Rough paths methods 1: Introduction

Samy Tindel

Purdue University

University of Aarhus 2016

Outline

1. Motivations for rough paths techniques

2. Summary of rough paths theory

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Equation under consideration

Equation:
Standard differential equation driven by fBm, $\mathbb{R}^n$-valued

$$Y_t = a + \int_0^t V_0(Y_s)\, ds + \sum_{j=1}^{d} \int_0^t V_j(Y_s)\, dB_s^j, \qquad (1)$$

with

- $t \in [0, 1]$.
- Vector fields $V_0, \ldots, V_d$ in $\mathcal{C}_b^\infty$.
- A d-dimensional fBm B with $1/3 < H < 1$.
- Note: some results will be extended to $H > 1/4$.

Fractional Brownian motion

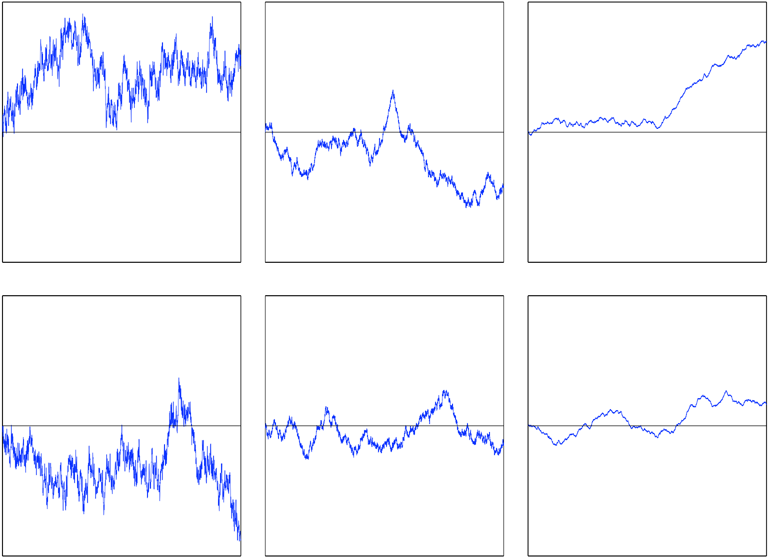
- $B = (B^1, \dots, B^d)$
- B^j centered Gaussian process, independence of coordinates
- Variance of the increments:

$$\mathbf{E}[|B_t^j - B_s^j|^2] = |t - s|^{2H}$$

- $H^- \equiv$ Hölder-continuity exponent of B
- If $H = 1/2$, $B =$ Brownian motion
- If $H \neq 1/2$ natural generalization of BM

Remark: FBm widely used in applications

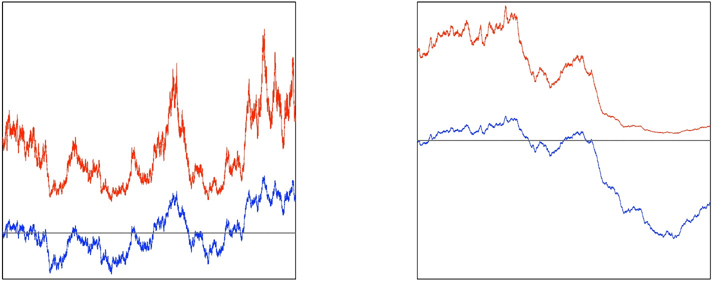
Examples of fBm paths



$H = 0.3$ $\qquad$ $H = 0.5$ $\qquad$ $H = 0.7$

Paths for a linear SDE driven by fBm

$$dY_t = -0.5Y_t dt + 2Y_t dB_t, \quad Y_0 = 1$$



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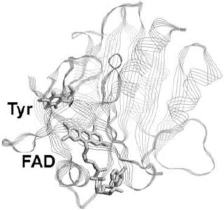
Blue: $(B_t)_{t\in[0,1]}$ Red: $(Y_t)_{t\in[0,1]}$

Some applications of fBm driven systems

Biophysics, fluctuations of a protein:

- New experiments at molecule scale
  $\hookrightarrow$ Anomalous fluctuations recorded
- Model: Volterra equation driven by fBm
  $\hookrightarrow$ Samuel Kou
- Statistical estimation needed



Finance:

- Stochastic volatility driven by fBm (Sun et al. 2008)
- Captures long range dependences between transactions

Outline

Rough paths assumptions

Context: Consider a Hölder path x and

- For $n \geq 1$, $x^n \equiv$ linearization of x with mesh $1/n$
  $\hookrightarrow$ x^n piecewise linear.
- For $0 \leq s < t \leq 1$, set

$$\mathbf{x}^{\mathbf{2},n,i,j}_{st} \equiv \int_{s<u<v<t} dx^{n,i}_u \, dx^{n,j}_v$$

Rough paths assumption 1:

- x is a $\mathcal{C}^\gamma$ function with $\gamma > 1/3$.
- The process $\mathbf{x}^{\mathbf{2},n}$ converges to a process $\mathbf{x}^{\mathbf{2}}$ as $n \to \infty$
  $\hookrightarrow$ in a $\mathcal{C}^{2\gamma}$ space.

Rough paths assumption 2:

- Vector fields $V_0, \ldots, V_j$ in $\mathcal{C}^\infty_b$.

Brief summary of rough paths theory

Main rough paths theorem (Lyons): Under previous assumptions
$\hookrightarrow$ Consider y^n solution to equation

$$y_t^n = a + \int_0^t V_0(y_u^n)\, du + \sum_{j=1}^d \int_0^t V_j(y_u^n)\, dx_u^{n,j}.$$

Then

- y^n converges to a function Y in $\mathcal{C}^\gamma$.
- Y can be seen as solution to
  $\hookrightarrow$ $Y_t = a + \int_0^t V_0(Y_u)\, du + \sum_{j=1}^d \int_0^t V_j(Y_u)\, dx_u^j$.

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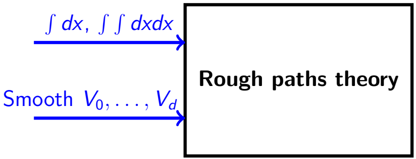

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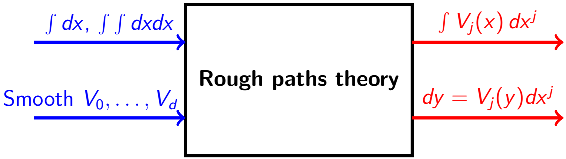


Iterated integrals and fBm

Nice situation: $H > 1/4$

$\hookrightarrow$ 2 possible constructions for geometric iterated integrals of B.

- Malliavin calculus tools
  $\hookrightarrow$ Ferreiro-Utzet
- Regularization or linearization of the fBm path
  $\hookrightarrow$ Coutin-Qian, Friz-Gess-Gulisashvili-Riedel

Conclusion: for $H > 1/4$, one can solve equation

$$dY_t = V_0(Y_t)\,dt + V_j(Y_t)\,dB_t^j,$$

in the rough paths sense.

Remark: Extensions to $H \leq 1/4$ (Unterberger, Nualart-T).

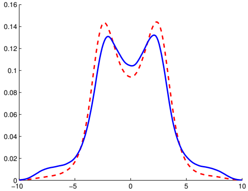
Study of equations driven by fBm

Basic properties:

1. Moments of the solution
2. Continuity w.r.t initial condition, noise

More advanced natural problems:

1. Density estimates
   ↪ Hu-Nualart + Lots of people
2. Numerical schemes
   ↪ Neuenkirch-T, Friz-Riedel
3. Invariant measures, ergodicity
   ↪ Hairer-Pillai, Deya-Panloup-T
4. Statistical estimation (H, coeff. V_j)
   ↪ Berzin-León, Hu-Nualart, Neuenkirch-T



Extensions of the rough paths formalism

Stochastic PDEs:

- Equation: $\partial_t Y_t(\xi) = \Delta Y_t(\xi) + \sigma(Y_t(\xi))\, \dot{x}_t(\xi)$
- $(t, \xi) \in [0,1] \times \mathbb{R}^d$
- Easiest case: x finite-dimensional noise
- Methods:
  $\hookrightarrow$ viscosity solutions or adaptation of rough paths methods

KPZ equation:

- Equation: $\partial_t Y_t(\xi) = \Delta Y_t(\xi) + (\partial_\xi Y_t(\xi))^2 + \dot{x}_t(\xi) - \infty$
- $(t, \xi) \in [0,1] \times \mathbb{R}$
- $\dot{x} \equiv$ space-time white noise
- Methods:
  - Extension of rough paths to define $(\partial_x Y_t(\xi))^2$
  - Renormalization techniques to remove ∞

Aim

1. Definition and properties of fractional Brownian motion
2. Some estimates for Young's integral, case $H > 1/2$
3. Extension to $1/3 < H \leq 1/2$

General strategy

1. In order to solve our equation, we shall go through the following steps:
   - Young integral for $H > 1/2$
   - Case $1/3 < H < 1/2$, with a semi-pathwise method
2. For each case, 2 main steps:
   - Definition of a stochastic integral $\int u_s dB_s$ for a reasonable class of processes u
   - Resolution of the equation by means of a fixed point method