Chapter 7

Rotational Motion

Angles, Angular Velocity and Angular Acceleration

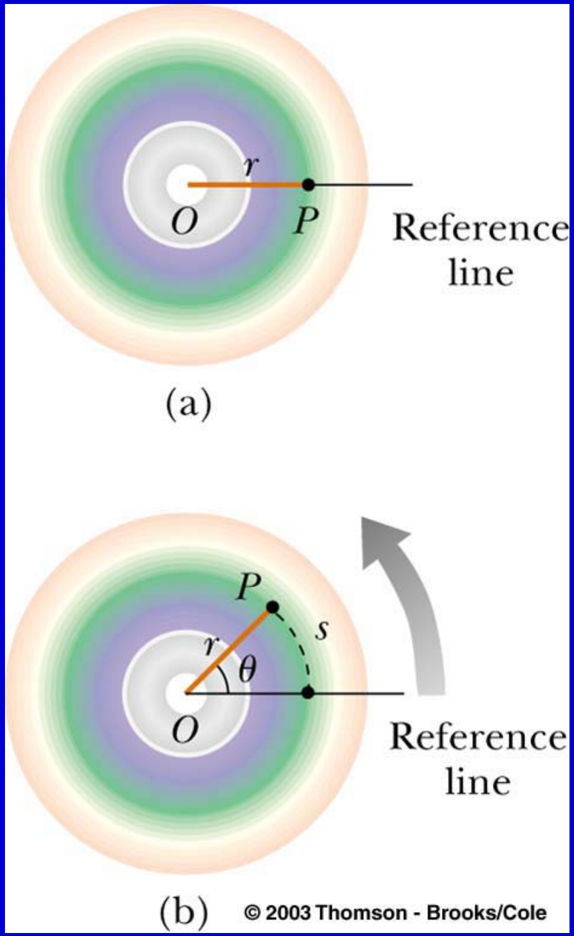
Universal Law of Gravitation

Kepler's Laws

Angular Displacement

- Circular motion about *AXIS*
- Three different measures of angles:

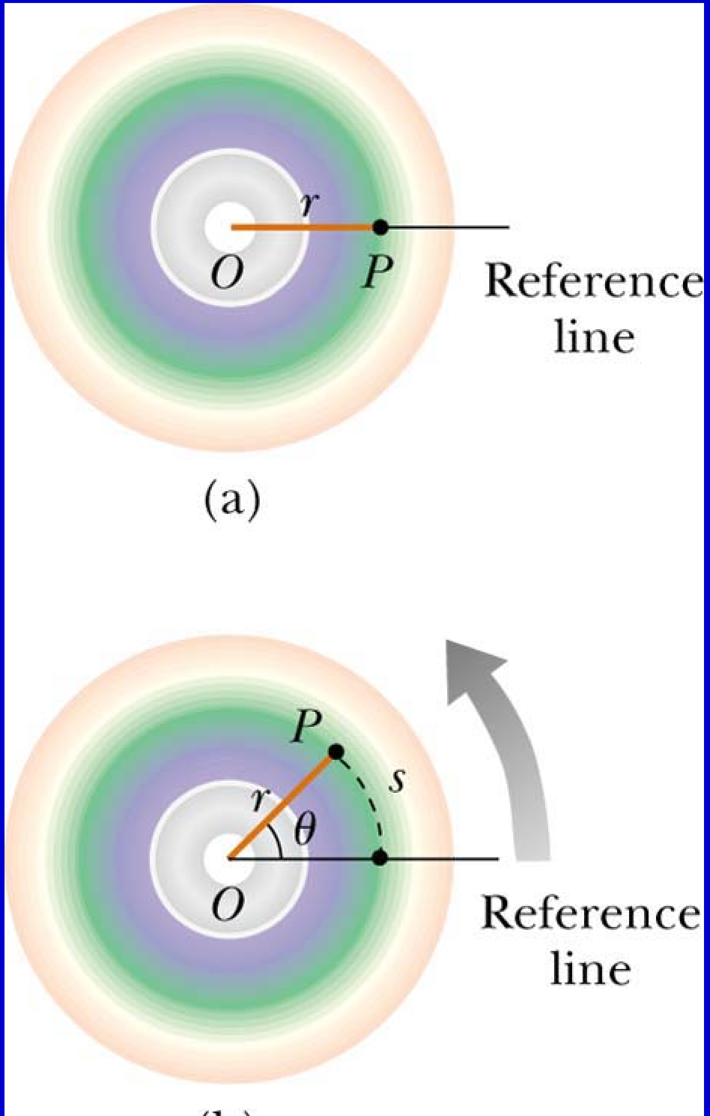
1. Degrees
2. Revolutions (1 rev. = 360 deg.)
3. Radians (2π rad.s = 360 deg.)



Angular Displacement, cont.

- Change in distance of a point:

$$s = 2\pi r\, N \quad \text{(N counts revolutions)}$$
$$= r\theta \quad (\theta \text{ is in radians})$$



Example

An automobile wheel has a radius of 42 cm. If a car drives 10 km, through what angle has the wheel rotated?

a) In revolutions

b) In radians

c) In degrees

Solution

Note distance car moves = distance outside of wheel moves

a) Find N:

Basic formula

$$s = 2\pi r N$$
$$= r\theta$$

Known: s = 10 000 m, r = 0.42 m

$$N = \frac{s}{2\pi r}$$ **= 3 789**

b) Find θ in radians

Known: N

$$\theta = 2\pi(\text{radians/revolution})\,N$$

θ = 2.38 x 10⁴ rad.

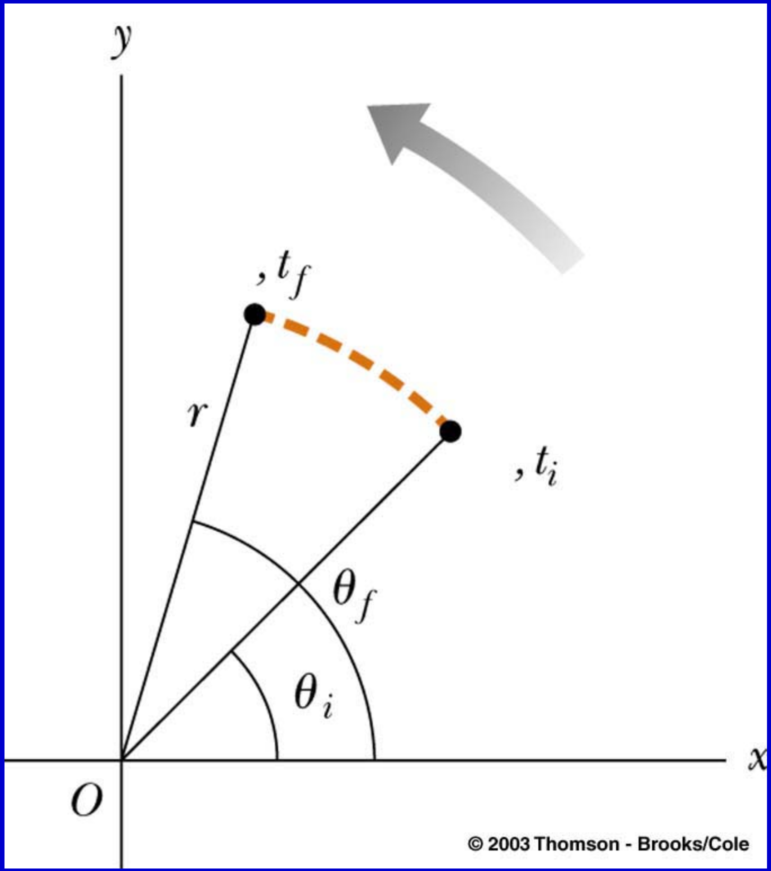
c) Find θ in degrees

Known: N

$$\theta = 360\,(\text{degrees/revolution})\,N$$

θ = 1.36 x 10⁶ deg

Angular Speed



- Can be given in
  - Revolutions/s
  - Radians/s --> Called ω
  - Degrees/s

$$\omega = \frac{\theta_f - \theta_i}{t} \text{ in radians}$$

- Linear Speed at *r*

$$v = 2\pi r \cdot \frac{\theta_f - \theta_i \text{ (in revolutions)}}{t}$$

$$= \frac{2\pi r}{360} \cdot \frac{\theta_f - \theta_i \text{ (in degrees)}}{t}$$

$$= \frac{2\pi r}{2\pi} \cdot \frac{\theta_f - \theta_i \text{ (in radians)}}{t} = \omega r$$

Example

A race car engine can turn at a maximum rate of 12 000 rpm. (revolutions per minute).

a) What is the angular velocity in radians per second.

b) If helipcopter blades were attached to the crankshaft while it turns with this angular velocity, what is the maximum radius of a blade such that the speed of the blade tips stays below the speed of sound.
DATA: The speed of sound is 343 m/s

Solution

a) Convert rpm to radians per second

$$\frac{12\,000\left(\dfrac{\text{rev.}}{\text{min}}\right)}{60\left(\dfrac{\text{sec}}{\text{min}}\right)}\cdot 2\pi\left(\frac{\text{rad}}{\text{rev}}\right)$$ **= 1256 radians/s**

b) Known: v = 343 m/s, ω = 1256 rad./s
Find r

Basic formula
$$v = \omega r$$

$$r = \frac{v}{\omega}$$ **= .27 m**

Angular Acceleration

- Denoted by α

$$\alpha = \frac{\omega_f - \omega_i}{t}$$

- ω must be in radians per sec.
- Units of angular acceleration are rad/s^2
- Every portion of the object has same angular speed and same angular acceleration

Analogies Between Linear and Rotational Motion

Rotational Motion	Linear Motion
$\Delta\theta = \frac{(\omega_i + \omega_f)}{2}t$	$\Delta x = \frac{(v_i + v_f)}{2}t$
$\Delta\theta = \omega_i t + \frac{1}{2}\alpha t^2$	$\Delta x = v_i t + \frac{1}{2}a t^2$
$\omega = \omega_i + \alpha t$	$v = v_i + at$
$\omega^2 = \omega_i^2 + 2\alpha\Delta\theta$	$v^2 = v_i^2 + 2a\Delta x$

Linear movement of a rotating point

- Distance

  $$s = \theta r$$

- Speed

  $$v = \omega r$$

- Acceleration

  $$a = \alpha r$$

- Different points on the same object have different linear motions!

Only works when θ, ω and α are in radians!

Example

A pottery wheel is accelerated uniformly from rest to a rate of 10 rpm in 30 seconds.

a.) What was the angular acceleration? (in rad/s^2)

b.)How many revolutions did the wheel undergo during that time?

Solution

First, find the final angular velocity in radians/s.

$$\omega_f = 10\left(\frac{\text{rev.}}{\text{min}}\right)\cdot\frac{1}{60(\text{sec/ min})}\cdot 2\pi\left(\frac{\text{rad}}{\text{rev}}\right) = 1.047\left(\frac{\text{rad}}{\text{sec}}\right)$$

a) Find angular acceleration

Basic formula

$$\omega_f = \omega_i + \alpha t$$

$$\alpha = \frac{\omega_f - \omega_i}{t}$$ **= 0.0349 rad./s^2**

b) Find number of revolutions: Known ω_i=0, ω_f =1.047, and t = 30

First find $\Delta\theta$ in radians

Basic formula

$$\Delta\theta = \frac{\omega_i + \omega_f}{2}t$$

$$\Delta\theta = \frac{\omega_f}{2}t = 15.7\ \text{rad.}$$

$$N = \frac{\Delta\theta(\text{rad.})}{2\pi(\text{rad./rev.})}$$ **= 2.5 rev.**

Solution

b) Find number of revolutions:

Known ω_i=0, ω_f =1.047, and t = 30,

First find $\Delta\theta$ in radians

Basic formula

$$\Delta\theta = \frac{\omega_i + \omega_f}{2} t$$

$$\Delta\theta = \frac{\omega_f}{2} t = 15.7 \text{ rad.}$$

$$N = \frac{\Delta\theta(\text{rad.})}{2\pi(\text{rad./rev.})}$$ = 2.5 rev.

Example

A coin of radius 1.5 cm is initially rolling with a rotational speed of 3.0 radians per second, and comes to a rest after experiencing a slowing down of $\alpha = 0.05\ \text{rad/s}^2$.

a.) Over what angle (in radians) did the coin rotate?

b.) What linear distance did the coin move?

Solution

a) Find $\Delta\theta$, Given ω_i= 3.0 rad/s, ω_f = 0, α = -0.05 rad/s^2

Basic formula

$$\omega_f^2 = \omega_i^2 + 2\alpha\Delta\theta$$

$$\Delta\theta = \frac{\omega_i^2}{-2\alpha}$$

= 90 radians = 90/2π revolutions

b) Find s, the distance the coin rolled

Given: r = 1.5 cm and $\Delta\theta$ = 90 rad

Basic formula

$$s = r\Delta\theta$$

$$s = r\Delta\theta, \ (\Delta\theta \text{ is in rad.s})$$ = 135 cm

Centripetal Acceleration

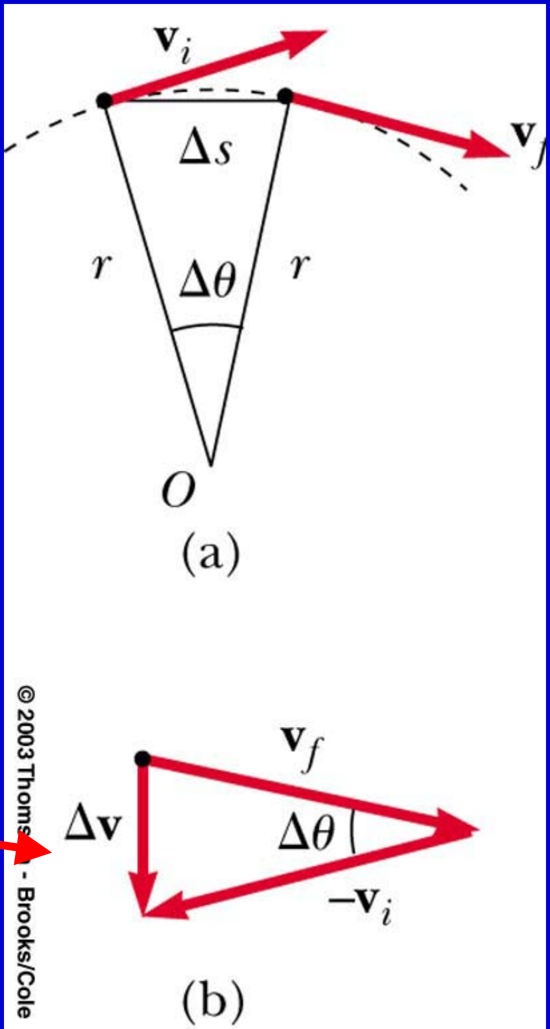
- Moving in circle at constant ***SPEED*** does not mean constant ***VELOCITY***
- Centripetal acceleration results from ***CHANGING DIRECTION*** of the velocity

Centripetal Acceleration, cont.

- Acceleration is directed toward the center of the circle of motion

Basic formula

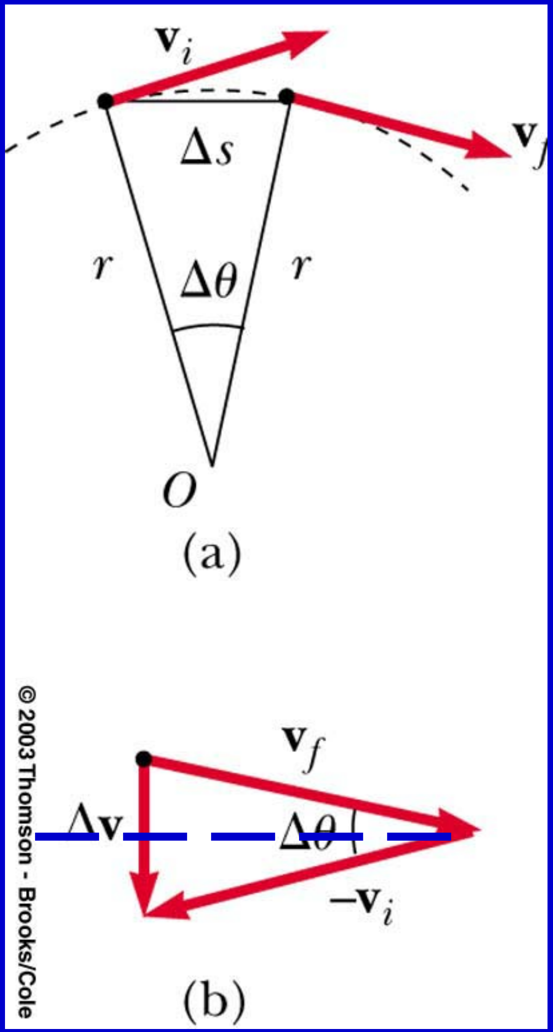
$$\vec{a} = \frac{\Delta \vec{v}}{t}$$



Derivation: a = ω^2r = v^2/r

From the geometry of the Figure

$$\Delta v = 2v\sin(\Delta\theta/2)$$
$$= v\Delta\theta \quad \text{for small } \Delta\theta$$



From the definition of angular velocity

$$\Delta\theta = \omega t$$
$$\Delta v = v\omega t$$
$$a = \frac{\Delta v}{t} = v\omega = \omega^2 r = \frac{v^2}{r}$$

Forces Causing Centripetal Acceleration

- Newton's Second Law

$$\vec{F} = m\vec{a}$$

- Radial acceleration requires radial force
- Examples of forces
  - Spinning ball on a string
  - Gravity
  - Electric forces, e.g. atoms

Example

A space-station is constructed like a barbell with two 1000-kg compartments separated by 50 meters that spin in a circle (r=25 m). The compartments spins once every 10 seconds.

a) What is the acceleration at the extreme end of the compartment? Give answer in terms of "g"s.

b) If the two compartments are held together by a cable, what is the tension in the cable?

Solution

a) Find acceleration a
Given: T = 10 s, r = 25 m

Basic formula

$$\omega = 2\pi\left(\frac{rad}{rev}\right)\cdot N\left(\frac{rev}{s}\right)$$

First, find ω in rad/s

$$\omega = 2\pi \cdot 0.1$$

Basic formula

$$a = \frac{v^2}{r} = \omega^2 r$$

Then, find acceleration

$a = \omega^2 r$ **= 9.87 m/s² = 1.006 g**

Solution

b) Find the tension

Given m = 1000 kg, a = 1.006 g

Basic formula

$$F = ma$$

$$T = ma$$ **= 9870 N**

Example

A race car speeds around a circular track.

a) If the coefficient of friction with the tires is 1.1, what is the maximum centripetal acceleration (in "g"s) that the race car can experience?

b) What is the minimum circumference of the track that would permit the race car to travel at 300 km/hr?

Solution

a) Find the maximum centripetal acceleration

Known: $\mu = 1.1$

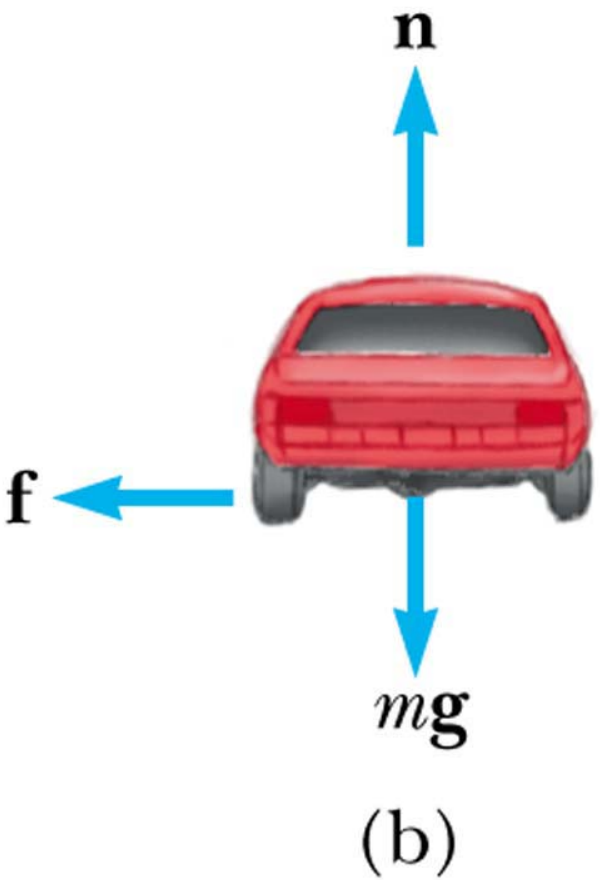
Remember, only consider forces towards center

Basic formula

$$f = \mu n$$
$$F = ma$$

$$f = \mu mg$$
$$ma = \mu mg$$
$$a = \mu g$$

Maximum a = 1.1 g



Solution

b) Find the minumum circumference

Known: v = 300 km/hr = 83.33 m/s, a = 1.1 g

Basic formula

$$a = \frac{v^2}{r} = \omega^2 r$$

First, find radius

$$r = \frac{v^2}{a}$$

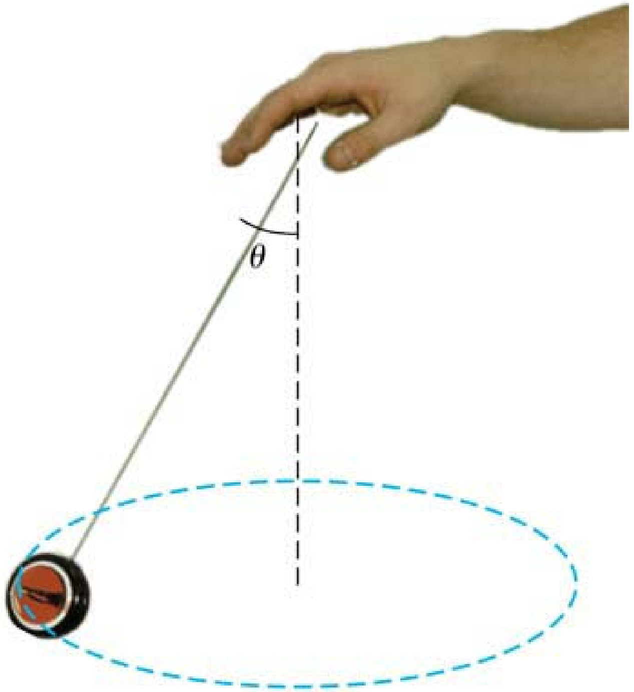
Then, find circumference

$$L = 2\pi r$$ **= 4 043 m**

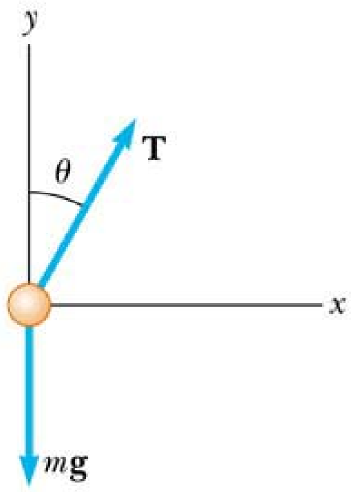
In the real world: tracks are banked

Example

A yo-yo is spun in a circle as shown. If the length of the string is L = 35 cm and the circular path is repeated 1.5 times per second, at what angle θ (with respect to the vertical) does the string bend?



(a)



(b)

Solution

Basic formula

$$\vec{F} = m\vec{a}$$

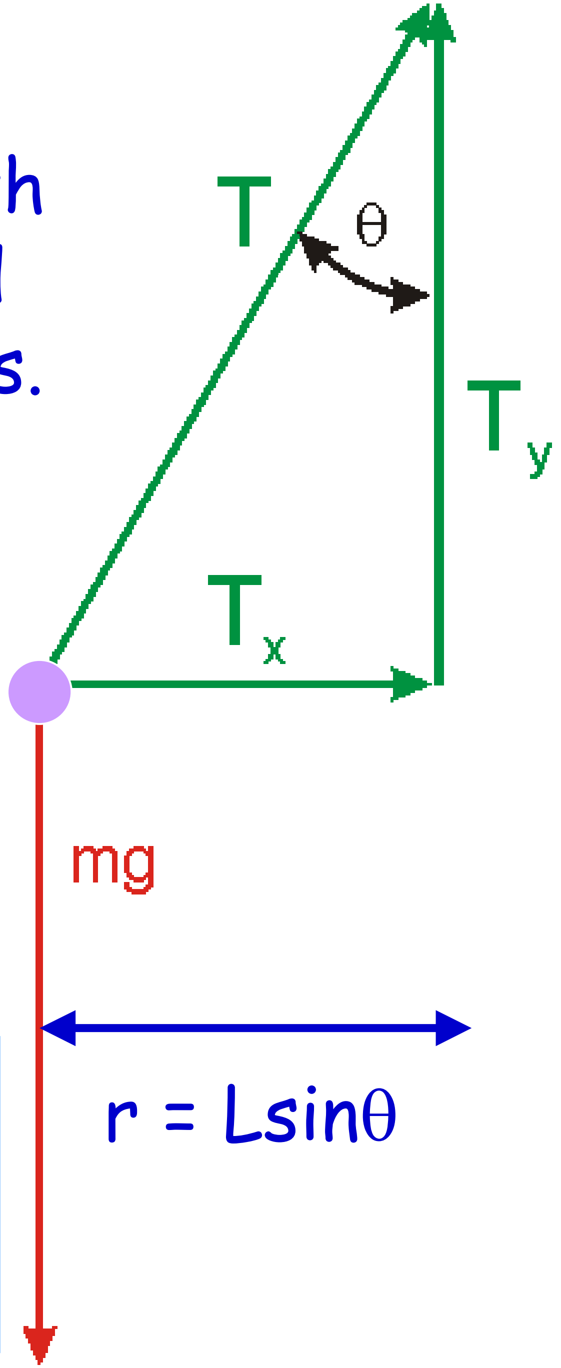
Apply F=ma for both the horizontal and vertical components.

$$ma_y = 0$$
$$\sum F_y = T\cos\theta - mg$$
$$T\cos\theta = mg$$

Basic formula

$$a = \omega^2 r$$

$$ma_x = m\omega^2 r = m\omega^2 L\sin\theta$$
$$\sum F_x = T\sin\theta$$
$$m\omega^2 L = T$$



Solution

We want to find θ, given ω=2π·1.5 & L=0.35

$$ma_x = m\omega^2 r = m\omega^2 L\sin\theta$$
$$\sum F_x = T\sin\theta$$
$$m\omega^2 L = T$$

$$ma_y = 0$$
$$\sum F_y = T\cos\theta - mg$$
$$T\cos\theta = mg$$

2 eq.s & 2 unknowns (T and θ)

$$\cos\theta = \frac{mg}{T} = \frac{g}{\omega^2 L}$$

θ = 71 degrees

Accelerating Reference Frames

Consider a frame that is accelerating with a_f

$$F = ma$$
$$F - ma_f = m(a - a_f)$$

Fictitious force

Looks like "gravitational" force

If frame acceleration = g,
fictitious force cancels real gravity.

Examples: Falling elevator, planetary orbit
rotating space stations

DEMO: FLYING POKER CHIPS

Example

Which of these astronauts experiences “zero gravity”?

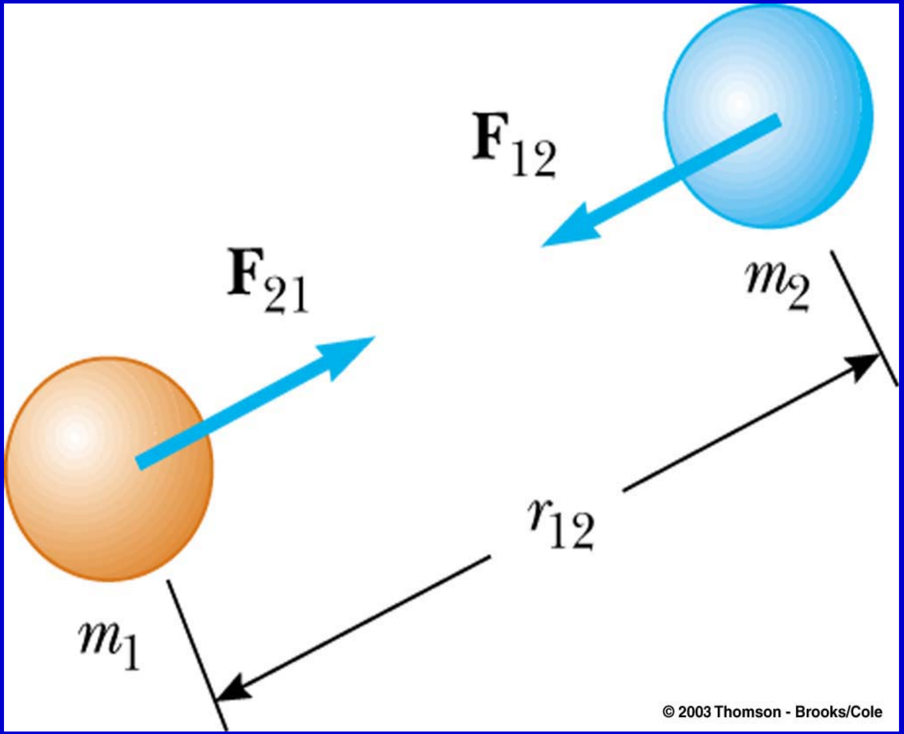
a) An astronaut billions of light years from any planet.
b) An astronaut falling freely in a broken elevator.
c) An astronaut orbiting the Earth in a low orbit.
d) An astronaut far from any significant stellar object in a rapidly rotating space station

Newton's Law of Universal Gravitation

- Force is always attractive
- Force is proportional to both masses
- Force is inversely proportional to separation squared

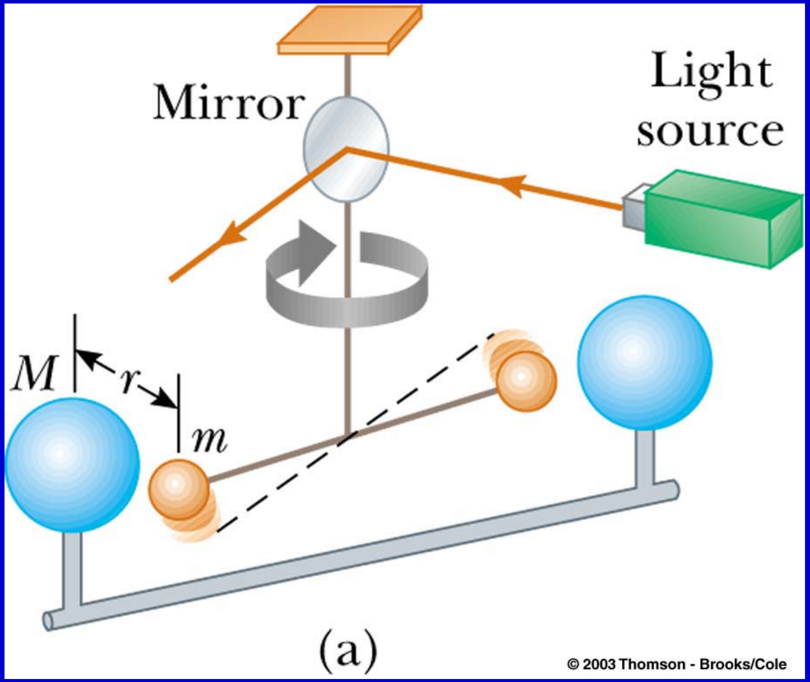
$$F = G\frac{m_1 m_2}{r^2}$$

$$G = 6.67\times10^{-11}\left(\frac{\mathrm{m}^3}{kg\cdot s^2}\right)$$



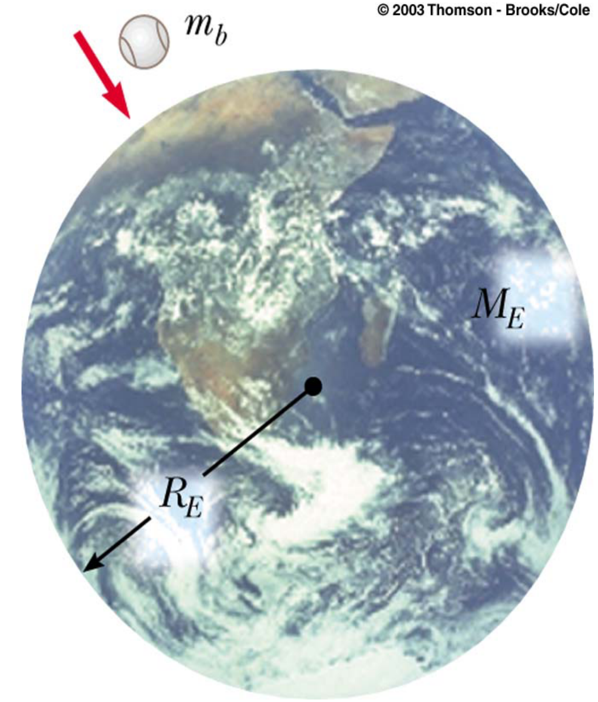
Gravitation Constant

- Determined experimentally
- Henry Cavendish, 1798
- Light beam / mirror amplify motion



Example

Given: In SI units, $G = 6.67 \times 10^{-11}$, $g=9.81$ and the radius of Earth is 6.38×10^{6}.
Find Earth's mass:



Basic formula

$$F = G\frac{Mm}{r^2} = mg$$

$$mg = G\frac{Mm}{R^2}$$

$$M = \frac{gR^2}{G} = 5.99 \times 10^{24}\ \text{kg}$$

Example

Given: The mass of Jupiter is 1.73×10^{27} kg
and Period of Io's orbit is 17 days

Find: Radius of Io's orbit

Solution

Given: T= 17·24·3600=1.47x10^6, M=1.73x10^{27}, G=6.67x10^{-11}

Find: r

Basic formula

$$\omega = 2\pi \frac{N}{t}$$

Basic formula

$$F = ma = m\omega^2 r$$

Basic formula

$$F = G\frac{Mm}{r^2}$$

First, find ω from the period

$$\omega = 2\pi \frac{1}{T} = 4.28\times10^{-6}\ \mathrm{s}$$

Next, solve for r

$$\omega^2 r = G\frac{M}{r^2}$$

$$r^3 = \left(\frac{GM}{\omega^2}\right)$$

r = 1.84x10^9 m

Tycho Brahe (1546-1601)

- Lost part of nose in a duel
- EXTREMELY ACCURATE astronomical observations, nearly 10X improvement, corrected for atmosphere
- Believed in *Retrograde* Motion
- Hired Kepler to work as mathematician

Johannes Kepler (1571-1630)

- First to:
  - Explain planetary motion
  - Investigate the formation of pictures with a pin hole camera;
  - Explain the process of vision by refraction within the eye
  - Formulate eyeglass designed for nearsightedness and farsightedness;
  - Explain the use of both eyes for depth perception.
  - First to describe: real, virtual, upright and inverted images and magnification

Johannes Kepler (1571-1630)

- First to:
  - explain the principles of **how** a telescope works
  - discover and describe total internal reflection.
  - explain that tides are caused by the Moon.
  - He tried to use stellar parallax caused by the Earth's orbit to measure the distance to the stars; the same principle as depth perception. Today this branch of research is called astrometry.
  - suggest that the Sun rotates about its axis
  - derive the birth year of Christ, that is now universally accepted.
  - derive logarithms purely based on mathematics,

Isaac Newton (1642-1727)

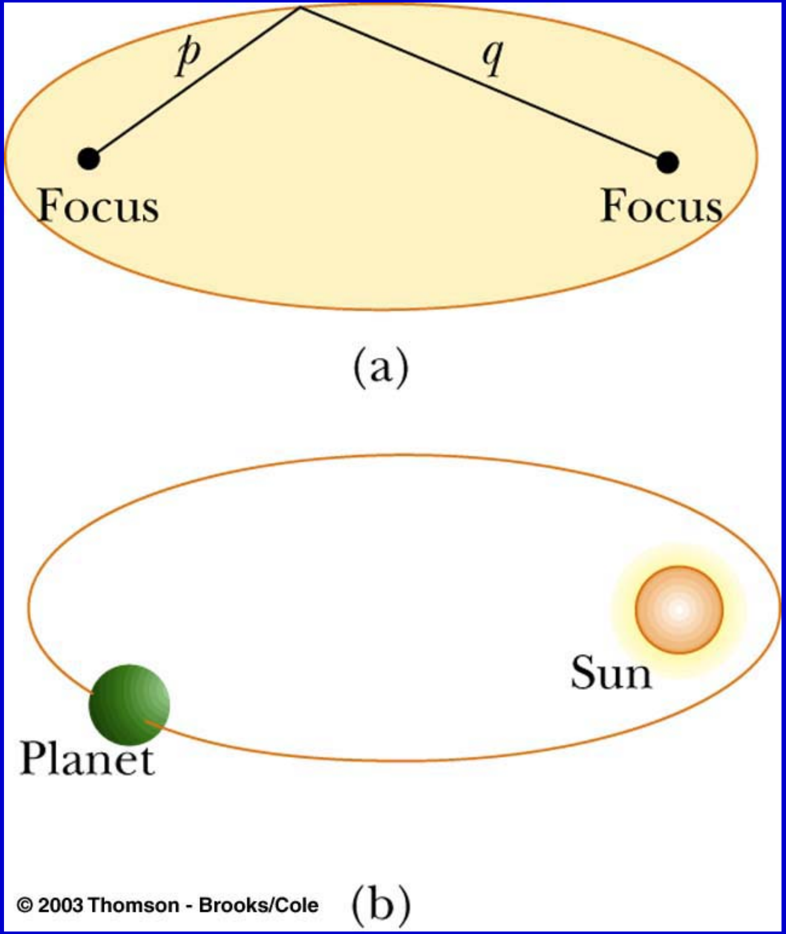
- Invented Calculus
- Formulated the universal law of gravitation
- Showed how Kepler's laws could be derived from an inverse-square-law force
- Invented Wave Mechanics
- Numerous advances to mathematics and geometry

Kepler's Laws

1. All planets move in elliptical orbits with the Sun at one of the focal points.
2. A line drawn from the Sun to any planet sweeps out equal areas in equal time intervals.
3. The square of the orbital period of any planet is proportional to cube of the average distance from the Sun to the planet.

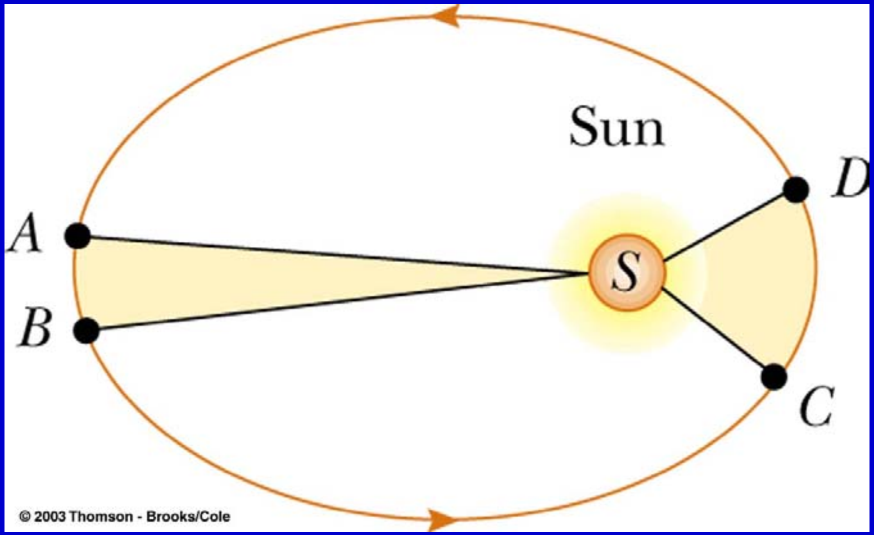
Kepler's First Law

- All planets move in elliptical orbits with the Sun at one focus.
  - Any object bound to another by an inverse square law will move in an elliptical path
  - Second focus is empty



Kepler's Second Law

- A line drawn from the Sun to any planet will sweep out equal areas in equal times
  - Area from A to B and C to D are the same



This is true for any central force due to angular momentum conservation (next chapter)

Kepler's Third Law

- The square of the orbital period of any planet is proportional to cube of the average distance from the Sun to the planet.

$$\frac{T^2}{r^3} = K_{\text{sun}}$$

- For orbit around the Sun, $K_S = 2.97 \times 10^{-19}$ s^2/m^3
- K is independent of the mass of the planet

Derivation of Kepler's Third Law

Basic formula

$$F = ma = G\frac{Mm}{r^2}$$

$$a = \omega^2 r$$

Basic formula

$$\omega = \frac{2\pi}{T}$$

$$ma = G\frac{Mm}{r^2}$$

$$\cancel{m}\omega^2 r = G\frac{M\cancel{m}}{r^2}$$

$$r^3 = \frac{GM}{4\pi^2}T^2$$

$$\frac{T^2}{r^3} = \frac{4\pi^2}{GM} = K_{\mathrm{sun}}$$

Example

Data: Radius of Earth's orbit = 1.0 A.U.
Period of Jupiter's orbit = 11.9 years
Period of Earth's orbit = 1.0 years
Find: Radius of Jupiter's orbit

Basic formula

$$\frac{T^2}{r^3} = K_{\text{sun}}$$

$$\frac{T_{Earth}^2}{r_{Earth}^3} = \frac{T_{Jupiter}^2}{r_{Jupiter}^3}$$

$$r_{Jupiter}^3 = r_{Earth}^3 \frac{T_{Jupiter}^2}{T_{Earth}^2}$$

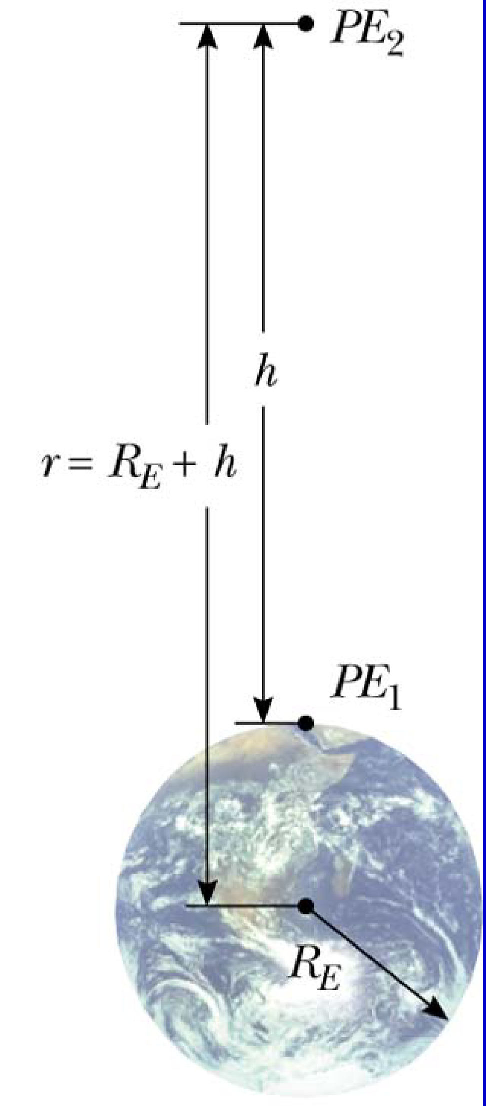
$$r_{Jupiter} = r_{Earth} \left(\frac{T_{Jupiter}}{T_{Earth}} \right)^{2/3} = 5.2 \text{ A.U.}$$

Gravitational Potential Energy

- PE = mgy is valid only near the Earth's surface
- For arbitrary altitude

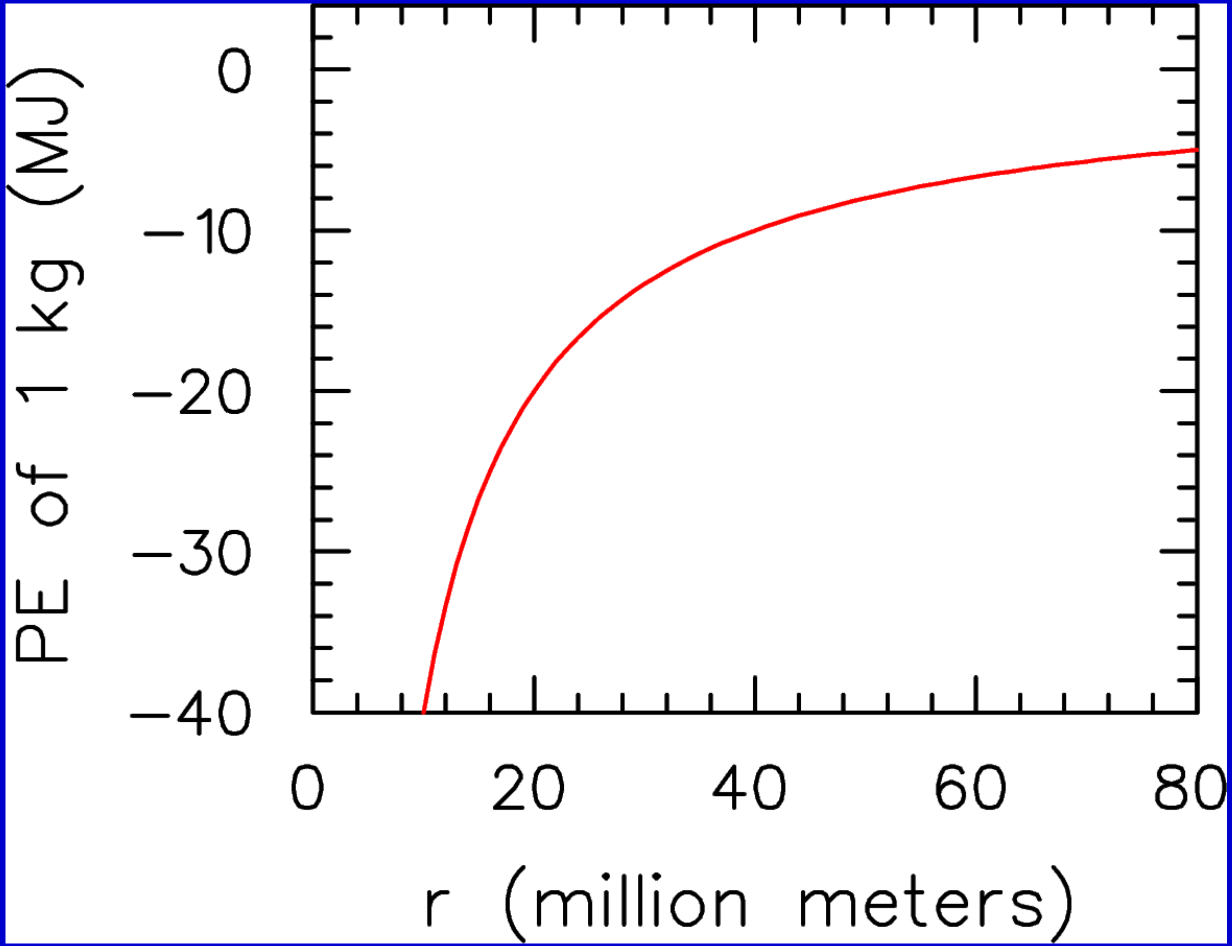
$$PE = -G\frac{M_E m}{r}$$

- Zero reference level is infinitely far from the earth



Graphing PE vs. position

$$PE = -G\frac{M_E m}{r}$$



Example

You wish to hurl a projectile is hurled from the surface of the Earth (R_e= 6.3x10^6 m) to an altitude of 20x10^6 m above the surface of the Earth. Ignore the rotation of the Earth and air resistance.

a) What initial velocity is required?

b) What velocity would be required in order for the projectile to reach infinitely high? I.e., what is the escape velocity?

Solution

Given: $R_0 = 6.3 \times 10^6$, $R = 26.3 \times 10^6$, G, $M = 6.0 \times 10^{24}$

Find: v_0

First, get expression for change in PE

Basic formula

$$PE = -G\frac{Mm}{r}$$

$$\Delta PE = GMm\left(\frac{1}{r_0} - \frac{1}{r}\right)$$

Then, apply energy conservation

Basic formula

$$KE = \frac{1}{2}mv^2$$

$$\Delta PE = \Delta KE$$

$$GM\cancel{m}\left(\frac{1}{r_0} - \frac{1}{r}\right) = \frac{1}{2}\cancel{m}v_0^2$$

Finally, solve for v_0 = **6 600 m/s**

Solution For Escape Velocity

Given: $R_0 = 6.3\times10^6$, $R = \infty$, G, $M = 6.0\times10^{24}$

Find: v_0

First, get expression for change in PE

Basic formula

$$PE = -G\frac{Mm}{r}$$

$$\Delta PE = GMm\frac{1}{r_0}$$

Then, apply energy conservation

Basic formula

$$KE = \frac{1}{2}mv^2$$

$$\Delta PE = \Delta KE$$

$$GM\cancel{m}\frac{1}{r_0} = \frac{1}{2}\cancel{m}v_0^2$$

Solve for v_0 **= 11 270 m/s**