Chapter 9: Applications | **Section 2**

Related rates problems

What you need to know already:	What you can learn here:
➢ All differentiation rules. ➢ The method of implicit differentiation. ➢ What a word problem is.	➢ How to use implicit differentiation to find a rate of change based on information about another rate of change related to it.

What am I going to learn here? That sentence is rather convoluted!

I agree! It is the best I could do to describe briefly what a related rates problem is. Let's see if this portrait helps to clarify that.

Quick portrait of a Related rates problem

A ***related rates*** problem is a word problem in which:

- Two quantities are ***identified*** and are ***related*** in some way that is usually not stated.
- The ***rate of change of one such quantity*** and some other information are ***given***
- The ***rate of change of the other quantity*** is ***required***.

As I mentioned when discussing word problems in general, I cannot give you a detailed strategy for how to solve a related rates problem, since that depends on the structure of the problem and your particular problem-solving style. However, I can give you some tips that will, hopefully, facilitate your job.

Knots on your finger

When solving a related rates problem:

- Follow all ***general tips*** for how to solve a word problem.
- Identify the ***existing relation*** between the two ***key quantities***, ignoring how their rates behave.
- As you do so, ***look up*** the correct formula you need, rather than ***guessing***.

- Notice that the two key variables may be related in ***more than one way***, with some such ways being more useful for the problem than others.
- Use ***letters*** to denote ***any*** variable involved. This includes the key variables as well as any other quantity that is relevant to the problem and changes during the process under investigation.
- Use the actual ***numerical values*** for any quantities that are ***constant*** throughout the process under investigation.
- Once the key relationship is established, apply ***implicit differentiation*** to it by using time as the independent variable.
- While performing such implicit differentiation, remember that ***the derivative of a constant is 0***.
- After implicit differentiation has been completed, use all numerical values you have, both for the quantities and their rates, and solve for the unknown rate.

Lots of advice: can we see it in action?

Here are some examples of possible ways to solve related rates problems.

Keep in mind that if you try to solve the same problems, you may be successful by using a different method, or you may follow a different train of thought, or you may place different priorities on different aspects of each problem. As long as you are following proper logical steps and they work, more power to you!

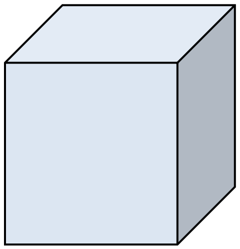
However, always reflect on what methods you chose, on how effective they were and on whether alternative methods are possible and how they compare to yours.

It is this reflective process that will allow you to become a good problem solver, not just a lot of unexamined practice.

Example:

An ice cube with a 10 cm side is melting, so that its volume decreases at 12 cm^3/min. How fast is its surface area changing?

We are dealing with a cube, so, we may want to start by drawing a simple picture representing it.



There are several quantities associated with this cube, among which are the two key quantities:

$$A = surface\ Area \quad ; \quad V = Volume$$

I say that these are the two *key* quantities because we are given the rate of change of one – volume – and are asked for the rate of change of the other – surface area. But we also have information about the side, which also changes during the process, so we assign a letter to it too:

$$s = Side$$

Notice that I am not using any of the numerical values given, since in the given context of a melting cube all these quantities are changing and therefore those values are only correct at one instant.

Now we want to relate the two key quantities; you may not know how to relate them directly, but you can look that up or you may know how to relate each of them to the side, so we start from that:

$$V = s^3 \quad ; \quad A = 6s^2$$

If we express the first relation in terms of s and use it in the second we obtain:

$$V = s^3 \quad \Rightarrow \quad s = V^{1/3} \quad \Rightarrow \quad A = 6s^2 = 6V^{2/3}$$

Now that we have the required relationship between the two key variables, we can differentiate both sides implicitly, with respect to time:

$$A = 6V^{2/3} \quad \Rightarrow \quad \frac{dA}{dt} = 6\frac{2}{3}V^{-1/3}\frac{dV}{dt} = \frac{4}{\sqrt[3]{V}}\frac{dV}{dt}$$

Now we can use the given numerical values:

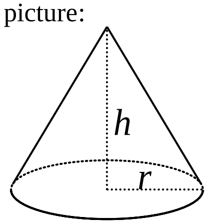
$$s = 10, \frac{dV}{dt} = -12 \quad \Rightarrow \quad \frac{dA}{dt} = \frac{4}{\sqrt[3]{10^3}}(-12) = -4.8$$

Therefore, the surface area is decreasing (notice the negative sign) at a rate of $4.8\ cm^2 / min$.

Example:

The height of a right circular cone is increasing at a rate of 1.5 cm/sec, while its volume is increasing at a rate of 2 cm^3/sec. How fast is the radius of the base of the cone changing when the radius is 6 cm and the height is 8 cm?

Again we start with a simple picture:



Before you dig any deeper, ask yourself whether know what a "*right circular cone*" is! It is a usual cone, with the vertex vertically on top of the centre of the circle that acts as a base, but better check than guess, eh?

We now identify the key variables:

$$V = Volume \quad ; \quad r = Radius \quad ; \quad h = Height$$

Notice that this time we have three variables that are changing and we are given information about the rate of change of two of them. That is fine, since we are left with only one quantity whose rate of change we need to compute.

Do you know which formula relates radius, height and volume in a cone? If not, look it up – don't *make* it up - and you will discover that it is given by:

$$V = \frac{1}{3}\pi r^2 h$$

We now take the derivative of both sides, implicitly, with respect to time:

$$V' = \frac{1}{3}\pi\left(2rr'h + r^2h'\right)$$

We are now ready to use the given values and arrive at the conclusion:

$$2 = \frac{1}{3}\pi\left(2\times6\times r'\times8 + 36\times1.5\right) \quad \Rightarrow \quad \frac{6}{\pi} = 96r' + 54$$

$$\Rightarrow \quad r' = \frac{\frac{6}{\pi} - 54}{96} \approx -0.54$$

Therefore, the radius is decreasing at a rate of approximately 0.54 cm/sec.

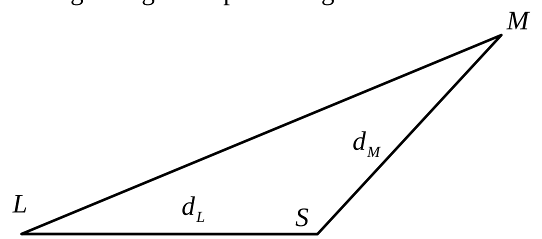
Example:

Larry and Moe have each bought recently a pickup truck and they meet to show them off to each other. As they leave, Larry heads West with a constant acceleration of $2\ m/\sec^2$*, while Moe looks at him for 3 seconds and then heads North-East also with a constant acceleration of* $2\ m/\sec^2$.

After 10 more seconds, how far is Larry from the departure point? And how fast is the distance between the two friends changing?

Lots of words here! They provide information, but also form confusion, so we may need to organize the information in a more user-friendly manner.

We start by sketching a diagram representing the situation:



Here L represents Larry's position, M is Moe's position and d_L, d_M are the corresponding distances travelled at any time t from the moment when Larry leaves. Think of S as the starting (or show off!) position.

Both questions are about the distance travelled, but the given information is about acceleration, so how do we connect the two? If you still remember your high school physics, or if you look it up or if you ask an expert, you will find that at time t, the distance d travelled by an object starting from rest and subject to an acceleration a is given by $d = \frac{at^2}{2}$. In our case, at any time $t \geq 3\,\text{sec}$, Larry and Moe will have travelled, respectively:

$$d_L = \frac{2t^2}{2} = t^2 \quad ; \quad d_M = \frac{2(t-3)^2}{2} = (t-3)^2 \ \ metres$$

Of course Moe's formula accounts for his waiting 3 seconds.

The first question is NOT a related rates problem, since the question is about distance and not rate of change. Therefore, its answer is a simple evaluation of the distance formula at $t = 3 + 10 = 13\,\text{sec}.$:

$$d_L(13) = 13^2 = 169 \ metres$$

The second question, however, is indeed a related rates problem, since we have to relate the rates of change of the two distances we have used so far and the distance $\overline{LM}$ between the two friends. But how do we link the three sides of a triangle that has no right angles? Again, through recollection or investigation you will end up relying on the *law of cosines*:

$$\overline{LM}^2 = d_L^2 + d_M^2 - 2d_L d_M \cos\hat{S}$$

Since Moe is travelling towards North-East, $\hat{S} = \frac{3\pi}{4}$ and we have:

$$LM^2 = t^2 + (t-3)^2 - 2(t^2 - 3t)\left(-\frac{\sqrt{2}}{2}\right)$$
$$= \left(2+\sqrt{2}\right)t^2 - \left(6 - 3\sqrt{2}\right)t + 9$$

To find the required rate of change we differentiate implicitly, even though the time variable is explicitly present, so as to avoid square roots:

$$2(LM)(LM)' = \left(4 + 2\sqrt{2}\right)t - \left(6 - 3\sqrt{2}\right)$$

At the given time, the distance between them is:

$$\overline{LM} = \sqrt{\left(2+\sqrt{2}\right)169 - \left(6-3\sqrt{2}\right)13 + 9} \approx 249.9$$

Therefore, the required rate is:

$$\left(\overline{LM}\right)' \approx \frac{\left(4+2\sqrt{2}\right)13 - \left(6-3\sqrt{2}\right)}{2 \times 249.9} \approx 42.5$$

This means that the two friends are moving away from each other at approximately 42.5 *m/sec*.

Notice how many steps needed to be done in each problem, especially in the last one, and how important it is to be clear on the meaning of each quantity considered and of each step taken. Attempting to do all this on auto-pilot is a great recipe for failure!

I could show you many more examples, but then I would rob you of the opportunity to experience the process yourself and of the benefits that come from practice. So, here are your learning questions: do as many as possible and don't neglect to reflect on each solution process you go through and to learn from it.

Summary

- In a related rates problem, two quantities are related through some formula to be determined, the rate of change of one is given and the rate of change of the other is required.
- Several steps can be taken to solve such a problem. Which ones apply varies from problem to problem and depending on the solver's style.
- Practice and reflection on the effectiveness of the method and steps used are essential to improve your problem solving skills, here as for any other type of word problem.

Common errors to avoid

- Don't approach a related rates problem in a random and disorganized manner, as that will not be effective and will not help you develop further problem solving skills.

Learning questions for Section D 9-2

Review questions:

1. Describe how to recognize a word problem as being a related rates problem.
2. Explain why and how implicit differentiation is important in related rates problems.
3. Identify the generic methods for solving word problems that you are already using and that can be useful in related rates problems.

Memory questions:

1. What feature of a word problem tells you that it is a related rates problem?
2. What should the initial equation of a related rates problem relate?

You may also want to review basic formulae from geometry and physics, as they are often used in this kind of problems.

Computation questions:

Questions 1-4 are not word problems, but their solutions require the same technical steps as a related rate problem. Determine why they are not word problems and find their solution anyway.

1. Two variable quantities P and Q are related through the equation $\sin^2 P + \cos^2 Q = 1$. Determine $\dfrac{dQ}{dt}$ when $P = \dfrac{\pi}{3}$ and $Q = -\dfrac{\pi}{3}$ assuming that at that time $\dfrac{dP}{dt} = 2$

2. An object is moving with a trajectory described by the function $y = \dfrac{4}{1+x^3}$. Knowing that when it reaches the point $\left(3, \dfrac{1}{7}\right)$ its horizontal velocity is ½ unit/min, determine its vertical velocity at the same moment.

3. A particle is moving on the circle $x^2 + y^2 = 25$ so that when it is at (3, 4) its vertical velocity is 4 (all in appropriate units). What is its horizontal velocity?

4. An object is travelling along the curve $y = x^2$. At the moment when it is at the point (3,9) its horizontal velocity is 2 units/sec. How fast is its vertical velocity changing at that moment?

Questions 5-8 present word problems, even though some of them do provide certain formulae. Solve each of them and describe what makes them related rates problems.

5. An object is travelling along the curve $y = x^2$. At the moment when it is at the point (9,3) its horizontal velocity is 2 units/sec. How fast is its distance from the origin changing at that moment?

6. An object is moving along the path traced by the curve $y = \sin x$ with a constant horizontal velocity of 2 units per second. A monitoring device is placed at the point (0, 1). How fast is the distance between the object and the point (0, 1) changing when the object is at the point of x-coordinate π?

7. The volume of a sphere is increasing at a rate of $2.0\, m^3 / min$. Find the rate at which the radius of the sphere is increasing when such radius is 1.5 meters.

8. The height of an isosceles triangle is decreasing at a rate of 3 cm/minute, while its area is increasing at a rate of 2 square cm/minute. If at that moment the area is 50 square cm and the base is 5 cm long, how fast is the base changing?

Theory questions:

1. Is it necessary to construct a function in order to solve a related rates problem?

2. When solving a related rates problem, at what stage do rates come in?

3. In a related rate problem which quantities are not labelled with letters?

4. What method of differentiation is usually needed for a related rates problem?

5. Which rule of differentiation is used in all related rates problems for which the independent variable does not appear in the equation?

6. In a related rates problem involving a right triangle, how do you decide whether to use the Pythagorean theorem or trigonometric ratios?

Application questions:

1. A funnel is being used to fill a bottle with wine. The diameter of the funnel is 8 cm and its depth is 10 cm. If the hole at the bottom of the funnel lets wine out at a rate of 2 cc/sec and the wine is being poured in at 3cc/sec, how fast is the wine level in the funnel rising when the funnel is filled to a depth of 5 cm? (Assume the hole to be of negligible width)

2. The entrance hall of a building is a 30x10 rectangle. A camera is placed at the centre of the 30 meter wall that is opposite the doors and must scan such opposite wall at a constant speed of 1 *m/sec* along the wall. Compute the angular speed at which the camera must be moving when it makes an angle of $\pi/4$ with the wall.

3. You are travelling at night on a highway at 72 km/hr when a deer that is crossing the road ahead of you stops and gets hypnotized by your headlights. If the deer is 2 m long, how fast is its angle of sight increasing when you are 100 m away?

4. Your U of A calculus instructor is doing a job on top of a 12 ft ladder. Hidden in a nearby bush, you start pulling a rope attached to the bottom of the ladder, at a rate of 1 ft/sec. How fast is the professor falling when he is 3 ft from the ground? With what speed will he hit the ground?

5. A 12 metre lamppost illuminates a basketball court. At a distance of 8 metres from the post a ball is thrown up vertically to a height of 6 metres. How fast is the shadow of the ball moving 1 second after it starts its descent?

6. The glass wall of a greenhouse forms an angle of 60° with the ground. A 12' plank is placed with one end (A) on the ground and the other (B) on the wall, but the slippery nature of the glass makes B slide towards the ground. If A is moving at 3" per second when the plank forms an angle of 45° with the ground, how fast is B sliding along the wall? At that same moment, how fast is the angle between the plank and the ground changing?

7. A construction worker is standing on top of a platform 10 metres above street level. By using a rope he pulls up one end of an 8 ft beam. Assuming that he is pulling the end straight up at a speed of 1/2 foot per second, how fast is the other end of the beam moving towards the platform 4 seconds after he starts pulling?

8. An observer is located 100m North of a runway and watches an airplane do its take-off run on an West-East runway. When the plane is 500m East (and 100m South) of the observer's position, it has a speed of 15 m/sec. At that moment how fast are the distance between observer and plane and the angle of observation changing?

9. You are tracking an airplane that is flying straight at you at a speed of 120km/hr and at an altitude of 300m. How fast is the angle of elevation from your perspective changing when the plane is 200m away from you? Careful: this wording contains an intentional ambiguity, which are expected to spot and resolve by requesting clarifications.

10. An airplane takes off from a runway and rises with an elevation angle of 30°, heading towards an observer located on the ground 1 km from the takeoff point. How fast is the distance between the plane and the observer changing when the plane is 400 metres high if at that time its speed is 250 km/hr?

11. The area of an isosceles trapezoid is increasing at the constant rate of 5 m^2/hour. Its shorter base is 6 m long, while its height is half as long as the longer base. How fast is the height of the trapezoid changing when the longer base is 16 m long?

12. A metallic cone has a base radius of 5 cm and a height of 10 cm. In order to obtain a frustum of a cone, a machine is shaving off its top portion. When the height of the frustum is 9 cm, it is decreasing at a rate of 0.2 cm per minute. How fast is the volume of the frustum decreasing?

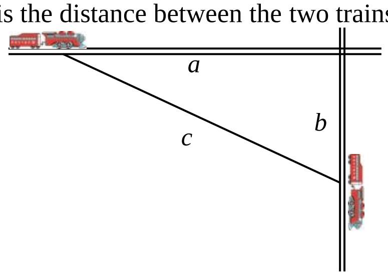
13. If the volume of an exploding supernova expands at a rate of a km^3/sec, how fast is its circumference expanding when the radius is p km?

14. A trough is 4 metres long and its ends have the shape of right trapezoids that are 90 cm across at the top and 60 cm across at the bottom and have a height of 50 cm. If the trough is being filled with water at the rate of 0.2 m^3/min, how fast is the water level rising when the water is 15 cm deep?

15. A trough is 10 metres long and its ends have the shape of isosceles trapezoids that are 3 metres across the top, 2 metres across the bottom and have a height of 1 metre. If the trough is being filled with water at a rate of 6 m^3/min, how fast is the water level rising when the water is 40 cm deep?

16. You are driving your 1986 Tercel, heading West on a rural Alberta road and just as you pass the top of a hill you see an intersection at the bottom of the valley, straight ahead of you, and a bright red Maserati exiting the intersection North bound on a secondary road. You are traveling at 100 km/hr, the Maserati is going at 80 km/hr (hey, this is a math problem, so things are not always reasonable) and the intersection is 1 km ahead of you.
    a) At what time will you be closest to the Maserati if neither changes velocity?
    b) How fast will your distance from the Maserati be changing when you are 500m from the intersection?

17. At 10:00 am, train A is 200 km west and 50 km north of train B, as indicated in the diagram. Train A is moving east at 70 km/h and train B is moving south at 80 km/h. How fast is the distance between the two trains changing at 11:30 am?



18. The rotating arm that controls the entrance to a college parking lot is 3 meters long. Just as the arm is coming down to close the entrance, a distracted professor is standing in its path, so that the tip of the arm will strike his head, at a height of 1.5 meters. If the angular speed is $\frac{\pi}{10}\,rad\,/\,sec$ with what vertical and horizontal speed will the arm hit the professor?

19. During a movie scene a camera is moving closer to a window that consists of a square topped by a semicircle. If on your monitor screen the side of the square portion appears to be 12 cm long and increasing at a rate of 2 cm/sec, how fast is the area of the whole window increasing?

20. A highway patrol plane is flying horizontally along a highway at an altitude of 200m and a speed of 180 km/hr when the radar equipment on board measures the straight distance from the plane to an incoming car at 5km and decreasing at 280 km/hr. How fast is the car travelling?

21. You are standing on an overpass 12 m above a highway and are watching a red Ferrari cruising the highway towards you at 120 km/hr. How fast is the distance between you and the Ferrari changing when it has 100 m to go before going under the overpass? Provide the exact value.

22. Two helicopters take off at low altitude from a Coast Guard support ship on a rescue mission. The first helicopter travels with a constant horizontal acceleration of 3 m/sec^2 (ignore the vertical component of the movement) heading West. The second takes off 5 seconds later and heads S 30° E. After 10 more seconds the computer of the first helicopter reports that the other helicopter is moving away from the first at a speed of 20 m/sec. At that time:
    a) How far is the first helicopter from the support ship, which has not moved in the meantime?
    b) How fast is the second helicopter moving?

23. A light placed at ground level illuminates the front entrance of a house, located 8 meters from it. A 50 cm tall dog gets out of the front door and runs straight towards the light at 2 meters per second. How fast is the shadow of the dog changing when his head is 3 meters from the light? ("*Who cares*" is not an acceptable answer ☺)

24. An absent-minded math teacher buys an ice cream cone, sets it on a holding ring and walks away without it. If the ice cream in the cone has the shape of a sphere

and melts at a rate of $15\,cm^3 / min$, how fast is its radius changing when there are $250\,cm^3$ of ice cream left?

25. Captain Kirk has just asked Mr. Spock to analyze a newly formed spherical asteroid that has been spotted by the Enterprise and that is shrinking under the effect of gravity. The ingenious Vulcan estimates that in *t* years the radius of the asteroid will be $r = e^{3-\frac{1}{2}t}$ km and that the acceleration of gravity on the surface of the asteroid is 2m/sec^2. How fast will the acceleration of gravity be changing at the next sighting, scheduled by Federation to occur in 10 years? According to your formulae, what will happen eventually to the acceleration of gravity? (Remember that according to Newton's law of gravitation $g = \dfrac{k}{r^2}$ for an object with fixed mass).

26. Air is being pumped into a spherical balloon, so that its volume increases at a rate of 60π cm^3/sec. How fast is the radius of the balloon changing when the surface area is 400π cm^2?

27. Abel and Boris are two friends who enjoy cross country skiing. One day they meet at the starting point of a set of trails, greet each other and then Abel starts skiing on the North-bound trail, which is a gradual downhill, at a constant speed of 21 km/hr. After watching him for 15 seconds, Boris starts on the East-bound trail, which is a slow uphill, at a constant speed of 9.6 km/hr. How fast are the two friends moving apart from each other after 1 minute?

28. Sand is being poured on the ground and it spreads around so as to form a hemispherical pile. If at a certain moment the radius of the pile is 3 meters and is increasing at a rate of 3 cm/sec, how fast is the sand is being poured on the pile?

29. In order to do a demonstration of static equilibrium and collect weather data at the same time, Wei and Nigel decide to fly a kite in the football field located West of campus. When the kite reaches a height of 20 m and 30 m of line have been released, the kite goes up at a rate of 0.5 m/sec while pulling 0.8 m of line per second. What is the horizontal speed of the kite at that moment?

30. A metallic cylinder is melting, while remaining in the shape of a cylinder. At a certain point its volume is decreasing at a rate of 3 cc/sec and its base radius is increasing at a rate of 1 cm/sec. If at that moment the radius is 50 cm and the height 15 cm, how fast is the height changing?

31. A right circular cylinder has a base radius that is half of its height. When the height is 3m, the volume is decreasing by 5 m^3/min. How fast is the base radius changing at the moment?

32. A highway patrol plane is flying horizontally along a highway at an altitude of 200m and a speed of 180 km/hr when the radar equipment on board spots an incoming car at an angle of depression on 9° and whose straight distance from the plane is decreasing at 280 km/hr. How fast is the car travelling?

33. At 10:00 am a train leaves Calgary and travels eastbound towards Medicine Hat. Shortly thereafter another train leaves Calgary and travels northbound towards Edmonton. At 10:40 am The first train is 45 km East of Calgary and travelling at 80 km/hr, while the second is 50 km North of Calgary and travelling at 90 km/hr. How fast is the distance between the two trains changing at that time?

34. A inverted right conical tank (same shape as an ice cream cone in use) has a base diameter of 3m, a depth of 4m and is full of liquid. If the liquid is draining our from the bottom at the rate of 5 litres per minute, how fast is the level of the liquid in the tank changing when the depth is 2m?

35. A tank in the shape of a cone is being filled with oil. If the height is 3/5 of the radius, find the rate of change of the volume when the radius is 10 cm, and is changing at a rate of 0.5 cm / min.

36. A water drop is sitting on a table in the shape of hemisphere. Heat is making the drop expand at a rate of $0.1\,mm^3 / hr$. How fast is the total surface area of the drop (top and bottom) changing when the radius is $5\,mm$?

37. Biker A is approaching an intersection at 36 km/hr, when he sees biker B leaving it at 18 km/hr. How fast is the angle at which A's head is turning to see B changing when A is 50 metres from the intersection and B is 10 metres from it?

Templated questions:

As you solve any related rates problem, consider the following additional activities:

1. Identify the units of the rate you computed.
2. Identify some features of your conclusion that support the fact that such conclusion is correct.
3. Check that the final answer makes sense in terms of size and sign.
4. Identify any assumptions that are used to solve the problem, but are not realistic.
5. Identify any piece of information provided in the main question, if any, that is not needed for the solution.
6. Identify any geometric relationships that can be useful to solve the given problem.

What questions do you have for your instructor?