Rotation Matrices

Suppose that $\alpha \in \mathbb{R}$. We let

$$R_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$$

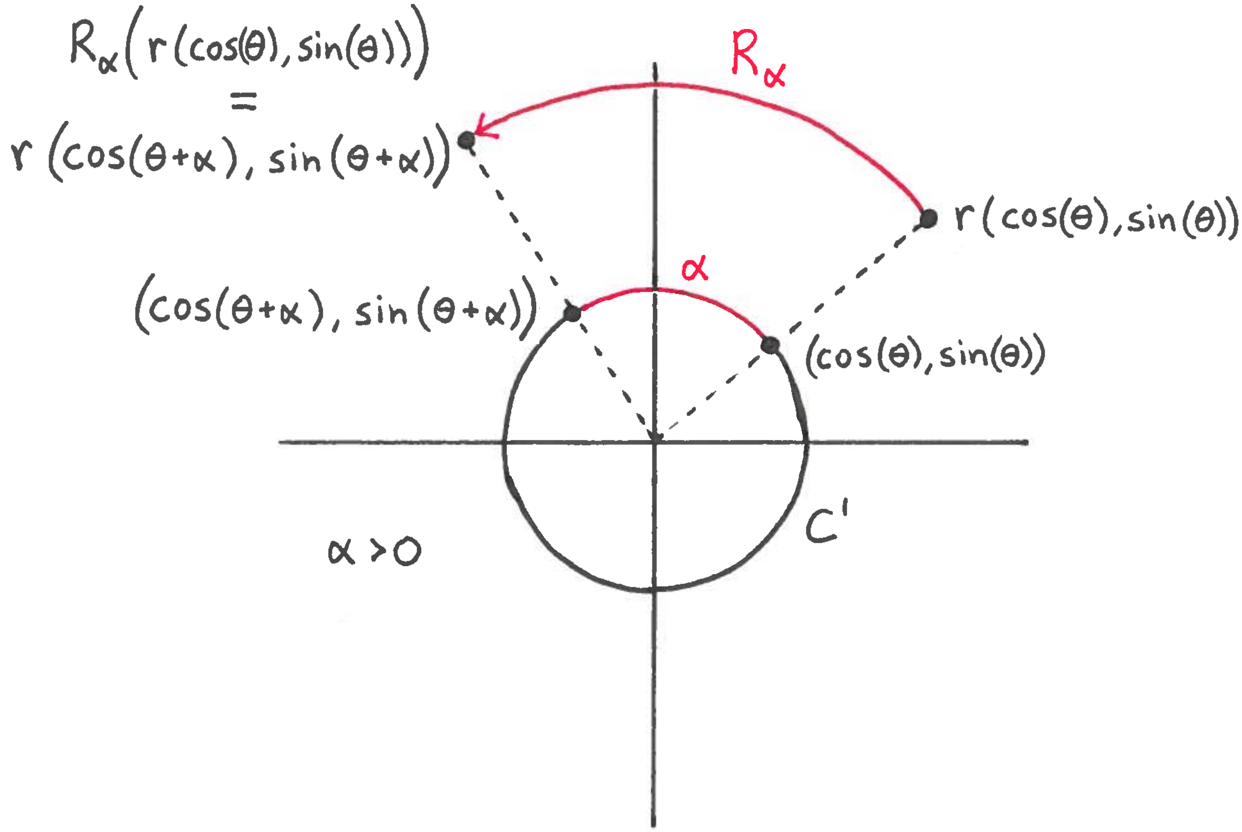
be the function defined as follows:

Any vector in the plane can be written in polar coordinates as $r(\cos(\theta), \sin(\theta))$ where $r \geq 0$ and $\theta \in \mathbb{R}$. For any such vector, we define

$$R_\alpha\big(\, r(\,\cos(\theta), \sin(\theta)\,)\,\big) = r(\,\cos(\theta + \alpha), \sin(\theta + \alpha)\,)$$

Notice that the function R_α doesn't change the norms of vectors (the number r), it just affects their direction, which is measured by the unit circle coordinate.

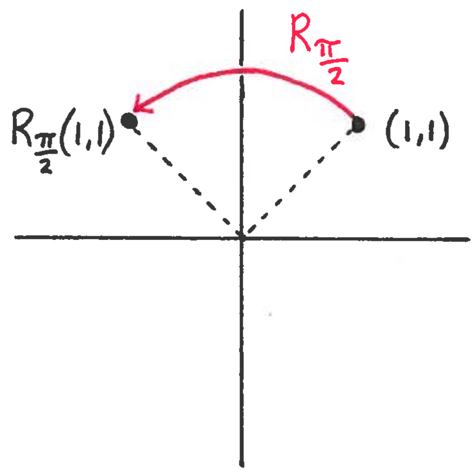
We call the function R_α *rotation of the plane by angle* α.



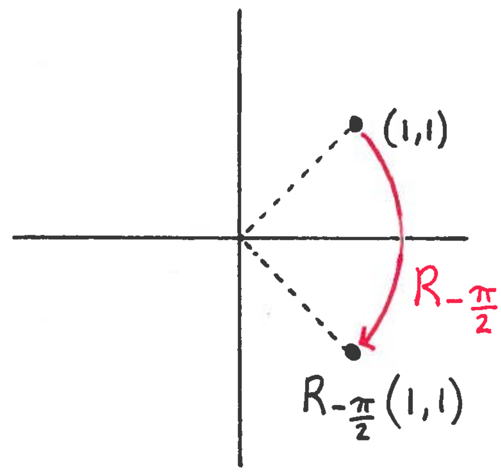
If $\alpha > 0$, then R_α rotates the plane counterclockwise by an angle of α. If $\alpha < 0$, then R_α is a clockwise rotation by an angle of $|\alpha|$. The rotation does not affect the origin in the plane. That is, $R_\alpha(0,0) = (0,0)$ always, no matter which number α is.

Examples.

• $R_{\frac{\pi}{2}}$ is the function that rotates the plane by an angle of $\frac{\pi}{2}$, or $90°$. Because $\frac{\pi}{2} > 0$, it is a counterclockwise rotation. Thus, $R_{\frac{\pi}{2}}(1,1)$ is the point in the plane that we obtain by rotating $(1,1)$ counterclockwise by an angle of $\frac{\pi}{2}$.

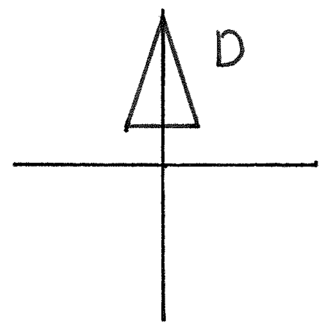


• Because $-\frac{\pi}{2} < 0$, $R_{-\frac{\pi}{2}}$ is a clockwise rotation. $R_{-\frac{\pi}{2}}(1,1)$ is the point in the plane obtained by rotating $(1,1)$ clockwise by an angle of $\frac{\pi}{2}$.

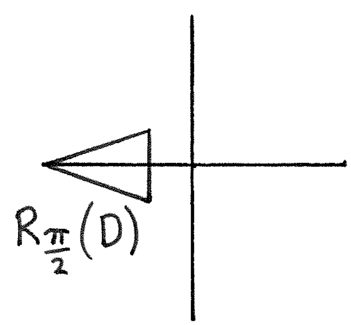


• The function $R_0 : \mathbb{R}^2 \to \mathbb{R}^2$ rotates the plane by an angle of 0. That is, it doesn't rotate the plane at all. It's just the identity function for the plane.

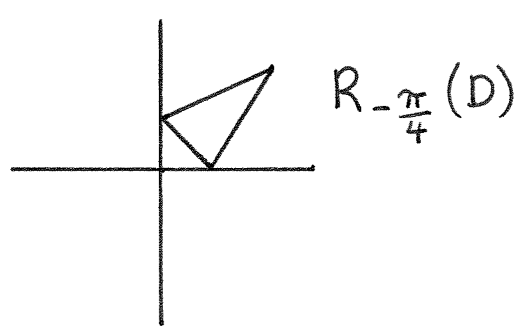
• Below is the picture of a shape in the plane. It's a triangle, and we'll call this subset of the plane D.



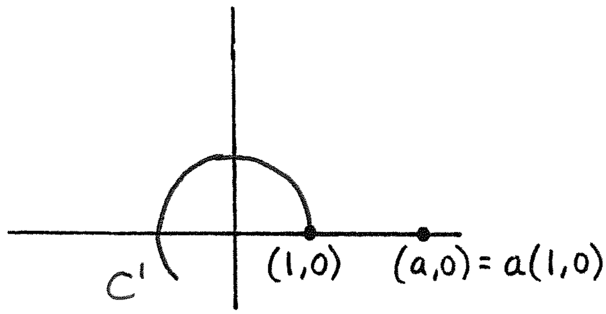
$R_{\frac{\pi}{2}}(D)$ is the set in the plane obtained by rotating D counterclockwise by an angle of $\frac{\pi}{2}$. (It's counterclockwise because $\frac{\pi}{2} > 0$.)



$R_{-\frac{\pi}{4}}(D)$ is D rotated clockwise by an angle of $\frac{\pi}{4}$.

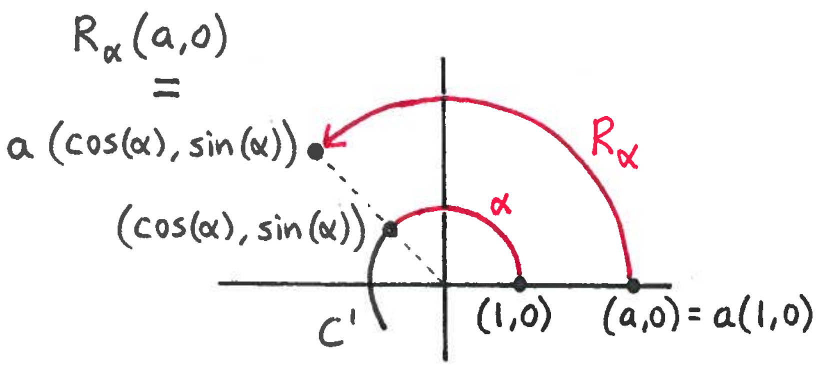


• Let's rotate the vector $(a, 0)$, where $a \geq 0$. This is a point on the x-axis whose norm equals a.

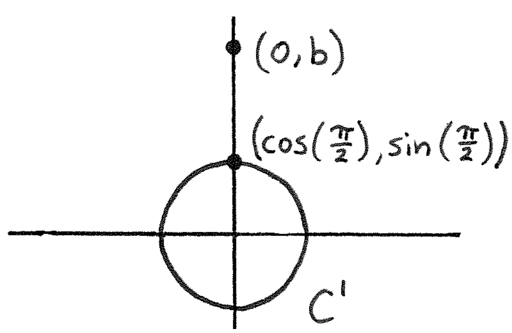


We can write this vector in polar coordinates as $a(1, 0)$, or equivalently, as $a(\cos(0), \sin(0))$. Now we can rotate the vector $(a, 0)$ by an angle α. That's the vector $R_\alpha(a, 0)$, which by the formula from the beginning of this chapter is

$$\begin{aligned} R_\alpha(a, 0) &= R_\alpha\big(a(\cos(0), \sin(0))\big) \\ &= a(\cos(0 + \alpha), \sin(0 + \alpha)) \\ &= a(\cos(\alpha), \sin(\alpha)) \end{aligned}$$



• In this example, we'll rotate a vector $(0, b)$, where $b \geq 0$. This is a vector whose norm equals b, and that points straight up. In polar coordinates, $(0, b) = b(\cos\left(\frac{\pi}{2}\right), \sin\left(\frac{\pi}{2}\right))$.



Now if we rotate $(0, b)$ by an angle α, then we have

$$\begin{aligned} R_\alpha(0,b) &= R_\alpha\Big(b\Big(\cos\left(\frac{\pi}{2}\right), \sin\left(\frac{\pi}{2}\right) \Big)\Big) \\ &= b\Big(\cos\left(\frac{\pi}{2}+\alpha\right), \sin\left(\frac{\pi}{2}+\alpha\right) \Big) \\ &= b\Big(\cos\left(\alpha+\frac{\pi}{2}\right), \sin\left(\alpha+\frac{\pi}{2}\right) \Big) \end{aligned}$$

There's a slightly better way to write the result above, but it requires a couple of the identities we learned in the chapter "Sine and Cosine". Specifically, Lemmas 8-10 tell us that

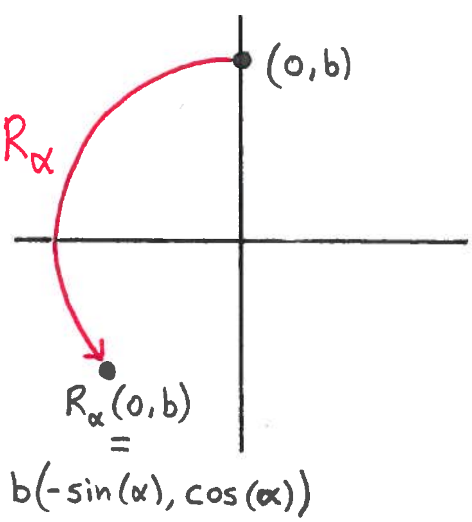
$$\begin{aligned} \cos\left(\alpha+\frac{\pi}{2}\right) &= \cos\left(\alpha+\pi-\frac{\pi}{2}\right) \\ &= \sin(\alpha+\pi) && \text{[Lemma 9]} \\ &= -\sin(\alpha) && \text{[Lemma 10]} \end{aligned}$$

and

$$\sin\left(\alpha+\frac{\pi}{2}\right) = \cos(\alpha) \qquad \text{[Lemma 8]}$$

Therefore,

$$R_\alpha(0,b) = b\Big(\cos\left(\alpha+\frac{\pi}{2}\right), \sin\left(\alpha+\frac{\pi}{2}\right) \Big) = b(-\sin(\alpha), \cos(\alpha))$$



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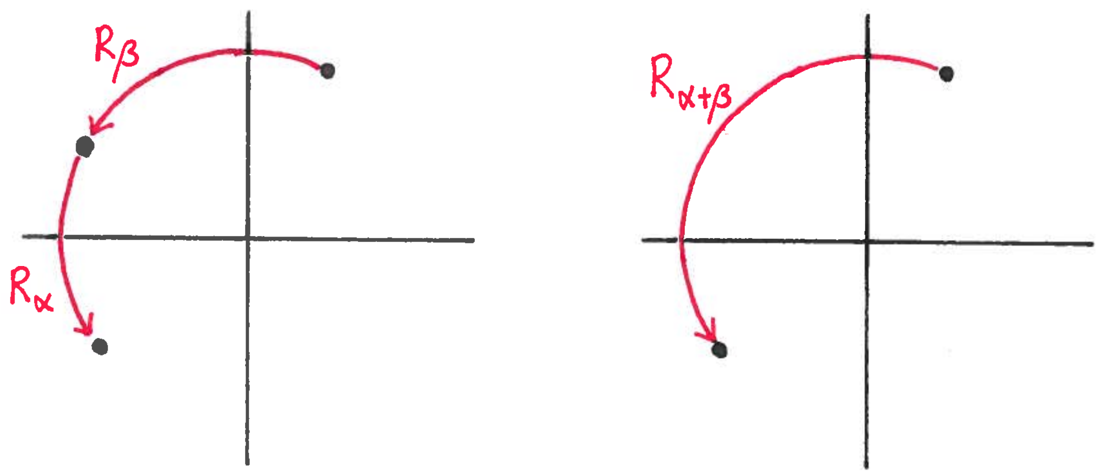
Composing rotations

It's rare for a function to satisfy any sort of nice algebraic rule. We know a few functions that do — powers ($x^n y^n = (xy)^n$), exponentials ($a^x a^y = a^{x+y}$), and logarithms ($\log_a(x) + \log_a(y) = \log_a(xy)$) — and rotations provide another example, as the following theorem states.

Theorem (14). $R_\alpha \circ R_\beta = R_{\alpha+\beta}$

Proof: If first we rotate the plane by an angle of β, and then we rotate the plane by an angle of α, we have rotated the plane by an angle of $\alpha + \beta$. That's what this theorem says.



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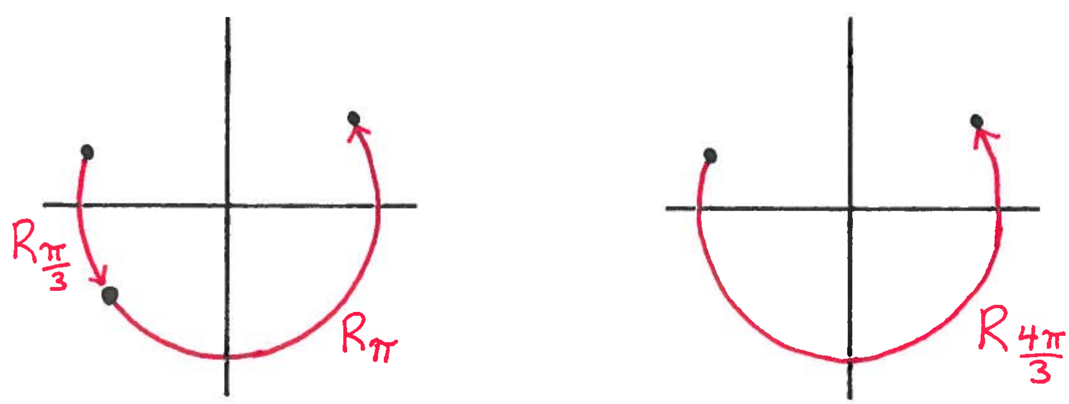
Example.

- If we rotate the plane counterclockwise by an angle of $\frac{\pi}{3}$, and then we rotate counterclockwise by an angle of π, we've rotated counterclockwise a total angle of $\pi + \frac{\pi}{3} = \frac{4\pi}{3}$. That's what Theorem 14 says, $R_\pi \circ R_{\frac{\pi}{3}} = R_{\frac{4\pi}{3}}$.



Corollary (15). $R_\alpha^{-1} = R_{-\alpha}$

Proof: As discussed at the bottom of page 259, the rotation R_0 is a rotation by an angle of 0, which means R_0 doesn't rotate anything at all. It's the identity function on the plane. That is, $R_0 = id$.

Using Theorem (14) we see that

$$R_\alpha \circ R_{-\alpha} = R_{\alpha-\alpha} = R_0 = id \quad \text{and} \quad R_{-\alpha} \circ R_\alpha = R_{-\alpha+\alpha} = R_0 = id$$

Summarizing the above line, we have

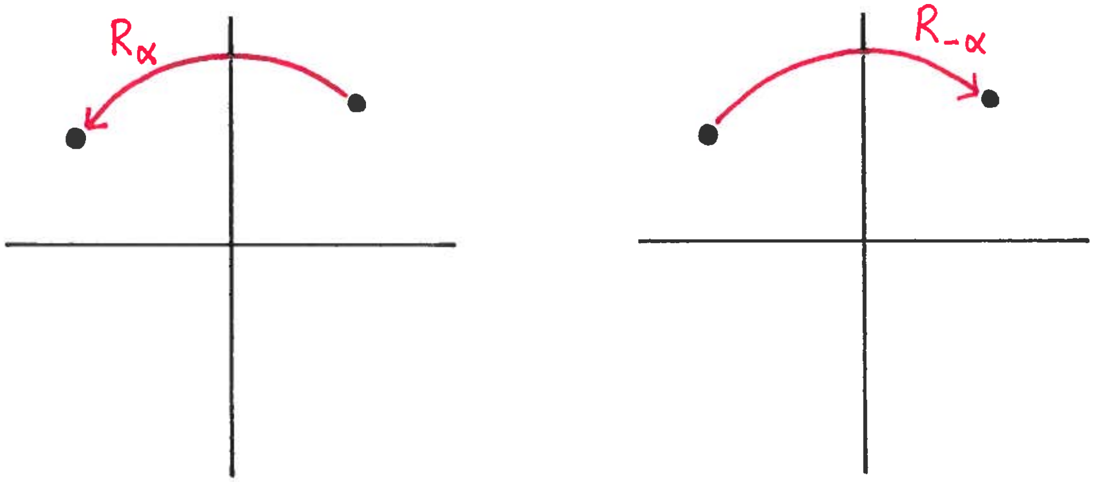
$$R_\alpha \circ R_{-\alpha} = id \qquad \text{and} \qquad R_{-\alpha} \circ R_\alpha = id$$

Recall that the definition of inverse functions is that they satisfy the relationship

$$f \circ f^{-1} = id \quad \text{and} \quad f^{-1} \circ f = id$$

We have seen that the functions R_α and $R_{-\alpha}$ satisfy this relationship, so they are inverse functions. That is, $R_\alpha^{-1} = R_{-\alpha}$ ■

Intuitively, Corollary 15 states that the opposite of rotating the plane by α, is rotating the plane by $-\alpha$.



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Rotations are matrices

We know what the rotation function $R_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ does to vectors written in polar coordinates. The formula is

$$R_\alpha\big(\, r(\, \cos(\theta), \sin(\theta)\,)\,\big) = r(\, \cos(\theta + \alpha), \sin(\theta + \alpha)\,)$$

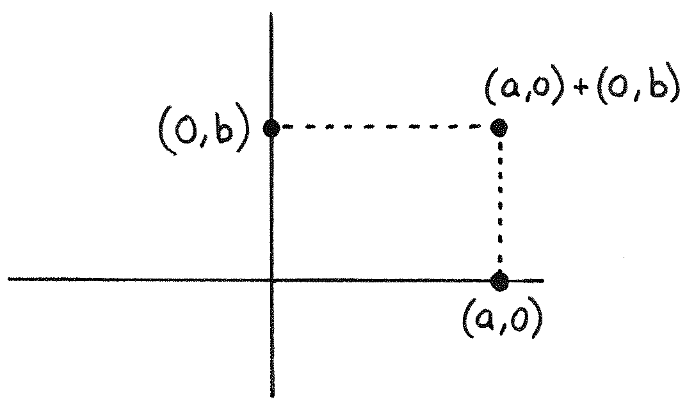
as we saw at the beginning of this chapter.

What's less clear is what the formula for R_α should be for vectors written in Cartesian coordinates. For example, what's $R_\alpha(3, 7)$? We'll answer this question below, in Theorem 17. Before that though, we need one more lemma.

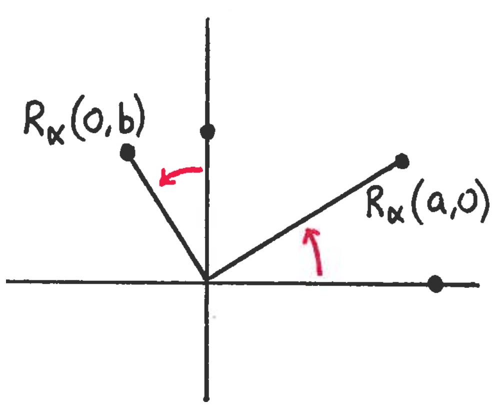
Lemma (16). For vectors $(a, 0)$ and $(0, b)$, we have

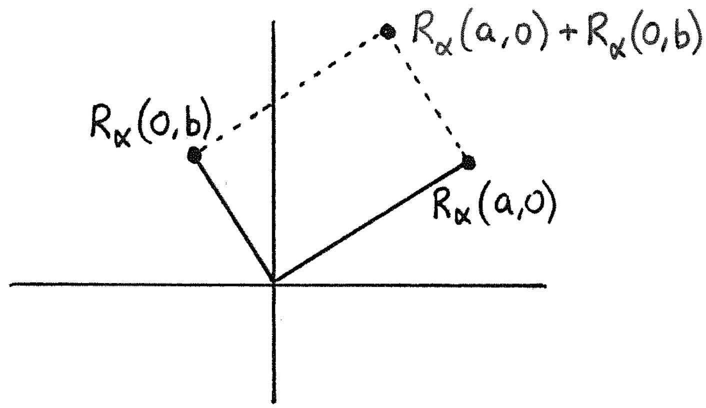
$$R_\alpha\big(\,(a, 0) + (0, b)\,\big) = R_\alpha(a, 0) + R_\alpha(0, b)$$

Proof: One way to find the sum of $(a, 0)$ and $(0, b)$ is draw the rectangle that they form. The sum of $(a, 0)$ and $(0, b)$ will be the corner of the rectangle that is opposite the corner at the origin.

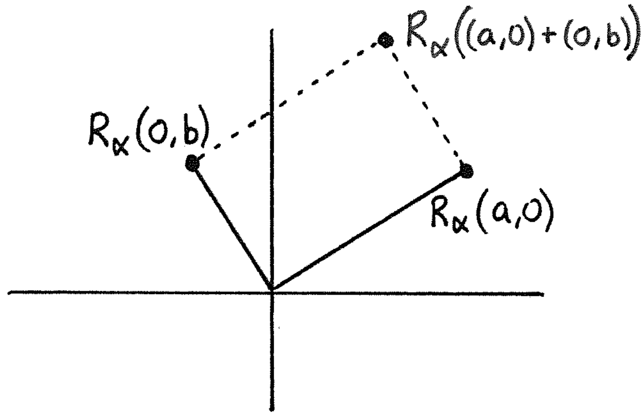


Similarly, the sum of $R_\alpha(a, 0)$ and $R_\alpha(0, b)$ is found by drawing the rectangle that they form.





Below is the first rectangle we drew in this proof, after being rotated by and angle of α.



Notice that it's the same rectangle as the second one that we drew, it's just that the corner opposite the origin is labeled differently. Since they are the same rectangle, the opposite corners are the same, so $R_\alpha\big(\,(a,0)+(0,b)\,\big) = R_\alpha(a,0)+R_\alpha(0,b)$, which is what we had wanted to show. ■

Now we are ready to describe the rotation function R_α using Cartesian coordinates. We will see that R_α can be written as a matrix, and we already know how matrices affect vectors written in Cartesian coordinates.

Theorem (17). $R_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ is the same function as the matrix function

$$\begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

For short,

$$R_\alpha = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

Proof: To show that R_α and the matrix above are the same function, we'll input the vector (a,b) into each function and check that we get the same output. First let's check the matrix function. Writing vectors interchangeably as row vectors and column vectors, we have that

$$\begin{aligned}
\begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}(a,b) &= \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} \\
&= \begin{pmatrix} a\cos(\alpha) - b\sin(\alpha) \\ a\sin(\alpha) + b\cos(\alpha) \end{pmatrix} \\
&= \big(\, a\cos(\alpha) - b\sin(\alpha)\,,\; a\sin(\alpha) + b\cos(\alpha)\,\big)
\end{aligned}$$

Now let's check that we obtain the same output if we input the vector (a, b) into the function R_α.

What we'll use is Lemma 16 and the two examples from pages 261 and 262 that showed us

$$R_\alpha(a, 0) = a(\cos(\alpha), \sin(\alpha)) \quad \text{and} \quad R_\alpha(0, b) = b(-\sin(\alpha), \cos(\alpha))$$

We can now check that

$$\begin{aligned} R_\alpha(a, b) &= R_\alpha\big((a, 0) + (0, b)\big) \\ &= R_\alpha(a, 0) + R_\alpha(0, b) && \text{[Lemma 16]} \\ &= a\Big(\cos(\alpha), \sin(\alpha)\Big) + b\Big(-\sin(\alpha), \cos(\alpha)\Big) && \text{[Examples]} \\ &= \Big(a\cos(\alpha), a\sin(\alpha)\Big) + \Big(-b\sin(\alpha), b\cos(\alpha)\Big) \\ &= \Big(a\cos(\alpha) - b\sin(\alpha)\,,\ a\sin(\alpha) + b\cos(\alpha)\Big) \end{aligned}$$

Because the matrix and the function R_α gave us the same output, they are the same function. ■

Examples.

- We can write R_π, rotation by π, as a matrix using Theorem 17:

$$R_\pi = \begin{pmatrix} \cos(\pi) & -\sin(\pi) \\ \sin(\pi) & \cos(\pi) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Counterclockwise rotation by $\frac{\pi}{2}$ is the matrix

$$R_{\frac{\pi}{2}} = \begin{pmatrix} \cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2}) \\ \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Because rotations are actually matrices, and because function composition for matrices is matrix multiplication, we'll often multiply rotation functions, such as $R_\alpha R_\beta$, to mean that we are composing them. Thus, we can write Theorem 14 as $R_\alpha R_\beta = R_{\alpha+\beta}$.

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Angle sum formulas

Theorem 14 tells us that $R_{\alpha+\beta} = R_\alpha R_\beta$. Theorem 17 allows us to write this equation with matrices, that we can then multiply:

$$\begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix} = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) & -\cos(\alpha)\sin(\beta) - \sin(\alpha)\cos(\beta) \\ \\ \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) & -\sin(\alpha)\sin(\beta) + \cos(\alpha)\cos(\beta) \end{pmatrix}$$

The top left entry of the matrix we started with has to equal the top left entry of the matrix that we ended with, since the two matrices are equal. Similarly, the bottom left entries are equal. That gives us two equations—two identities that are known as the *angle sum formulas*:

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

Double angle formulas

If we write the angle sum formulas with $\alpha = \beta$ then we'd have two more identities, called the *double angle formulas*:

$$\cos(2\alpha) = \cos(\alpha)^2 - \sin(\alpha)^2$$

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$$

More identities encoded in matrix multiplication

The angle sum and double angle formulas are encoded in matrix multiplication, as we saw above. In fact all but one of the identities for sine and cosine that we've see so far are encoded in matrix multiplication. Lemma 8 can be seen in the matrix equation $R_{\theta+\frac{\pi}{2}} = R_\theta R_{\frac{\pi}{2}}$; Lemma 9 in the matrix equation $R_{\theta-\frac{\pi}{2}} = R_\theta R_{-\frac{\pi}{2}}$; Lemma 10 in the matrix equation $R_{\theta+\pi} = R_\theta R_\pi$; Lemmas 11 and 12 in the equation $R_{-\theta} = R_\theta^{-1}$; and the period identities for sine and cosine in the equation $R_{\theta+2\pi} = R_\theta R_{2\pi}$.

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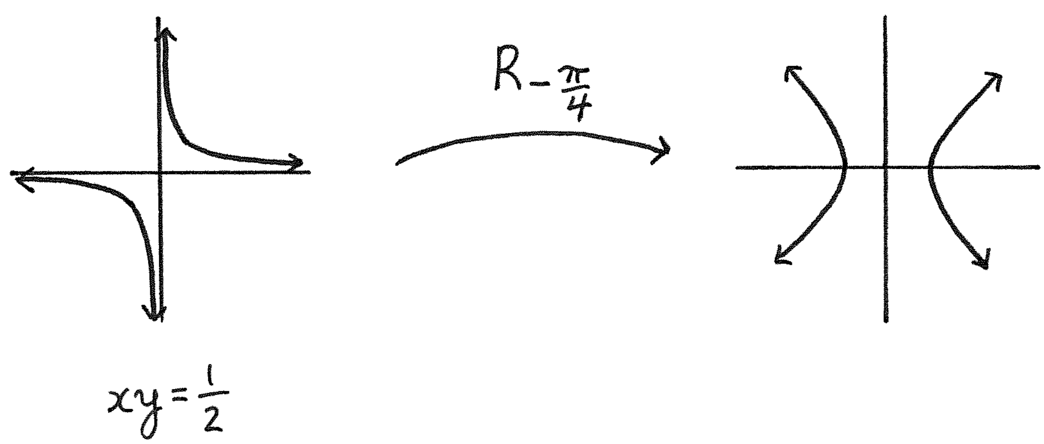
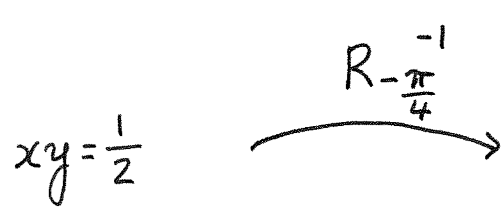
Rotating a conic

Let's rotate the hyperbola $xy = \frac{1}{2}$ clockwise by an angle of $\frac{\pi}{4}$. That means, that we will apply the rotation matrix $R_{-\frac{\pi}{4}}$ to the hyperbola.



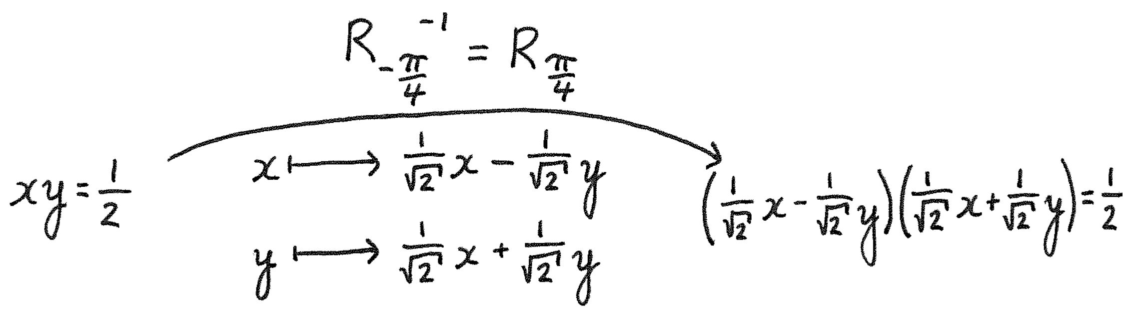
To find the new equation for our rotated hyperbola, we'll precompose the equation of the original hyperbola — the equation $xy = \frac{1}{2}$ — with the inverse of $R_{-\frac{\pi}{4}}$.



Notice that

$$R_{-\frac{\pi}{4}}{}^{-1} = R_{\frac{\pi}{4}} = \begin{pmatrix} \cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right) \\ \\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

is the function that replaces x with $\frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y$ and replaces y with $\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y$.



Therefore, the equation for the rotated hyperbola is

$$\left(\frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y\right)\left(\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y\right) = \frac{1}{2}$$

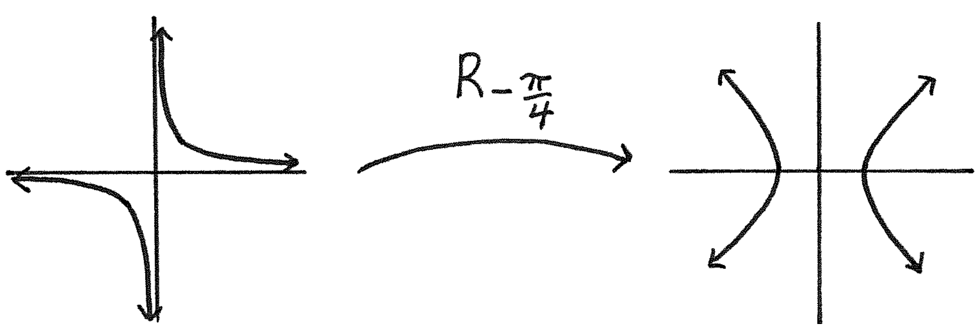
Using the distributive law, the above equation is the same as

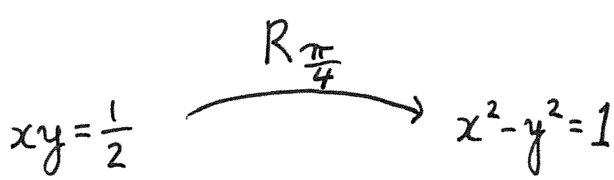
$$\frac{1}{2}x^2 - \frac{1}{2}y^2 = \frac{1}{2}$$

Multiplying both sides by 2 leaves us with the equivalent equation

$$x^2 - y^2 = 1$$

This is the equation for the rotated hyperbola.





Exercises

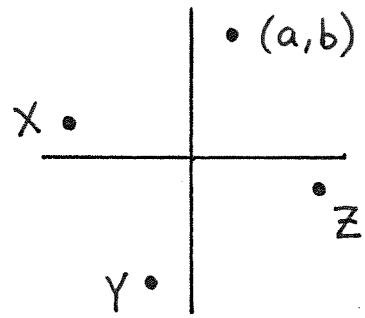
One of the vectors drawn on the right is (a, b). For #1-4, match the rotated vectors with the vectors drawn on the right. Remember that if $\alpha > 0$, then R_α is a counterclockwise rotation by the angle α. If $\alpha < 0$, then R_α is a clockwise rotation by the angle $|\alpha|$.

1.) $R_{\frac{\pi}{2}}(a, b)$

2.) $R_{-\frac{\pi}{2}}(a, b)$

3.) $R_{-\pi}(a, b)$

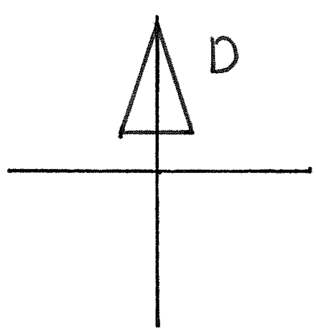
4.) $R_{\pi}(a, b)$



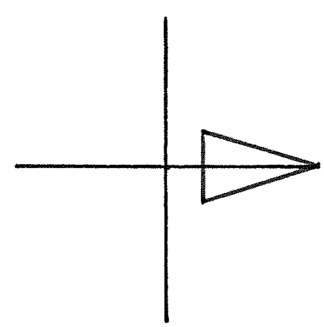
Below are some shapes in the plane. All the shapes drawn are rotations of the first shape, D. For #5-9, match the rotated shape with the correct picture.

5.) $R_{\frac{\pi}{4}}(D)$ 6.) $R_{-\frac{\pi}{6}}(D)$ 7.) $R_{2\pi}(D)$ 8.) $R_{-\frac{7\pi}{4}}(D)$ 9.) $R_{-\frac{\pi}{2}}(D)$

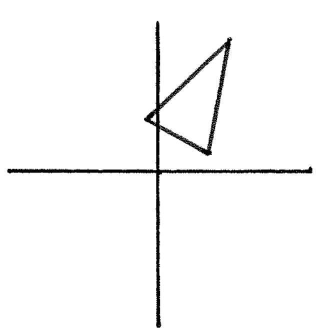
A.)



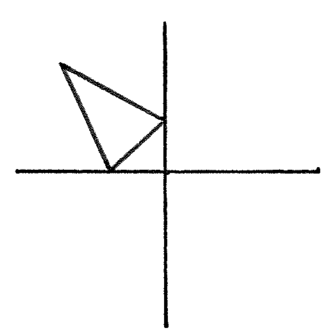
B.)



C.)



E.)



For some of the remaining exercises you'll need to use the rotation matrix

$$R_\alpha = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

For #10-13 use the values of $\sin(\alpha)$ and $\cos(\alpha)$ that you know to write the given matrices using exact numbers that do not involve trigonometric functions such as sin or cos. For example, as we saw when rotating the hyperbola at the end of the chapter,

$$R_{\frac{\pi}{4}} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

10.) $R_{\frac{3\pi}{2}}$ 11.) $R_{-\pi}$ 12.) $R_{\frac{\pi}{3}}$ 13.) $R_{2\pi}$

For #14-17, use the matrices $R_{\frac{\pi}{4}}$ and $R_{\frac{\pi}{3}}$ from above to write the given vectors in Cartesian coordinates.

14.) $R_{\frac{\pi}{4}}(3,5)$ 15.) $R_{\frac{\pi}{4}}(4,-8)$ 16.) $R_{\frac{\pi}{3}}(-2,-7)$ 17.) $R_{\frac{\pi}{3}}(5,-9)$

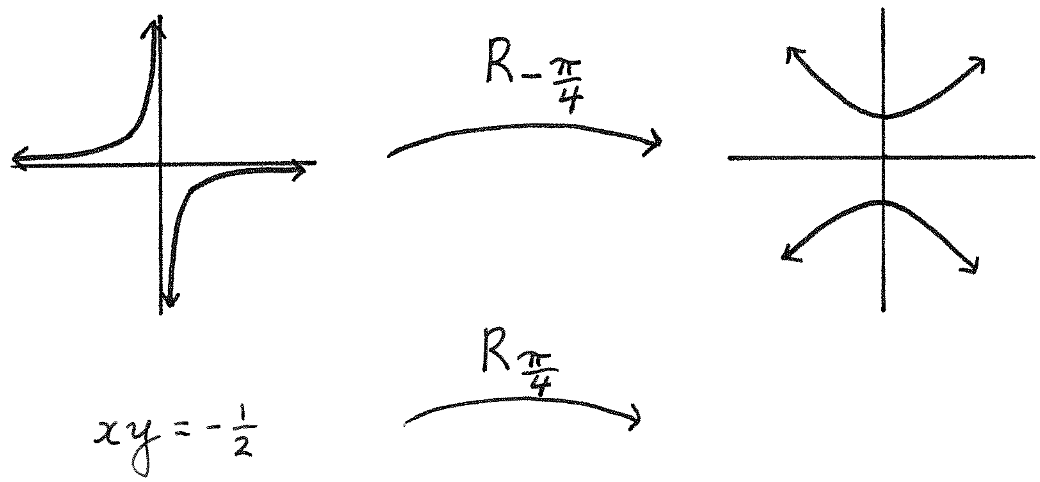
18.) Use the Pythagorean Identity to find the determinant of the rotation matrix R_α.

19.) Use an angle sum formula to find $\cos\left(\frac{\pi}{12}\right)$. (Hint: $\frac{\pi}{12} = \frac{\pi}{3} + (-\frac{\pi}{4})$.)

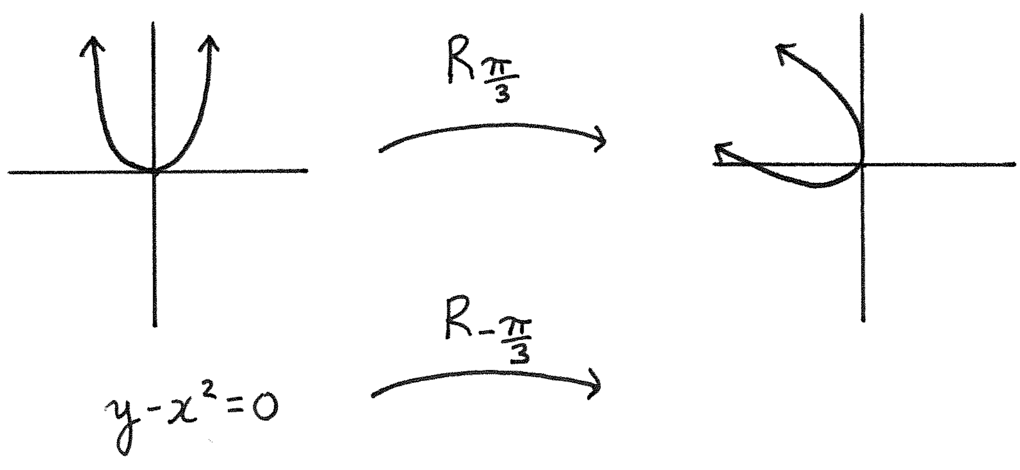
20.) Use an angle sum formula to find $\sin\left(\frac{\pi}{12}\right)$.

21.) Suppose that $(\cos(\theta), \sin(\theta)) = \left(\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\right)$. Use the double angle formulas to find $(\cos(2\theta), \sin(2\theta))$.

22.) Find an equation for the hyperbola $xy = -\frac{1}{2}$ rotated clockwise by an angle of $\frac{\pi}{4}$.



23.) The solution of $y - x^2 = 0$ is a parabola. Find an equation for the parabola rotated counterclockwise by an angle of $\frac{\pi}{3}$. Write your answer in the form $x^2 + Bxy + Cy^2 + Dx + Ey = 0$.



For #24-26, find the solutions of the given equations.

24.) $\frac{x}{x-1} + x = 1$ 25.) $\ln(x) + 2\ln(x) = 7$ 26.) $(x+1)^2 = 16$