Rotation

Moment of inertia of a rotating body:

$$I = \int r^2 dm$$

Usually reasonably easy to calculate when
- Body has symmetries
- Rotation axis goes through Center of mass

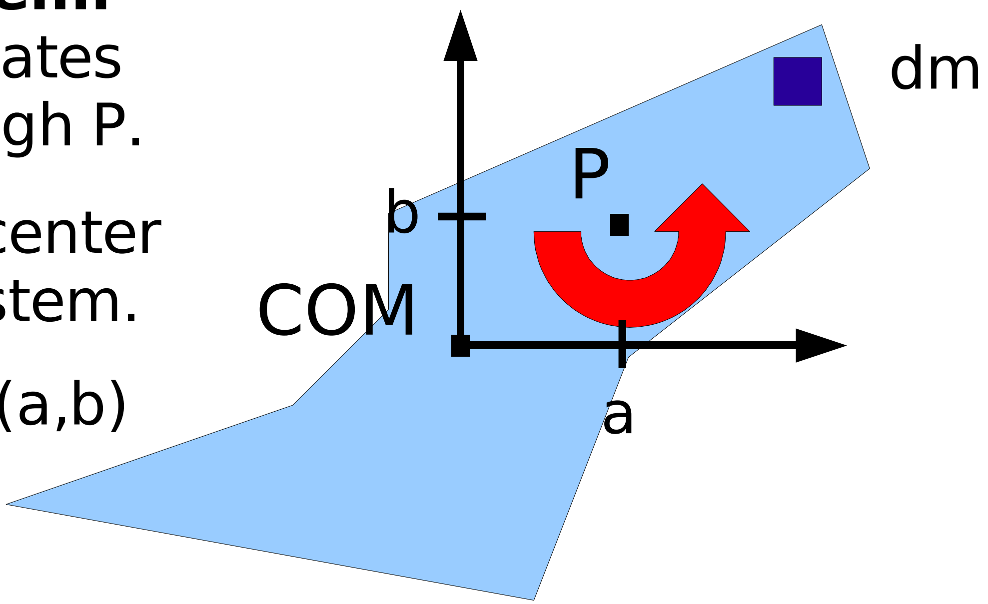
Exams: All moment of inertia will be given!
No need to copy the table from the book.

Rotation

Parallel axis theorem:
Assume the body rotates around an axis through P.

Let the COM be the center of our coordinate system.

P has the coordinates (a,b)



$$I = \int r^2 dm = \int (x-a)^2 + (y-b)^2 dm$$

$$= \int (x^2+y^2) dm - 2a \int x dm - 2b \int y dm + (a^2+b^2) \int dm$$

$$= I_{COM} - 0 - 0 + h^2 M$$

$$I = I_{COM} + h^2 M$$

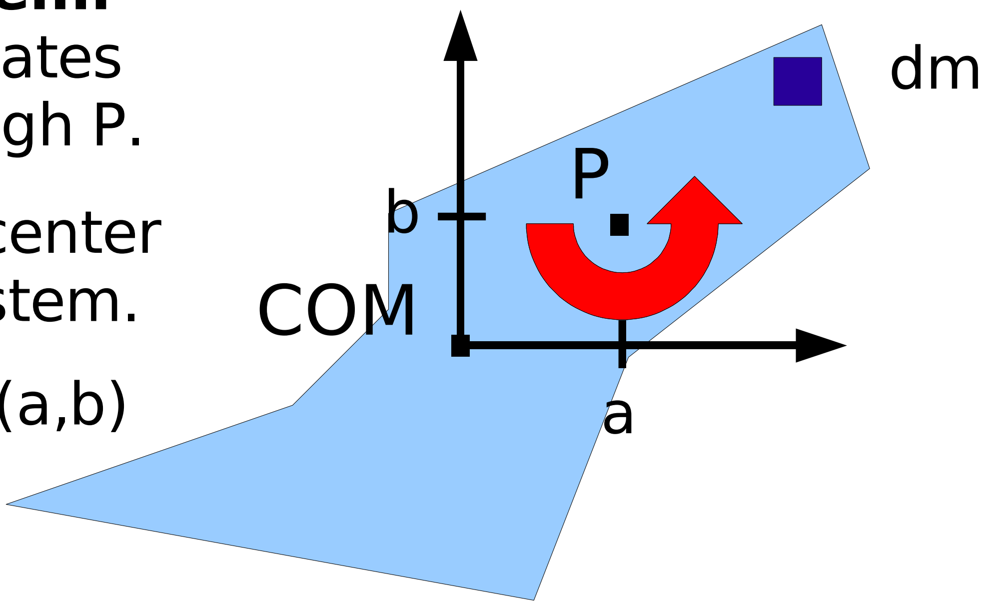
This is something you might need

Rotation

Parallel axis theorem:
Assume the body rotates around an axis through P.

Let the COM be the center of our coordinate system.

P has the coordinates (a,b)

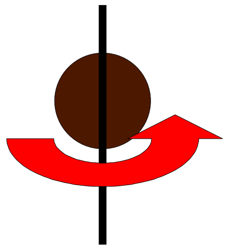


$$I = I_{COM} + Mh^2$$

The moment of inertia of a body rotating around an arbitrary axis is equal to the moment of inertia of a body rotating around a **parallel axis through the center of mass** plus the mass times the perpendicular distance between the axes h squared.

Example: Moment of inertia

Solid sphere of radius R rotating around symmetry axis: $I = 2MR^2/5$



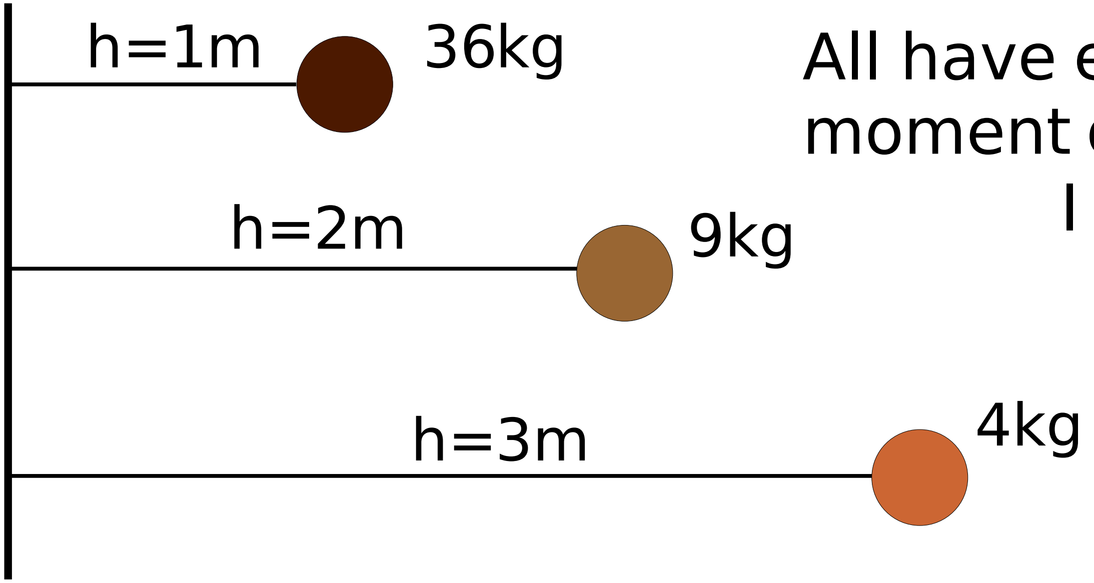
$I = I_{COM} + Mh^2$

If $R << h$ → $I_{COM} <<< Mh^2$

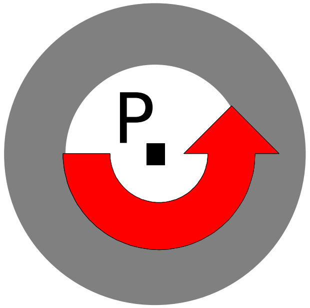
→ All have essentially the same moment of inertia:

$I \sim 36\ kg\ m^2$

$I_{COM,A}$ > $I_{COM,B}$ > $I_{COM,C}$



Example: Moment of inertia



$$I = \int r^2 dm$$

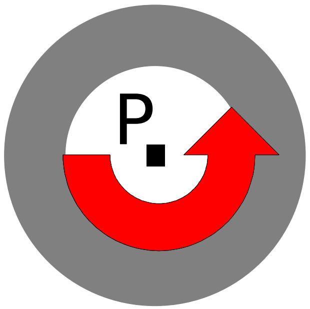
Lets calculate the moment of inertia for an annular homogeneous cylinder rotating around the central axis:
Parameters: Mass M, Length L
Outer and Inner Radii R_1, R_2

Step1: Replace dm with an integration over a volume element dV.

Step 2: Express the volume element in useful coordinates and find the boundaries for the integration.

Step 3: Integrate

Example: Moment of inertia



Lets calculate the moment of inertia for an annular homogeneous cylinder rotating around the central axis:
Parameters: Mass M, Length L
Outer and Inner Radii R_1, R_2

$$I = \int r^2 dm$$

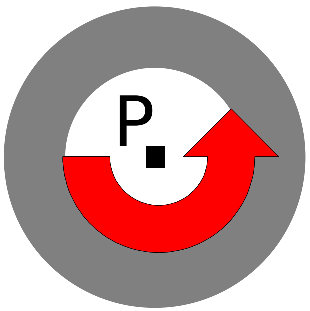
Step1: Replace dm with an integration over a volume element dV.

$$V_{cyl} = \pi L \left(R_1^2 - R_2^2\right)$$

$$I = \rho \int r^2 \, dV$$

Homogeneous:
Density $\rho = M/V$ $\Rightarrow$ $dm = \rho \, dV$
and ρ is independent of the coordinates
(As long as we only integrate over the body itself and not over empty space)

Example: Moment of inertia



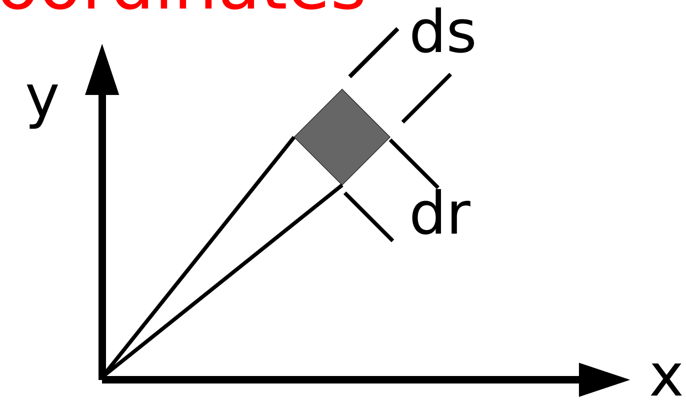
Lets calculate the moment of inertia for an annular homogeneous cylinder rotating around the central axis:
Parameters: Mass M, Length L
Outer and Inner Radii R_1, R_2

$$I = \rho \int r^2 \, dV$$

Step 2: Express the volume element in useful coordinates and find the boundaries for the integration.

$dV = dx dy dz$ in cartesian coordinates

Better coordinates:
cylindrical coordinates



Volume element in cylindrical coordinates

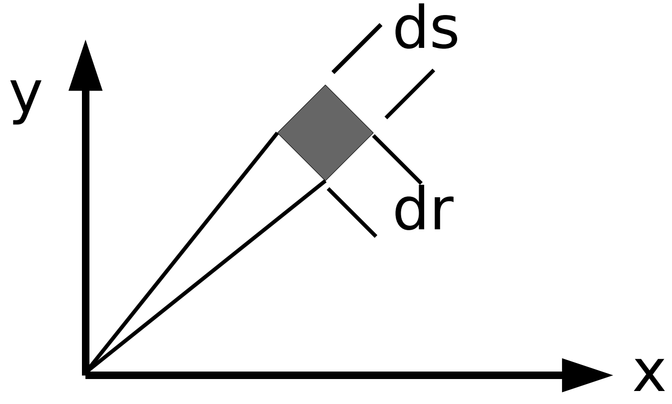
$$I = \rho \int r^2 \, dV$$

dV = dxdydz in cartesian coordinates

Better coordinates for this problem:
Cylindrical Coordinates

- dz is the same in both
- only change how we describe things in the x-y plane

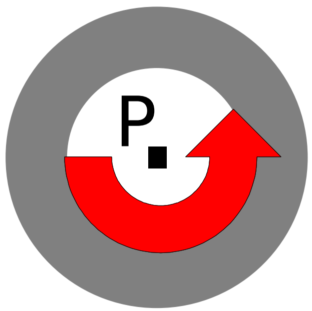
Area element in x-y plane



$ds\,dr = r\,d\phi\,dr$

$dV = dx\,dy\,dz = r\,d\phi\,dr\,dz$

Example: Moment of inertia



Step 2: Express the volume element in useful coordinates and find the boundaries for the integration.

$$I = \rho \int r^2 \, dV$$

$$I = \rho \int r^2 \, r\,d\phi\,dr\,dz$$

Integration boundaries:
z: -L/2 to L/2, ϕ: 0 to 2π
r: R_2 to R_1

$$I = \rho \, L \, 2\pi \, (R_1^4 - R_2^4)/4$$

z-integration, ϕ-integration, r-integration

Use $(R_1^4 - R_2^4) = (R_1^2 + R_2^2)(R_1^2 - R_2^2)$

Recall: $V_{cyl} = \pi L (R_1^2 - R_2^2)$

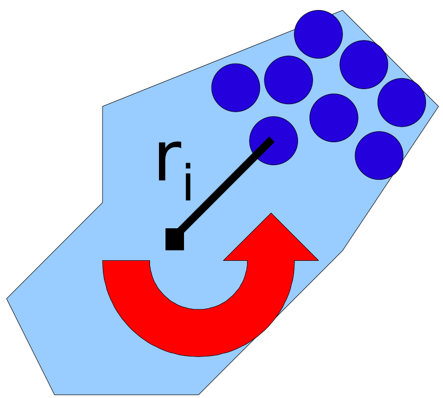
$$I = M (R_1^2 + R_2^2)/2$$

See Table in book, also Sample Problem 10-7

Kinetic Energy

Kinetic energy of a rotating body:

- Lets split up the body into a collection of particles

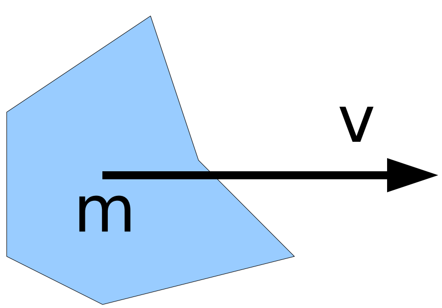


$K = 0.5\,(\Sigma m_i\, r_i^{\,2})\,\omega^2$

Rotational Inertia:

$I = (\Sigma m_i\, r_i^{\,2})$

$K = 0.5\, I\, \omega^2$



Compare with translation:

$K = 0.5\, m\, v^2$

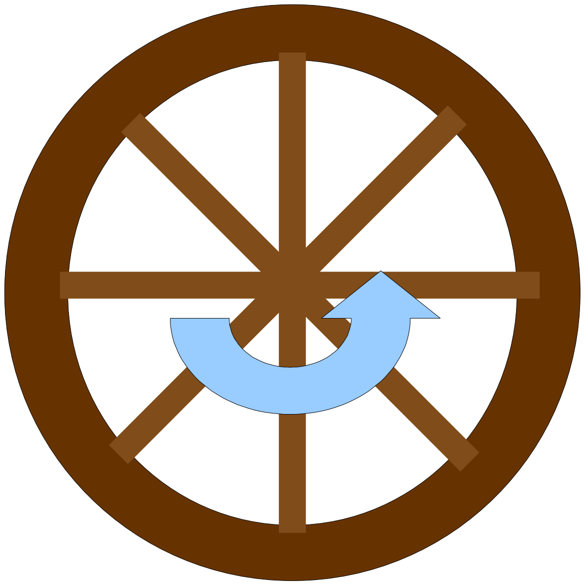
Correspondences:

- v <-> ω
- I <-> m

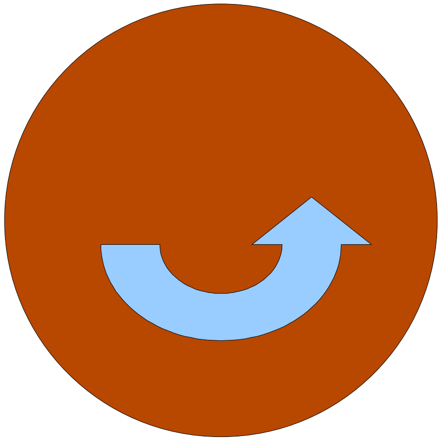
Example: Kinetic Energy

Energy storage in an electric flywheel:

$$K_{rot} = 0.5\ I\ \omega^2$$



To reduce mass w/o reducing I
it might be useful to use an
annular cylinder.
But material stress favors
solid wheels for high end
applications.



http://en.wikipedia.org/wiki/Flywheel#Physics

Example: Kinetic Energy

Energy storage in an electric flywheel:

$K_{rot} = 0.5\, I\, \omega^2$

Example (solid wheel):

M = 600kg, R=0.5m

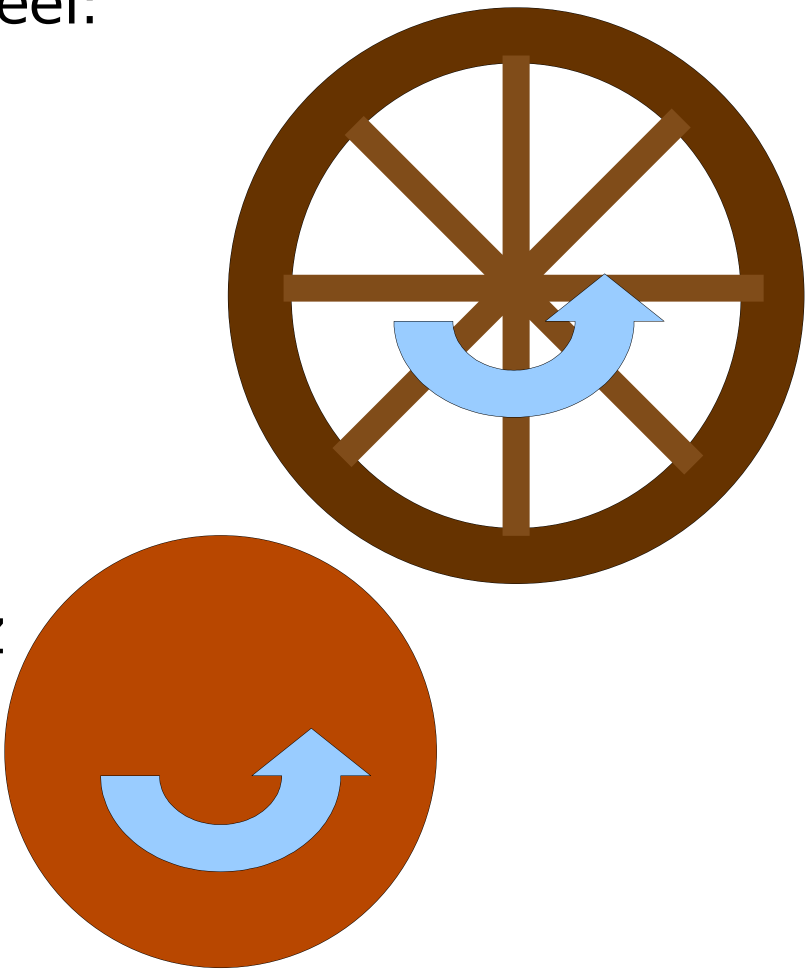
$\Rightarrow$ $I = 75 \text{kg m}^2$

Rotates 30000rpm:

f = 30000rpm/60s/min = 500Hz

$\omega = 2\pi f = \pi * 1000$ rad/s

$K_{rot} = 370$MJ



A lot of energy which can be released very fast in fusion reactors, in braking systems used in trains etc.

Torque

We have

- discussed the equations describing the dynamical variables of rotation (θ, ω, α)

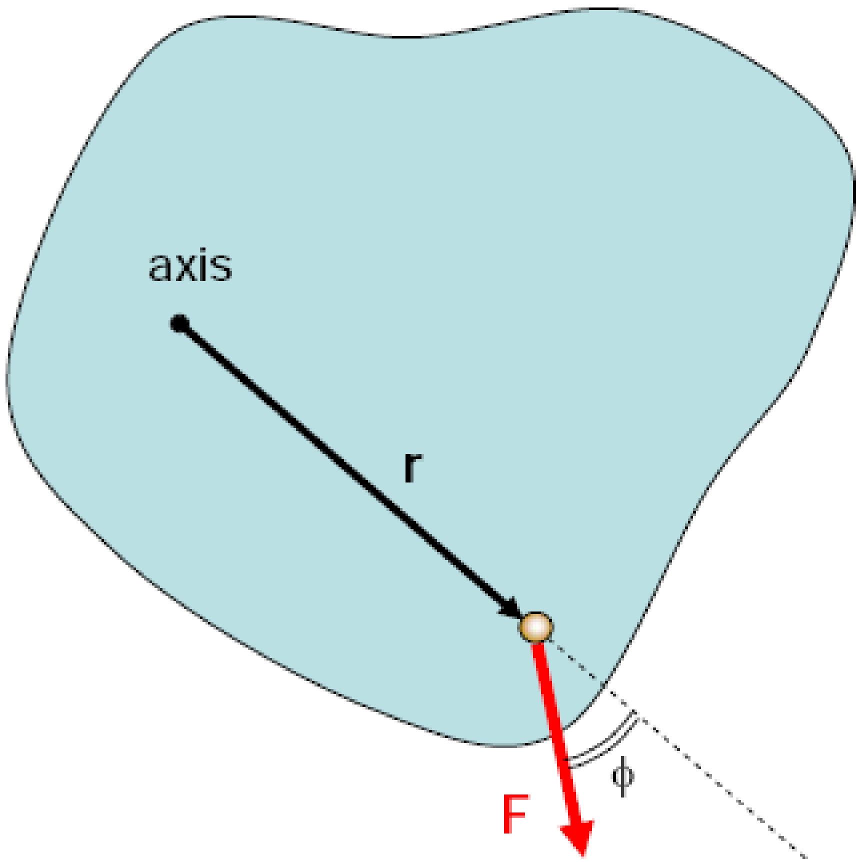
- calculated the kinetic energy associated with a rotating body

- pointed out the similarities with translations

What is missing: The 'force' of rotation, the torque

Torque (Latin for "twist"): $\vec{\tau} = \vec{r} \times \vec{F}$

$$\tau = Fr\sin\phi$$



To twist an object, one needs to apply force

The ease of twisting will depend on

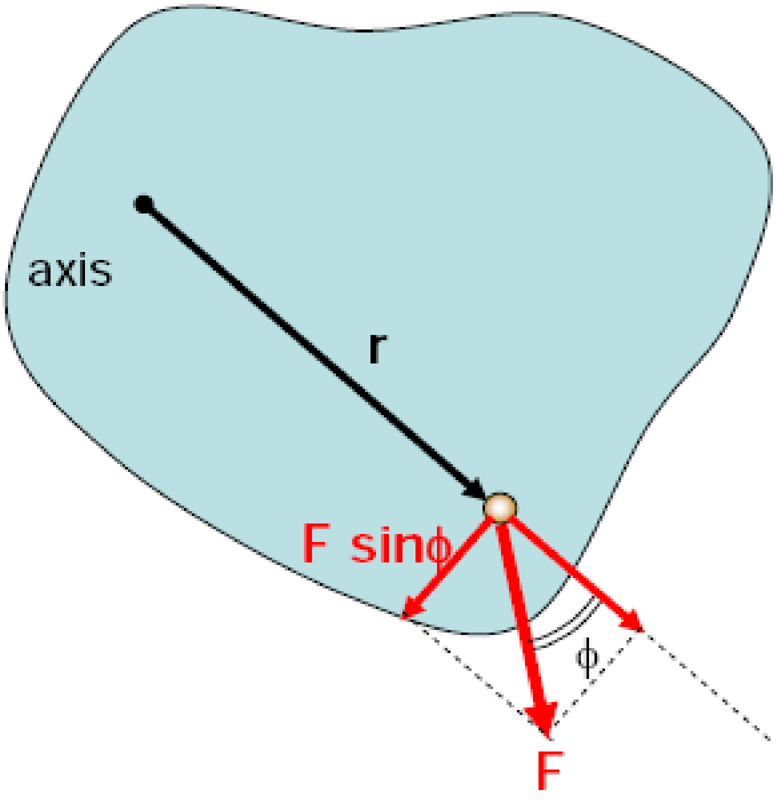
- force
- distance between the force and the hinge
- direction of the force

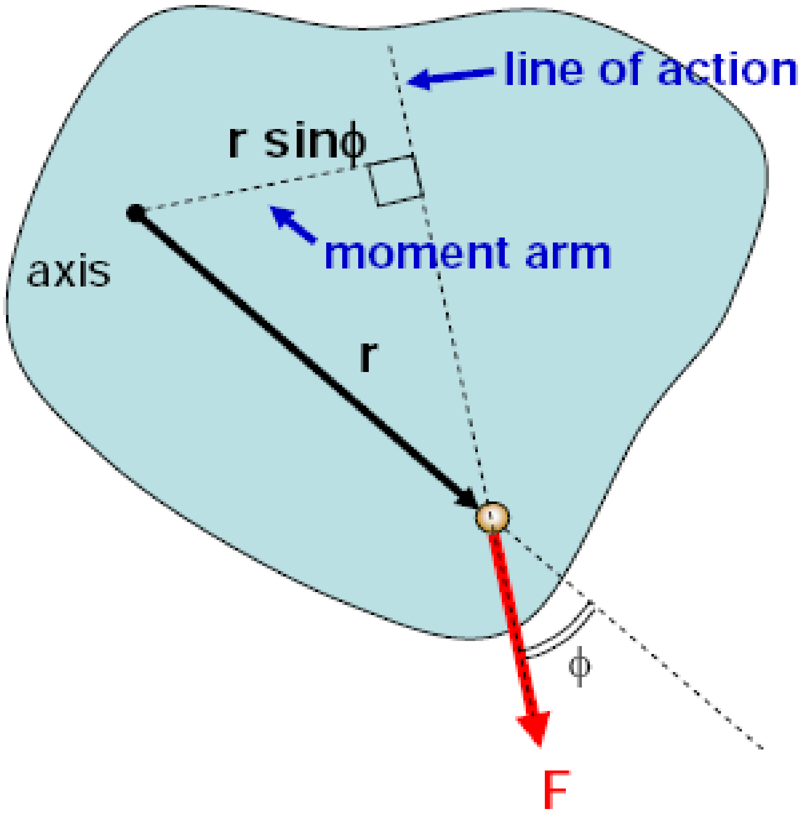
Torque

$$\tau = Fr\sin\phi$$

$$\tau = (F\sin\phi)\,r$$

$$\tau = F(r\sin\phi)$$

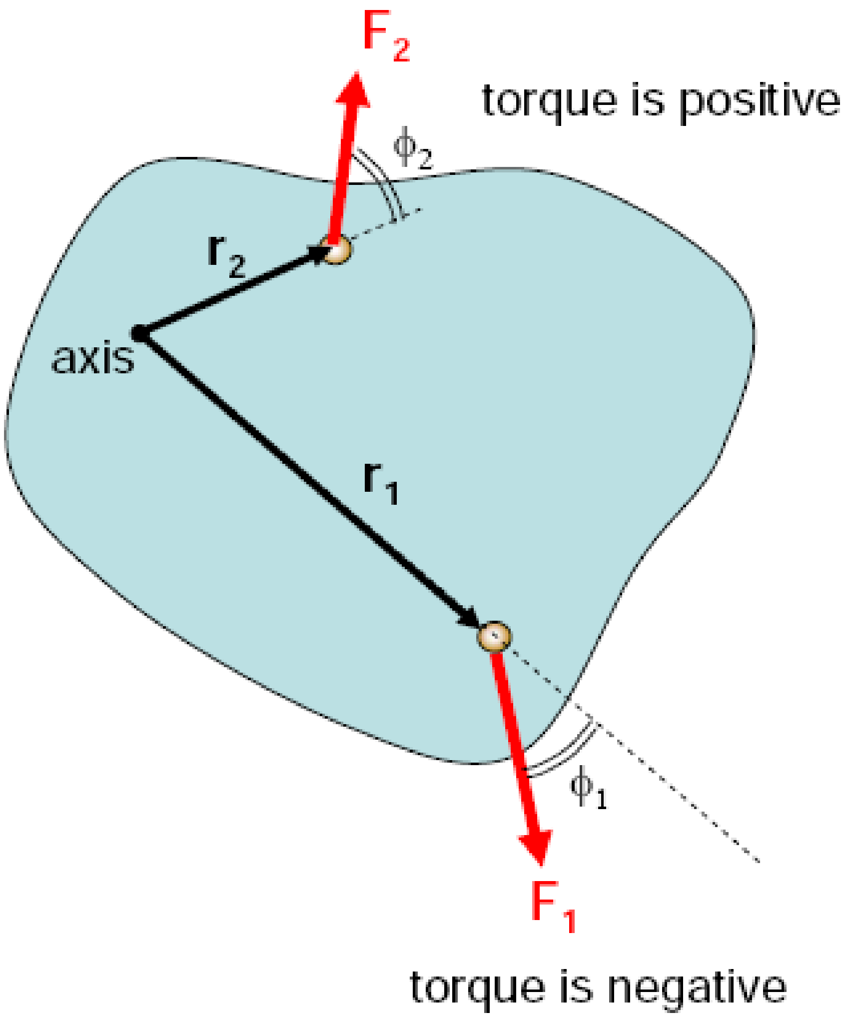




Torque: $\vec{\tau} = \vec{r} \times \vec{F}$

Sign convention:

- into screen
(clockwise): negative
- out of screen
(counter-clockwise): positive



Multiple forces:

- $\vec{\tau}_{net} = \vec{\tau}_1 + \vec{\tau}_2 + \ldots$

Units:

- N-m

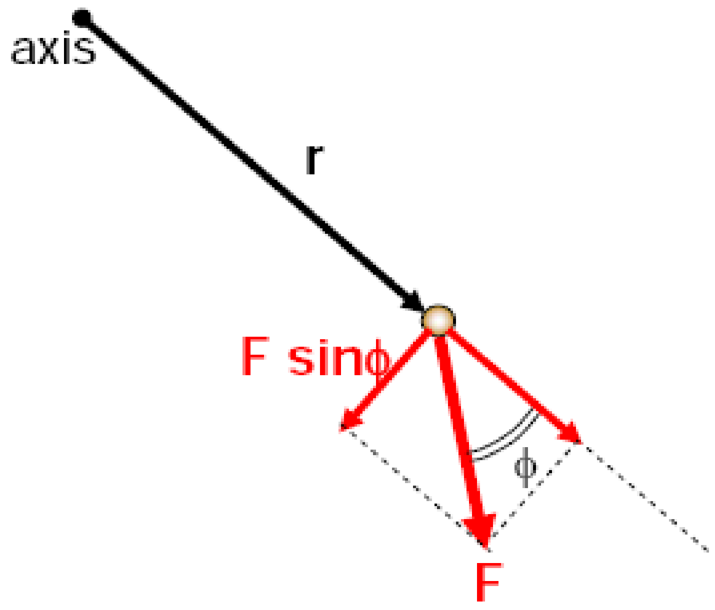
Note: Nm = J but J is only used for energies and Nm only for torques

Torque and angular acceleration

$$\tau = I\alpha$$

Note analogy with F=ma

Proof for simple case of one particle



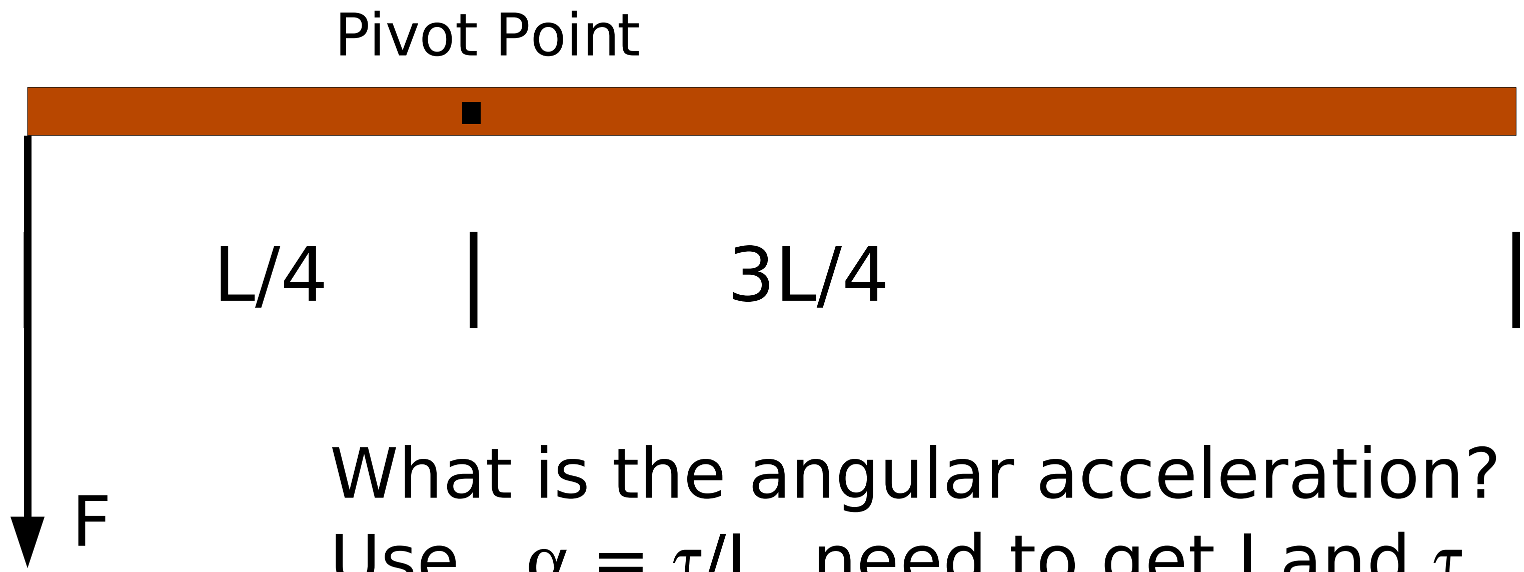
$$F\sin\phi = ma_t$$

$$F\sin\phi = m(\alpha r)$$

$$(F\sin\phi)\,r = m(\alpha r)\,r = (mr^2)\,\alpha$$

$$\tau = I\alpha$$

Example



What is the angular acceleration?

Use $\alpha = \tau/I$ need to get I and τ

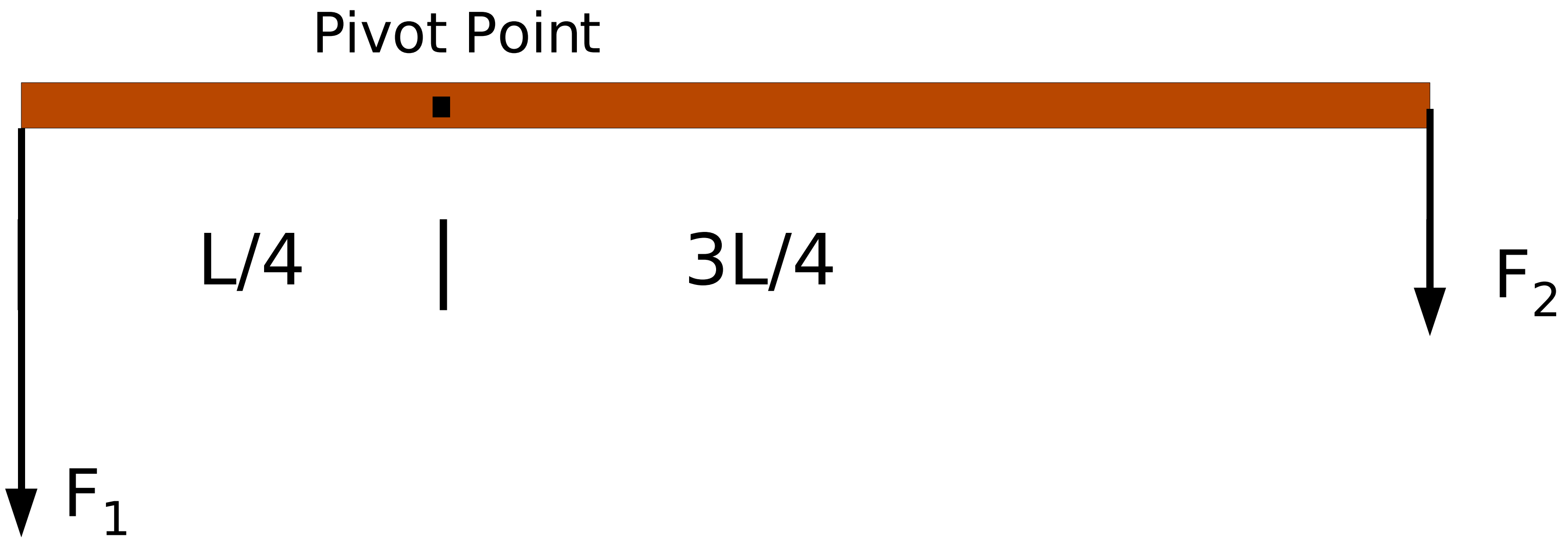
Moment of inertia: $I = I_{com} + M(L/4)^2 = ML^2/12 + ML^2/16$

$I = 7ML^2/48$

Torque: $\tau = F\ L/4$ (Note F is perpendicular to lever arm)

$\alpha = \tau/I = (FL/ML^2)\ (48/(7*4)) = 12 * F/(7*M*L)$

Example



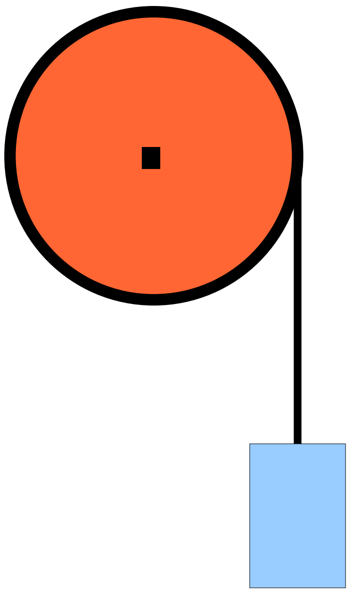
Assume a second force is now applied at the other end of the stick. What would be its magnitude and direction if it prevents the stick from rotating?

No rotation: $\tau_{Net} = 0$ $\tau_1 = F_1 L/4 = -\tau_2 = F_2\, 3L/4$

Direction of the **force** will be the **same** (both down)

Magnitude: $F_2 = F_1/3$

Example



A uniform disk with mass M = 2.5kg and radius R=20cm is mounted on a horizontal axle. A block of mass m=1.2kg hangs from a massless cord that is wrapped around the rim of the disk.

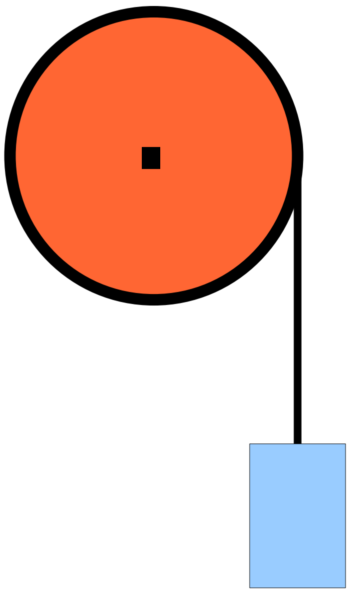
Q1:

Find the acceleration of the falling block.

Notice:

- We have two forces acting on mass m: Gravity and tension from the string
- We have one torque caused by the tension in the string acting on the disk
- The linear motion of the mass is linked to the circular motion of the disk via the cord.

Example



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Q1:

Find the acceleration of the falling block.

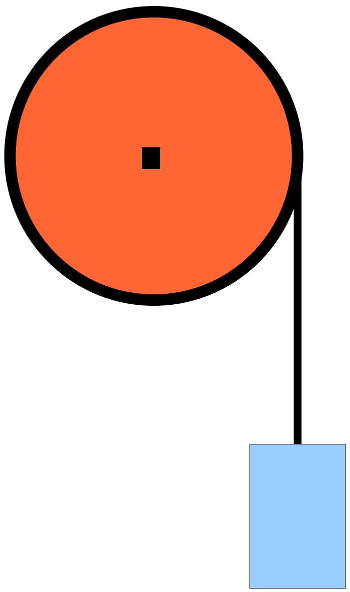
Notice:

- We have two forces acting on mass m: Gravity and tension from the string

ma = T-mg   (equation of motion of the falling block)

(Unknowns: a and T)

Example



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Q1:

Find the acceleration of the falling block.

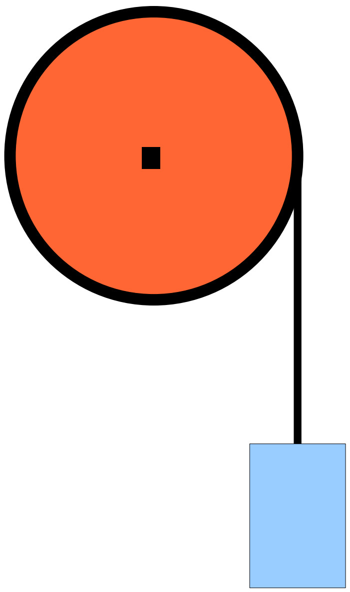
Notice:

- We have one torque caused by the tension in the string acting on the disk

$\tau = -TR$ (force T perpendicular to lever arm negative, because of clockwise rotation)

$= I\alpha = 0.5\ MR^2\alpha$ $T = -0.5MR\alpha$

Example



A uniform disk with mass M = 2.5kg and radius R=20cm is mounted on a horizontal axle. A block of mass m=1.2kg hangs from a massless cord that is wrapped around the rim of the disk.

Q1:

Find the acceleration of the falling block.

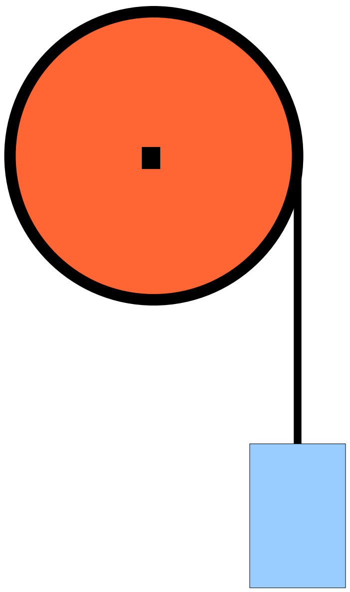
Notice:

$ma = T - mg$ and $T = -0.5MR\alpha$

- The linear motion of the mass is linked to the circular motion of the disk via the cord.

→ $a = R\alpha$ 3 equations, 3 unknowns: T, a, α

Example



A uniform disk with mass M = 2.5kg and radius R=20cm is mounted on a horizontal axle. A block of mass m=1.2kg hangs from a massless cord that is wrapped around the rim of the disk.

$ma = T - mg$, $T = -0.5MR\alpha$, $a = R\alpha$

<u>Some math:</u>

$ma = -0.5Ma - mg$ ⇨ $a = -2mg/(M+2m) = -4.8 m/s^2$

Angular accel.: $\alpha = a/R = -4.8/0.2 rad/s^2 = -24 rad/s^2$

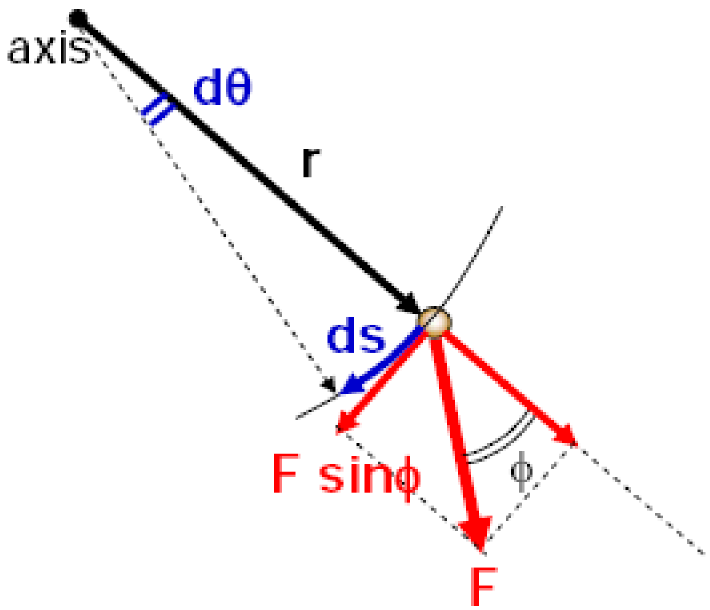
Tension in the cord: $T = -0.5MR\alpha = 6.0N$

Torque and work

$$dW = \tau\, d\theta$$

Note analogy with $dW = F dx$

Proof for simple case of one particle



$$dW = \left(F \sin\phi\right) ds$$

$$dW = \left(F \sin\phi\right) r \frac{ds}{r}$$

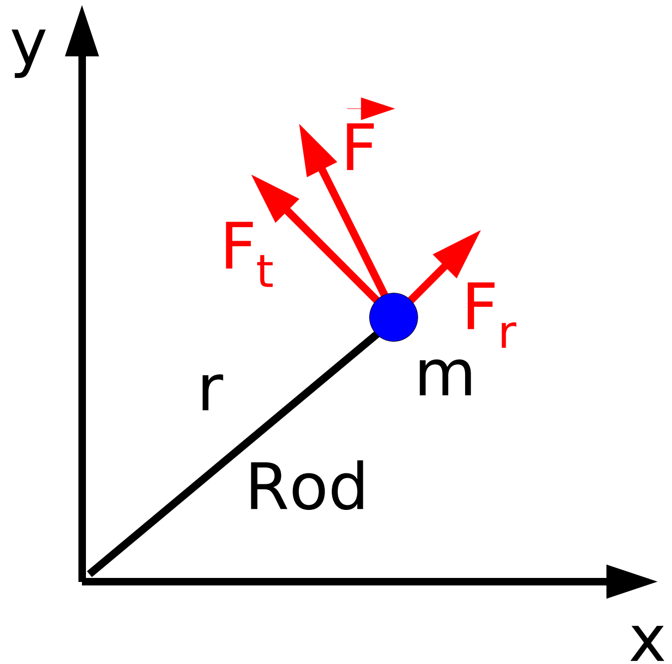
$$dW = \left(F \sin\phi\ r\right) \frac{ds}{r}$$

$$dW = \tau\ d\theta$$

$$\text{Power:} \quad P = \frac{dW}{dt} = \frac{\tau\, d\theta}{dt} = \tau\omega$$

Check out Table 10.3
for analogy with translation

Work and kinetic Energy



The radial component of F does not do any work as it does not displace the mass m

The tangential component of F does displace the mass by ds and does work:

$$dW = F_t ds = F_t \, r d\theta = \tau d\theta$$

Work and kinetic Energy

$$W = \ dW = \int_{\theta_i}^{\theta_f} \tau \, d\theta = \tau \, (\theta_f - \theta_i)$$

Work done during displacement

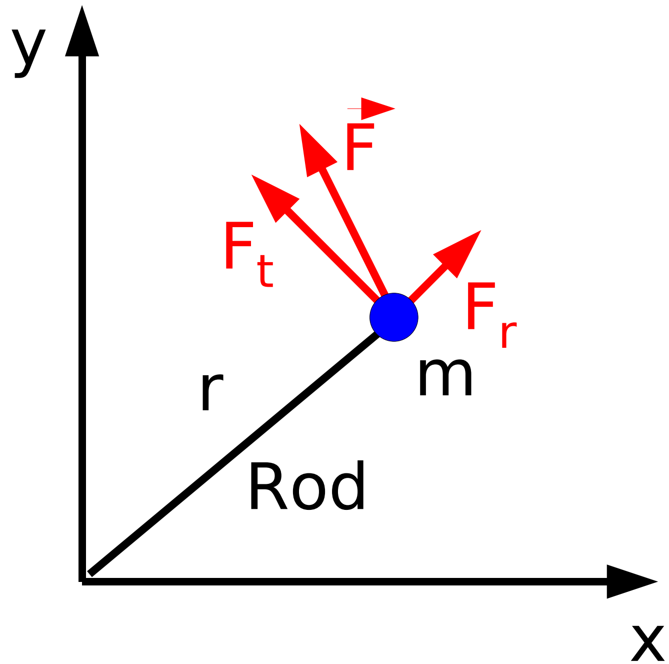
for constant torque

Recall Work for constant force and translation:

$$W = \int dW = \int_{x_i}^{x_f} F dx = F(x_f - x_i)$$

Power: $P = dW/dt = \tau\omega$ (rotation about fixed axis)

Work and kinetic Energy



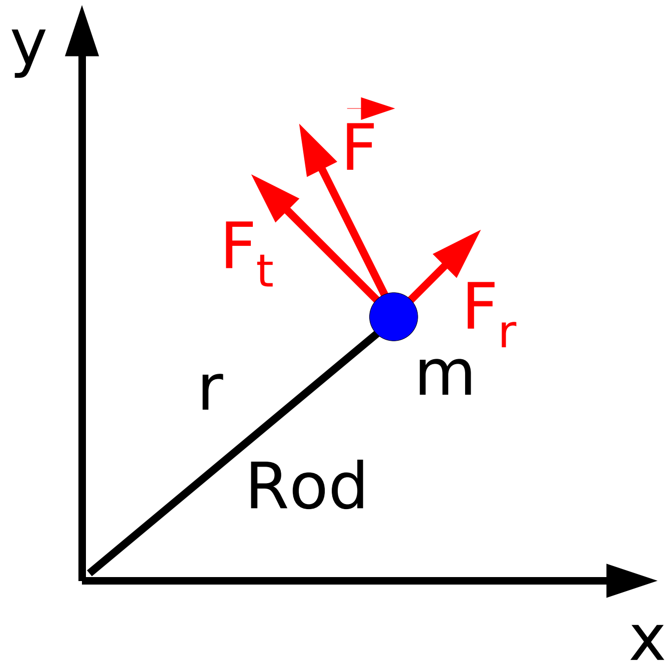
The tangential component of F does increase the velocity of the mass:

$$K_f - K_i = 0.5mv_f^2 - 0.5mv_i^2 = W$$

use $v = \omega r$ $\quad K_f - K_i = 0.5mr^2\omega_f^2 - 0.5mr^2\omega_i^2$

$$K_f - K_i = 0.5I\omega_f^2 - 0.5I\omega_i^2 = W$$

Work and kinetic Energy

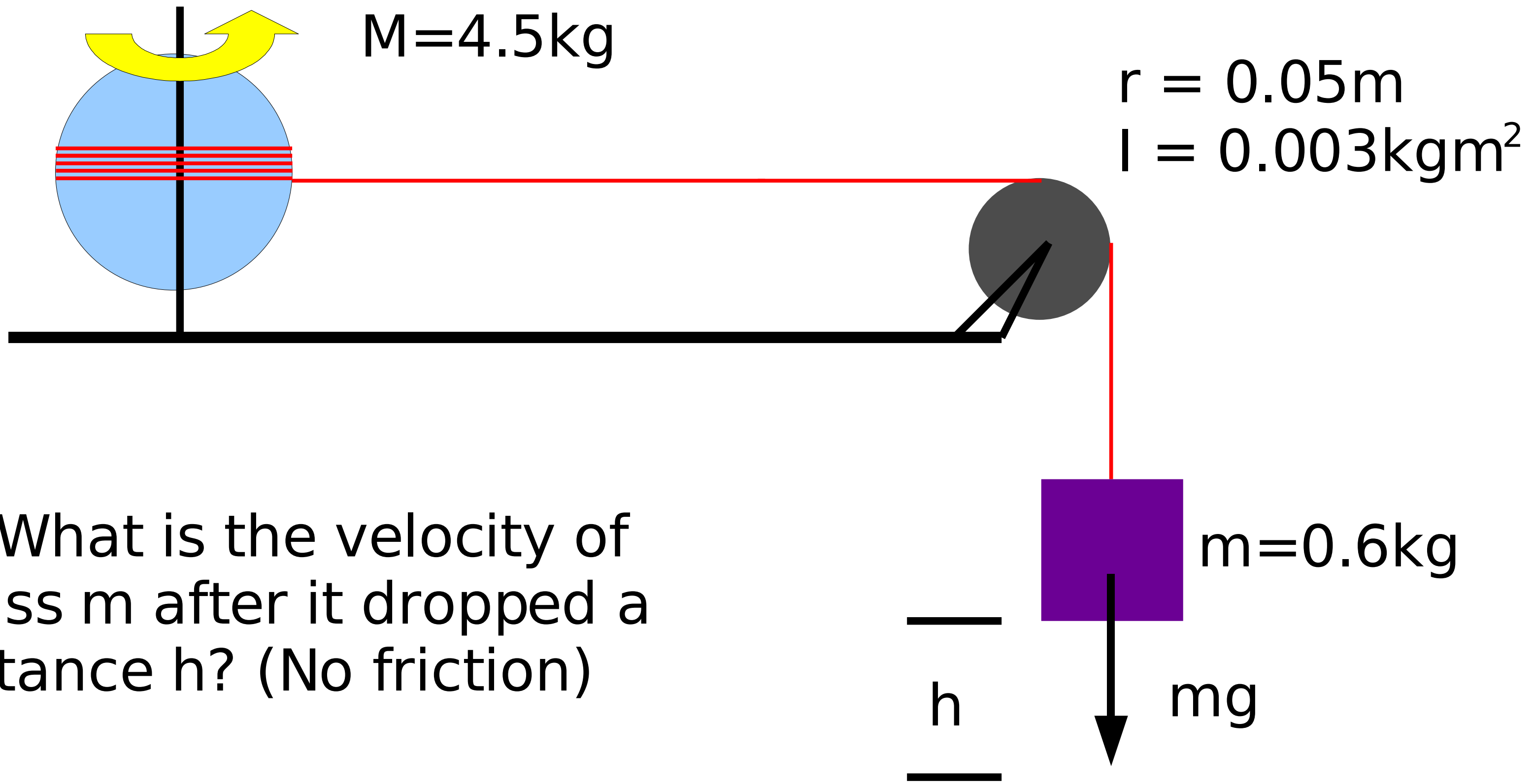


We just derived the relation between work done by a net torque and the change in kinetic energy for a single particle.

If we assume that the particle is part of a solid body then we would have to repeat this for all particles in the body. Adding up all these changes in kinetic energy and the work would show that this also works for a solid body.

Work and kinetic Energy

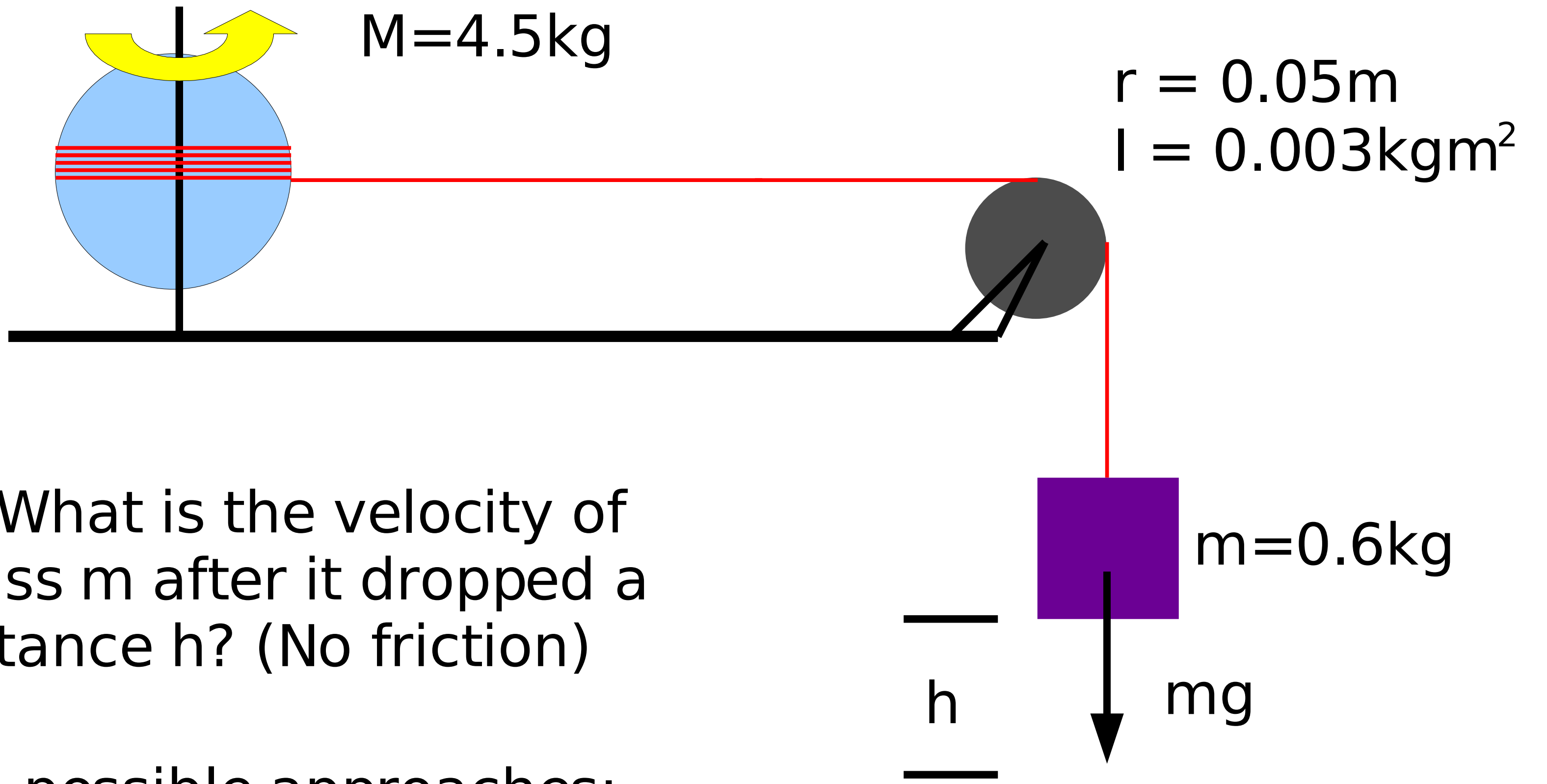
Problem 66.



Q: What is the velocity of mass m after it dropped a distance h? (No friction)

Work and kinetic Energy

Problem 66.

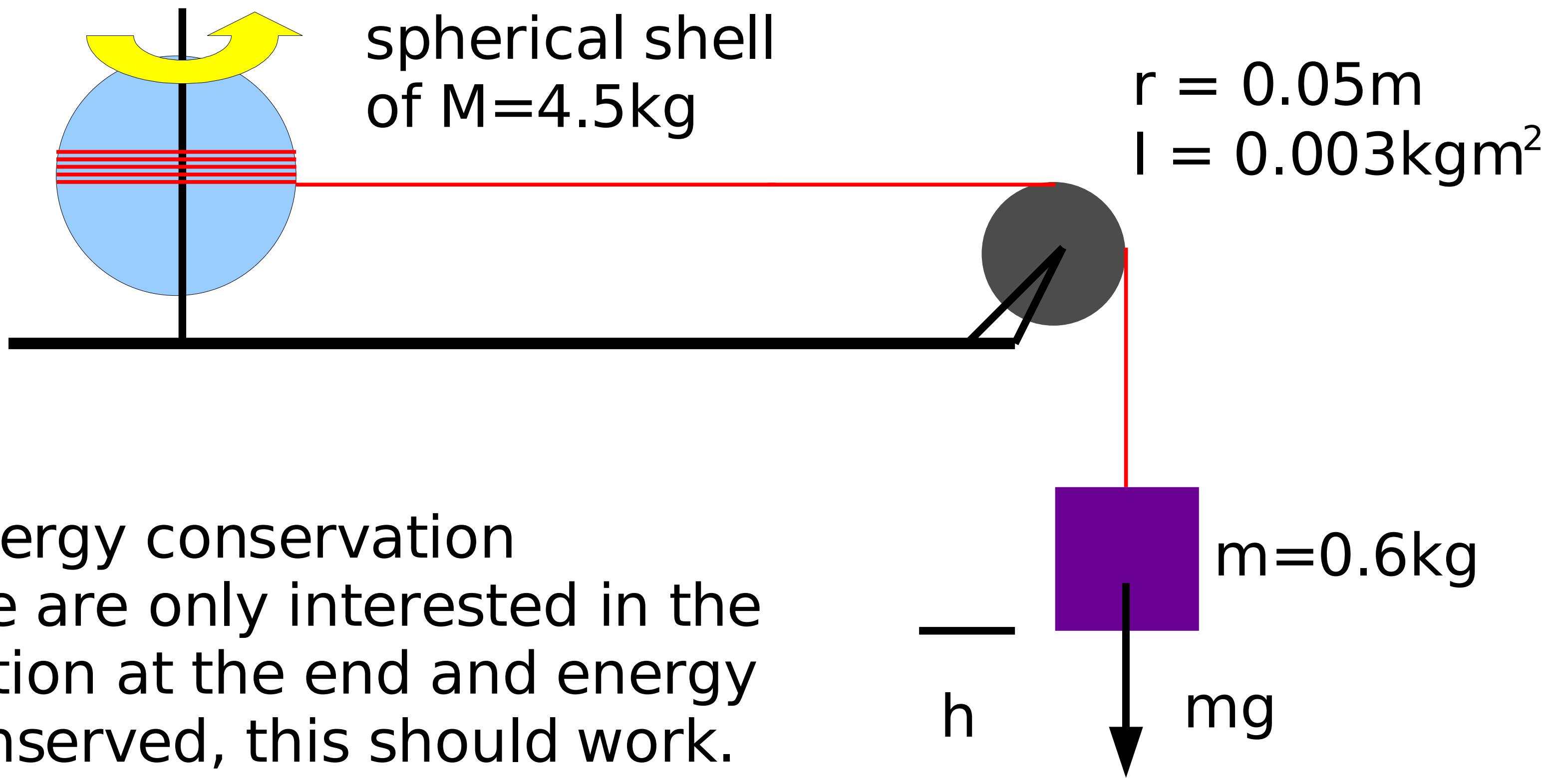


Q: What is the velocity of mass m after it dropped a distance h? (No friction)

Two possible approaches:
a) Use forces (gravity and tension) and torques
b) Energy conservation

Work and kinetic Energy

Problem 66.

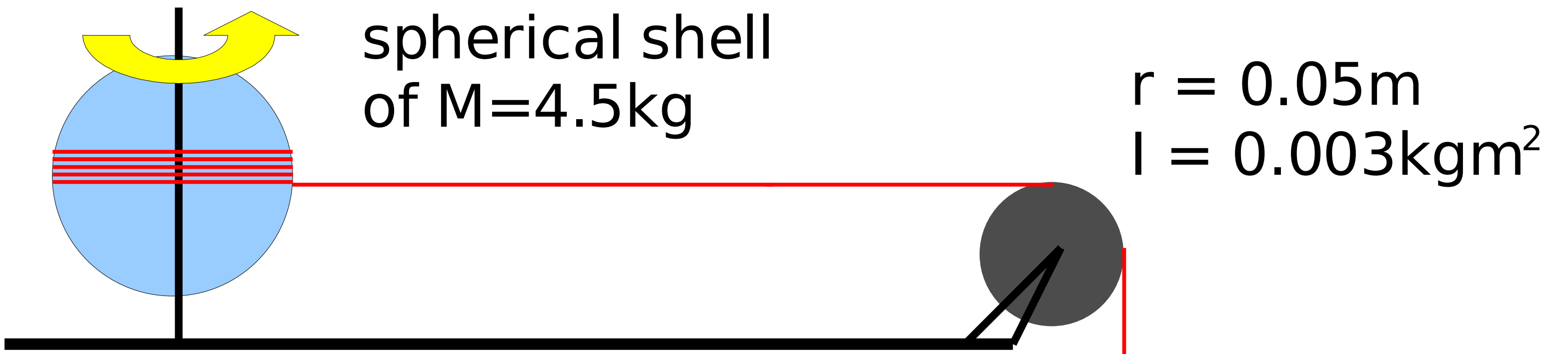


b) Energy conservation
As we are only interested in the situation at the end and energy is conserved, this should work.

Initial: $U=mgh$, Final: $K_{shell} + K_{pulley} + K_{mass}$

Work and kinetic Energy

Problem 66.

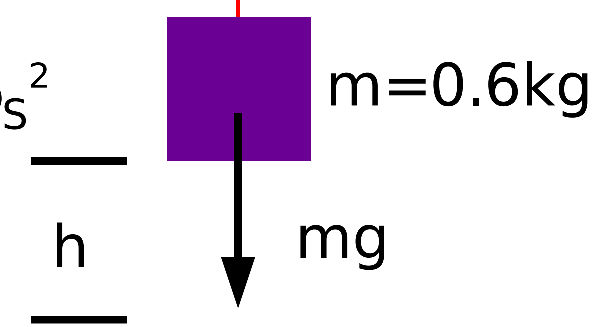


Initial: U=mgh,

Final: $K_{shell}=0.5I_S\omega_S{}^2=0.5(2MR^2/3)\omega_S{}^2$

$K_{pulley} = 0.5I_p\omega_P{}^2$

$K_{mass} = 0.5mv^2$



Also know that the 'length of the string is conserved':

$v=\omega_P r$, $v=\omega_S R$ Use this to replace the ω's in the kinetic energies

$mgh = Mv^2/3 + 0.5I_p v^2/r^2 + 0.5mv^2$ ➡ Solve for v = 1.4m/s