Roll and Yaw Moments and Stability

Yaw Moment Equation

The yaw moment is the moment about the z^{body} axis and is positive if it moves the nose of the plane to the right. The big contributor to the yaw moment is the vertical tail. We can write the yaw moment equation in a similar manner to the way we wrote the pitch-moment equation by considering the contributions from the wing-body combination and from the vertical tail. If we take moments about the center of gravity we have:

$$N = N_{wb} - L_{vt} l_{vt} \cos\alpha_{vt} - D_{vt} l_{vt} \sin\alpha_{vt} \tag{1}$$

where l_{vt} is the vertical tail length, the distance from the cg to the aerodynamic center of the vertical tail, L_{vt}, D_{vt} are the lift and drag of the vertical tail, and α_{vt} is the angle of attack of the vertical tail measured positive so as to create a positive side force. If we make the usual assumptions such as that α_{vt} is a small angle, and that $D_{vt} << L_{vt}$, we can reduce Eq. (1) to the form:

$$N = N_{wb} - L_{vt} l_{vt} \tag{2}$$

and

$$N = N_{wb} - C_{L_{vt}} 1/2\, \rho\, V_{vt}^2 S_{vt} l_{vt} \tag{3}$$

where S_{vt} is the vertical tail area. Dividing by $1/2\, \rho\, V^2 S\, b$ we obtain the yaw-moment equation in coefficient form:

$$\begin{aligned} C_n &= C_{n_{wb}} - C_{L_{vt}} \eta_{vt} \left(\frac{S_{vt} l_{vt}}{S\, b} \right) \\ &= C_{n_{wb}} - C_{L_{vt}} \eta_{vt} \forall_{vt} \end{aligned} \tag{4}$$

If we let the vertical tail lift coefficient depend on a vertical tail lift curve slope and a rudder deflection we can write it as:

$$C_{L_{vt}} = a_{vt} \alpha_{vt} + a_r \delta_r \tag{5}$$

where a_{vt} and a_r are the vertical tail lift curve slope and rudder “lift curve slope” respectively. In addition we can write the vertical tail angle of attack in terms of the vehicle side slip angle, β, and the side-wash angle, σ, as:

$$\alpha_{vt} = -\beta + \sigma \tag{6}$$

If we make the substitutions, the yawing moment equation takes the form:

$$C_n = C_{n_{wb}} - \eta_{vt} \forall_{vt} [a_{vt} (\sigma - \beta) + a_r \delta_r] \qquad (7)$$

We can put this equation in a more useful form by determining the stability and control derivatives C_{n_β} and $C_{n_{\delta_r}}$. Taking the derivative of Eq. (7) with respect to β and δ_r, we have

$$C_{n_\beta} = C_{n_{\beta_{wb}}} + a_{vt} \eta_{vt} \forall_{vt} \left(1 - \frac{\partial \sigma}{\partial \beta} \right) \qquad (8)$$

and

$$C_{n_{\delta_r}} = -a_r \eta_{vt} \forall_{vt} \qquad (9)$$

With these expressions for the stability and control derivatives, Eq. (7) can be rewritten in the convenient form:

$$C_n = C_{n_\beta} \beta + C_{n_{\delta_r}} \delta_r \qquad (10)$$

Rudder Considerations

What is a rudder for? There are several purposes for a rudder on an aircraft. These usually drive the sizing of the rudder. The four dominant needs for a rudder are:

1) To counter yawing moments due to propeller slip stream effects, especially at high angle of attack and low speed. This effect is sometimes called the “p” effect.

The “p” effect is caused by the asymmetric loading of the propeller due to the angle of attack. The down moving blade (on the right side) sees a bigger angle of attack than the up-moving blade on the left side. Consequently there is more thrust on the right side than the left. In addition, the torque of the engine driving the propeller requires more lift on the left wing then the right, causing more induced drag on the left wing adding an additional left yaw moment. Further, the rotation of the air behind the propeller causes more of a sidewash on the vertical tail from the left, causing an additional nose left yaw moment. All these yaw moments must be countered with the rudder.

2) To counter adverse yaw. When turning to the right, the left aileron is down, and the right aileron is up. This alignment causes the left wing to have more lift and hence more

drag. Hence even though one wants to turn right, the ailerons cause a left yaw moment. This moment must be countered by the rudder.

3) To counter asymmetric thrust. If one engine goes out on a multi-engine aircraft, then the rudder is used to counter the resulting thrust.

4) To assist in cross wind landings. The one time that it is desired to fly with a sideslip angle is during a cross-wind landing. Here the yawing moment due to the sideslip angle is countered by using the rudder.

Tentative Rudder Sizing for Asymmetric Thrust

A quick and dirty method for sizing the rudder for asymmetric thrust can be achieved by using just the yaw moment equation and ignoring the roll and sideforce equations. Under these circumstances we can consider countering the yawing moment due to asymmetric thrust with just the rudder. The yaw moment due to asymmetric thrust is given by

$$N_T = -T y_p \tag{11}$$

where T is the thrust of the good engine, and y_p is the moment arm along the y axis to that engine. The associated yawing moment coefficient is given by

$$C_{n_T} = -\frac{T y_p}{1/2\,\rho\, V^2 S\, b} = -C_T \frac{y_p}{b}\frac{S_p}{S} \tag{12}$$

where S_p is the reference area for the thrust coefficient, C_T. (Note that since engine manufacturers do not necessarily know in which vehicles their engines will be installed, they often use their own reference area, S_p, to base their thrust coefficient).

The yaw-moment equation can be modified to include this additional term.

$$C_n = C_{n_\beta}\beta + C_{n_{\delta_r}}\delta_r + C_{n_T} = 0 \tag{13}$$

Generally, we will require the sideslip angle, β, to be zero (to minimize drag with the engine out), so we end up with the equation for sizing the rudder as:

$$C_{n_{\delta_r}} = \frac{|\,C_{n_T}\,|}{|\,\delta_r\,|_{\max}} \tag{14}$$

Free Rudder (Stick Free)

If the rudder control is released, the rudder is free to swing if the control system is reversible. If the system is irreversible, the rudder will stay where it is put. In a reversible system, freeing the "stick" is equivalent to setting the hinge moment equal to zero. The hinge moment is given by:

$$\begin{aligned} C_{h_r} &= C_{H_0} + C_{H_{\alpha_{vt}}} \alpha_{vt} + C_{H_{\delta_r}} \delta_r + C_{H_{t_r}} \delta_{t_r} \\ &= b_{0_{vt}} + b_{1_{vt}} \alpha_{vt} + b_{2_{vt}} \delta_r + b_{3_{vt}} \delta_{t_r} \end{aligned} \tag{15}$$

If we assume $b_{0_{vt}} = 0$, and ignore the rudder trim tab, then we can determine the free rudder angle:

$$\delta_{r_{free}} = -\frac{b_{1_{vt}}}{b_{2_{vt}}} \alpha_{vt} \tag{16}$$

Then we can arrive at the free rudder factor in a manner similar to how we developed the free elevator factor for the longitudinal case. The result is:

$$F_r = 1 - \frac{a_r b_{1_{vt}}}{a_{vt} b_{2_{vt}}} \tag{17}$$

We can now replace the vertical tail lift-curve-slope with the modified value $a_{vt} \Rightarrow F_r a_{vt}$ for the case where the rudder is free. The resulting equations are then for the stick-free case. Hence we have the stick free weather-cock stability parameter:

$$\hat{C}_{n_\beta} = C_{n_{\beta_{wb}}} + F_r a_{vt} \eta_{vt} \forall_{vt} \left(1 - \frac{\partial \sigma}{\partial \beta} \right) \tag{18}$$

The stick force is given by:

$$F_{s_r} = -G\, 1/2\, \rho V^2 \eta_{vt} S_r \bar{c}_r [(b_{1_{vt}} + b_{2_{vt}} \frac{C_{n_\beta}}{C_{n_{\delta_r}}}) \beta - b_1 \sigma] \tag{19}$$

and the stick force gradient by:

$$\frac{dF_{s_r}}{d\beta} = -\frac{G_r\, 1/2\, \rho V^2 \eta_{vt} S_r \bar{c}_r b_2}{C_{n_{\delta_r}}} \hat{C}_{n_\beta} \tag{20}$$

Estimating C_{n_β}

The expression for the weathercock stability parameter is:

$$C_{n_\beta} = C_{n_{\beta_{wb}}} + a_{vt}\,\eta_{vt}\,\frac{S_{vt}\,l_{vt}}{S\,b}\left(1 - \frac{\partial\sigma}{\partial\beta}\right) \qquad (21)$$

We can separate the wing-body contributions into two parts, the contribution from the fuselage, and the contribution from the wings. From Perkins and Hage, we can write the contribution from the fuselage as:

$$C_{n_{\beta_{fuse}}} = -0.96\,K_B\,\frac{S_s}{S}\,\frac{l_f}{b}\left(\frac{h_1}{h_2}\right)^{1/2}\left(\frac{w_2}{w_1}\right)^{1/3} \text{ /rad} \qquad (23)$$

where: S_s = projected side area of fuselage

l_f = overall fuselage length

$h_1,\ w_1$ = height and width of fuselage at $\frac{l_f}{4}$ (quarter fuselage location)

$h_2,\ w_2,$ = height and width of fuselage at $\frac{3\,l_f}{4}$ (3/4 fuselage location)

$K_B = (k_B' - 0.0285) + 0.2857\,\frac{d}{l_f}$, where d = distance from nose to cg location

Further, the following table gives values for k_B' as the ratio of the length of the fuselage over its greatest height. Here, *h* is the greatest height of the fuselage measured to a "faired curve" neglecting the "bump" due to the canopy.

$\frac{l_f}{h}$	k_B'
2.5	0.175
3.0	0.150
4.0	0.125
5.0	0.080
6.0	0.055
7.0	0.038
8.0	0.025
10.0	0.005

The wing contribution to the weathercock stability parameter is given by a DATCOM expression (also found in Etkin and Reid) as: [for low speed (incompressible) flight]

$$\frac{C_{n_\beta}}{C_L^2} = \frac{1}{4\pi AR} - \frac{\tan\Lambda_{1/4}}{\pi AR(AR + 4\cos\Lambda_{1/4})}\left(\cos\Lambda_{1/4} - \frac{AR}{2} - \frac{AR^2}{8\cos\Lambda_{1/4}} + 6(h_{n_w} - h)\frac{\sin\Lambda_{1/4}}{AR}\right) \quad (24)$$

The units in this equation are /rad>

To account for compressibility effects, we must correct the above expression by multiplying it by the correction factor:

$$\frac{C_{n_\beta}}{C_L^2} = \left(\frac{AR + 4\cos\Lambda_{1/4}}{AR\,\beta + 4\cos\Lambda_{1/4}}\right)\left(\frac{AR^2\beta^2 + 4AR\,\beta\cos\Lambda_{1/4} - 8\cos^2\Lambda_{1/4}}{AR^2 + 4AR\cos\Lambda_{1/4} - 8\cos^2\Lambda_{1/4}}\right)\left(\frac{C_{n_\beta}}{C_L^2}\right)_{incomressible} \quad (25)$$

where $\beta = \sqrt{1 - M_a^2\cos^2\Lambda_{1/4}}$.

Vertical Tail Contribution

The vertical tail properties such as the lift curve slope are determined in the same manner as for the wing and horizontal tail. The only difficulty is arriving at an aspect ratio for the vertical tail. We can define two aspect ratios, a geometric aspect ratio and an aerodynamic effective aspect ratio.

The dimensions of the vertical tail can be measured in several ways. For uniformity we can define the root chord to be the chord of the vertical tail if it were projected to the reference line (typically the centerline) of the fuselage. The vertical tail span is then the distance from the centerline to the tip of the vertical tail and is designated as b_{vt}. The area of the vertical tail, S_{vt} is then the area between the root chord and the tip chord. We can now define the two aspect ratios indicated previously.

The ***Geometric Aspect Ratio*** is determined by reflecting the tail about the reference line and calculating the aspect ratio of the (imaginary) resulting symmetric vertical surface. Using the definitions of the area and span of the vertical given previously, we have the geometric aspect ratio given as:

$$AR_G = \frac{(2\,b_{vt})^2}{2\,S_{vt}} \quad (26)$$

This is the aspect ratio that should be used in all calculations that involve the geometry of the vertical tail such as sweep angles, and location of mean aerodynamic chord.

The ***Effective or Aerodynamic Aspect Ratio*** is the aspect ratio on which aerodynamic properties are based such as the lift curve slope. An approximation to the aerodynamic aspect ratio is given by (from Perkins & Hage)

$$AR_{eff} = 1.55 \frac{b_{vt}^2}{S_{vt}} \tag{27}$$

This value is smaller than the geometric value and reflects the fact that the half span is less efficient than the full span. However it is not as inefficient as a half span lifting surface by itself would be because the fuselage acts as an "end plate" making the surface slightly more effective. Hence this value represents a compromise. This is the value to be used in calculating the vertical tail lift curve slope:

$$a_{vt} = \frac{2\,\pi\,AR}{2 + \sqrt{\frac{AR_{eff}^2 (1 - M_a^2)}{k^2}\left(1 + \frac{\tan^2 \Lambda_{c/2}}{(1 - M_a^2)}\right) + 4}} \tag{28}$$

The remaining part of the vertical tail contribution can be determined from the DATCOM. The term we need to estimate is:

$$\left(1 - \frac{\partial \sigma}{\partial \beta}\right)\eta_{vt} = 0.724 + 3.06\,\frac{\frac{S_{vt}}{S}}{1 + \cos\Lambda_{1/4}} + 0.4\,\frac{z_w}{h} + 0.009\,AR_{wing} \tag{29}$$

where z_w = distance from wing root quarter chord to fuselage centerline in feet, (z_w positive down, the wing would be below the centerline for positive z_w.

h = maximum fuselage height

Roll Moment

The roll moment is designate with an L. Sometimes to avoid confusion with lift we will designate it with an L_{roll}. The roll moment coefficient is defined as:

$$C_l = \frac{L_{roll}}{1/2\, \rho\, V^2\, S\, b} \tag{30}$$

If we consider static stability in roll, we seek a restoring force for a displacement in the roll angle. If we look real carefully we find there is none. ***Hence there is no direct static stability in roll!*** However, we can look a a secondary effect due to roll. If we think of a pure disturbance in roll, we will note that the aircraft rolls, and the lift vector rolls with it (assuming angle of attack and sideslip angle are unchanged- otherwise it would be a pitch or yaw disturbance!). In the new configuration there is an unbalance side force due to the gravity component along the y axis. This force initiates a sideslip, a positive roll will cause an unbalanced force that will produce a positive sideslip and consequently a positive sideslip angle. For stability (secondary at that), we require the positive sideslip to generate a negative roll moment to restore the aircraft to level flight. Consequently, for static stability in roll (a slight misnomer) we require:

$$\frac{\partial C_l}{\partial \beta} = C_{l_\beta} < 0 \tag{31}$$

This stability parameter is called the "***dihedral effect***."

Estimating the dihedral effect

We can estimate the dihedral effect using techniques from the DATCOM and from previous experience. We can compute the value of C_{l_β} fom the following equation:

$$C_{l_\beta} = C_{L_{\beta_{wb}}} + C_{l_{\beta_{vt}}} \tag{32}$$

The contribution from the wing-body (mostly from the wing) can be determined from charts and graphs in the DATCOM and in Etkin and Reid. The main equation is:
For AR > 1

$$C_{l_{\beta_{wb}}} = C_L \left[\left(\frac{C_{l_\beta}}{C_L} \right)_{\Lambda_{1/2}} K_{M_a} + \left(\frac{C_{l_\beta}}{C_L} \right) \right]_a + \Gamma \left(\frac{C_{l_\beta}}{\Gamma} K_{m\Gamma} \right) + \theta \tan\Lambda_{1/4} \left(\frac{\Delta C_{l_\beta}}{\theta \tan\Lambda_{1/4}} \right) \tag{33}$$

where θ = the wing twist, negative for washout
Γ = the wing dihedral angle

The four ratios that appear in the ( ) along with the two parameters K_x are determined from charts in the DATCOM or Etking and Reid.

The vertical tail contribution to C_{l_β}

The vertical tail contribution to the dihedral effect can be established with simple geometry. The rolling moment due to the vertical tail is given by:

$$\begin{aligned} L_{roll_{vt}} &= -L_{vt} z_{vt} \\ &= -a_{vt} \alpha_{vt} 1/2 \rho V_{vt}^2 S_{vt} z_{vt} \end{aligned} \tag{34}$$

where z_{vt} is measured positive down, and the vertical tail lift is positive in the positive y direction. If we divide by $\bar{q} S b$ and take the derivative with respect to the sideslip angle, β, we get the desired expression:

$$C_{l_{\beta_{vt}}} = a_{vt} \eta_{vt} \frac{S_{vt} l_{vt}}{S b} \left(1 - \frac{\partial \sigma}{\partial \beta} \right) \frac{z_{vt}}{l_{vt}} = C_{n_\beta} \frac{z_{vt}}{l_{vt}} \tag{35}$$

Roll Control

For basic calculations, we can assume that the rolling motion is purely about the x axis. It that's the case, we can estimate the maximum sustained roll rate by looking at a simple roll moment balance. In general, the roll moment equation of motion is given by:

$$L_{roll} = I_x \dot{p} = 0 \tag{36}$$

where it is equal to zero for the steady state case. Dividing through by $\bar{q} S b$ we see that the requirement is that $C_l = 0$. We can write this expression in terms of the aileron deflection and roll rate as

$$C_l = C_{l_p} \hat{p}_{ss} + C_{l_{\delta_a}} \delta_a = 0 \tag{37}$$

where $\hat{p} = \frac{p b}{2 V}$ is the non-dimensional roll rate. Then the steady state roll rate for a given aileron deflection is given by:

$$\begin{aligned} \hat{p}_{ss} &= -\frac{C_{l_{\delta_a}}}{C_{l_p}} \delta_a \\ p_{ss} &= \hat{p}_{ss} \frac{2 V}{b} \end{aligned} \tag{38}$$