Chapter 11: Vector spaces — **Section 1**

Vector space axioms

What you need to know already:	*What you can learn here:*
➢ How Euclidean vectors "work". ➢ What linear combinations are and why they are important.	➢ How mathematicians have generalized the idea of Euclidean vectors to more general structures that have the same key properties.

Over the years, mathematicians reflected on the following observations, which by now should be familiar to you, although not in the same words or as fast!

Knot on your finger

The features that characterize Euclidean vectors as special objects are:

- The presence of the two operations of ***addition*** and a ***scalar multiplication***.
- The possibility of constructing ***linear combinations*** with them.
- Certain basic ***algebraic properties*** ensuring that some computations follow familiar patterns.

By using these observations, we started from geometric vectors – which can be considered as very concrete and visible objects – and generalized them to Euclidean vectors, which are more abstract mathematical constructs, but with very similar properties (we even use arrows to draw them!) and the possibility of both theoretical and practical applications. Well, those mathematicians wondered, can we extend the notion of vectors to totally different sets of objects that still rely on similar properties?

It turns out that yes, we can, but only after we have answered the obvious question: *"What are the basic properties that make vectors a special kind of object worth studying and generalizing?"* I will first give you a general idea of what the accepted answer to this question is and then provide you with its technical details.

That looks pretty simple.

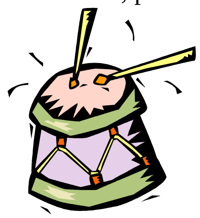
Warning bells

You are now entering what is probably the ***most abstract*** area of mathematics you have seen so far!

Behind some statements that look simple and familiar lurk ***logical pitfalls*** that are hidden by a number of ***assumptions and beliefs*** that are true for usual numbers, but may not be true for other objects!

Where we are going, a familiar language is spoken, but those who live there (abstract vectors) are strange things indeed and follow the letter of the logical law. So, tread carefully, but get ready to be excited by a brave new world!

Here is the formal definition: drum roll, please!



Definition of vector space

A ***vector space*** consists of ***a set*** $\mathfrak{V}$ ***of objects*** (any objects, but usually mathematical ones) ***called vectors*** (surprise!), together with ***two operations***:

- An operation ***called*** "*addition*" that associates to two vectors **u** and **v** in $\mathfrak{V}$ a vector, usually denoted by $\mathbf{u} \oplus \mathbf{v}$.
- An operation ***called*** "*scalar product*" that associates to a vector **v** in $\mathfrak{V}$ and a scalar *c* a vector, usually denoted by $c \otimes \mathbf{v}$.

These two operations must also satisfy the set of 10 ***properties***, usually called ***vector space axioms***, listed in the following two definition boxes.

So far, we have a set of items, things, that we call *vectors* and two operations. Let's now see how these operations are supposed to behave in order for the whole structure to earn the title of vector space.

Definition of the addition axioms

In a vector space, the ***addition*** operation, usually denoted by $\oplus$, must satisfy the following axioms:

1. ***Closure***: The addition (or *sum*) $\mathbf{u} \oplus \mathbf{v}$ of ***any two*** vectors **u** and **v** of $\mathfrak{V}$ ***exists*** and is a ***unique*** vector of $\mathfrak{V}$.
2. ***Commutativity***: For ***any two*** vectors **u** and **v** of $\mathfrak{V}$, $\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}$.
3. ***Associativity***: For ***any three*** vectors **u**, **v** and **w** of $\mathfrak{V}$, $\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}$
4. ***Existence of the zero vector***: There ***exists*** a vector in $\mathfrak{V}$, called the ***zero vector*** and denoted by $\mathbf{0}$, such that for any vector **u** of $\mathfrak{V}$:
$$\mathbf{0} \oplus \mathbf{u} = \mathbf{u} \oplus \mathbf{0} = \mathbf{u}$$
5. ***Existence of all negative vectors***: For *each* vector **u** of $\mathfrak{V}$ there ***exists*** a vector, called its ***negative*** and denoted by "– **u**", such that:
$$\mathbf{u} \oplus (-\mathbf{u}) = \mathbf{0}$$

Five down, five to go, namely the axioms for the scalar product or scalar multiplication.

Definition of the scalar product axioms

In a vector space, the ***scalar product***, or scalar multiplication operation, usually denoted by $\otimes$, must satisfy the following axioms:

6. ***Closure***: The product of ***any*** scalar c with ***any*** vector $\mathbf{u}$ of $\mathfrak{V}$ ***exists*** and is a ***unique*** vector of $\mathfrak{V}$, denoted by $c \otimes \mathbf{u}$.

7. ***Associativity***: For ***any*** scalars c and d and ***any*** vector $\mathbf{u}$ of $\mathfrak{V}$:

$$(cd) \otimes \mathbf{u} = c \otimes (d \otimes \mathbf{u})$$

8. ***Neutrality of 1***: For ***any*** vector $\mathbf{u}$ of $\mathfrak{V}$:

$$1 \otimes \mathbf{u} = \mathbf{u}$$

9. ***Distributivity over vector addition***: For ***any*** scalar c and ***any*** two vectors $\mathbf{u}$ and $\mathbf{v}$ of $\mathfrak{V}$:

$$c \otimes (\mathbf{u} \oplus \mathbf{v}) = (c \otimes \mathbf{u}) \oplus (c \otimes \mathbf{v}).$$

10. ***Distributivity over scalar addition***: For ***any*** scalars c and d and ***any*** vector $\mathbf{u}$ of $\mathfrak{V}$:

$$(c + d) \otimes \mathbf{u} = (c \otimes \mathbf{u}) \oplus (d \otimes \mathbf{u}).$$

Wow! What a mess of symbols, words and formulae! And you expect me to memorize all this?!

First of all, there are *only* 10 axioms and I am sure you know way more than 10 rules of the road! Moreover, the more important thing is to understand what each axiom is trying to guarantee, because then it will become easy to remember them, almost second nature. To help you do that, here is a quick portrait of the vector space axioms, organized in terms of what they try to accomplish.

Quick portrait of the vector space axioms

Axioms 1) and 6) are ***closure axioms***, meaning that when we combine vectors and scalars in the prescribed way, we do not stray outside of $\mathfrak{V}$. That is, they keep the results within the vector space, rather than ending up somewhere else.

Axioms 2), 3), 7), 9) and 10) are ***algebraic axioms***, meaning that certain basic properties of usual algebraic operations of numbers still hold within $\mathfrak{V}$ and we need not worry about these formulae falling apart. We have seen other operations (such as matrix multiplication) for which the usual rules of algebra do not hold. Here we want to make sure that *these* rules do.

Axioms 4), 5) and 8) are ***special items axioms***, meaning that we want **0** (as a vector) and 1 (as a scalar) to maintain the properties that make them special values among usual numbers.

Given the importance of understanding these axioms and their purposes, here is another way to list and label them that reflects the above protrait. In what follows, I will use either list and I encourage you to use whichever list you find easier to remember. Also, I will now make the statements of the axioms shorter, trusting that you remember the notation and details of the original definitions.

Definition of vector space

A ***vector space*** consists of a set $\mathfrak{V}$ of ***objects*** called vectors, together with ***two operations*** – an "*addition*" $\oplus$ and a "*scalar product*" $\otimes$ – which satisfy the following ten properties, usually called vector space ***axioms***:

Closure axioms:

C1) For *any two* vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathfrak{V}$, $\mathbf{u} \oplus \mathbf{v}$ is unique and is also in $\mathfrak{V}$.

C2) For *any* scalar c and *any* vector $\mathbf{u}$ of $\mathfrak{V}$, $\mathbf{c} \otimes \mathbf{u}$ is unique and is also in $\mathfrak{V}$.

Algebraic axioms:

A1) Addition is commutative:
$$\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}.$$

A2) Addition is associative:
$$\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}$$

A3) Scalar product is associative:
$$(cd) \otimes \mathbf{u} = c \otimes (d \otimes \mathbf{u}).$$

A4) Scalar product is distributive for addition:
$$c \otimes (\mathbf{u} \oplus \mathbf{v}) = (c \otimes \mathbf{u}) \oplus (c \otimes \mathbf{v}).$$

A5) Scalar product is distributive for usual addition:
$$(c + d) \otimes \mathbf{u} = (c \otimes \mathbf{u}) \oplus (d \otimes \mathbf{u})$$

Special items axioms:

S1) There is a zero vector such that, for every $\mathbf{u}$:
$$\mathbf{0} \oplus \mathbf{u} = \mathbf{u} \oplus \mathbf{0} = \mathbf{u}$$

S2) Each vector $\mathbf{u}$ has a negative "$-\mathbf{u}$", such that:
$$\mathbf{u} \oplus (-\mathbf{u}) = \mathbf{0}.$$

S3) For any vector $\mathbf{u}$, $1 \otimes \mathbf{u} = \mathbf{u}$.

Too much abstract talk! Can we see some examples?

Told ya! Of course, geometric and Euclidean vectors satisfy all the axioms. I will prove this for $\mathbb{R}^2$ and you may want to generalize this proof to any $\mathbb{R}^n$

Technical fact

The set $\mathbb{R}^2$ of 2D vectors with the usual operations of vector addition and scalar multiplication, forms a ***vector space***.

Proof

In this case $\mathbf{u} \oplus \mathbf{v} = \mathbf{u} + \mathbf{v}$ and $c \otimes \mathbf{v} = c\mathbf{v}$. Even though in this context the statements of the axioms seem rather obviously true, I will go through their formal proof to give you an idea of how this is done in general.

C1) $[a_1 \;\; a_2] + [b_1 \;\; b_2] = [a_1 + b_1 \;\; a_2 + b_2]$ and this is a 2D vector.

C2) $c[a_1 \;\; a_2] = [ca_1 \;\; ca_2]$ and this is a 2D vector.

A1) $[a_1 \;\; a_2]+[b_1 \;\; b_2]=[a_1+b_1 \;\; a_2+b_2]$

$$=[b_1+a_1 \;\; b_2+a_2]=[b_1 \;\; b_2]+[a_1 \;\; a_2]$$

This is the commutativity required by the axiom.

A2) $[a_1 \;\; a_2]+([b_1 \;\; b_2]+[c_1 \;\; c_2])=[a_1 \;\; a_2]+[b_1+c_1 \;\; b_2+c_2]$

$$=[a_1+b_1+c_1 \;\; a_2+b_2+c_2]$$

$$=[a_2+b_2 \;\; a_1+b_1]+[c_1 \;\; c_2]=([a_1 \;\; a_2]+[b_1 \;\; b_2])+[c_1 \;\; c_2]$$

This is the associativity required by the axiom.

A3) $(cd)[a_1 \;\; a_2]=[cda_1 \;\; cda_2]=c[da_1 \;\; da_2]=c(d[a_1 \;\; a_2])$

This is the associativity required by the axiom.

A4) $c([a_1 \;\; a_2]+[b_1 \;\; b_2])=c[a_1+b_1 \;\; a_2+b_2]$

$$=[c(a_1+b_1) \;\; c(a_2+b_2)]=[ca_1+cb_1 \;\; ca_2+cb_2]$$

$$=[ca_1 \;\; ca_2]+[cb_1 \;\; cb_2]=c[a_1 \;\; a_2]+c[b_1 \;\; b_2]$$

This is the distributivity required by the axiom.

A5) $(c+d)[a_1 \;\; a_2]=[(c+d)a_1 \;\; (c+d)a_2]$

$$=[ca_1+da_1 \;\; ca_2+da_2]=[ca_1 \;\; ca_2]+[da_1 \;\; da_2]$$

$$=c[a_1 \;\; a_2]+d[a_1 \;\; a_2]$$

This is the distributivity required by the axiom.

S1) $[a \;\; b]+[0 \;\; 0]=[a \;\; b]$ so that $\mathbf{0}=[0 \;\; 0]$ is the required zero vector.

S2) $[a \;\; b]+[-a \;\; -b]=[0 \;\; 0]$

Therefore, all negative vectors exist.

S3) $1[a \;\; b]=[1a \;\; 1b]=[a \;\; b]$

Therefore, 1 does act as required.

Hmmmm. So far, this just seems like a useless complication of a simple concept!

Yes, but pretty soon our vector spaces will consist of objects and operations that are much less obvious and familiar and you will need to patiently go through each of these steps. The next section will offer two examples of possibly mind-boggling vector spaces, so, you better start from the familiar.

Before showing you some initial mind-boggling facts, let me try to simplify the notation.

Knot on your finger

When working in a specific vector space whose operations have been identified, and ***if no further confusion arises***, we shall denote the two operations by using the usual notation. That is, we shall write:

- $\mathbf{u}+\mathbf{v}$ to denote $\mathbf{u}\oplus\mathbf{v}$
- $c\mathbf{v}$ to denote $c\otimes\mathbf{v}$

Finally, some common sense!

We all like simplicity, but do not underestimate the highlighted caution: "*if no further confusion arises!*" And, believe me, it will likely arise!

Just to show you how hairy things can become (and because we are going to need this soon), here are two facts that seem obvious, are true, but need checking.

Technical fact

If $\mathbf{u}$ is a vector in vector space $\mathcal{V}$, then:

- $0\mathbf{u} = \mathbf{0}$
- $(-1)\mathbf{u} = -\mathbf{u}$

Oh, come on! This is definitely obvious!

Are you sure? It is obvious for usual numbers and Euclidean vectors, here I claim that they are true no matter what objects we use, as long as the axioms hold! That means that we cannot use any familiar property of Euclidean vectors, but only the axioms. So, here is the proof: notice that it is NOT simple!

Proof

To prove that $0\mathbf{u}$ is the $\mathbf{0}$ vector, we notice that:

- $0\mathbf{u} = (0+0)\mathbf{u}$ because 0+0=0 as number.
- $(0+0)\mathbf{u} = 0\mathbf{u} + 0\mathbf{u}$ by axiom A5.
- Since $0\mathbf{u} = 0\mathbf{u} + 0\mathbf{u}$, axioms S2 (the negative vector exists) and C1 (the sum of two vectors is unique) allow us to claim that
- $0\mathbf{u} + (-0\mathbf{u}) = [0\mathbf{u} + 0\mathbf{u}] + (-0\mathbf{u})$
- Now, axioms S2 (what the negative vector does) and A2 (associativity) imply that $\mathbf{0} = 0\mathbf{u} + [0\mathbf{u} + (-0\mathbf{u})]$
- So, finally axiom S1 allows us to confirm that $\mathbf{0} = 0\mathbf{u} + \mathbf{0} = 0\mathbf{u}$.

You will probably have to go through the above steps several times to convince yourself that they are correct and, more importantly, each of them is needed, since we can use no other property of the mysterious objects involved!

Now we are ready for the second part. To prove that $(-1)\mathbf{u}$ is the negative of $\mathbf{u}$, we notice that:

- BY axiom S3, $\mathbf{u} + (-1)\mathbf{u} = 1\mathbf{u} + (-1)\mathbf{u}$.
- Next, A5 implies that $\mathbf{u} + (-1)\mathbf{u} = (1-1)\mathbf{u} = 0\mathbf{u}$
- The fact we just proved tells us that this is the zero vector and hence that $(-1)\mathbf{u}$ is the negative of $\mathbf{u}$.

Wow! Are you sure that there is no shorter way to do this?

Believe me: mathematicians have tried, but have come to the conclusion that all axioms are needed for this and other proofs, as well as the fact that they are sufficient to obtain what we need from a vector space.

But since I appreciate the desire to look for shorter ways to do something, here is one property that you may like and that we shall use often in the future.

Technical fact

If all algebraic and special items axioms are true, the two closure axioms C1 and C2 together are equivalent to the single ***linear combination axiom***:

LC) For *any two* object $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{V}$ and *any* scalars c and d, the linear combination $(c \otimes \mathbf{u}) \oplus (d \otimes \mathbf{v}) = c\mathbf{u} + d\mathbf{v}$ is also in $\mathcal{V}$.

Proof

If C1 and C2 are valid, then, by C2, for *any two* $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{V}$ and *any* scalars c and d, $c\mathbf{u}$ and $d\mathbf{v}$ are in $\mathcal{V}$. But in that case, by C1, so is $c\mathbf{u} + d\mathbf{v}$, thus implying LC.

On the other hand, if LC is valid, then for *any two* **u** and **v** in $\mathcal{V}$,
$\mathbf{u}+\mathbf{v}=1\mathbf{u}+1\mathbf{v}$ is also in $\mathcal{V}$, thus implying C1, and
$c\mathbf{u}=c\mathbf{u}+\mathbf{0}=c\mathbf{u}+0\mathbf{u}$ is also in $\mathcal{V}$, thus implying C2.

Again, you may want to go through these two lines several times, maybe together with another student to convince yourself of what they say and that their logic is correct and needed.

This is going to take some getting used to! Is it worth it?

Only if you plan to pass the course ☺ . But I guess that what you are wondering is whether the notion of vector spaces is useful to analyze sets that are not made up of Euclidean vectors and their usual operations.

No, I was wondering if it was worth our efforts, but I will trust you for the time being.

Thank you! In the next few sections we shall see many such examples, but I want to finish this section by pointing out how we can prove that a set with two operations like the one we are dealing with is NOT a vector space, since this will be important information as well. It turns out that this is an easier task than the long and tedious job of checking all 10 axioms.

Knot on your finger

In order to prove that a set with two operations is ***NOT*** a vector space, it is sufficient to check that ***one*** of the axioms is not true.

In order to prove that one of the axioms is not true, it is sufficient to prove that it does not work in ***one instance***.

In other words, ***a single counterexample*** is sufficient to show that the given set is not a vector space.

Example: $\mathbb{Z}=\{0,\pm 1,\pm 2,\ldots\}$

This strange version of the capital letter Z is the universal symbol used to identify the set of integer numbers with the usual addition and multiplication. Well, it turns out that this is NOT a vector space, since the product of an integer and a non-integer scalar is usually not an integer. But all we need is one counterexample! So, by noticing that the product of the scalar π and the integer 2 is not an integer, we have proved that $\mathbb{Z}$ is not a vector space.

Similarly, the set of rational numbers with the usual operations is not a vector space, since, say, $\sqrt{3}\times\frac{1}{2}$ is not a rational number. One counterexample is enough to disprove the vector space structure.

Example: $\mathbb{Q}=\left\{\pm\frac{n}{m};\, n, m\in\mathbb{Z}\right\}$

This is the set of rational numbers, that is, all fractions. This is also NOT a vector space and for the same reason: the fact that $\sqrt{3}\times\frac{1}{2}$ is not a rational number provides one counterexample and that is enough to disprove the vector space structure.

And you are sure there are other useful examples of sets that ARE vector spaces, eh?

Positive! Just look at the next few sections.

Fine, but why is it important to know if some set with operations is a vector space or not?

Because there are several nice and useful properties that are consequences of the axioms and are therefore true for ALL vector spaces. So, when we know that we are dealing with a vector space, we can assume that those properties hold, while if we know that we are not dealing with one, we cannot count on those properties.

Time for a break and review.

Summary

- By identifying the key properties that make Euclidean vectors tick, we can use them as the defining features of a general structure called a vector space.
- In order to have a vector space we need a set of objects endowed with two operations, with the whole setup satisfying ten specific axioms.
- In order to verify that a set with two operations is a vector space, we need to verify each and every one of the axioms. However, to show that it is NOT a vector space all we need is one counterexample that violates one of the axioms.
- Once we know that a set with two operations is a vector space, we can apply to it any knowledge that applies to vector spaces in general. These properties are yet to be identified, but the go along the lines of what we learned about Euclidean vectors.

Common errors to avoid

- This is a tremendously theoretical, abstract and logical concept. It will take a while to get used to the axioms, what they say, why they are important, how to use them and how to ensure that good logic is used and no facts are assumed without checking that they are true.
- This will take a lot of working time, that is, work on it and don't expect great leaps of understanding in a short time. But that understanding will come, if you are willing to put in that working time.

Learning questions for Section LA 11-1

Review questions:

1. Describe the underlying concepts for the idea of vector spaces.

2. List the 10 axioms of a vector space in your own words, but be accurate!

Memory questions:

The key point of this section is the conceptual goal we are trying to achieve. While memorizing the ten axioms is a worthwhile endeavor, it is not the main point.

1. What is required in order to have a vector space?

Computation questions:

1. Check that all vector space axioms are valid in $\mathbb{R}^3$ with the usual operations of vector addition and scalar multiplication.

2. Check that all vector space axioms are valid in $\mathbb{R}^n$ with the usual operations of vector addition and scalar multiplication.

3. Consider the set of real numbers with the usual addition and with scalar multiplication given by $a \otimes b = \dfrac{a+b}{a-b}$. With this strange operation, compute $(6 \otimes 4) \otimes 3$ and find one vector space axiom that does not work for this scalar multiplication.

4. Consider the set of real numbers with the usual multiplication, but the addition defined by: $a \oplus b = ab - a - b$. With this strange operation, compute $((5 \oplus 4) \oplus 3) \oplus 2$ and explain why this operation cannot be used as the addition for a vector space structure.

In each of questions 5-10, determine whether the given set, together with the usual operations of addition and scalar product, form a vector space. In each case, provide sufficient arguments in support of your conclusion.

5. The set of all even numbers.

6. The set of all irrational numbers.

7. The set of all orthogonal 4×4 matrices.

8. The set of all 4×4 matrices whose entries add up to 0.

9. The set of all 4×4 matrices whose diagonal entries are all 0.

10. The set of all elementary 4×4 matrices

11. Consider the following pairs of operations on vectors in $\mathbb{R}^2$:

$$[x_1 \quad y_1] \oplus [x_2 \quad y_2] = \big[|x_1| + |x_2| \quad |y_1| + |y_2|\big]$$
$$k \otimes [x \quad y] = \big[k|x| \quad k|y|\big]$$

    i) Determine if the associativity axiom for addition holds.
    ii) Pick one distributivity axiom and determine if it holds.
    iii) Explain why these operations do not endow $\mathbb{R}^2$ with a vector space structure.

12. Consider the following pairs of operations on vectors in $\mathbb{R}^2$:

$$[x_1 \quad y_1] + [x_2 \quad y_2] = [x_1 x_2 \quad y_1 y_2]$$
$$k[x \quad y] = \big[|x|^k \quad |y|^k\big]$$

    i) Determine if the associativity axiom for addition holds.
    ii) Pick one distributivity axiom and determine if it holds.
    iii) Explain why these operations do not endow $\mathbb{R}^2$ with a vector space structure.

13. Consider the set of 2×2 matrices of the form $\begin{bmatrix} a & 2a \\ a+1 & a \end{bmatrix}$ with the usual operations. Explain why this set together with these operations does not form a vector space and show that the set satisfies at least two of the vector space axioms.

Theory questions:

1. Which axioms of a vector space are called *closure* axioms?
2. A vector space must be closed under which operations?
3. What is the main mathematical purpose of studying vector spaces?
4. Is the dot product a required part of the definition of a vector space?
5. What does the associativity axiom for addition require?
6. How many axioms must be contradicted to show that a set with two operations is not a vector space?
7. In a vector space, is it always true that $(\mathbf{u}+\mathbf{v})^2 = \mathbf{u}^2 + \mathbf{uv} + \mathbf{vu} + \mathbf{v}^2$?

Proof questions:

1. We have seen that for any element $\mathbf{u}$ of a vector space $\mathcal{V}$, $0\mathbf{u} = \mathbf{0}$ (the zero vector), but we have also seen that this fact needs a proof, since it is not one of the axioms. Another true fact that is not an axiom and must be proved is that if k is any scalar, then $k\mathbf{0} = \mathbf{0}$.
   Below is a proof of the fact that if $k\mathbf{u} = \mathbf{0}$, then either $k = 0$ or $\mathbf{u} = \mathbf{0}$.
   For each step of this proof, identify whether the step is based on a vector space axiom, a property of scalars, one of the two above facts, or the starting assumption of the claim.
   a) If k is 0 the claim is true
   b) Otherwise, $\mathbf{u} = 1\mathbf{u}$
   c) But $1\mathbf{u} = \left(\frac{1}{k}k\right)\mathbf{u}$
   d) Moreover, $\left(\frac{1}{k}k\right)\mathbf{u} = \frac{1}{k}(k\mathbf{u})$
   e) Therefore, $\frac{1}{k}(k\mathbf{u}) = \frac{1}{k}\mathbf{0}$
   f) So that $\frac{1}{k}\mathbf{0} = \mathbf{0}$, and the original vector $\mathbf{u}$ is the zero vector.

What questions do you have for your instructor?