ROOK NUMBER INTERPRETATIONS OF GENERALIZED CENTRAL FACTORIAL AND GENOCCHI NUMBERS

FERYAL ALAYONT, RACHEL MOGER-REISCHER, AND RUTH SWIFT

Abstract. Rook polynomials count the number of ways of placing non-attacking rooks on a chess board. One way to generalize the classical 2-dimensional rook polynomial theory to higher dimensions is by letting rooks attack along hyperplanes. In this paper, we use this generalization to provide rook number interpretations of generalized central factorial and Genocchi numbers. Finally, we describe a list of permutations that are counted by generalized Genocchi numbers.

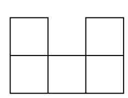
Introduction

The classical 2-dimensional rook theory was generalized to three dimensions in [5]. The boards in three dimensions can be pictured as consisting of cubes that can be stacked one on top of another and next to each other, with a rook placed in a cube attacking along the planes containing the cube. This theory is then further developed and generalized to include dimensions of four and higher in [1], along with generalizations of the properties of classical rook polynomials to higher dimensions. [1] also investigates three-dimensional generalizations of families of boards in two dimensions whose rook numbers correspond to famous known number sequences such as Stirling numbers and numbers of Latin squares.

In this paper, we generalize the family of three-dimensional triangular and Genocchi boards, which were defined in [1], to higher dimensions. These higher dimensional generalizations of the triangular and Genocchi boards are shown to correspond with the generalized central factorial and generalized Genocchi numbers as defined by [3] and investigated in [2]. The first two sections of this paper are devoted to a review of the relevant properties of the classical rook theory and higher dimensional rook theory [1]. In section 2, we also include a review of the results on three-dimensional triangular and Genocchi boards. In sections 3 and 4, we generalize these results to four and higher dimensions. In the last section, we provide an interpretation of the rook numbers of these higher dimensional boards in terms of certain lists of permutations.

1. Classical Rook Theory

Given a natural number m, let $[m]$ denote the set $\{1, 2, ..., m\}$. In two dimensions, we define a *board* B with m rows and n columns to be a subset of $[m] \times [n]$. We call such a board an $m \times n$ board if m and n are the smallest such natural numbers. Each of the elements in the board is referred to as a *cell* of the board. The set $[m] \times [n]$ is called the *full* $m \times n$ board. An example of how we visualize a board is as follows:



Numbering the rows from top to bottom and columns from left to right, the above picture corresponds to the 2×3 board $B = \{(1, 1), (1, 3), (2, 1), (2, 2), (2, 3)\}$. We sometimes highlight the cells missing from the board by shading them in gray.

The *rook polynomial* $R_B(x) = r_0(B) + r_1(B)x + ... + r_k(B)x^k + ...$ of a board B represents the number of ways that one can place various numbers of non-attacking rooks on B, i.e. no two rooks can lie in the same column or row. More specifically, $r_k(B)$ is equal to the number of ways of placing k non-attacking rooks on B. For any board, $r_0(B) = 1$ and $r_1(B)$ is equal to the number of cells in B. The rook polynomial of the above example is $R_B(x) = 1 + 5x + 4x^2$. Note that the rook polynomial of a board is invariant under translating the board, and permuting the rows and columns of the board.

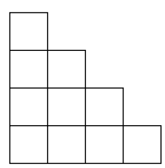
One result from classical rook theory that we will use a generalization of is the relationship between the rook numbers of a board and those of its complement. Given an $m \times n$ board B, we define the *complement* of B, denoted $\bar{B}$ to consist of all cells missing from B so that the disjoint union of B and $\bar{B}$ is the full $m \times n$ board. In other words $\bar{B} = [m] \times [n] \backslash B$. If needed, we explicitly indicate with respect to which full board the complement is taken.

Theorem (Complementary Board Theorem)**.** *Let $\bar{B}$ be the complement of B inside $[m] \times [n]$ and $R_B(x) = \sum r_i(B)x^i$ the rook polynomial of B. Then the number of ways to place k non-attacking rooks on $\bar{B}$ is*

$$r_k(\bar{B}) = \sum_{i=0}^{k} (-1)^i \binom{m-i}{k-i} \binom{n-i}{k-i} (k-i)! r_i(B)$$

taking r_i to be 0 for i greater than the degree of $R_B(x)$.

As we will show later, the generalized central factorial numbers and generalized Genocchi numbers are the rook numbers of two complementary boards whose shapes generalize the triangle boards in two dimensions. In two dimensions, a *triangle board* of size m consists of all cells (i, j) such that $1 \leq j \leq i \leq m$. The triangle board of size 4 is shown below.



The rook numbers of this family are known to correspond with the Stirling numbers of the second kind. Recall that the Stirling numbers of the second kind $S(n, k)$ count the number of ways to partition a set of size n into k non-empty sets, and can be defined recursively by

$$S(n, k) = S(n-1, k-1) + kS(n-1, k)$$

with $S(n, 1) = 1$ and $S(n, n) = 1$.

Theorem. *The number of ways to place k non-attacking rooks on a triangle board of size m is equal to $S(m+1, m+1-k)$, where $0 \leq k \leq m$.*

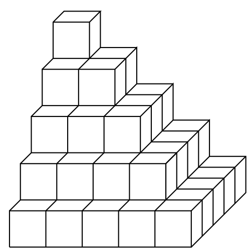
2. Rook Theory in Three and Higher Dimensions

A board in d dimensions is a subset of $[m_1] \times [m_2] \times ... \times [m_d]$ with cells corresponding to d-tuples $(i_1, i_2, ..., i_d)$ with $1 \leq i_j \leq m_j$. A *full board* is the whole set $[m_1] \times [m_2] \times \ldots \times [m_d]$. In three and higher dimensions, rooks attack along hyperplanes which consist of cells with one fixed coordinate. In particular, in three dimensions, when we place a rook in cell (i_1, i_2, i_3), we can no longer place a rook in another cell with i_1 in the first coordinate, i_2 in the second coordinate, or i_3 in the third coordinate. In d dimensions, once we place a rook in cell $(i_1, i_2, ..., i_d)$, we can no longer place a rook in any cell with first coordinate i_1 or second coordinate i_2 or so on to dth coordinate i_d.

Theorem (Complementary Board Theorem)**.** [1] *Let $\bar{B}$ be the complement of B inside $[m_1] \times [m_2] \times \cdots \times [m_d]$ and $R_B(x) = \sum_i r_i(B)x^i$ the rook polynomial of B. Then the number of ways to place $0 \leq k \leq \min_i m_i$ non-attacking rooks on B is*

$$r_k(\bar{B}) = \sum_{i=0}^{k} (-1)^i \binom{m_1 - i}{k - i}\binom{m_2 - i}{k - i} \cdots \binom{m_d - i}{k - i}(k-i)!^{d-1} r_i(B).$$

[1] generalizes the two-dimensional triangle boards to three dimensions as follows. A size 1 triangle board is simply one cell. The size 2 triangle board is obtained by placing a 2×2 layer below the size 1 triangle board. Continuing recursively in a similar way, the size m board is obtained by adding an $m \times m$ layer at the bottom of a size $m-1$ triangle board. In terms of coordinates, the cells included in the size m triangle are (i, j, k) with $1 \leq i, j \leq k$ and $1 \leq k \leq m$. The size 5 three-dimensional triangle board is depicted below.



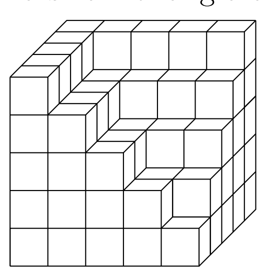
The rook numbers of these boards can be expressed in terms of central factorial numbers. Recall that the *central factorial numbers* are defined recursively by

$$T(n, k) = T(n-1, k-1) + k^2 T(n-1, k)$$

with $T(n, 1) = 1$ and $T(n, n) = 1$.

Theorem. [1] *The number of ways to place k rooks on a size m triangle board in three dimensions is equal to $T(m+1, m+1-k)$, where $0 \leq k \leq m$.*

An interesting result is obtained when the complement of a triangle board is considered. The picture below shows the complement of a size 4 triangle board inside $[5] \times [5] \times [5]$:



The number of ways to place the maximum number of rooks in this board is the unsigned 6th Genocchi number of even index (sequence A110501 in [4]). More generally, by taking the complement of the size $m-1$ triangle board inside $[m] \times [m] \times [m]$ to be the size m *Genocchi board*, [1] shows that the number of ways of placing m rooks on this board is given by the $(m+1)$th unsigned Genocchi number of even index, which we denote by G_{m+1}. By convention, the Genocchi board of size 1 consists of a single cell. The rook numbers of the Genocchi boards are found using the complementary board theorem and the relationship

$$G_m = (-1)^m \sum_{j=1}^{m-1} (-1)^{j+1} j!^2 T(m-1, j)$$

between the Genocchi numbers and the central factorial numbers [1].

3. Triangle boards in Four and Higher Dimensions

In d dimensions, a size 1 triangle board is a single cell, $(1, 1, \ldots, 1)$. A size 2 board consists of a single cell on top of a layer of cells with coordinates of the form $(i_1, i_2, \ldots, i_{d-1}, 2)$, where $1 \leq i_j \leq 2$. As m increases, additional layers are added to the bottom. Hence, a size m triangle board has an mth layer with cells described by d-tuples of the form $(i_1, i_2, \ldots, i_{d-1}, m)$ for $1 \leq i_j \leq m$. We let $\Delta_m^{(d)}$ to denote a size m triangle in d dimensions. We will omit the superscript if no confusion will arise. As in two and three dimensions, the only way to place m rooks on a size m triangle is along the diagonal, in the cells of the form $(i, i, \ldots, i)$.

To describe the rook numbers of the d-dimensional triangle boards, we will use the generalized central factorial numbers T_k defined in [3] as

$$T_k(s, t) = T_k(s-1, t-1) + t^k \cdot T_k(s-1, t)$$

with $T_k(s, s) = 1$ and $T_k(s, 1) = 1$.

Theorem. *The number of ways to arrange k rooks on a size m triangle board in d dimensions is $T_{d-1}(m+1, m+1-k)$ where $T_{d-1}(s, t)$ are the generalized central factorial numbers.*

Proof. The proof is similar to the proof in [1] of the corresponding result in three dimensions, and proceeds by induction on n.

Consider $m = 1$. The rook polynomial of the size 1 board is $1 + x$. By definition, $T_{d-1}(2, 2) = T_{d-1}(2, 1) = 1$. Thus, the result is true for $m = 1$.

In the general case, consider first $k = m$ and $k = 1$. As we noted above, for a size m triangle board in d dimensions, there is only one way to place m non-attacking rooks. This corresponds to $T_{d-1}(m+1, 1) = 1$. Also, by convention there is only one way to put zero rooks on any board, which corresponds with $T_{d-1}(m+1, m+1) = 1$.

Suppose now $1 < k < m$. We wish to show that the number of ways of placing k rooks on the size m board is $T_{d-1}(m+1, m+1-k)$. We will consider two cases, the case where no rook is placed in the bottom layer (cells with coordinates of the form $(*, \cdots, *, m)$) and the case where one rook is placed in the bottom layer.

The number of ways in the first case is the number of placing k rooks in a size $m-1$ triangle board, which by induction is $T_{d-1}(m, m-k)$. In the second case, we first find the number of ways of placing $k-1$ rooks in the a size $m-1$ triangle boards, which is $T_{d-1}(m, m-(k-1)) = T_{d-1}(m, m+1-k)$. We then find the number of ways of placing the last rook. In the bottom layer, there are m^{d-1} cells. Once the first $k-1$ rooks have been introduced on the size $m-1$ triangle board above the bottom layer, for each coordinate, $k-1$ entries become restricted. Hence, there are $m-(k-1)$ allowed entries for each coordinate, resulting in $(m-(k-1))^{d-1}$ possible cells for the last rook. So there are $(m+1-k)^{d-1} T_{d-1}(m, m+1-k)$ ways in the second case. Therefore, the total number of placing k rooks on the size n board is

$$T_{d-1}(m, m-k) + (m+1-k)^{d-1} \cdot T_{d-1}(m, m+1-k)\,,$$

which by the recurrence relation for generalized central factorial numbers equals $T_{d-1}(m+1, m+1-k)$. □

4. Genocchi Boards in Four and Higher Dimensions

Generalizing the definition of the three-dimensional Genocchi boards, we define the size m Genocchi board in d dimensions to be the complement of size $m-1$ triangle board inside the d-dimensional cube of size m. Let $\Gamma_m^{(d)}$ denote the size m Genocchi board in d dimensions. Using a similar rotation to the three-dimensional Genocchi board, we can express the cells in $\Gamma_m^{(d)}$ as $(i_1, i_2, \ldots, i_d)$ where $1 \le i_j \le m$ and $\min\{i_1, i_2, \ldots, i_{d-1}\} \le i_d$. In particular, $\Gamma_m^{(4)}$ contains cells of the form (i, j, k, ℓ) with $1 \le i, j, k, \ell \le m$ and $\min\{i, j, k\} \le \ell$. The rook numbers associated with these Genocchi boards correspond to a generalization of the Genocchi numbers defined in [2].

Let $A_{n+1}^{(k)}(x)$ be the *Ghandi polynomials* defined by

$$A_{n+1}^{(k)}(x) - x^k A_n^{(k)}(x+1) - (x-1)^k A_n^{(k)}(x)$$

where $n \ge 0$ and $A_0^{(k)}(x) = 1$. The *k-th generalized Genocchi numbers* are defined as $G_{2n}^{(k)} = A_{n-1}^{(k)}(1)$ [2]. For $k = 2$, these numbers correspond to the unsigned Genocchi numbers of even index as described in Section 2.

Because the complement of a d-dimensional Genocchi board is the triangle board, we are again able to find the rook numbers of the Genocchi board using the complementary board formula.

Theorem. *The number of ways to place m rooks on a size m Genocchi board in d dimensions is given by the $(m+1)$th number among the $(d-1)$-th generalized Genocchi numbers.*

Proof. By the complementary board formula, the number of ways to place m rooks on the Genocchi board of size m is

$$r_m(\Gamma_m^{(d)}) = \sum_{i=0}^{m} (-1)^i \cdot (m-i)!^{d-1} \cdot r_i(\Delta_{m-1})$$

where $r_i(\Delta_{m-1})$ denotes the number of ways to place i non-attacking rooks on a size $m-1$ triangle board in d dimensions. However, because it is impossible to place m rooks on a size $m-1$ triangle board, the m^{th} term in the sum is zero. Also, for $1 \le i \le m-1$, $r_i(\Delta_{m-1}) = T_{d-1}(m, m-i)$. Thus,

$$r_m(\Gamma_m^{(d)}) = \sum_{i=0}^{m-1} (-1)^i \cdot (m-i)!^{d-1} \cdot T_{d-1}(m, m-i).$$

Using the change of variable $j = m - i$, we obtain

$$r_m(\Gamma_m^{(d)}) = \sum_{j=1}^{m} (-1)^{m-j} \cdot (j!)^{d-1} \cdot T_{d-1}(m, j).$$

It has been shown in [3] that the generalized Genocchi numbers are related to the generalized central factorial numbers by the formula

$$G_{m+1}^{(k)} = \sum_{j=1}^{m} (-1)^{m-j} \cdot (j!)^k \cdot T_k(m, j).$$

Therefore, $r_m(\Gamma_m^{(d)}) = G_{m+1}^{(d-1)}$, as desired. □

5. A Combinatorial Interpretation

The classical rook theory was developed to study permutations with restrictions. A rook placement of n rooks on an $n \times n$ rook board in two dimensions corresponds to a permutation σ of n elements as follows. If a rook is placed in position (i, j), we then have $\sigma(i) = j$. This can also be thought of as ordering the rook positions based on their first coordinate (in increasing order) and then reading the numbers in the second coordinate to create a permutation. Restrictions in the permutations are realized by removing certain tiles on the board. In a similar way, rook placements in three and higher dimensions correspond to lists of permutations.

In three dimensions, rooks are placed in positions (i, j, k). Consider now placing n rooks on an $n \times n \times n$ board. Based on how rooks attack in three dimensions, there has to be one rook per layer, per wall and per slab. If we order the rook positions from first (top) layer to the last (bottom) layer, we obtain two permutations by reading the first two coordinates of these positions. Similarly, in dimension d with $d \geq 4$, we obtain lists of $d-1$ permutations by reading coordinates of the rook positions from layer 1 to layer n.

For triangle boards in any dimension, there is only one way to place the maximum number of rooks. For example, the only way to place 3 rooks on a size 3 triangle board in four dimensions is achieved by placing the rooks in positions $(1, 1, 1, 1), (2, 2, 2, 2), (3, 3, 3, 3)$. This placement corresponds to the list where the identity permutation is repeated three times, which is not an interesting list.

For Genocchi boards, however, the results are more interesting. For example, in four dimensions, there are 145 ways to place 3 rooks on a size 3 Genocchi board. Recall that the cells in a Genocchi board of size m in four dimensions are of the form (i, j, k, ℓ) with $1 \leq i, j, k, \ell \leq m$ and $\min\{i, j, k\} \leq \ell$. One of the 145 placements of 3 rooks on size 3 Genocchi board is as follows: $(3, 2, 1, 1), (1, 3, 2, 2), (2, 1, 3, 3)$. This placement corresponds to the following list of 3 permutations: $312, 231, 123$. In general, a rook placement of m rooks on size m Genocchi board in d dimensions corresponds to a list of $d-1$ permutations satisfying the property that the smallest number in the ith position of all permutations is not larger than i. Thus the $(d-1)$-th generalized Genocchi numbers count such lists of permutations.

References

[1] Feryal Alayont, Nicholas Krzywonos, "Rook Polynomials in Three and Higher Dimensions", to appear in *Involve*

[2] Michael Domaratzki, "Combinatorial Interpretations of a Generalization of the Genocchi Numbers"', *Journal of Integer Sequences*, Volume 7 (2004), Article 04.3.6.

[3] Guo-Niu Han, "Escaliers Évalués et Nombres Classiques", *Publ. I.R.M.A. Strasbourg, Actes 24e Séminaire Lotharingien*, 461/S24 (1993), 77-85.

[4] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, published electronically at http://www.research.att.com/~njas/sequences/

[5] Benjamin Zindle, "Rook Polynomials for Chessboards of Two and Three Dimensions", School of Mathematical Sciences, Rochester Institute of Technology, New York, 2007.

Department of Mathematics, Grand Valley State University, Allendale, MI 49401, e-mail: alayontf@gvsu.edu

Mathematics Department, Bucknell University, Lewisburg, PA 17837, e-mail: remr001@bucknell.edu

Yale Law School, New Haven, Connecticut, 06520, e-mail: ruthmswift@gmail.com