Laboratoire d'Automatique

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

Robust Control

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Laboratoire d'Automatique

Course Program

Laboratoire d'Automatique

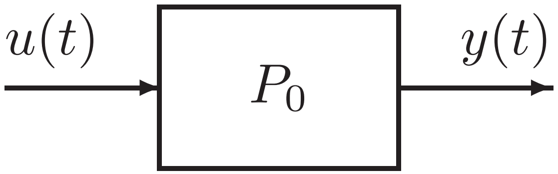
1. Introduction
2. Norms for Signals and Systems
3. Basic Concepts (stability and performance)
4. Uncertainty and Robustness
5. Stabilization (coprime factorization)
6. Design Constraints
7. Loopshaping
8. Model Matching
9. Design for Performance
10. Linear Fractional Transformation
11. H_∞ Control
12. Model and Controller Reduction
13. Robust Control by Convex Optimization
14. LMIs in Robust Control
15. Robust Pole Placement
16. Parametric uncertainty

References:

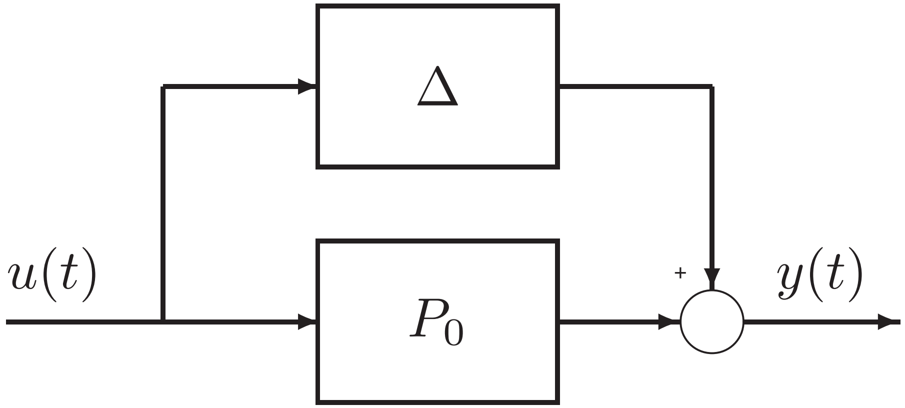
- **Feedback Control Theory** by Doyle, Francis and Tannenbaum (*on the website of the course*)
- **Essentials of Robust Control** by Kemin Zhou with Doyle, Prentice-Hall, 1998

Introduction

Classical Control



Robust Control



$$P = P_0 + \Delta$$

P_0: nominal model Δ: plant uncertainty

Uncertainty sources:

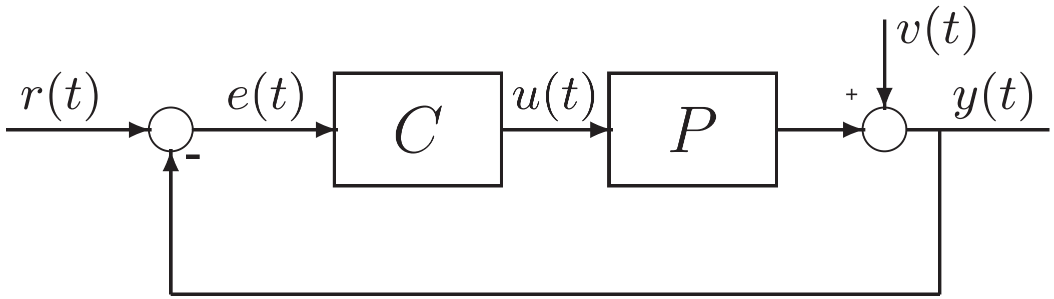
Structured: parametric uncertainty, multimodel uncertainty

Unstructured: frequency-domain uncertainty, unmodeled dynamics, nonlinearity

Robust Control Objective: Design a controller satisfying stability and performance for a set of models

Model Uncertainty and Feedback

Feedback control has been used for the first time to overcome the model uncertainty



$T = \frac{CP}{1+CP}$ Transfer function between r and y

$S = \frac{1}{1+CP}$ Transfer function between v and y

For very large CP, $T \approx 1$ (tracking) and $S \approx 0$ (disturbance rejection) whatever the plant model is.

For an open-loop stable system:

$C = 0$ robust stability → $C \to \infty$ robust performance

Loopshaping: $|C(j\omega)P(j\omega)|$ *should be large in the frequencies where good performances are desired and small where the stability is critical*

Norms for Signals and Systems

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Norms for signals: Consider piecewise continuous signals mapping $(-\infty, +\infty)$ to $\mathbb{R}$. A norm must have the following four properties:

1. $\|u\| \geq 0$ (positivity)
2. $\|au\| = |a| \, \|u\|, \forall a \in \mathbb{R}$ (homogenity)
3. $\|u\| = 0 \iff u(t) = 0 \quad \forall t$ (positive definiteness)
4. $\|u + v\| \leq \|u\| + \|v\|$ (triangle inequality)

1-Norm: $\|u\|_1 = \int_{-\infty}^{\infty} |u(t)| dt$

2-Norm: $\|u\|_2 = \left(\int_{-\infty}^{\infty} u^2(t) dt \right)^{1/2}$ $\quad$ $\|u\|_2^2$ is the total signal energy

∞-Norm: $\|u\|_\infty = \sup_t |u(t)|$

p-Norm: $\|u\|_p = \left(\int_{-\infty}^{\infty} |u(t)|^p dt \right)^{1/p} \quad 1 \leq p \leq \infty$

Norms for Signals

Average power of a signal is denoted by:

$$pow(u) = \left(\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} u^2(t)dt \right)^{1/2}$$ pow is not a norm (it is not positive definite)

Remark: One says $u(t) \in \mathcal{L}_p$ if $\|u\|_p < \infty$ where $\mathcal{L}_p$ is an infinite-dimensional *Banach space* ($\mathcal{L}_2$ is an infinite-dimensional *Hilbert space* as well).

Recall:

- A Banach space is a complete vector space with a norm.
- A Hilbert space is a complete vector space with an inner product $< x, y >$ such that the norm defined by $\|x\| = \sqrt{< x, x >}$. A Hilbert space is always a Banach space, but the converse need not hold.

Examples: $1(t) \in \mathcal{L}_\infty$ but $\notin \mathcal{L}_1, \notin \mathcal{L}_2$ $\quad$ $u(t) = (1 - e^{-t})1(t) \in \mathcal{L}_\infty$ but $\notin \mathcal{L}_1, \notin \mathcal{L}_2$

$u(t) = e^{-t}1(t) \in \mathcal{L}_\infty, \in \mathcal{L}_1, \in \mathcal{L}_2$ (Besides, $u(t)$ is a power signal $pow(u) = 0$)

$u(t) = \sin(t) \in \mathcal{L}_\infty$ but $\notin \mathcal{L}_1, \notin \mathcal{L}_2$ ($u(t)$ is a power signal)

Norms for Systems

We consider linear, time-invariant, causal and usually finite-dimensional systems.

$$y(t) = G(t) * u(t), \qquad y(t) = \int_{-\infty}^{\infty} G(t-\tau)u(\tau)d\tau, \qquad \hat{G}(s) = \mathcal{L}[G]$$

Some definitions:

- $\hat{G}(s)$ is *stable* if it is analytic in the closed RHP (Re $s \geq 0$)
- $\hat{G}(s)$ is *proper* if $\hat{G}(j\infty)$ is finite (deg den $\geq$ deg num)
- $\hat{G}(s)$ is *strictly proper* if $\hat{G}(j\infty) = 0$ deg den $>$ deg num
- $\hat{G}(s)$ is *biproper* if (deg den = deg num)

Norms for SISO systems:

2-Norm: $\|\hat{G}\|_2 = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{G}(j\omega)|^2 d\omega \right)^{1/2}$ $\qquad$ **∞-Norm:** $\|\hat{G}\|_\infty = \sup_\omega |\hat{G}(j\omega)|$

Parsval's theorem: (for stable systems)

$$\|\hat{G}\|_2 = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{G}(j\omega)|^2 d\omega \right)^{1/2} = \left(\int_{-\infty}^{\infty} |G(t)|^2 dt \right)^{1/2}$$

Norms for Systems

Remarks:

- $\mathcal{L}_2$ is a *Hilbert space* of scalar-valued functions on $j\mathbb{R}$. The inner product for this Hilbert space is defined as:

$$< \hat{F}, \hat{G} >= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{F}^*(j\omega)\hat{G}(j\omega)d\omega$$

- We say $\hat{G}(s) \in \mathcal{L}_2$ if $\|\hat{G}\|_2 < \infty$. It is the case iff $\hat{G}$ is strictly proper and has no poles on the imaginary axis.

- We say $\hat{G}(s) \in \mathcal{L}_\infty$ if $\|\hat{G}\|_\infty < \infty$. It is the case iff $\hat{G}$ is proper and has no poles on the imaginary axis. $\mathcal{L}_\infty$ is a *Banach space* of scalar-valued functions on $j\mathbb{R}$.

- $\|\hat{G}\|_\infty$ is the peak value of the Bode magnitude plot of $\hat{G}$. It is also the distance from the origin to the farthest point on the Nyquist plot of $\hat{G}$.

- $\mathcal{H}_2$ and $\mathcal{H}_\infty$ are respectively subspaces of $\mathcal{L}_2$ and $\mathcal{L}_\infty$ with $\hat{G}(s)$ stable ($\mathcal{H}_p$ spaces are usually called *Hardy spaces*).

Examples: $\frac{1}{s-1} \in \mathcal{L}_2, \mathcal{L}_\infty$ but $\notin \mathcal{H}_2, \mathcal{H}_\infty$ $\quad$ $\frac{s+1}{s+2} \in \mathcal{H}_\infty$ but $\notin \mathcal{H}_2$ $\quad$ $\frac{1}{s^2+1} \notin \mathcal{L}_2, \mathcal{L}_\infty$

Norms for Systems

Norms for matrices:

1-Norm: The maximum absolute column sum norm is defined as $\|A\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}|$.

2-Norm: The spectral norm or simply *the norm* of A is defined as: $\|A\|_2 = \sqrt{\lambda_{\max}(A^* A)}$.

∞-Norm: The maximum absolute row sum norm is defined as $\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$.

F-Norm: *Frobenius norm* is defined as $\|A\|_F = \sqrt{\text{trace}(A^* A)}$

Induced p-norm: The induced p-norm is defined from a vector p-norm: $\|A\|_p = \max_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}$

Remarks:

- The induced 2-norm and the norm of A are the same and called also the natural norm. This norm is also equal to the maximum *singular value* of A. $\|A\| = \bar{\sigma}(A)$.
- The spectral radius $\rho(A) = |\lambda_{\max}(A)|$ is not a norm.

Norms for Systems

Norms for MIMO systems: Given $\hat{G}(s)$ a multi-input multi-output system

2-Norm: This norm is defined as

$$\|\hat{G}\|_2 = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \text{trace}\left[\hat{G}^*(j\omega)\hat{G}(j\omega)\right] d\omega\right)^{1/2}$$

∞-Norm: The $\mathcal{H}_\infty$ norm is defined as

$$\|\hat{G}\|_\infty = \sup_\omega \|\hat{G}(j\omega)\| = \sup_\omega \bar{\sigma}[\hat{G}(j\omega)]$$

Remark: The infinity norm has an important property (submultiplicative)

$$\|\hat{G}\hat{H}\|_\infty \leq \|\hat{G}\|_\infty \, \|\hat{H}\|_\infty$$

Computing the Norms

How to compute the 2-norm: Suppose that $\hat{G} \in \mathcal{L}_2$, we have:

$$\|\hat{G}\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{G}(j\omega)|^2 d\omega = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \hat{G}(-s)\hat{G}(s)ds = \frac{1}{2\pi j}\oint \hat{G}(-s)\hat{G}(s)ds$$

Then by the residue theorem, $\|\hat{G}\|_2^2$ equals the sum of the residues of $\hat{G}(-s)\hat{G}(s)$ at its poles in the left half-plane (LHP).

How to compute the ∞-norm:

Choose a fine grid of frequency point $\{\omega_1, \ldots, \omega_N\}$, then an estimate for $\|\hat{G}\|_\infty$ is:

$$\max_{1 \leq k \leq N} |\hat{G}(j\omega_k)|$$

or for the MIMO systems:

$$\max_{1 \leq k \leq N} \bar{\sigma}[\hat{G}(j\omega_k)]$$

Matlab commands: `norm, bode, frsp, vsvd`

Computing the Norms

State-space methods for 2-norm: Consider a SISO state-space model $\in \mathcal{H}_2$

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \quad \xrightarrow{\mathcal{L}} \quad \begin{aligned} s\hat{x}(s) &= A\hat{x}(s) + B\hat{u}(s) \\ \hat{y}(s) &= C\hat{x}(s) \end{aligned}$$

$$\hat{G}(s) = C(sI - A)^{-1}B \quad \Longrightarrow \quad \text{impulse response : } G(t) = Ce^{tA}B$$

From Parseval's theorem:

$$\|\hat{G}\|_2^2 = \|G\|_2^2 = \int_0^\infty \left(Ce^{tA}B\right)\left(B^T e^{tA^T} C^T\right) dt = CLC^T$$

where

$$L = \int_0^\infty e^{tA}BB^T e^{tA^T} dt$$

is the observability Gramian and can be obtained from following Lyapunov equation:

$$AL + LA^T + BB^T = 0 \text{ and the 2-norm is } \|\hat{G}\|_2 = (CLC^T)^{1/2}$$

For MIMO systems we have $\|\hat{G}\|_2 = \left[\text{trace}(CLC^T)\right]^{1/2}$

Computing the Norms

State-space methods for ∞-norm: Consider a SISO strictly proper state-space model $\in \mathcal{L}_\infty$.

Theorem: $\|\hat{G}\|_\infty < \gamma$ iff the *Hamiltonian matrix* H has no eigenvalues on the imaginary axis:

$$H = \begin{pmatrix} A & \gamma^{-2}BB^T \\ -C^TC & -A^T \end{pmatrix}$$

Bisection algorithm:

1. Select γ_u and γ_l such that $\gamma_l \le \|\hat{G}\|_\infty \le \gamma_u$;
2. If $(\gamma_u - \gamma_l)/\gamma_l \le$ specified level., stop and $\|\hat{G}\|_\infty \approx (\gamma_u + \gamma_l)/2$. Otherwise continue;
3. Set $\gamma = (\gamma_u + \gamma_l)/2$ and test if $\|\hat{G}\|_\infty < \gamma$
4. If H has no eigenvalue on $j\mathbb{R}$, set $\gamma_u = \gamma$ otherwise set $\gamma_l = \gamma$; go back to step 2.

For MIMO biproper ($D \neq 0$) systems:

$$H = \begin{pmatrix} A + BR^{-1}D^TC & BR^{-1}B^T \\ -C^T(I + DR^{-1}D^T)C & -(A + BR^{-1}D^TC)^T \end{pmatrix}$$ and $R = \gamma^2 I - D^TD$

Input-output relationships

If we know how big the input is, how big is the output going to be?

Output Norms for Two Inputs

$u(t)$	$\delta(t)$	$\sin(\omega t)$
$\|y\|_2$	$\|\hat{G}\|_2$	∞
$\|y\|_\infty$	$\|G\|_\infty$	$\|\hat{G}(j\omega)\|$

Proofs:

- If $u(t) = \delta(t)$ then $y(t) = \int_{-\infty}^{\infty} G(t-\tau)\delta(\tau)d\tau = G(t)$, so $\|y\|_2 = \|G\|_2 = \|\hat{G}\|_2$
- If $u(t) = \delta(t)$ then $y(t) = G(t)$, so $\|y\|_\infty = \|G\|_\infty$
- If $u(t) = \sin(\omega t)$ then $y(t) = |\hat{G}(j\omega)| \sin(\omega t + \phi)$, so $\|y\|_2 = \infty$ and $\|y\|_\infty = |\hat{G}(j\omega)|$

Input-output relationships

System Gain: $\sup\{\|y\| : \|u\| \leq 1\}$

	$u(t) \in \mathcal{L}_2$	$u(t) \in \mathcal{L}_\infty$
$\|y\|_2$	$\|\hat{G}\|_\infty$	∞
$\|y\|_\infty$	$\|\hat{G}\|_2$	$\|G\|_1$

Proofs:

Entry (1,1): We have

$$\|y\|_2^2 = \|\hat{y}\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{G}(j\omega)|^2 |\hat{u}(j\omega)|^2 d\omega \leq \|\hat{G}\|_\infty^2 \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{u}(j\omega)|^2 d\omega$$

$$= \|\hat{G}\|_\infty^2 \|\hat{u}\|_2^2 = \|\hat{G}\|_\infty^2 \|u\|_2^2$$

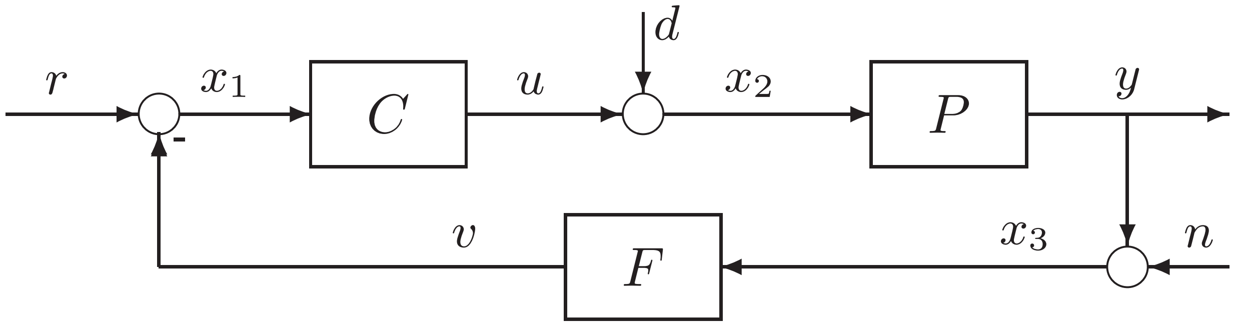
Entry(2,1): According to the Cauchy-Schwartz inequality

$$|y(t)| = \left|\int_{-\infty}^{\infty} G(t-\tau)u(\tau)d\tau\right| \leq \left(\int_{-\infty}^{\infty} G^2(t-\tau)d\tau\right)^{1/2} \left(\int_{-\infty}^{\infty} u^2(\tau)d\tau\right)^{1/2}$$

$$= \|G\|_2\|u\|_2 = \|\hat{G}\|_2\|u\|_2 \Rightarrow \|y\|_\infty \leq \|\hat{G}\|_2\|u\|_2$$

Basic Concepts

Basic Feedback Loop:



$$x_1 = r - Fx_3$$
$$x_2 = d + Cx_1 \implies \begin{pmatrix} 1 & 0 & F \\ -C & 1 & 0 \\ 0 & -P & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} r \\ d \\ n \end{pmatrix}$$
$$x_3 = n + Px_2$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & F \\ -C & 1 & 0 \\ 0 & -P & 1 \end{pmatrix}^{-1} \begin{pmatrix} r \\ d \\ n \end{pmatrix} = \frac{1}{1+PCF} \begin{pmatrix} 1 & -PF & -F \\ C & 1 & -CF \\ PC & P & 1 \end{pmatrix} \begin{pmatrix} r \\ d \\ n \end{pmatrix}$$

Well-posedness: The system is *well-posed* iff the above matrix is nonsingular. A stronger notion of well-posedness is that all the nine transfer functions be proper. A necessary and sufficient condition for this is that $1 + PCF$ not be strictly proper ($PCF(\infty) \neq -1$)

Internal Stability

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If the following nine transfer functions are stable then the feedback system is *internally stable*.

$$\frac{1}{1+PCF}\begin{pmatrix} 1 & -PF & -F \\ C & 1 & -CF \\ PC & P & 1 \end{pmatrix} \qquad P=\frac{N_P}{M_P}, \quad C=\frac{N_C}{M_C}, \quad F=\frac{N_F}{M_F}$$

Theorem: The feedback system is internally stable iff

- there are no zeros in Re $s \geq 0$ in the characteristic polynomial

$$N_P N_C N_F + M_P M_C M_F = 0$$

  or

- the following two conditions hold:

  (a) The transfer function $1+PCF$ has no zeros in Re $s \geq 0$.

  (b) There is no pole-zero cancellation in Re $s \geq 0$ when the product PCF is formed.

Asymptotic Tracking

Internal Model Principle: For perfect asymptotic tracking of $r(t)$, the loop transfer function $\hat{L} = \hat{P}\hat{C}$ (with $\hat{F} = 1$) must contain the unstable poles of $\hat{r}(s)$.

Theorem: Assume that the feedback system is internally stable and n=d=0.

(a) If $r(t)$ is a step, then $\lim_{t\to\infty} e(t) = r(t) - y(t) = 0$ iff $\hat{S} = (1 + \hat{L})^{-1}$ has at least one zero at the origin.

(b) If $r(t)$ is a ramp, then $\lim_{t\to\infty} e(t) = 0$ iff $\hat{S}$ has at least two zeros at the origin.

(c) If $r(t) = \sin(\omega t)$, then $\lim_{t\to\infty} e(t) = 0$ iff $\hat{S}$ has at least one zero at $s = j\omega$.

Final-Value Theorem: If $\hat{y}(s)$ has no poles in Re $s \geq 0$ except possibly one pole at $s = 0$ then:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\hat{y}(s)$$

Proof (a): We have $\hat{r}(s) = \frac{c}{s}$ and $\hat{e}(s) = \hat{S}(s)\frac{c}{s}$ then $\lim_{t\to\infty} e(t) = \lim_{s\to 0} s\hat{S}(s)\frac{c}{s}$. The limit is zero iff $\hat{S}$ has at least one zero at origin. For this, $\hat{P}$ or $\hat{C}$ should have a pole at origin.

Performance

Tracking performance can be quantified in terms of a weighted norm of the sensitivity function

Sensitivity Function: Transfer function from r to tracking error e : $S = \frac{1}{1+PC}$

Complementary Sensitivity Function: Transfer function from r to y : $T = \frac{PC}{1+PC}$

S is the relative sensitivity of T with respect to relative perturbations in P:

$$S = \lim_{\Delta P \to 0} \frac{\Delta T/T}{\Delta P/P} = \frac{dT}{dP}\frac{P}{T} = \frac{C(1+PC) - PC^2}{(1+PC)^2}\frac{P(1+PC)}{PC} = \frac{1}{1+PC}$$

Performance Specification:

1. $r(t)$ is any sinusoid of amplitude ≤ 1 filtered by W_1, then the max. amp. of e is $\|W_1 S\|_\infty$.
2. Suppose that $\{r = W_1 r_{pf}, \|r_{pf}\|_2 \leq 1\}$, then $\sup_r \|e\|_2 = \|W_1 S\|_\infty$.
3. In some applications good performance is achieved if $|S(j\omega)| < |W_1(j\omega)|^{-1}, \quad \forall\omega$ or

$$\|W_1 S\|_\infty < 1 \Leftrightarrow |W_1(j\omega)| < |1 + L(j\omega)|, \forall\omega$$

Uncertainty and Robustness

Plant Uncertainty: We cannot exactly model the physical systems so there is always the modeling errors. The best technic is to define a model set which can be *structured* or *unstructured*.

- **Structured model set** such as parametric uncertainty or multiple model set

$$\mathcal{P} = \{\frac{1}{s^2+as+1} : a_{\min} \leq a \leq a_{\max}\} \quad \text{or} \quad \mathcal{P} = \{P_0, P_1, P_2, P_3\}$$

- **Unstructured model set** such as unmodeled dynamics or disk uncertainty

$$\mathcal{P} = \{P_0 + \Delta : \quad \|\Delta\|_\infty \leq \gamma\}$$

Conservatism: Controller design for a model set greater than the real model set leads to a *conservative* design.

Uncertainty Models (unstructured):

Additive uncertainty	**Multiplicative uncertainty**	**Feedback uncertainty**
$\tilde{P} = P + \Delta W_2$	$\tilde{P} = P(1+\Delta W_2)$	$\tilde{P} = \frac{P}{1+\Delta W_2 P}$ or $\tilde{P} = \frac{P}{1+\Delta W_2}$

$\tilde{P}$: true model $\quad P$: nominal model $\quad \Delta$: norm-bounded uncertainty $\quad W_2$: weighting filter

Examples

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Example 1: k frequency-response models are identified. Find the multiplicative uncertainty model and the weighting filter.

$$\tilde{P} = P(1 + \Delta W_2) \Rightarrow \frac{\tilde{P}}{P} - 1 = \Delta W_2$$

$$\text{if} \quad \|\Delta\|_\infty \leq 1 \Rightarrow \left| \frac{\tilde{P}(j\omega)}{P(j\omega)} - 1 \right| \leq |W_2(j\omega)|$$

Let (M_{ik}, ϕ_{ik}) be the magnitude-phase at ω_i in k-th experiment and (M_i, ϕ_i) that of the nominal model (e.g. the mean value).

$$\max_k \left| \frac{M_{ik} e^{\phi_{ik}}}{M_i e^{\phi_i}} - 1 \right| \leq |W_2(j\omega_i)| \qquad \forall i$$

Example 2: Suppose that $\tilde{P}(s) = \{\frac{k}{s-2} : 0.1 \leq k \leq 10\}$. Represent this model by the multiplicative uncertainty.

$$P(s) = \frac{k_0}{s-2} \Rightarrow \left| \frac{\tilde{P}(j\omega)}{P(j\omega)} - 1 \right| \leq |W_2(j\omega)| \Rightarrow \max_{0.1 \leq k \leq 10} \left| \frac{k}{k_0} - 1 \right| \leq |W_2(j\omega)|$$

The best value for k_0 is 5.05 which gives $W_2(s) = 4.95/5.05$

Examples

Example 3: Assume that $P(s) = \frac{1}{s^2}$ and $\tilde{P}(s) = e^{-\tau s}\frac{1}{s^2}$ where $0 \le \tau \le 0.1$. Find $W_2(s)$ for the multiplicative uncertainty model.

$$\left|\frac{\tilde{P}(j\omega)}{P(j\omega)} - 1\right| \le |W_2(j\omega)| \Rightarrow |e^{-\tau j\omega} - 1| \le |W_2(j\omega)| \quad \forall \omega, \tau$$

Using the Bode diagram we can find $W_2(s) = \frac{0.21s}{0.1s+1}$.

Example 4: Consider the model set $\{\frac{1}{s^2+as+1} : 0.4 \le a \le 0.8\}$. Find $W_2(s)$ for Feedback uncertainty model.

Take:

$$a = 0.6 + 0.2\Delta, \quad -1 \le \Delta \le 1$$

So

$$\tilde{P}(s) = \frac{1}{s^2 + 0.6s + 0.2\Delta s + 1} = \frac{P(s)}{1 + \Delta W_2(s) P(s)}$$

where

$$P(s) = \frac{1}{s^2 + 0.6s + 1}, \quad W_2(s) = 0.2s$$

Robust Stability

Robustness: A controller is robust with respect to a closed-loop characteristic, if this characteristic holds for every plant in $\mathcal{P}$

Robust Stability: A controller is robust in stability if it provides internal stability for every plant in $\mathcal{P}$

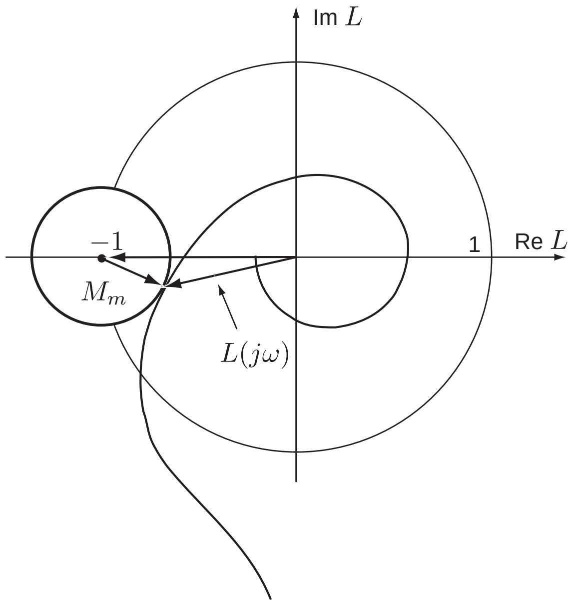
Stability margin: For a given model set with an associate size, it can be defined as the largest model set stabilized by a controller.

Stability margin for an uncertainty model: Given $\tilde{P} = P(1 + \Delta W_2)$ with $\|\Delta\|_\infty \leq \beta$, the stability margin for a controller C is the least upper bound of β.

Modulus margin: The distance from -1 to the open-loop Nyquist curve.

$$M_m = \inf_\omega |-1 - L(j\omega)| = \inf_\omega |1 + L(j\omega)|$$

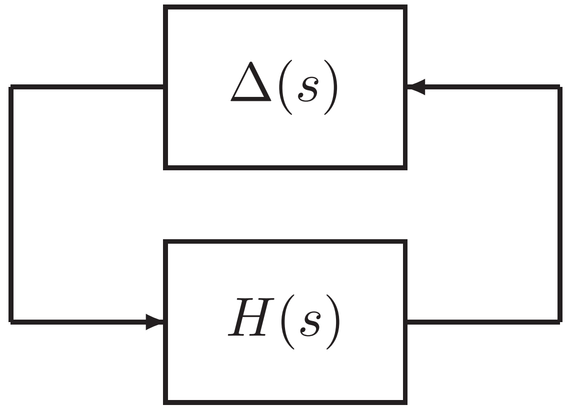
$$= \left[\sup_\omega \frac{1}{1 + L(j\omega)}\right]^{-1} = \|S\|_\infty^{-1}$$



Robust Stability

Small Gain Theorem: Suppose $H \in \mathcal{RH}_\infty$ and let $\gamma > 0$. The following feedback loop is internally stable for all $\Delta(s) \in \mathcal{RH}_\infty$ with

$$\|\Delta\|_\infty \leq 1/\gamma \quad \text{if and only if} \quad \|H\|_\infty < \gamma$$

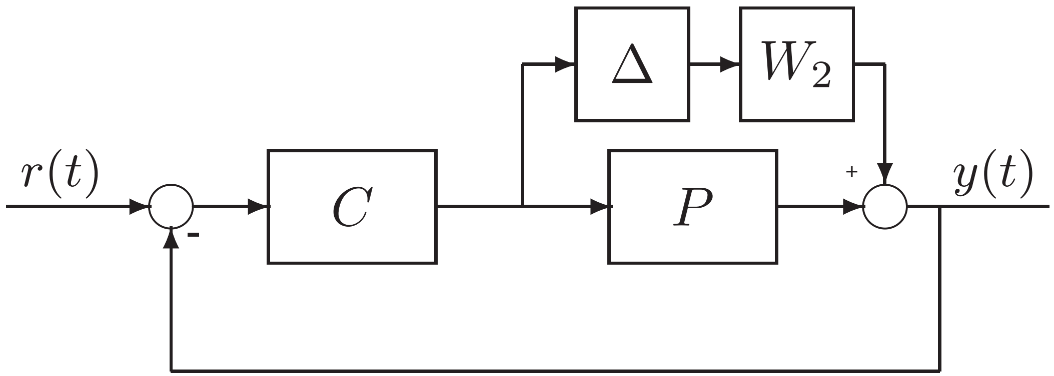


Remark: For a given Δ with $\|\Delta\|_\infty \leq 1/\gamma$ the condition $\|H\|_\infty < \gamma$ is only sufficient and very conservative. However for all $\Delta \in \mathcal{RH}_\infty$, it is a necessary and sufficient condition.

Robust stability condition for plants with additive uncertainty:

$$\tilde{P} = P + \Delta W_2 \Rightarrow H = W_2 \frac{-C}{1 + CP}$$

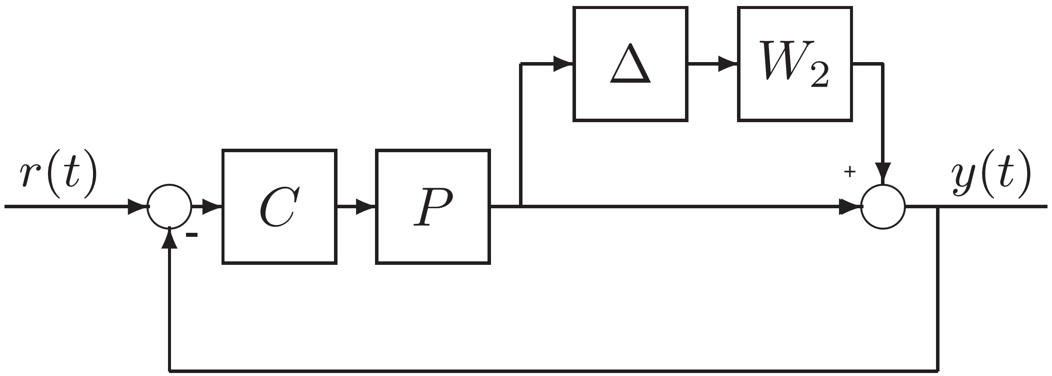
Closed-loop system is internally stable for all $\|\Delta\|_\infty \leq 1$ *iff* $\|W_2 C S\|_\infty < 1$



Robust Stability

Robust stability condition for plants with multiplicative uncertainty:

$$\tilde{P} = P(1+\Delta W_2) \Rightarrow H = W_2 \frac{-CP}{1+CP}$$



Closed-loop system is internally stable for all $\|\Delta\|_\infty \leq 1$ *iff* $\|W_2 T\|_\infty < 1$

Proof: Assume that $\|W_2 T\|_\infty < 1$. We show that the winding number of $1 + CP$ around zero is equal to that of $1 + C\tilde{P}$.

$$1 + C\tilde{P} = 1 + CP(1 + \Delta W_2) = 1 + CP + CP\Delta W_2 = 1 + CP + (1 + CP)T\Delta W_2$$

$$1 + C\tilde{P} = (1 + CP)(1 + \Delta W_2 T)$$

so Wno $\{ (1 + C\tilde{P})\} =$ Wno$\{(1 + CP)\} +$ Wno$\{(1 + \Delta W_2 T)\}$.

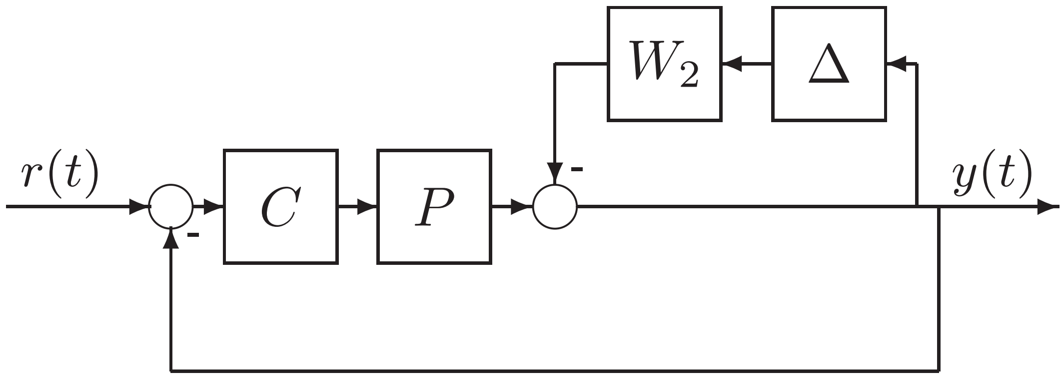
But Wno $\{(1 + \Delta W_2 T)\} = 0$ because $\|\Delta W_2 T\|_\infty < 1$

Robust Stability

Robust stability condition for plants with feedback uncertainty (1):

$$\tilde{P} = \frac{P}{1 + \Delta W_2} \Rightarrow H = W_2 \frac{-1}{1 + CP}$$

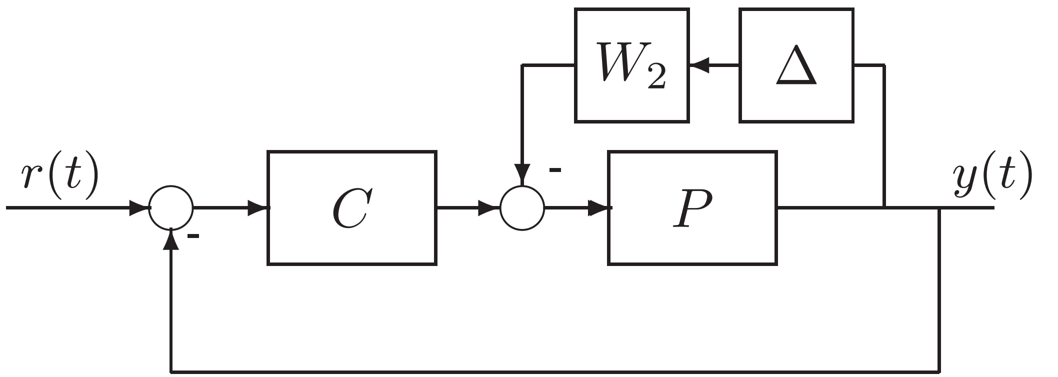
Closed-loop system is internally stable for all $\|\Delta\|_\infty \leq 1$ *iff* $\|W_2 S\|_\infty < 1$



Robust stability condition for plants with feedback uncertainty (2):

$$\tilde{P} = \frac{P}{1 + \Delta W_2 P} \Rightarrow H = W_2 \frac{-P}{1 + CP}$$

Closed-loop system is internally stable for all $\|\Delta\|_\infty \leq 1$ *iff* $\|W_2 P S\|_\infty < 1$



Robust Performance

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Nominal performance condition: $\|W_1 S\|_\infty < 1$

Robust stability condition for multiplicative uncertainty: $\|W_2 T\|_\infty < 1$

Robust performance for multiplicative uncertainty: $\|W_2 T\|_\infty < 1$ and $\|W_1 \tilde{S}\|_\infty < 1$ where:

$$\tilde{S} = \frac{1}{1 + C\tilde{P}} = \frac{1}{1 + CP(1 + \Delta W_2)} = \frac{1}{(1 + CP)(1 + \Delta W_2 T)} = \frac{S}{1 + \Delta W_2 T}$$

Robust performance conditions: $\|W_2 T\|_\infty < 1$ and $\left\| \frac{W_1 S}{1 + \Delta W_2 T} \right\|_\infty < 1$

Theorem: A necessary and sufficient condition for robust performance is

$$\| \, |W_1 S| + |W_2 T| \, \|_\infty < 1$$

Robust performance for additive uncertainty: $\|W_2 C S\|_\infty < 1$ and $\|W_1 \tilde{S}\|_\infty < 1$ where:

$$\tilde{S} = \frac{1}{1 + C\tilde{P}} = \frac{1}{1 + CP + C\Delta W_2} = \frac{S}{1 + \Delta W_2 C S} \Rightarrow \left\| \frac{W_1 S}{1 + \Delta W_2 C S} \right\|_\infty < 1$$

Or equivalently in one inequality condition: $\| \, |W_1 S| + |W_2 C S| \, \|_\infty < 1$

Stabilization

The main objective is to parameterize all of the controllers which provide internal stability for a given plant

Theorem: Assume that $P \in \mathcal{RH}_\infty$ (P is stable). The set of all stabilizing controllers is given by:

$$C := \left\{ \frac{Q}{1 - PQ} \middle| Q \in \mathcal{RH}_\infty \right\}$$

Proof: ($F = 1$)

$$\frac{1}{1 + PCF} \begin{pmatrix} 1 & -PF & -F \\ C & 1 & -CF \\ PC & P & 1 \end{pmatrix} = \begin{pmatrix} 1 - PQ & -P(1 - PQ) & -1(1 - PQ) \\ Q & 1 - PQ & -Q \\ PQ & P(1 - PQ) & 1 - PQ \end{pmatrix} \in \mathcal{RH}_\infty$$

On the other hand, suppose that C stabilizes P then define

$$Q := \frac{C}{1 + CP} \in \mathcal{RH}_\infty \text{ which leads to } C = \frac{Q}{1 - PQ}$$

In this parameterization sensitivity and complementary sensitivity are

$$S = 1 - PQ \qquad T = PQ$$

Coprime Factorization

Objective: Given P, find M, N, X and $Y \in \mathcal{RH}_\infty$ such that:

$$P = \frac{N}{M} \qquad NX + MY = 1$$

Remarks:

- N and M are called coprime factors of G over $\mathcal{RH}_\infty$
- N and M can have no common zeros in Re $s \geq 0$ nor at $s = \infty$

$$N(s_0)X(s_0) + M(s_0)Y(s_0) = 0 \neq 1$$

- If P is stable we have : $M = 1, N = P, X = 0, Y = 1$
- It is easy to obtain N and M, for example:

$$P(s) = \frac{1}{s-1} = \frac{N(s)}{M(s)} \quad \Rightarrow N(s) = \frac{1}{(s+1)^k}, \quad M(s) = \frac{s-1}{(s+1)^k}$$

if $k > 1$ then M and N have a common zero at $s = \infty$, so $k = 1$

How to compute $X(s)$ and $Y(s)$?

Coprime Factorization

Euclid's algorithm: Given polynomials $m(\lambda)$ and $n(\lambda)$ (deg $n \leq$ deg m) find polynomials $x(\lambda)$ and $y(\lambda)$ such that $nx + my = 1$.

Step 1: Divide m into n to get quotient q_1 and remainder r_1: $n = mq_1 + r_1$, deg $r_1 <$ deg m

Step 2: Divide r_1 into m to get quotient q_2 and remainder r_2: $m = r_1 q_2 + r_2$, deg $r_2 <$ deg r_1

Step 3: Divide r_2 into r_1 to get quotient q_3 and remainder r_3: $r_1 = r_2 q_3 + r_3$, deg $r_3 <$ deg r_2

Continue Stop at step k when r_k is a nonzero constant.

Find r_3 as a function of m, n and q_i:

$$r_3 = \underbrace{(n - mq_1)}_{r_1} - \overbrace{(m - \underbrace{(n - mq_1)}_{r_1} q_2)}^{r_2} q_3 = n(1 + q_2 q_3) + m(-q_3 - q_1 - q_1 q_2 q_3)$$

which gives:

$$x = \frac{1}{r_3}(1 + q_2 q_3) \quad \text{and} \quad y = \frac{1}{r_3}(-q_3 - q_1 - q_1 q_2 q_3)$$

Coprime Factorization

Procedure to find M, N, X and Y for an unstable plant G:

Step 1: Transform $G(s)$ to $\tilde{G}(\lambda)$ under the mapping $s = (1-\lambda)/\lambda$. Write $\tilde{G} = \dfrac{n(\lambda)}{m(\lambda)}$

Step 2: Using Euclid's algorithm, find $x(\lambda)$ and $y(\lambda)$ such that: $nx + my = 1$

Step 3: Find M, N, X and Y from m, n, x and y under the mapping $\lambda = 1/(s+1)$

State-Space Method:

Step 1: Transform $G(s)$ to A, B, C and D (state space realization)

Step 2: Compute F and H so that $A + BF$ and $A + HC$ are stable (`F=-place(A,B,Pf)`)

Step 3: Compute M, N, X and Y as follows:

$$M(s) := \left[\begin{array}{c|c} A+BF & B \\ \hline F & 1 \end{array}\right] \qquad N(s) := \left[\begin{array}{c|c} A+BF & B \\ \hline C+DF & D \end{array}\right]$$

$$X(s) := \left[\begin{array}{c|c} A+HC & H \\ \hline F & 0 \end{array}\right] \qquad Y(s) := \left[\begin{array}{c|c} A+HC & -B-HD \\ \hline F & 1 \end{array}\right]$$

Controller Parametrization

Theorem: The set of all Cs for which the feedback system is internally stable equal:

$$C = \left\{ \frac{X + MQ}{Y - NQ} : \quad Q \in \mathcal{RH}_\infty \right\}$$

Proof: For $C = \dfrac{N_c}{M_c}$, the stability condition is: $(NN_c + MM_c)^{-1} \in \mathcal{RH}_\infty$, but we have:

$$N(X + MQ) + M(Y - NQ) = NX + MY = 1 \Rightarrow (NN_c + MM_c)^{-1} \in \mathcal{RH}_\infty$$

Conversely, if C stabilizes the closed-loop system we should show that it belongs to the above set.

C is stabilizing $\Rightarrow V := (NN_c + MM_c)^{-1} \in \mathcal{RH}_\infty \Rightarrow NN_cV + MM_cV = 1$

Let Q be the solution of $M_cV = Y - NQ$. From the above equation and $NX + MY = 1$ we find that $N_cV = X + MQ$ so the controller $C = \dfrac{N_cV}{M_cV} \in$ the set of all stabilizing controller. It is easy to verify that $Q \in \mathcal{RH}_\infty$

Remark: The sensitivity functions are:

$$S = \frac{1}{1 + CP} = M(Y - NQ) \qquad T = \frac{CP}{1 + CP} = N(X + MQ)$$

Example

Let

$$P(s) = \frac{1}{(s-1)(s-2)}$$

Compute a proper controller C so that:

1. The feedback system is internally stable.

2. Perfect asymptotic tracking of step reference ($d = 0$).

3. Perfect asymptotic disturbance rejection when $d = \sin 10t$ ($r = 0$).

Procedure:

- Parameterize all stabilizing controllers.

- Reduce the asymptotic specs to interpolation constraints on the parameters.

- Find (if possible) a parameter to satisfy these constraints.

- Back-substitute to get the controller.

Design Constraints

Algebraic Constraints:

- $S+T=1$ so $|S(j\omega)|$ and $|T(j\omega)|$ cannot both be less than $1/2$ at the same frequency.
- A necessary condition for robust performance is that:

$$\min\{|W_1(j\omega)|, |W_2(j\omega)|\} < 1, \quad \forall\omega$$

  So at every frequency either $|W_1|$ or $|W_2|$ must be less than 1. Typically $|W_1|$ is monotonically decreasing and $|W_2|$ is monotonically increasing.

- If p is a pole and z a zero of L both in Re $s \geq 0$ then:

$$S(p)=0 \quad S(z)=1 \quad T(p)=1 \quad T(z)=0$$

Analytic Constraints:

- Bounds on the weights W_1 and W_2:

$$\|W_1 S\|_\infty \geq |W_1(z)| \qquad \|W_2 T\|_\infty \geq |W_2(p)|$$

  Proof from the Maximum Modulus Theorem: $\|F\|_\infty = \sup_{\text{Re } s>0} |F(s)|$

Analytic Constraints

All-Pass and Minimum-Phase Transfer Functions:

- $F(s) \in \mathcal{RH}_\infty$ is *all-pass* if $|F(j\omega)| = 1 \quad \forall\omega$
- $G(s) \in \mathcal{RH}_\infty$ is *minimum-phase* if it has no zeros in Re $s > 0$. It has the minimum phase among all transfer functions with the same magnitude (FG where F is all-pass).
- Every function G in $\mathcal{RH}_\infty$ can be presented as $G = G_{ap}G_{mp}$
- Suppose that $L = CP$ has no poles on the imaginary axis, so $S = (1+L)^{-1} = S_{ap}S_{mp}$ and S_{mp} has no zeros on the imaginary axis. Thus $S_{mp}^{-1} \in \mathcal{RH}_\infty$.
- Suppose that z and p are the only zero and pole of P in the closed RHP and C has neither poles nor zeros there. Then:

$$S_{ap} = \frac{s-p}{s+p} \qquad S(z) = 1 \Rightarrow S_{mp}(z) = S_{ap}^{-1}(z) = \frac{z+p}{z-p}$$

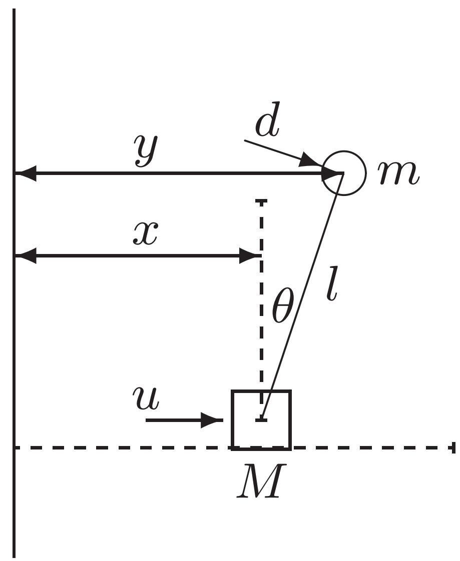
Then: $\|W_1S\|_\infty = \|W_1S_{mp}\|_\infty \geq |W_1(z)S_{mp}(z)| = \left|W_1(z)\frac{z+p}{z-p}\right|$

Similarly: $T_{ap} = \dfrac{s-z}{s+z}$ and $T(p) = 1 \Rightarrow \|W_2T\|_\infty \geq \left|W_2(p)\dfrac{p+z}{p-z}\right|$

Analytic Constraints

Example: Consider the inverse pendulum problem.

$$\begin{aligned}(M+m)\ddot{x}+ml(\ddot{\theta}\cos\theta-\dot{\theta}^2\sin\theta) &= u\\ m(\ddot{x}\cos\theta+l\ddot{\theta}-g\sin\theta) &= d\end{aligned}$$



Linearized model:

$$\begin{pmatrix} x\\ \theta\end{pmatrix} = \frac{1}{s^2[Mls^2-(M+m)g]}\begin{pmatrix} ls^2-g & -ls^2\\ -s^2 & \frac{M+m}{m}s^2\end{pmatrix}\begin{pmatrix} u\\ d\end{pmatrix}$$

$$T_{ux}=\frac{ls^2-g}{s^2[Mls^2-(M+m)g]}$$ RHP poles and zeros: $z=\sqrt{g/l}\quad p=0,0,\sqrt{\frac{(M+m)g}{Ml}}$

$$T_{u\theta}=\frac{-1}{Mls^2-(M+m)g}\qquad T_{uy}=\frac{-g}{s^2[Mls^2-(M+m)g]}$$ no RHP zero

For T_{ux} if $m\ll M\Rightarrow\|W_2T\|_\infty\gg 1$ ($|W_2(p)|$ is an increasing function) the system is difficult to control. The best case is m/M and l large.

For $T_{u\theta}$ and T_{uy} a larger l gives a smaller p so the system is easier to stabilize.

Analytic Constraints

Laboratoire d'Automatique

The Waterbed Effect

Lemma: For every point $s_0 = \sigma_0 + j\omega_0$ with $\sigma_0 > 0$,

$$\log |S_{mp}(s_0)| = \frac{1}{\pi} \int_{-\infty}^{\infty} \log |S(j\omega)| \frac{\sigma_0}{\sigma_0^2 + (\omega - \omega_0)^2} d\omega$$

Theorem: Suppose that P has a zero at z with Re $z > 0$ and:

$$M_1 := \max_{\omega_1 \leq \omega \leq \omega_2} |S(j\omega)| \qquad M_2 := \|S\|_\infty$$

Then there exist positive constants c_1 and c_2, depending only on ω_1, ω_2 and z, such that :

$$c_1 \log M_1 + c_2 \log M_2 \geq \log |S_{ap}^{-1}(z)| \geq 0$$

Theorem (The Area Formula): Assume that the relative degree of L is at least 2. Then

$$\int_0^{\infty} \log |S(j\omega)| d\omega = \pi(\log \mathrm{e}) \sum_i \mathrm{Re}\, p_i$$

where $\{p_i\}$ denotes the set of poles of L in Re $s > 0$.

Loopshaping

Objective: Given P, W_1 and W_2 find controller C providing internal stability and robust performance:

$$\| \, |W_1 S| + |W_2 T| \, \|_\infty < 1 \quad \text{or} \quad \Gamma(j\omega) := \left| \frac{W_1(j\omega)}{1 + L(j\omega)} \right| + \left| \frac{W_2(j\omega) L(j\omega)}{1 + L(j\omega)} \right| < 1 \qquad \forall \omega$$

Idea: Find graphically $L(j\omega)$ satisfying the above condition and then compute $C = L/P$

Note that we assume P is minimum phase and stable.

We have: $\Gamma|1 + L| = |W_1| + |W_2 L|$ and $|1 - |L|\,| \leq |1 + L| \leq 1 + |L|$

$$\Rightarrow \frac{|W_1| + |W_2 L|}{1 + |L|} \leq \Gamma \leq \frac{|W_1| + |W_2 L|}{|1 - |L|\,|}$$

So if $|W_1| + |W_2 L| < |1 - |L|\,| \Rightarrow \Gamma < 1$:

In low frequencies $\quad |L| > 1 \Rightarrow \quad |L| > \dfrac{|W_1| + 1}{1 - |W_2|} \simeq \dfrac{|W_1|}{1 - |W_2|} \quad |W_1| \gg 1 > |W_2|$

In high frequencies $\quad |L| < 1 \Rightarrow \quad |L| < \dfrac{1 - |W_1|}{1 + |W_2|} \simeq \dfrac{1 - |W_1|}{|W_2|} \quad |W_2| \gg 1 > |W_1|$

Procedure

step 1: Plot two curves on log-log scale:

at LF $(|W_1| > 1 > |W_2|)$ $\dfrac{|W_1|}{1 - |W_2|}$ and at HF $(|W_2| > 1 > |W_1|)$ $\dfrac{1 - |W_1|}{|W_2|}$

step 2: Fit the graph of $|L|$ on the same plot such that:

- at low frequency it lies above the first curve and also $\gg 1$
- at high frequency it lies below the second curve and $\ll 1$
- at very high frequency let it roll off at least as fast as does $|P|$ (so C is proper)
- near crossover frequency do a smooth transition, keeping the slope as gentle as possible. Because the slope of $|L|$ determines the phase of L (Bode's integral):

$$\angle L(j\omega_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d\,\ln|L|}{d\,\nu} \ln\coth\frac{|\nu|}{2} d\nu \qquad \text{where } \nu = \ln(\omega/\omega_0)$$

  The steeper the graph of L near the crossover frequency, the smaller the value of $\angle L$ and larger the phase margin

step 3: Get a stable, minimum-phase TF for L such that $L(0) > 0$ and compute $C = L/P$

Example

Assume that the relative degree of P equals 1. Find L for robust performance if the objective is to track sinusoidal signals over the frequency range from 0 to 1 rad/s and the weighting function W_2 is:

$$W_2(s) = \frac{s+1}{20(0.01s+1)}$$

We can define W_1 as follows (in loopshaping design it is not necessary to have a rational TF for W_1):

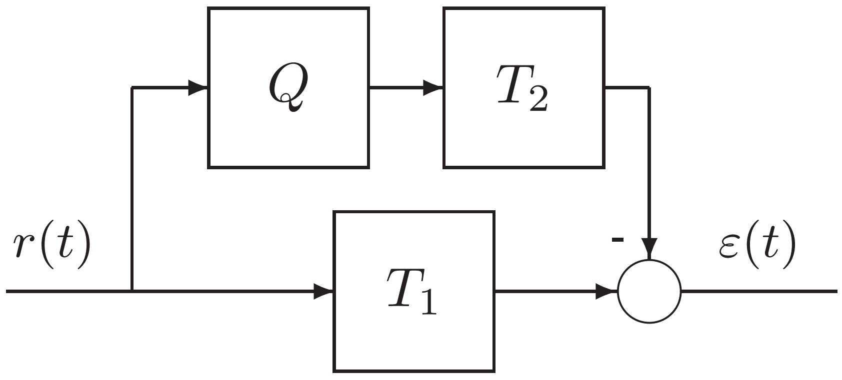
$$|W_1(j\omega)| = \begin{cases} a & 0 \le \omega \le 1 \\ 0 & \text{else} \end{cases}$$

The larger the value of a, the smaller the tracking error

- LF ($|W_1| > 1$): $\omega < 1$ HF ($|W_2| > 1$): $\omega \ge 20$
- Plot $\dfrac{|W_1|}{1-|W_2|}$ in LF ($\omega < 1$) and $\dfrac{1-|W_1|}{|W_2|}$ in HF ($\omega > 20$)
- Choose $L = \dfrac{b}{s+1}$ and find b such that in HF $|L| \le \dfrac{1-|W_1|}{|W_2|} = \dfrac{1}{|W_2|}$ ($\Rightarrow |b| \le 20$)
- Find the maximum value of a such that in LF $|L| \ge \dfrac{|W_1|}{1-|W_2|} = \dfrac{a}{1-|W_2|} \Rightarrow a = 13.15$

Model Matching

Objective: Given $T_1(s)$ and $T_2(s)$, stable proper transfer functions, find a stable $Q(s)$ to minimize $\|T_1 - T_2Q\|_\infty$



Trivial case: If T_1/T_2 is stable then the unique optimal Q is T_1/T_2 and

$$\gamma_{\text{opt}} = \min \|T_1 - T_2Q\|_\infty = 0$$

Simplest nontrivial case: T_2 has only one RHP zero at $s = s_0$. Then by the maximum modulus theorem:

$$\|T_1 - T_2Q\|_\infty \geq |T_1(s_0) - T_2(s_0)Q(s_0)| = |T_1(s_0)| \Rightarrow \gamma_{\text{opt}} \geq |T_1(s_0)|$$

Note that $Q = \dfrac{T_1 - T_1(s_0)}{T_2}$ is stable and leads to $\gamma_{\text{opt}} = |T_1(s_0)|$.

Example: $T_1(s) = \frac{4}{s+3}, \quad T_2(s) = \frac{s-2}{(s+1)^3} \Rightarrow Q = \frac{T_1 - T_1(2)}{T_2} = -\frac{4(s+1)^3}{5(s+3)}$

Nevanlinna-Pick Problem

Problem: Let $\{a_1, \ldots, a_n\}$ be a set of points in the open RHP and $\{b_1, \ldots, b_n\}$ a set of distinct points in complex plane. Find a stable, proper, complex-rational function G satisfying:

$$\|G\|_\infty \leq 1 \quad \text{and} \quad G(a_i) = b_i, \quad i = 1, \ldots, n$$

Solvability: The NP problem is solvable iff the $n \times n$ Pick matrix Q, whose ijth element is $\dfrac{1 - b_i\overline{b_j}}{a_i + \overline{a_j}}$ is positive semidefinite ($Q \geq 0$). Note that Q is Hermitian ($Q = Q^*$ where Q^* is the complex conjugate transpose of Q). $Q \geq 0$ iff all its eigenvalues are ≥ 0.

Mobius Function: A Mobius function has the form:

$$M_b(z) = \frac{z - b}{1 - z\overline{b}} \qquad \text{where } |b| < 1$$

- M_b has a zero at $z = b$ and a pole at $z = 1/\overline{b}$ so M_b is analytic in open unit disk..
- M_b maps the unit disk onto the unit disk and the unit circle onto the unit circle.
- The inverse map $M_b^{-1} = \dfrac{z + b}{1 + z\overline{b}} = M_{-b}$ is a Mobius function too.

Nevanlinna-Pick Problem

NP problem for $n = 1$: Find a stable, proper G(s) such that $\|G\|_\infty \leq 1$ and $G(a_1) = b_1$ where $|b_1| \leq 1$ and Re $a_1 > 0$.

Case 1 $|b_1| = 1$: The unique solution is $G(s) = b_1$.

Case 2 $|b_1| < 1$: The set of all solutions is:

$$\{G : G(s) = M_{-b_1}[G_1(s)A_{a_1}(s)], G_1 \in \mathcal{CRH}_\infty, \|G_1\|_\infty \leq 1]\}$$

where the all-pass function $A_a(s) := \dfrac{s-a}{s+\overline{a}}$

Example: For $a_1 = 2$ and $b_1 = 0.6$ we have: $G(s) = \dfrac{G_1(s)\frac{s-2}{s+2} + 0.6}{1 + 0.6G_1(s)\frac{s-2}{s+2}}$

$G_1(s) = 1$ results in $G(s) = \dfrac{s-0.5}{s+0.5}$

Remark 1: If G_1 is an all-pass function, so is G

Remark 2: When a_i are the complex-conjugate pairs, if $G = G_R + jG_I$ is the solution of the NP problem then G_R is also a solution to the NP problem.

Nevanlinna-Pick Problem

Laboratoire d'Automatique

Consider the NP problem with n points:

Case 1 $|b_1| = 1$**:** $G(s) = b_1$ is the unique solution (and hence $b_1 = b_2 = \cdots = b_n$).

Case 2 $|b_1| < 1$**:** Pose the NP' problem with $n-1$ data points: $\{a_2, \ldots a_n\}$ and $\{b'_2, \ldots, b'_n\}$ where $b'_i := M_{b_1}(b_i)/A_{a_1}(a_i) \quad i = 2, \ldots, n$

Lemma: The set of all solutions to the NP problem is $G(s) = M_{-b_1}[G_1(s)A_{a_1}(s)]$ where $G_1(s)$ ranges over the solutions to the NP' problem.

Example: Consider the NP problem with $a = \{1, 2\}$ and $b = \{1/2, 1/3\}$.

Solvability: The problem is solvable, because

$$Q = \begin{pmatrix} \frac{1-b_1^2}{2a_1} & \frac{1-b_1 b_2}{a_1+a_2} \\ \frac{1-b_2 b_1}{a_2+a_1} & \frac{1-b_2^2}{2a_2} \end{pmatrix} = \begin{pmatrix} 3/8 & 5/18 \\ 5/18 & 2/9 \end{pmatrix} \Rightarrow eig(Q) = [0.5867 \quad 0.0105] \Rightarrow Q \geq 0$$

NP' problem: $a_2 = 2$, $b'_2 = \dfrac{\frac{b_2-b_1}{1-b_2 b_1}}{\frac{a_2-a_1}{a_2+a_1}} = \dfrac{-0.2}{1/3} = -0.6 \Rightarrow G_1(s) = \dfrac{\frac{s-2}{s+2} - 0.6}{1 - 0.6\frac{s-2}{s+2}} = \dfrac{s-8}{s+8}$

NP problem: $G(s) = \dfrac{\frac{s-8}{s+8}\frac{s-1}{s+1} + \frac{1}{2}}{1 + \frac{1}{2}\frac{s-8}{s+8}\frac{s-1}{s+1}} = \dfrac{s^2 - 3s + 8}{s^2 + 3s + 8}$

Model Matching Problem

Find Q such that

$$\gamma_{\text{opt}} = \min_{\gamma}\{\|T_1 - T_2 Q\|_\infty \leq \gamma\} \qquad \text{Define:} \quad G = \frac{1}{\gamma}(T_1 - T_2 Q)$$

We find first G such that $\|G\|_\infty \leq 1$ then we compute $Q = \dfrac{T_1 - \gamma G}{T_2}$. However, to ensure the stability of Q, $T_1 - \gamma G$ should contain the RHP zeros of T_2 (i.e. z_i), that is:

$$\gamma G(z_i) = T_1(z_i) \Rightarrow G(z_i) = \frac{1}{\gamma} T_1(z_i)$$

This is a NP problem and γ_{opt} is the smallest γ for which the problem has a solution. That is, the associated Pick matrix is positive semidefinite. $A - \gamma^{-2} B \geq 0$ where :

$$A_{ij} = \frac{1}{z_i + \overline{z_j}} \qquad B_{ij} = \frac{T_1(z_i)\overline{T_1(z_j)}}{z_i + \overline{z_j}}$$

Lemma: γ_{opt} equals the square root of the largest eigenvalue of the matrix $A^{-1/2}\, B\, A^{-1/2}$.

Model Matching Problem

Procedure: Given T_1 and T_2 find a stable Q to minimize $\|T_1 - T_2Q\|_\infty$ (`T1=tf(num,den)`)

Step 1: Determine z_i the zeros of T_2 in Re$s > 0$.

```
zz=zero(T2);z=zz(find(real(zz)>0))
```

Step 2: Form the matrices A and B:

$$A_{ij} = \frac{1}{z_i + \overline{z_j}} \qquad B_{ij} = \frac{T_1(z_i)\overline{T_1(z_j)}}{z_i + \overline{z_j}}$$

Step 3: Compute γ_{opt} as the square root of the largest eigenvalue of the matrix $A^{-1/2}\, B\, A^{-1/2}$.

```
gamma=sqrt(max(eig(inv(sqrtm(A))*B*inv(sqrtm(A)))))
```

Step 4: Find G, the solution of the NP problem with data:

$$\begin{array}{ccc} z_1 & \dots & z_n \\ \gamma_{\text{opt}}^{-1} T_1(z_1) & \dots & \gamma_{\text{opt}}^{-1} T_1(z_n) \end{array}$$

Step 5: Set $$Q = \frac{T_1 - \gamma_{\text{opt}} G}{T_2}$$

```
Q=minreal((T1-gamma*G)/T2,0.01)
```

Model Matching Problem

State-Space Procedure:

Step 1: Factor T_2 as the product of an all-pass T_{2ap} and a minimum phase factor T_{2mp}

Step 2: Define $R := \dfrac{T_1}{T_{2ap}}$ and factor R as $R = R_1 + R_2$ with R_1 strictly proper with all poles in RHP and $R_2 \in \mathcal{H}_\infty$ and find a minimum realization of $R_1(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$

Step 3: Solve the Lyapunove equations:

$$\begin{aligned} AL_c + L_cA' &= BB' \\ A'L_o + L_oA &= C'C \end{aligned}$$

Step 4: Find the maximum eigenvalue λ^2 of L_cL_o and a corresponding eigenvector w.

Step 5: Define: $f(s) = \left[\begin{array}{c|c} A & w \\ \hline C & 0 \end{array}\right]$ $\quad g(s) = \left[\begin{array}{c|c} -A' & \lambda^{-1}L_ow \\ \hline B' & 0 \end{array}\right]$

Step 6: Then $\gamma_{\text{opt}} = \lambda$ and $Q = (R - \lambda\frac{f(s)}{g(s)})/T_{2mp}$

Design for Performance

Objective: Find a proper C for which the feedback system is internally stable and $\|W_1 S\|_\infty < 1$

Lemma: If G is stable and strictly proper, then $\lim_{\tau \to 0} \|G(1-J)\|_\infty = 0$ where $J(s) = \dfrac{1}{(\tau s + 1)^k}$

P **and** P^{-1} **stable:** In this case the set of all stabilizing controller is:

$$C = \frac{Q}{1 - PQ} \qquad Q \in \mathcal{H}_\infty \quad \text{and} \quad W_1 S = W_1(1 - PQ)$$

Clearly, $Q = P^{-1}$ is stable but not proper, so let's try $Q = P^{-1}J$ to make it proper. Then $W_1 S = W_1(1 - J)$ whose ∞-norm is less than 1 for sufficiently small τ.

P^{-1} **stable:**

- Do a coprime factorization of $P = N/M, \quad NX + MY = 1$
- Set $J = (\tau s + 1)^{-k}$ with $k =$ the relative degree of P
- Choose τ so small that $\|W_1 MY(1 - J)\|_\infty < 1$
- Set $Q = YN^{-1}J$ and $C = (X + MQ)/(Y - NQ)$

P^{-1} Unstable (General Case)

Assumptions: P has no poles or zeros on the imaginary axis, only distinct poles and zeros in the RHP and at least one zero in the RHP. W_1 is stable and strictly proper.

Procedure:

Step 1: Do a coprime factorization of $P = N/M, \quad NX + MY = 1$

Step 2: Find a stable improper Q_{im} such that:

$$\|W_1 S\|_\infty = \|W_1 M(Y - NQ_{\text{im}})\|_\infty < 1$$

It is a standard model matching problem that can be solved using the NP algorithm.

Step 3: Set $J = \dfrac{1}{(\tau s + 1)^k}$ with $k =$ large enough that Q is proper and τ small enough that

$$\|W_1 M(Y - NQ_{\text{im}} J)\|_\infty < 1$$

Step 4: Set $Q = Q_{\text{im}} J$

Step 5: Set $C = (X + MQ)/(Y - NQ)$

Design Example

Flexible Beam: Consider the following simplified plant transfer function:

$$P(s) = \frac{-6.47s^2 + 4.03s + 176}{s(5s^3 + 3.57s^2 + 140s + 0.093)} \quad \begin{cases} \text{zeros} & -4.91 \quad 5.53 \\ \text{poles} & 0 \quad -0.0007 \quad -0.356 \pm 5.27j \end{cases}$$

Performance Specification: Settling time ≈ 8s and overshoot $\leq$ 10%

Assume that the ideal $T(s)$ is a standard second-order system:

$$T_{\text{id}}(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad \frac{4.6}{\zeta\omega_n} \approx 8 \qquad \exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right) = 0.1 \Rightarrow \zeta = 0.6 \quad \omega_n = 1$$

Then the ideal sensitivity function is $S_{\text{id}}(s) = 1 - T_{\text{id}}(s) = \dfrac{s(s+1.2)}{s^2 + 1.2s + 1}$

We take the weighting function $W_1(s)$ to be $S_{\text{id}}^{-1}(s)$:

$$W_1(s) = \frac{s^2 + 1.2s + 1}{s(s+1.2)} \overset{\text{stable, strictly proper}}{\Longrightarrow} W_1(s) = \frac{s^2 + 1.2s + 1}{(s+0.0001)(s+1.2)(0.0001s+1)}$$

Design Example

Step 1: $P(s)$ has a pole on the imaginary axis ($s = 0$) so we perturb P to fix the problem (we add 10^{-6} to the denominator)

Step 2: The model matching problem is to minimize: $\|W_1 S\|_\infty = \|W_1(1 - PQ_{\text{im}})\|_\infty$

P has only one RHP zero at 5.53, thus $\min \|W_1(1 - PQ_{\text{im}})\|_\infty = |W_1(5.53)| = 1.02$ and the specification is not achievable.

Step 3: Let us scale W_1 as $W_1 := \frac{0.9}{1.02} W_1$. Then the optimal $Q_{\text{im}} = \frac{W_1 - 0.9}{W_1 P}$

Step 4: Set $J(s) = \frac{1}{(\tau s + 1)^3}$ and compute $\|W_1(1 - PQ_{\text{im}} J)\|_\infty$ for decreasing values of τ

τ	$\infty -$ norm
0.1	1.12
0.05	1.01
0.04	0.988

take $\tau = 0.04$ and set $Q = Q_{\text{im}} J$

Step 5: $C = \frac{Q}{1 - PQ} = \frac{(W_1 - 0.9)J}{W_1(1 - J) + 0.9J} P^{-1}$

2-Norm Minimization

Objective: Given P and W, find a proper stabilizing controller to minimize the 2-norm of a weighted closed-loop transfer function: e.g. $\min \|WPS\|_2$

Define: The subspace of functions in $\mathcal{L}_2$ that are analytic in the open RHP (all poles with Re$s \geq 0$) is the orthogonal complement of $\mathcal{H}_2$ and is denoted by $\mathcal{H}_2^{\perp}$. Every function $F \in \mathcal{L}_2$ can be expressed as $F = F_{\text{st}} + F_{\text{un}}$ where $F_{\text{st}} \in \mathcal{H}_2, F_{\text{un}} \in \mathcal{H}_2^{\perp}$

Lemma: If $F \in \mathcal{H}_2$ and $G \in \mathcal{H}_2^{\perp}$, then $\|F+G\|_2^2 = \|F\|_2^2 + \|G\|_2^2$

Problem: Obtain $Q \in \mathcal{H}_\infty$ to minimize $\|WPS\|_2 = \|WNY - WN^2Q\|_2$

Idea: Factor $U := WN^2 = U_{\text{ap}}U_{\text{mp}}$, then we have:

$$
\begin{aligned}
\|WNY - WN^2Q\|_2^2 &= \|WNY - U_{\text{ap}}U_{\text{mp}}Q\|_2^2 = \|U_{\text{ap}}^{-1}WNY - U_{\text{mp}}Q\|_2^2 \\
&= \|(U_{\text{ap}}^{-1}WNY)_{\text{un}} + (U_{\text{ap}}^{-1}WNY)_{\text{st}} - U_{\text{mp}}Q\|_2^2 \\
&= \|(U_{\text{ap}}^{-1}WNY)_{\text{un}}\|_2^2 + \|(U_{\text{ap}}^{-1}WNY)_{\text{st}} - U_{\text{mp}}Q\|_2^2
\end{aligned}
$$

which leads to: $Q_{\text{im}} = U_{\text{mp}}^{-1}(U_{\text{ap}}^{-1}WNY)_{\text{st}}$ and the minimum of the criterion: $\|(U_{\text{ap}}^{-1}WNY)_{\text{un}}\|_2$

To get a proper suboptimal Q, Q_{im} should be rolled off at high frequency.

Optimal Robust Stability

Objective: Given $P_\epsilon = (1 + \Delta W_2)P$ where $\|\Delta\|_\infty \leq \epsilon$, find the controller C that stabilizes every plant in P_ϵ and maximizes the stability margin:

$$\gamma_{\inf} := \inf_C \|W_2 T\|_\infty \qquad \epsilon_{\sup} = 1/\gamma_{\inf}$$

Procedure: Input P and W_2

Step 1: Do a coprime factorization of $P = N/M$, $NX + MY = 1$

Step 2: Solve the model-matching problem:

$$\|W_2 T\|_\infty = \|W_2 N(X + MQ)\|_\infty \qquad \text{with} \quad T_1 = W_2 N X \quad T_2 = -W_2 N M$$

and find Q_{im} and $\epsilon_{\sup} = 1/\gamma_{\text{opt}}$

Step 3: Let $\epsilon < \epsilon_{\sup}$ and set $J(s) = (\tau s + 1)^{-k}$ where k is large enough that $Q_{\text{im}} J$ is proper and τ small enough that:

$$\|W_2 N(X + M Q_{\text{im}} J)\|_\infty < \frac{1}{\epsilon}$$

Step 4: Set $Q = Q_{\text{im}} J$ and $C = (X + MQ)/(Y - NQ)$

Robust Performance Problem

Objective: Given P, W_1, W_2 find a proper controller C so that the feedback system for the nominal plant is internally stable and that:

$$\| \, |W_1 S| + |W_2 T| \, \|_\infty < 1$$

This problem cannot be solved !

Modified Problem: Consider the following inequality:

$$\| \, |W_1 S|^2 + |W_2 T|^2 \, \|_\infty < 1/2$$

The robust performance problem with this inequality can be converted to a model matching problem (See Feedback Control Theory chapter 12.3)

This inequality is a sufficient condition for the inequality in the exact problem.

General framework: The inequality in the modified problem can be presented also as:

$$\left\| \begin{matrix} W_1 S \\ W_2 T \end{matrix} \right\|_\infty = \max_\omega \sigma_{\max} \begin{bmatrix} |W_1 S(j\omega)| \\ |W_2 T(j\omega)| \end{bmatrix} < \frac{1}{\sqrt{2}}$$