n = 1$). Let $\tilde{L} := KL^{-1}$ and $\tilde{n} = h^0\tilde{L}$. The Riemann–Roch theorem then states that

$$n = d + 1 - g + \tilde{n}\,.$$

Hence $n \geq d + 1 - g$. Moreover, since $\deg \tilde{L} = \deg KL^{-1} = 2g - 2 - d$, ($c$) and ($d$) follow from ($a$) and ($b$) applied to $\tilde{L}$. □

A less obvious bound is the following.

Theorem 9.4.6 *For $d \geq -1$, we have $n \leq d + 1$.*

Proof. For $n = 0$ the inequality holds. Suppose that $n > 0$. For $p_1, \ldots, p_{n-1} \in M$ pairwise distinct, $p_i \neq p_j$ for $i \neq j$, and $U_0 := H^0 L$. Define recursively

$$U_i := \{\psi \in U_{i-1} \mid \psi_{p_i} = 0\}\,.$$

Then $\dim U_i \geq n - i$. In particular, $\dim U_{n-1} \geq 1$. Hence there is a holomorphic section with at least $n - 1$ zeros. The Poincaré–Hopf index theorem then yields $d \geq n - 1$. □

9.4.2 Clifford's Theorem—A Tighter Upper Bound

Let $L \to M$ be a holomorphic line bundle of degree d over a compact Riemann surface of genus g and let $n = h^0 L$. We have seen that $n = 0$ for $d < 0$ and that $n = d + 1 - g$ for $d > 2g - 2$. Moreover, for $0 \leq d \leq 2g - 2$, we have shown that

$$d + 1 - g \leq n \leq d + 1 .$$

A tighter upper bound is given by the following theorem.

Theorem 9.4.7 (Clifford's Theorem) *Suppose* $n > 0$ *and* $n > d - (g - 1)$. *Then*

$$n \leq \frac{d}{2} + 1 .$$

Its proof is mainly based on the following dimensionality relation between three finite-dimensional complex vector spaces U, V, W on which exists a *bi-injective* complex bilinear pairing $*U \times V \to W$, i.e. $u * v = 0$ only if $u = 0$ or $v = 0$. The dimension of W is bound form below by the dimensions of U and V.

Example 9.4.1 Let $\mathbb{C}_{\leq d}[z]$ denote the space of polynomials of degree at most d. Then multplication of functions yields a biinjective bilinear pairing $\mathbb{C}_{\leq d_1}[z] \times \mathbb{C}_{\leq d_2}[z] \to \mathbb{C}_{\leq d_1 + d_2}[z]$ and we have

$$\dim \mathbb{C}_{\leq d_1 + d_2}[z] = d_1 + d_2 + 1 = \dim \mathbb{C}_{\leq d_1}[z] + \dim \mathbb{C}_{\leq d_2}[z] - 1 .$$

The example shows that the inequality from the next theorem is tight.

Theorem 9.4.8 *Let* U, V, W *be finite-dimensional complex vector spaces, such that* $\dim U + \dim V \leq \dim W$, *and* $*\colon U \times V \to W$ *bilinear, such that* $u * v = 0 \Rightarrow u = 0$ *or* $v = 0$. *Then*

$$\dim W \geq \dim U + \dim V - 1 .$$

We state the theorem here without proof—its proof is much harder and more interesting than one might think at first. A proof can be found in [12]. A more geometric proof is given in Appendix A.3.

Theorem 9.4.9 $h^0 L > 0,\ h^0 \tilde{L} > 0 \Rightarrow h^0 L\tilde{L} \geq h^0 L + h^0 \tilde{L} - 1$

Proof. Apply Theorem 9.4.8 to the map $*\colon H^0 L \times H^0 \tilde{L} \to H^0 L\tilde{L}$ defined by the tensor product. □

Proof of Theorem 9.4.7. Assume that $h^0 L, h^0 \tilde{L} > 0$, where $\tilde{L} = KL^{-1}$. Then, by Theorem 9.4.9 and the Riemann–Roch theorem,

$$g = h^0 K \geq n + \underbrace{\tilde{n}}_{n-d+g-1} - 1 = 2n - d + g - 2 .$$

Hence $2n \leq d + 2$ and thus $n \leq d/2 + 1$. □

9.4.3 Rational Curves—Holomorphic Line Bundles over $\mathbb{S}^2$

Over a compact Riemann surface of genus zero the inequality of the previous proposition becomes tight.

Theorem 9.4.10 *If $g = 0$ and $d \geq -1$, then $n = d + 1$.*

Proof. Exercise. □

Without loss of generality, $M = \mathbb{CP}^1$. The tautological line bundle $\mathcal{T} = \mathrm{Taut}(\mathbb{CP}^1)$ is of degree -1. By Corollary 9.1.17, there is only one holomorphic bundle for each degree d over the sphere. Thus, if $L \to M$ is a holomorphic line bundle of degree d, then

$$L \cong \mathcal{T}^{-d} .$$

Recall that a homogeneous polynomial of degree d on $\mathbb{C}^2$ is a polynomial $P \colon \mathbb{C}^2 \to \mathbb{C}$ such that $P(\lambda\psi) = \lambda^d P(\psi)$ for all $\lambda \in \mathbb{C}$ and $\psi \in \mathbb{C}^2$. Let V_d denote the space of homogeneous polynomials on $\mathbb{C}^2$.

Each $P \in V_d$ can be seen as a constant map defined on $\underline{\mathbb{C}}_M$. In particular, each P restricts to a map $\alpha \colon \mathcal{T} \to \mathbb{C}$. For $\lambda \in \mathbb{C}$ and $\psi \in \mathcal{T}_p$, we have

$$\alpha_p(\lambda\psi) = \lambda^d \alpha_p(\psi)$$

and thus $\alpha_p \in \mathcal{T}_p^{-d} = L_p$. Clearly, $\alpha \in H^0 L$. As $V_d \to \Gamma L$ is injective, we have $V_d \subset H^0 L$. Moreover, since there is a one to one correspondence between homogeneous polynomials of degree d on $\mathbb{C}^2$ and polynomials of degree d on $\mathbb{C}$, we know $\dim V_d = d + 1 = h^0 L$ and thus $V_d = H^0 L$.

To write the Kodaira correspondence $f_{H^0 L}$ in coordinates, consider the basis $\{\alpha_0, \ldots, \alpha_d\} \subset H^0 L$ given by

$$w^d, zw^{d-1}, \ldots, z^{d-1}w, z^d ,$$

where (z, w) are the standard coordinates of $\mathbb{C}^2$. Let $\{\alpha_0^*, \ldots, \alpha_d^*\} \subset V_d^*$ denote its dual basis. Then, for $\xi \in L_p^*$,

$$\delta_\xi|_{V_d} = \sum_{i=0}^{d} \langle \delta_\xi | \alpha_i \rangle \, \alpha_i^* .$$

Let $p \in \mathbb{CP}^1$ and $\psi = (\psi_1, \psi_2) \in \mathbb{C}^2$ such that $\pi(\psi) = p$. Then $\psi \in \mathcal{T}_p$ and hence $\psi^d \in \mathcal{T}_p^d = L_p^*$. Thus $\langle \delta_\xi | \alpha_i \rangle = \psi_1^i \psi_2^{d-i}$. The Kodaira correspondence $f \colon \mathbb{CP}^1 \to \mathbb{CP}^d$ is then given by

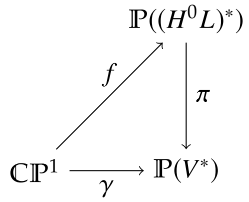
$$f([z, w]) = [(w^d, zw^{d-1}, \ldots, z^{d-1}w, z^d)] .$$

or, in affine coordinates, $z \mapsto (z, z^2, \ldots, z^d)$—the *rational normal curve*.

Definition 9.4.3 (Rational Curve) *The image of a non-constant holomorphic $\gamma \colon \mathbb{CP}^1 \to \mathbb{CP}^{m-1}$ of degree d, i.e. $d = -\deg \gamma^* \mathrm{Taut}(\mathbb{CP}^{m-1})$, is called a rational curve in $\mathbb{CP}^{m-1}$.*

A rational curve $\gamma \colon \mathbb{CP}^1 \to \mathbb{CP}^{m-1}$ of degree d is the same thing as an m-dimensional baspoint-free linear system $V \subset H^0 L$, where $L = \mathcal{T}^{-d} \cong \gamma^* \mathrm{Taut}(\mathbb{CP}^{m-1})^{-1}$.

Since the inclusion $V \to H^0 L$ is injective, its dual map $(H^0 L)^* \to V^*$ is surjective and we get the following commuting diagram:



Consider $\gamma\colon \mathbb{C} \to \mathbb{C}^n$ given by $\gamma(z) = \big(P_1(z)/Q_1(z), \dots, P_{m-1}(z)/Q_{m-1}(z)\big)$, where $P_i, Q_i\colon \mathbb{C} \to \mathbb{C}$ are polynomial. Then γ is an affine version of $\hat{\gamma}\colon \mathbb{C} \to \mathbb{C}^m$ given by

$$\hat{\gamma}(z) = \begin{pmatrix} Q_1(z)Q_2(z)\cdots Q_{m-1}(z) \\ P_1(z)Q_2(z)\cdots Q_{m-1}(z) \\ \vdots \\ Q_1(z)Q_2(z)\cdots P_{m-1}(z) \end{pmatrix} = \underbrace{\begin{pmatrix} a_{00} & \cdots & a_{0d} \\ \vdots & & \vdots \\ a_{m-1,0} & \cdots & a_{m-1,d} \end{pmatrix}}_{=A} \begin{pmatrix} 1 \\ z \\ \vdots \\ z^d \end{pmatrix}.$$

Hence, if $[A]$ denote the projective map given by A, then $\gamma = [A]f$.

9.4.4 Curves in $\mathbb{CP}^{m-1}$

Let M be a compact Riemann surface of genus g. The Kodaira correspondence then identifies pairs (L, V) consisting of a holomorphic line bundle $L \to M$ and an m-dimensional basepoint-free linear systems $V \subset H^0L$ with linearly full holomorphic maps $f\colon M \to \mathbb{CP}^{m-1}$:

$$(L, V) \quad \longleftrightarrow \quad f_V\,.$$

This understood, let us fix notation for degree and the dimensions of the space of holomorphic sections and the linear system:

$$d = \deg L\,, \quad n = h^0L\,, \quad m = \dim V\,.$$

Clearly, $m \leq n$. We are interested in further relations between d, n, and m an the genus g of M.

Theorem 9.4.11 *If $d = 1$ and $m \geq 2$, then $g = 0, m = 2$ and f is a biholomorphism.*

Proof. Corollary 9.3.12 we obtain that $g = 0$. In particular, if $m > 2$, then $n \geq m > 2 = d - (g-1)$ and Clifford's theorem yields that $m \leq n \leq d/2 + 1 < 2$—a contradiction. ☐

Corollary 9.4.12 *The only holomorphic curves of degree 1 in projective space are projective lines.*

Theorem 9.4.13 *If $d = 2$, then either*

- *$m = 2$, or*
- *$g = 0$ and $m = 3$.*

Proof. Suppose that $m \neq 2$. Then $n > 2$. If we assume that $g > 0$, then $n > d - (g-1)$ and, by Clifford's theorem, we get $m \leq n \leq d/2 + 1 = 2$—a contradiction. Thus $g = 0$. Now, if $n > 3$, then Clifford's theorem yields $n \leq 2$—again a contradiction. Thus $2 < m \leq n \leq 3$ and so $m = 3$. ☐

Theorem 9.4.14 *If $d = 3$ and $m \geq 3$, then either*

- *$g = 0$ and $m \in \{3, 4\}$, or*
- *$g = 1$ and $m = 3$.*

Proof. If $n > d - (g-1) = 4 - g$, then Clifford's theorem yields $2 \leq m \leq n < 3/2 + 1 < 3$—a contradiction. Thus $3 \leq m \leq n \leq 4 - g$. Thus, either $g = 0$ and $m \in \{3, 4\}$, or $g = 1$ and $m = 3$. ☐

Recall: Any rational curve defined on $\mathbb{CP}^1$ is a projection of the rational normal curve. For $d = 3$ the rational normal curve $f\colon \mathbb{CP}^1 \to \mathbb{CP}^3$ is given by

$$f([(z,w)]) = [(z^3, z^2 w, z w^2, w^3)]\,.$$

Corollary 9.4.15 *Cubic plane curves, i.e. $d = 3, m = 3$ are either rational, i.e. $g = 0$ and $n = 4$, or elliptic, i.e. $g = 1$.*

Proof. This follows from the previous theorem with $d = 3, m = 3$. □

9.4.5 The Kodaira Embedding Theorem

Let $L \to M$ be a holomorphic line bundle over a compact Riemann surface of genus g. We have seen that the Kodaira corresponce $f\colon M \to \mathbb{CP}^{m-1}$ of a basepoint-free linear system V of dimension $\dim V = m \geq 1$ is holomorphic. We want to see now that, provided the degree of L is large enough, H^0L is a basepoint-free linear system and that its Kodaira correspondence f actually provides an embedding of M into $\mathbb{CP}^{m-1}$.

The key to show the existence of a basepoint-free linear system is the Mittag–Leffler theorem.

Proposition 9.4.16 *For $\deg L > \deg K + 1$ the space of holomorphic sections H^0L is a basepoint-free linear system.*

Proof. Let $p \in M$. Since $\deg L = \deg K + 1$, we have $h^0KL^{-1}[p] = h^0KL^{-1} = 0$. The Riemann–Roch theorem then yields

$$h^0L[-p] = \deg L - g < \deg L + 1 - g = h^0L\,,$$

which, by Theorem 9.2.13, is equivalent to the existence of $\psi \in \Gamma L$ with $\psi_p \neq 0$. □

Since M is compact, injective immersions are embeddings. For the Kodaira correspondence both injectivity and the immersion property translate into existence of certain holomorphic sections.

Lemma 9.4.17 *Let $L \to M$ be a holomorphic line bundle, H^0L basepoint-free, f its Kodaira correspondence. Then:*

$$f \text{ injective} \iff \forall\, p,q \in M,\ p \neq q\ \exists \psi \in H^0L\colon\ \ \psi_p \neq 0\,,\ \ \psi_q = 0\,.$$

Proof. Let $p \neq q$ and let $\xi \in (L_p^*)^\times$ and $\zeta \in (L_q^*)^\times$ such that $f(p) = [\delta_\xi]$ and $f(q) = [\delta_\zeta]$. If $f(p) \neq f(q)$, then δ_ξ, δ_ζ are linearly independent and extend to a basis of (H^0L). The first vector ψ of the dual basis of H^0L then satisfies $1 = \delta_\xi(\psi) = \langle \xi | \psi_p \rangle$ and $0 = \delta_\zeta(\psi) = \langle \zeta | \psi_q \rangle = 0$. Thus $\psi_p \neq 0$ and, since $\zeta \neq 0$, $\psi_q = 0$. Conversely, if there is $\psi \in H^0L$ such that $\psi_p \neq 0$ and $\psi_q = 0$, then $\langle \xi | \psi_p \rangle \neq 0$ and $\langle \zeta | \psi_q \rangle = 0$. From this one easily concludes that δ_ξ and δ_ζ are linearly independent. Thus $f(p) \neq f(q)$. □

Lemma 9.4.18 *Let $L \to M$ be a holomorphic line bundle, H^0L basepoint-free, f its Kodaira correspondence. Then f is an immersion if and only if for each $p \in M$ there exists $\psi \in H^0L$ with a simple zero at p.*

Proof. Let $p \in M$. Suppose there is $0 \neq \psi \in H^0L$ with simple zero at p. Since H^0L is basepoint-free we can extend ψ to a basis $\psi_1 = \psi, \psi_2, \ldots, \psi_n \in H^0L$ with $\psi_{n,p} \neq 0$. Then, locally at p, with respect to the dual basis, f is given in homogeneous coordinates by

$$f = [f_1, f_2, \ldots, f_{n-1}, 1]\,,$$

where $f_i = \psi_i/\psi_n$. Then, if z is a complex coordinate at p with $z(p) = 0$, $f_1 = zg$ for some holomorphic function h such that $h(p) \neq 0$. Thus $d_p f_1 = h(p) d_p z \neq 0$ and f is an immersion. The converse is left as an exercise. □

This in turn can be transformed into a statement on the dimension of spaces of holomorphic sections.

Theorem 9.4.19 *Let $L \to M$ be a holomorphic line bundle, H^0L basepoint-free, f its Kodaira correspondence. Then:*

$$f \text{ embedding} \iff \forall\, p, q \in M: \quad h^0L[-p-q] < h^0L[-q]\,.$$

Proof. By Lemma 9.4.17, f is injective if and only if for any two distinct points $p, q \in M$ there is $\psi \in H^0L$ such that $\psi_p \neq 0$ but $\psi_q = 0$. By Theorem 9.2.13 this is equivalent to

$$h^0L[-p-q] < h^0L[-q]\,.$$

The very same equation with $p = q$ is equivalent to the existence of $\psi \in H^0L$ with a simple zero at p, which, by Lemma 9.4.18, is equivalent to f being an immersion. Since M is compact injective immersions are embeddings. □

Theorem 9.4.20 (Kodaira Embedding Theorem) *Let $L \to M$ be a compact Riemann surface of genus g of degree $d > 2g$, then the Kodaira correspondence $f: M \to \mathbb{P}((H^0L)^*) \cong \mathbb{CP}^{d-g}$ is an embedding.*

Proof. Since $d > 2g = \deg K + 2$, we have $h^0KL^{-1} = 0$. Thus the Riemann–Roch theorem yields

$$h^0L = d + 1 - g > 2g + 1 \geq 1\,.$$

Thus, by Proposition 9.4.16, H^0L is basepoint-free and the Kodaira correspondence is a well-defined map $f: M \to \mathbb{CP}^{d-g}$. Since $d > \deg K + 2$, we have $h^0KL^{-1}[p] = h^0KL^{-1}[p+q] = 0$ and thus

$$h^0L[-p-q] = d - 1 - g < d - g = h^0L[-p]$$

for all $p, q \in M$, by the Riemann–Roch theorem. Thus the claim follows from Theorem 9.4.19. □

Remark 9.4.2 Since on every compact Riemann surface there exist holomorphic line bundles of arbitrary degree, every compact Riemann surfaces can be embedded into some complex projective space.

The following important theorem is out of the scope of the lecture (cf. [11]).

Theorem 9.4.21 (Chow's Theorem) *Each compact complex submanifold of complex projective space is algebraic, i.e. it is the zero set of homogeneous polynomials.*

Corollary 9.4.22 *Every compact Riemann surface is algebraic.*

9.4.6 Canonical Curves and Hyperellipticity

Let M be a compact Riemann surface of genus $g \geq 2$. By Theorem 9.3.13 the holomorphic differentials H^0K forms a g-dimensional basepoint-free linear system. Its Kodaira correspondence is a holomorphic map

$$\iota_K: M \to \mathbb{CP}^{g-1}\,,$$

which is called the *canonical mapping of M*. Its image $\iota_K(M) \subset \mathbb{CP}^{g-1}$ is called the *canonical curve of M*.

Theorem 9.4.23 *The canonical mapping ι_K is an embedding if and only if $h^0[p+q] = 1$ for all $p, q \in M$.*

Proof. By Theorem 9.4.19, ι_K is an embedding if and only if $h^0K[-p-q] < h^0K[-p]$ for all $p, q \in M$. Moreover, by Corollary 9.2.12 and the fact that H^0K is basepoint-free, we get $h^0K[-p] = g-1$. Thus

$$h^0K[-p-q] < h^0K[-p] = g-1\,.$$

Moreover, the Riemann–Roch theorem yields

$$h^0K[-p-q] = g-3+h^0[p+q]\,.$$

Thus ι_K is an embedding if and only if

$$h^0K[-p-q] < h^0K[-p] \iff h^0[p+q] = 1\,.$$

for all $p, q \in M$. □

Definition 9.4.4 (Hyperelliptic Riemann Surface) *A Riemann surface is called hyperelliptic if there are $p, q \in M$ such that $h^0[p+q] > 1$.*

Theorem 9.4.24 *M is hyperelliptic if and only if it is a 2-sheeted branch covering of $\mathbb{CP}^1$.*

Remark 9.4.3 Hyperelliptic Riemann surfaces have a holomorphic involution.

Corollary 9.4.25 *The canonical mapping is an embedding if and only if M is not hyperelliptic.*

Theorem 9.4.26 *If $g = 2$, then M is hyperelliptic.*

Proof. This follows from the Riemann–Roch theorem—the details are left as exercise. □

For $p, q \in M$ the Riemann–Roch theorem yields

$$h^0[p+q] = 3-g+h^0K[-p-q]$$

Moreover, by Corollary 9.2.12, we have $h^0K[-p] = h^0K[-p-q]+b$ with $b \in \{0,1\}$ and $h^0K[-p] = g-1$. Hence

$$h^0[p+q] = h^0K[-p-q]+3-g = 2-b\,.$$

Thus M is hyperelliptic if there are p, q such that $b = 0$. By Theorem 9.2.13 this means that there is no $\psi \in H^0K[-p]$ such that $\psi_q \neq 0$. Since $H^0K[-p]$ is isomorphic to the holomorphic differentials which vanish at p, we find that

$$M \text{ hyperelliptic} \iff \exists\, p, q \in M \;\; \forall \omega \in H^0K: \;\; \omega_p = 0 \Rightarrow \omega_q = 0\,.$$

—a rather special situation. That the generic Riemann surface of genus $g \geq 3$ is nonhyperelliptic is shown by Riemann's count.

Appendix

A Miscellaneous

A.1 Paracompactness of Manifolds

Let X be a topological space and $\mathscr{A}$ be an open cover. A refinement of $\mathscr{A}$ is an open cover $\mathscr{B}$ such that each $U \in \mathscr{A}$ is contained in some $V \in \mathscr{B}$. A collection $\mathscr{C}$ of subsets of X is called *locally finite*, if each point $x \in X$ has a neighborhood V such that $V \cap U \neq \emptyset$ for only finitely many $U \in \mathscr{C}$. If each open cover of X has a locally finite refinement, then X is called *paracompact*. In the end we want to show that manifolds are paracompact.

We start spelling out an immediate consequence of being locally euclidean.

Corollary A.1.1 *Every manifold is locally compact, i.e. each point has a compact neighborhood.*

A *precompact subset* is a subset whose closure is compact.

Lemma A.1.2 *On a locally compact Hausdorff space each basis of topology has a refinement to a basis, which only consists of precompact sets.*

Proof. Let $\mathscr{B}$ be a basis of a locally compact Hausdorff space X. Let $\mathscr{B}_0 := \{U \in \mathscr{B} \mid U \text{ precompact}\}$. We claim that $\mathscr{B}$ still forms a basis. Let $V \overset{\circ}{\subset} X$. Every point $p \in V$ has a compact neighborhood $K_p \subset V$. The interior $\mathring{K}_p$ then contains an open neighborhood $U_p \in \mathscr{B}$ of p. Since $K_p \subset X$ is compact and X is Hausdorff, K_p is closed. In particular, the closure of U_p in X coincides with the closure of U_p in K_p, which—as closed subset of a compact space—is compact. Hence U_p is precompact in X and thus $U_p \in \mathscr{B}_0$. Moreover, $V = \bigcup_{p \in V} U_p$. □

Lemma A.1.3 (Existence of Compact Exhaustion) *Every locally compact 2nd-countable Hausdorff space X admits a compact exhaustion, i.e. a countable collection of compact sets $\{K_i \subset X\}_{i \in \mathbb{N}}$ such that $\mathring{K}_i \subset K_{i+1}$ and $X = \bigcup_{i \in \mathbb{N}} K_i$.*

Proof. By 2nd-countability and Lemma A.1.2, there is a basis of topology $\{U_i\}_{i \in \mathbb{N}}$, such that U_i is precompact for each $i \in N$. Then we define a sequence of compact sets K_i recursively by setting $K_0 = \overline{U}_0$, $n_0 = 0$ and

$$K_{i+1} := \bigcup_{j=1}^{n_{i+1}} \overline{U_j}\,, \quad n_{i+1} := \min\left\{n \in \mathbb{N} \mid \bigcup_{j=1}^{n} U_i \supset K_i, \quad n > n_i\right\}.$$

Note, since K_i is compact, we have $n_i \in \mathbb{N}$. In particular, K_{i+1} is compact. $\mathring{K}_i \subset K_{i+1}$ for all $i \in \mathbb{N}$ and $\bigcup_{i=0} K_i = X$ follows from construction. □

Theorem A.1.4 (Paracompactness of Manifolds) *Every locally compact 2nd-countable Hausdorff space is paracompact. In fact, given a locally compact 2nd-countable Hausdorff space X, an open cover $\mathscr{C}$ of X, and any basis $\mathscr{B}$ of the topology of X, there exists a countable, locally finite refinement of $\mathscr{C}$ consisting of elements of $\mathscr{B}$.*

Proof. Let X be a locally compact 2nd-countable Hausdorff space and let $\mathscr{C}$ be an open cover of X. Let further $\mathscr{B}$ denote a basis of topology. By Lemma A.1.2, there is a refinement $\mathscr{B}_0$ of $\mathscr{B}$ which consists only of precompact sets. By Lemma A.1.3, there exists a compact exhaustion $\{K_i\}_{i\in\mathbb{N}}$ of X. Now, set

$$V_j := K_{j+1} \setminus \mathring{K}_j\,, \quad W_j := \mathring{K}_{j+2} \setminus K_{j-1}\,, \quad \text{for} \;\; j \in \mathbb{N}\,,$$

where we used the convention that $K_{-1} = \emptyset$. Note that V_j is compact, contained in $W_j \overset{\circ}{\subset} X$. For each $x \in V_j$ there is some $U_x \in \mathscr{C}$ containing x. Thus there is $U_{x,j} \in \mathscr{B}$ such that $x \in U_{x,j} \subset U_x \cap W_j$. These $U_{x,j}$ form an open cover for V_j and hence there is a finite subcover. The union of these subcovers over $j \in \mathbb{N}$ forms a countable cover of X, which refines $\mathscr{C}$. Furthermore,

$$W_j \cap W_k \neq \emptyset \quad \Longleftrightarrow \quad |k - j| \leq 2\,.$$

Since each subcover of V_j is finite, we get that the constructed cover is locally finite. □

A.2 Riemannian Immersions and Submersions

Definition A.2.1 (Riemannian Immersion) *A Riemannian immersion is an immersion $f\colon S \to M$ between Riemannian manifolds (S, g) and $(M, \langle .,. \rangle)$ such that*

$$g(X, Y) = f^*\langle .,. \rangle(X, Y) := \langle df(X), df(Y)\rangle\,, \quad \textit{for all} \;\; X, Y \in \Gamma TS\,.$$

Remark A.2.1 Suppose S has no metric, then $g := f^*\langle .,. \rangle$ defines a metric on S, which turns f into a Riemannian immersion. In this context g is called the *first fundamental form*.

Given an immersion $f\colon S \to M$ we obtain an orthogonal splitting of f^*TM into TS and normal bundle $\perp_f S$ with fiber $\perp_f S_p := (\operatorname{im} d_f)^\perp$ over $p \in S$—the identification is given by

$$\phi\colon TS \oplus \perp_f S \to f^*TM\,, \quad (X, \xi) \mapsto df(X) + \xi\,.$$

Clearly, ϕ is an isomorphism of vector bundles. Hence $f^*\nabla$ induces a connection $\tilde{\nabla}$ on $TS \oplus \perp_f S$ which then splits in four parts:

$$\tilde{\nabla} = \begin{pmatrix} \nabla^1 & A \\ B & \nabla^2 \end{pmatrix}\,,$$

where $A \in \Omega^1(S; \operatorname{Hom}(\perp_f S; TS))$, $B \in \Omega^1(S; \operatorname{Hom}(TS; \perp_f S))$ and ∇^1 resp. ∇^2 are connections on TS resp. $\perp_f S$.

Now, if S has a Riemannian metric g, then we can build the orthogonal sum $TS \oplus_\perp \perp_f S$ and, if f is a Riemannian immersion, Φ becomes an isomorphism of euclidean vector bundles. In particular,

$$\tilde{\nabla} = \begin{pmatrix} \nabla^1 & -\mathrm{II}^* \\ \mathrm{II} & \nabla^2 \end{pmatrix}\,.$$

with ∇^1 and ∇^2 metric and $\mathrm{II} \in \Omega^1(TS; \operatorname{Hom}(TS; \perp_f S))$. The tensor II is called the *second fundamental form*.

Let $\omega_S \in \Omega^1(S; TS)$ and $\omega_M \in \Omega^1(M; TM)$ denote the tautological 1-forms and consider ϕ as element of $\Gamma\operatorname{Hom}(TS \oplus \perp_f S; f^*TM)$, i.e. as a homomorphism-valued 0-form. Note, that

$$f^*\omega_M = df = \phi \begin{pmatrix} \omega_S \\ 0 \end{pmatrix}$$

By the definition of $\tilde{\nabla}$, ϕ is parallel. Hence we have

$$0 = f^*(T^\nabla) = f^*(d^\nabla \omega_M) = d^{f^*\nabla}\left(f^*\omega_M\right) = d^{f^*\nabla}\left(\phi \begin{pmatrix} \omega_S \\ 0 \end{pmatrix}\right) = \phi\, d^{\tilde{\nabla}} \begin{pmatrix} \omega_S \\ 0 \end{pmatrix} = \begin{pmatrix} d^{\nabla^1}\omega_S \\ \mathrm{II} \wedge \omega_S \end{pmatrix} = \begin{pmatrix} T^{\nabla^1} \\ \mathrm{II} \wedge \omega_S \end{pmatrix}.$$

Note that $\mathrm{II} \wedge \omega_S(X, Y) = \mathrm{II}_X Y - \mathrm{II}_Y X$. Thus $\mathrm{II}(X, Y) = \mathrm{II}_X Y$ is symmetric in X, Y and ∇^1 is torsion-free, i.e. ∇^1 is the Levi–Civita connection of S. In particular we have shown the following useful theorem.

Theorem A.2.1 *If $f\colon S \to M$ is a Riemannian immersion, then*

$$(f^*\nabla)_X df(Y) = df(\nabla_X Y) + \mathrm{II}(X, Y),$$

where $\mathrm{II} \in \Gamma\mathrm{Sym}^2(TS; \perp_f S)$ denotes the second fundamental form.

Now, let $f\colon \hat{M} \to M$ be a submersion. Then the the *vertical bundle* V is defined to be the kernel of df, $V := \ker df$, i.e. it consists of all the vector in $T\hat{M}$ which are tangent to the fibers of f. Now, given a Riemannian metric on $\hat{M}$, we can define the *horizontal bundle* H to be the orthogonal bundle of V, $H := V^\perp$. Thus we have an orthogonal splitting

$$T\hat{M} = V \oplus_\perp H.$$

Note also that the horizontal bundle is isomorphic to f^*TM. The isomorphism is just given by the restriction of df to H,

$$H \ni \hat{X} \mapsto df(\hat{X}) \in f^*TM.$$

In particular, for each vector field $X \in \Gamma TM$ there is a horizontal vector field $X_H \in \Gamma H \subset \Gamma T\hat{M}$ such that

$$f^*X = df \circ X_H.$$

We call X_H the *horizontal lift* of X.

Definition A.2.2 (Riemannian Submersion) *A Riemannian submersion is a submersion $f\colon \hat{M} \to M$ between Riemannian manifolds $\hat{M}$ and M such that the restriction*

$$df\colon H \to f^*TM$$

is an isomorphism of euclidean vector bundles.

Remark A.2.2 In particular, along a fiber, the horizontal spaces are isometrically identified.

Now, given a Riemannian submersion, we would like to know how the Levi–Civita connections are related.

Let $X, Y, Z \in \Gamma TM$. Then, by Theorem 3.3.7, we have $f^*[Y, Z] = df([Y_H, Z_H])$ and thus

$$f^*\langle X, [Y, Z]\rangle = \langle f^*X, f^*[Y, Z]\rangle = \langle df(X_H), df([Y_H, Z_H])\rangle = \langle X_H, [Y_H, Z_H]\rangle.$$

Moreover, we have

$$f^*(X\langle Y, Z\rangle) = (f^*X)\langle Y, Z\rangle = df(X_H)\langle Y, Z\rangle = X_H(f^*\langle Y, Z\rangle) = X_H\langle df(Y_H), df(Z_H)\rangle = X_H\langle Y_H, Z_H\rangle.$$

Together with the Koszul formula, this proves the following theorem.

Theorem A.2.2 *Let $f\colon \hat{M} \to M$ be a Riemannian submersion, then*

$$f^*\langle \nabla_X Y, Z\rangle = \langle \nabla_{X_H} Y_H, Z_H\rangle.$$

A.3 A Proof of Theorem 9.4.8

In order to show Theorem 9.4.8 we apply Theorem A.3.1 to the so-called *Segré embedding*: Let U, V be complex vector spaces, $\dim U = r$, $\dim V = s$, the *Segré embedding* is given by

$$\sigma\colon \mathbb{CP}^{r-1} \times \mathbb{CP}^{s-1} \to \mathbb{CP}^{rs-1}, \quad ([\psi],[\varphi]) \mapsto [\psi \otimes \varphi] \in \mathbb{P}(U \otimes V)\,.$$

To formulate Theorem A.3.1 we still need the notion of sectional curvature: Let M be a Riemannian manifold, $p \in M$ and $E \subset T_pM$ be a 2-dimensional subspace. Then

$$K_E := \frac{\langle R(X,Y)Y, X\rangle}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2}\,,$$

where $E = \operatorname{span}\{X, Y\}$, is called the *sectional curvature of* E. Using the symmetries of the Riemannian curvature tensor one easily checks that K_E is well-defined.

Theorem A.3.1 *Let $M_0, M_1 \subset M$ be compact complex submanifolds of a Kähler manifold with positive sectional curvature,* $\dim M_0 + \dim M_1 \geq \dim M$. *Then*

$$M_0 \cap M_1 \neq \emptyset\,.$$

The proof of Theorem A.3.1 relies on the *first variational formula of length*: Let

$$\gamma\colon (-\varepsilon,\varepsilon) \times [0, L] \to M, \quad \gamma_t(s) = \gamma(t,s)\,,$$

be a smooth variation of the curve γ_0 parametrized by arclength, $|\gamma_0'| = 1$. Then, with $Y = \frac{\partial\gamma}{\partial t}\big|_{(0,s)}$. we have

$$\tfrac{d}{dt}\big|_{t=0} L(\gamma_t) = \langle Y, \gamma_0'\rangle\big|_0^L - \int_0^L \langle Y, \gamma_0''\rangle\,.$$

Proof. Since ∇ is torsion-free we have $|\gamma_t'|^{\cdot} = \langle(\gamma')^{\cdot}, \gamma'\rangle/|\gamma'| = \langle\dot\gamma', \gamma'\rangle/|\gamma'|$. Thus

$$\tfrac{d}{dt}\big|_{t=0} L(\gamma_t) = \int_0^L \langle Y', \gamma_0'\rangle = \int_0^L \langle Y, \gamma_0'\rangle - \langle Y, \gamma_0''\rangle$$

Since the distance $d\colon M \times M \to [0,\infty)$ is a continuous function it has a minimum L on a compact set. Thus for two compact submanifolds $M_0, M_1 \subset M$ of a complete Riemannian manifold there is always a geodesic γ_0 connecting them $\gamma_0(0) \in M_0, \gamma_0(L) \in M_1$ which realizes this distance. Since γ_0 was a minimum of the distance, we have $\frac{d}{dt}|_0 L(\gamma_t) = 0$ for all variations γ of γ_0 with $\gamma_t(0) \in M_0$ and $\gamma_t(L) \in M_1$ for all $t \in (-\varepsilon,\varepsilon)$. Since γ_0 is a geodesic, we obtain from the first variational formula

$$0 = \langle Y(L), \gamma_0'(L)\rangle - \langle Y(0), \gamma_0'(0)\rangle\,.$$

Since we can achieve variations with arbitrary vectors $Y(0), Y(L)$ we conclude that $\gamma_0'(0)$ is perpendicular to $T_{\gamma_0(0)}M_0$ and $\gamma_0'(1)$ is perpendicular to $T_{\gamma_0(1)}M_1$.

Lemma A.3.2 *Let $M_0, M_1 \subset M$ be two compact submanifolds of a complete Riemannian manifold and $\gamma\colon (-\varepsilon,\varepsilon) \times [0, L] \to M$ be a variation of the shortest curve γ_0 connecting M_0 to M_1 such that $\eta_0(t) := \gamma_t(0) \in M_0$ and $\eta_1(t) := \gamma_t(L) \in M_1$. Then*

$$\tfrac{d^2}{dt^2}\big|_{t=0} L(\gamma_t) = \langle \eta_1''(0), \gamma_0'(L)\rangle - \langle \eta_0''(0), \gamma_0'(0)\rangle + \big\langle \hat Y, \hat Y'\big\rangle\Big|_0^L - \int_0^L \langle \hat Y'' + R(\gamma_0', \hat Y)\gamma_0', \hat Y\rangle\,,$$

where $\hat Y$ denotes the normal part of the variational vector field of γ.

Proof. As before, we have $|\gamma_t'|^{\cdot} = \langle \dot\gamma', \gamma' \rangle / |\gamma'| = \langle \dot\gamma', \gamma'/|\gamma'| \rangle$. Thus

$$\begin{aligned} |\gamma_t'|^{\cdot\cdot} &= \left\langle (\dot\gamma')^{\cdot}, \tfrac{\gamma'}{|\gamma'|} \right\rangle + \left\langle \dot\gamma', (\tfrac{\gamma'}{|\gamma'|})^{\cdot} \right\rangle \\ &= \left\langle (\dot\gamma')^{\cdot}, \tfrac{\gamma'}{|\gamma'|} \right\rangle + \left\langle \dot\gamma', \tfrac{1}{|\gamma'|}\left((\gamma')^{\cdot} - \tfrac{\langle (\gamma')^{\cdot}, \gamma' \rangle}{\langle \gamma' \gamma' \rangle} \gamma' \right) \right\rangle \\ &= \left\langle (\dot\gamma')^{\cdot}, \tfrac{\gamma'}{|\gamma'|} \right\rangle + \left\langle \dot\gamma', \tfrac{1}{|\gamma'|} (\dot\gamma')^{\perp} \right\rangle, \end{aligned}$$

where $(.)^{\perp}$ denotes the projection to the normal space of γ. Furthermore,

$$((\dot\gamma)')^{\cdot} = R(\dot\gamma, \gamma')\dot\gamma + \ddot\gamma' .$$

So we get for $t = 0$

$$\left|\gamma_t'\right|^{\cdot\cdot} = \left\langle R(Y, \gamma')Y, \gamma_0' \right\rangle + \left\langle \dot Y', \gamma_0' \right\rangle + \left\langle Y', (Y')^{\perp} \right\rangle .$$

Note that, since $\gamma_0'' = 0$, we have $(Y^{\perp})' = (Y')^{\perp}$ and by the symmetries of the curvature tensor

$$\left\langle R(Y, \gamma_0')Y, \gamma_0' \right\rangle = \left\langle R(Y^{\perp}, \gamma_0')Y^{\perp}, \gamma_0' \right\rangle .$$

Thus,

$$\begin{aligned} \tfrac{d^2}{dt^2}\Big|_{t=0} L(\gamma_t) &= \int_0^L \left\langle R(Y^{\perp}, \gamma')Y^{\perp}, \gamma_0' \right\rangle + \left\langle (Y^{\perp})', (Y^{\perp})' \right\rangle + \left\langle \dot Y', \gamma_0' \right\rangle \\ &= -\int_0^L \left\langle Y^{\perp}, (Y^{\perp})'' + R(Y^{\perp}, \gamma')\gamma_0' \right\rangle + \left(\left\langle (Y^{\perp}), (Y^{\perp})' \right\rangle + \left\langle \dot Y, \gamma_0' \right\rangle \right)' . \end{aligned}$$

□

Furthermore, given a Riemannian immersion $f\colon M \to \tilde M$, then for $X, Y \in \Gamma TM$, we have

$$(f^*\tilde\nabla)_X df(Y) = df(\nabla_X Y) + \mathrm{II}(X, Y),$$

where II denotes the second fundamental form (cf. Appendix A.2). Using this one easily shows the following

Theorem A.3.3 *If $\tilde M$ is Kähler, M complex and $f\colon M \to \tilde M$ a holomorphic immersion. Then, for $X, Y \in \Gamma TM$,*

$$\mathrm{II}(X, JY) = J\mathrm{II}(X, Y) = \mathrm{II}(JX, Y) .$$

Proof. Exercise. □

Corollary A.3.4 *In the above situation we have,* $\mathrm{II}(JX, JX) = -\mathrm{II}(X, X)$.

Remark A.3.1 In particular, in this situation the mean curvature of M is zero, i.e. M is minimal.

Also note that if $\gamma\colon I \to M$ is a smooth curve and $\tilde\gamma = f \circ \gamma$, then

$$\tilde\gamma'' = df(\gamma'') + \mathrm{II}(\gamma', \gamma') .$$

Proof of Theorem A.3.1. Let γ denote the shortest curve connecting M_0 to M_1. Assume that $L(\gamma) = L > 0$, i.e. $M_0 \cap M_1 \neq \emptyset$. Then we know that $N_0 := \gamma'(0)$ is a normal vector of M_0 and $N_1 := \gamma'(L)$ is a normal vector of M_1. Let

$$P_\gamma\colon T_{\gamma(0)}M \to T_{\gamma(L)}M$$

denote the parallel transport along γ. Since ∇ is metric and complex, P_γ is orthogonal and a complex linear isomorphism. Since γ is a geodesic, γ' is parallel and hence

$$P_\gamma(N_0) = N_1 \,.$$

Now, let us consider $U = P_\gamma(T_{\gamma(0)}M_0)$ and $V = T_{\gamma(L)}M_1$. Then, since

$$\dim U + \dim V \geq m = \dim M$$

and

$$0 \neq N_1 \in (U+V)^\perp$$

we have

$$1 \leq \dim(U+V)^\perp = m - (\dim U + \dim V - \dim(U \cap V)) \leq \dim(U \cap V)\,.$$

So there is a parallel vector field Y along γ such that $Y(0)$ is tangent to M_0 and $Y(L)$ is tangent to M_1. If we now vary γ such that $\dot{\gamma} = Y$ and apply Lemma A.3.2, we get

$$0 \leq \langle \ddot{\eta}_1(L), \gamma'(L)\rangle - \langle \ddot{\eta}_0(0), \gamma'(0)\rangle - \int_0^L K_{\operatorname{span}\{Y,\gamma'\}}\,.$$

Since $K > 0$, we have then

$$0 < \langle \ddot{\eta}_1(L), \gamma'(L)\rangle - \langle \ddot{\eta}_0(0), \gamma'(0)\rangle = \langle \mathrm{II}(Y_L, Y_L), \gamma'(L)\rangle - \langle \mathrm{II}(Y_0, Y_0), \gamma'(0)\rangle\,.$$

Since U and V were complex, JY has the same properties as Y, but $\mathrm{II}(JY, JY) = -\mathrm{II}(Y, Y)$ by the last corollary. This yields the above inequality with opposite sign, which is a contraction. □

Theorem A.3.5 *Let U, V, W be finite-dimensional complex vector spaces, such that $\dim U + \dim V \leq \dim W$, and $*\colon U \times V \longrightarrow W$ bilinear, such that $u * v = 0 \Rightarrow u = 0$ or $v = 0$. Then*

$$\dim W \geq \dim U + \dim V - 1\,.$$

Proof. Let $\dim U = r$, $\dim V = s$, $\dim W = t$ and consider the *Segré embedding* by

$$\sigma\colon \mathbb{CP}^{r-1} \times \mathbb{CP}^{s-1} \longrightarrow \mathbb{CP}^{rs-1}, \quad ([\psi],[\varphi]) \mapsto [\psi \otimes \varphi] \in \mathbb{P}(U \otimes V)\,.$$

We leave as an exercise to show that σ defines a holomorphic embedding.
Now choose bases $\psi_1, \ldots, \psi_r$ of U and $\varphi_1, \ldots, \varphi_s$ of V and define a linear map

$$f\colon U \otimes V \longrightarrow W \quad \text{by} \quad f(\psi_i \otimes \varphi_j) = \psi_i * \varphi_j\,.$$

Then

$$\dim \ker f = rs - \dim \operatorname{im} f \geq rs - t\,.$$

Then $\tilde{M} := \mathbb{P}(\ker f) \subset \mathbb{CP}^{rs-1}$ is of dimension $\geq rs - t - 1$. Furthermore, we have that $\hat{M} = \operatorname{im} \sigma$ is of dimension $r + s - 2$. Suppose now that $t < r + s - 1$, i.e. $t \leq r + s - 2$. Then

$$\dim \tilde{M} + \dim \hat{M} \geq r + s - 2 + rs - t - 1 \geq rs - 1 = \dim \mathbb{CP}^{rs-1}\,.$$

Thus, by Theorem A.3.1, $\hat{M} \cap \tilde{M} \neq \emptyset$ and hence there are $0 \neq \psi \in U$ and $0 \neq \varphi \in V$ with $\psi * \varphi = f(\psi \otimes \varphi) = 0$, which is a contradiction to the *"zero-divisor free"* property of $*$. □

Bibliography

Here are the references in citation order.

[1] Franz Pedit. $\mathbb{C}$-*Seminar*. Lecture Notes. 2006 (cited on page iii).

[2] Ulrich Pinkall. *Riemann Surfaces*. Lecture Notes. 2018 (cited on page iii).

[3] Hermann Weyl. *Die Idee der Riemannschen Fläche*. B. G. Teubner, 1913 (cited on page v).

[4] Lars V. Ahlfors. *Complex Analysis: An Introduction to the Theory of Analytic Functions of One Complex Variable*. International series in pure and applied mathematics. McGraw-Hill, 1979 (cited on page v).

[5] Frank W. Warner. *Foundations of Differentiable Manifolds and Lie Groups*. Graduate Texts in Mathematics. Springer, 1983 (cited on pages 22, 97).

[6] Jerry L. Kazdan. *Applications of Partial Differential Equations to Problems in Geometry*. Lecture Notes (last revised 2016). 1983 (cited on page 43).

[7] Shoshichi Kobayashi. *Differential Geometry of Complex Vector Bundles*. Princeton University Press, 1987 (cited on pages 93, 94).

[8] Raymond O. Wells. *Differential Analysis on Complex Manifolds*. Graduate Texts in Mathematics. Springer New York, 1980 (cited on page 97).

[9] Morris .W. Hirsch. *Differential Topology*. Graduate Texts in Mathematics. Springer New York, 1976 (cited on page 116).

[10] Victor Guillemin and Alan Pollack. *Differential Topology*. AMS Chelsea Publishing. AMS Chelsea Pub., 2010 (cited on page 116).

[11] Phillip Griffiths and Joseph Harris. *Principles of Algebraic Geometry*. Wiley, 1994 (cited on pages 117, 126).

[12] W.J. Gordon. 'Linear algebra proof of CLifford's theorem'. In: *L'Enseignement Mathématique* 30 (1984), pp. 85–94 (cited on page 122).

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