S. Mishra
L. Scarabosio
J. Šukys

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Numerical Methods for Partial Differential Equations

ETH Zürich
D-MATH

Exam Winter 2015

The exam consists of 100 points + 10 bonus points.
This means that there are 110 total points but the mark is computed on a basis of 100 points.
Duration: 3 hours.
Slightly harder sub-problems are marked with an asterisk symbol *.

Problem 1 Stationary Diffusion problem [40 points]

We consider the following two-dimensional diffusion problem:

$$-\nabla\cdot\big((x^2+y^2+1)\nabla u(\boldsymbol{x})\big)=f(\boldsymbol{x}),\qquad \boldsymbol{x}\in\Omega=(0,1)^2, \tag{1.1}$$

$$u|_{\Gamma_D}(\boldsymbol{x})=0,\qquad \boldsymbol{x}\in\Gamma_D, \tag{1.2}$$

$$\left.\frac{\partial u}{\partial \boldsymbol{n}}\right|_{\Gamma_N}(\boldsymbol{x})=0,\qquad \boldsymbol{x}\in\Gamma_N, \tag{1.3}$$

where $\boldsymbol{x}=(x,y)$, $f\in L^2(\Omega)$ and the domain boundary $\partial\Omega$ is the (disjoint) union of a Dirichlet part Γ_D and a Neumann part Γ_N, as depicted in Figure 1.1.

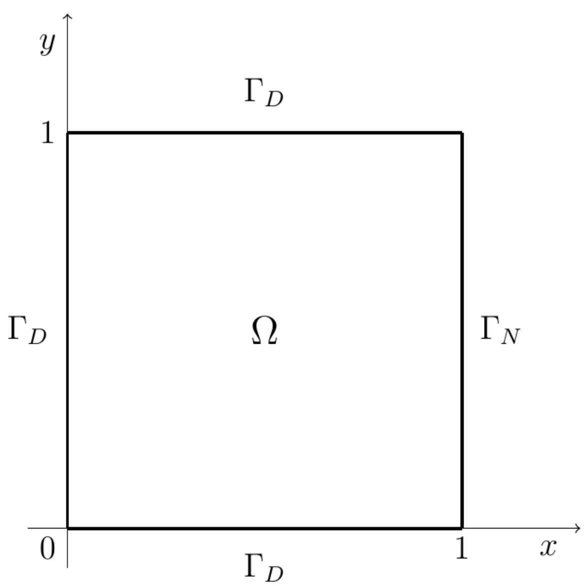

Figure 1.1: Domain for Problem 1.

(1a) Write the variational formulation for (1.1)-(1.3).

HINT: Don't forget to specify trial and test spaces.

(1b) Specify the bilinear form and linear form in the variational formulation obtained in the previous subtask.

(1c) Show that the variational formulation from subproblem (1a) admits a solution and the solution is unique.

HINT: Use the Lax-Milgram lemma.

HINT: The Poincaré inequality holds.

(1d) Give an a priori upper bound for $|u|_{H^1(\Omega)}$, where u is the solution to the variational formulation from subsproblem (1a).

HINT: Use the Lax-Milgram lemma.

We now want to compute an approximate solution to (1.1)-(1.3) using linear finite elements (LFE) on a triangular mesh.

(1e) Compute the stiffness element matrix (i.e. the element matrix associated to the bilinear form) on the reference element $\hat{K}$, that is, on the triangle with vertices $\hat{\boldsymbol{x}}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\hat{\boldsymbol{x}}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\hat{\boldsymbol{x}}_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

For the computation of the integrals, use the 2$^{\text{nd}}$-order quadrature rule on triangles:

$$\int_T h(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \approx \frac{\mathrm{Vol}(T)}{3}(h(\boldsymbol{x}_1) + h(\boldsymbol{x}_2) + h(\boldsymbol{x}_3)), \tag{1.4}$$

where T is a generic triangle, h is a continuous function on T and $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ are the three vertices of T.

(1f) Consider now the assembly of the stiffness matrix. Denote by N_{El} the number of triangles in the mesh, by `idx` the global indices of the vertices of a generic triangle K_i ($i = 1, \ldots, N_{El}$), by A_{K_i} the element stiffness matrix for the triangle K_i and by A the global stiffness matrix. Complete line 4 of the following routine that assembles the stiffness matrix: which entries of the global matrix A have to be selected? which are the addends on the right-hand side of this line?

1: **for** i=1:N_{El} **do**
2:     Extract global indices of vertices of element K_i $\rightarrow$ `idx`
3:     Compute element stiffness matrix for K_i $\rightarrow A_{K_i}$
4:     $A(\ldots, \ldots) = \ldots + \ldots$
5: **end for**

In line 4, the notation $A(\ldots, \ldots)$ can be used not only to extract single entries of the matrix, but also to extract a submatrix: for example, if `vec=[1,10,20]`, then $A($`vec`$,$`vec`$)$ denotes the 3×3 submatrix of A containing the elements with row and column index contained in `vec`.

(1g) Suppose now that the boundary condition (1.2) is replaced by

$$u|_{\Gamma_D}(\boldsymbol{x}) = g, \boldsymbol{x} \in \Gamma_D, \tag{1.5}$$

with g a continous function on Γ_D. In other words, we replace the homogeneous Dirichlet boundary conditions with inhomogeneous Dirichlet boundary conditions.

Let N_V be the total number of mesh vertices and $N_{V,D}$ the number of vertices which are on the Dirichlet boundary Γ_D. Let $\boldsymbol{D}$ be the array of length $N_{V,D}$ containing the indices of the vertices which are on the Dirichlet boundary and **FreeDofs** the array of length $N_V - N_{V,D}$ containing the indices of all the other vertices, that is the vertices which are not on Γ_D.
We denote by $\boldsymbol{\mu} \in \mathbb{R}^{N_V}$ the *column* vector containing the coefficients of the discrete solution u_N with respect to the basis functions associated to all the mesh vertices. Then we have $\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_{\textbf{FreeDofs}} \\ \boldsymbol{\mu}_{\boldsymbol{D}} \end{pmatrix}$, with $\boldsymbol{\mu}_{\boldsymbol{D}} \in \mathbb{R}^{N_{V,D}}$ the coefficients with respect to the basis functions associated to nodes on Γ_D, and $\boldsymbol{\mu}_{\textbf{FreeDofs}} \in \mathbb{R}^{N_V - N_{V,D}}$ the coefficients with respect to the remaining basis functions. Finally, let $\boldsymbol{x}_D \in \mathbb{R}^{N_{V,D} \times 2}$ the matrix containing the coordinates of the Dirichlet nodes (each row corresponding to a node).
Let us denote by $A \in \mathbb{R}^{N_V \times N_V}$ the global stiffness matrix and by $L \in \mathbb{R}^{N_V}$ the global load vector, both computed considering the basis functions of $H^1(\Omega)$ associated to *all* the mesh vertices.

The following pseudocode computes the discrete finite element solution to (1.1) (i.e. the coefficient vector μ), with nonhomogeneous boundary conditions given by (1.5) and Neumann boundary conditions as before ((1.3)). This pseudocode contains an error: which one? Write clearly the correction that has to be made in order to get the correct discrete solution.

Compute global stiffness matrix $\to A$
Compute global load vector $\to L$
Extract indices on $\Gamma_D \to \boldsymbol{D}$
Extract coordinates of vertices with index in $\boldsymbol{D} \to \boldsymbol{x}_D$
Initialize $\boldsymbol{\mu}$: $\boldsymbol{\mu} = \mathbf{0} \in \mathbb{R}^{N_V}$
Initialize $\boldsymbol{\mu}_D$: $\boldsymbol{\mu}_D = g(\boldsymbol{x}_D)$
Solve $A(\textbf{FreeDofs}, \textbf{FreeDofs})\boldsymbol{\mu}_{\textbf{FreeDofs}} = L(\textbf{FreeDofs})$

(In the last line $A(\textbf{FreeDofs}, \textbf{FreeDofs})$ denotes the part of the stiffness matrix associated to both test and trial functions relative to the nodes **FreeDofs**, and similarly $L(\textbf{FreeDofs})$ denotes the part of the load vector associated to the test functions relative to the nodes **FreeDofs**.)

(1h) Suppose that the exact solution to the variational formulation from subproblem (1a) is smooth. The two double logarithmic (loglog) plots in Figure 1.2 show respectively the results of the convergence studies of the L^2-norm and H^1-seminorm of the error with respect to the meshwidth h. For each of the two plots, which is the line showing the right convergence rate?

Problem 2 Convergence of Finite Element Solutions [15 points]

A student is testing his implementation of a finite element method. On the square domain $\Omega = (0,1)^2$ he considers the 2nd-order elliptic boundary value problem

$$\begin{aligned} -\Delta u &= 1 && \text{in } \Omega, \\ u &= \frac{1}{4}\left(1 - \|\boldsymbol{x}\|^2\right) && \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

He computes an approximate solutions u_N by means of a finite element Galerkin method using linear (piecewise first order polynomials) and quadratic (piecewise second order polynomials) finite elements, denoted by LFE and QFE respectively, on a sequence of triangular meshes $\mathcal{M}$.

The following table lists the measured $H^1(\Omega)$-seminorm of the discretization error as a function of the meshwidth h.

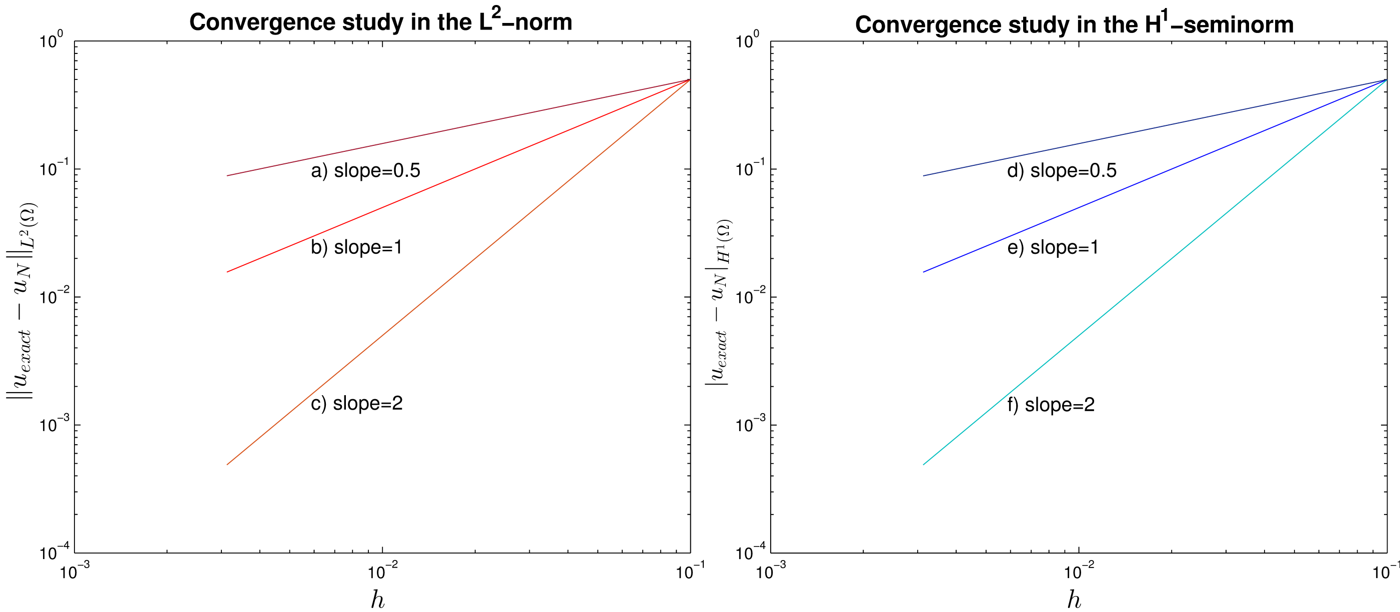

Figure 1.2: Convergence plots for subproblem (1h).

h	0.70	0.35	0.17	0.088	0.044	0.022	0.011
LFE	0.10	0.051	0.025	0.012	0.0064	0.0032	0.0008
QFE	$1.75\cdot10^{-16}$	$1.24\cdot10^{-15}$	$5.71\cdot10^{-15}$	$2.29\cdot10^{-14}$	$8.91\cdot10^{-14}$	$3.53\cdot10^{-13}$	$1.41\cdot10^{-12}$

(2a) Show that $u(\boldsymbol{x}) = \frac{1}{4}(1 - \|\boldsymbol{x}\|^2)$ is the exact solution of (2.1)

(2b) What kind of convergence (qualitative and quantitative) for linear Lagragian finite elements can be inferred from the error table?

(2c) *Explain the striking difference between the behavior of the discretization error for linear and quadratic Lagrangian finite elements.

Problem 3 Radiative Cooling in 1D [15 points]

This problem is dedicated to the full spatial and temporal discretization of a 2nd-order parabolic evolution problem.

In a homogeneous "1D body" occupying the space $\Omega := (0, 1)$, the evolution of the temperature distribution $u = u(x, t)$ with convective cooling is modelled by the linear second-order parabolic initial-boundary value problem (IBVP) with flux (spatial) boundary conditions

$$
\begin{aligned}
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= 0 && \text{in } \Omega \times (0, T], \\
\frac{\partial u}{\partial \boldsymbol{n}} + \gamma u &= 0 && \text{on } \partial\Omega \times (0, T], \\
u(x, 0) &= u_0(x) && \text{on } \Omega \times \{0\},
\end{aligned}
\tag{3.1}
$$

with $\gamma > 0$ constant.

Notice that the boundary condition given in the second equation in (3.1) can be rewritten as

$$\begin{aligned} \frac{\partial u}{\partial \boldsymbol{n}} + \gamma u &= \frac{\partial u}{\partial x} + \gamma u &&= 0 \quad \text{on } \{1\} \times (0,T] \\ \frac{\partial u}{\partial \boldsymbol{n}} + \gamma u &= \frac{\partial u}{\partial (-x)} + \gamma u = -\frac{\partial u}{\partial x} + \gamma u &&= 0 \quad \text{on } \{0\} \times (0,T]. \end{aligned} \tag{3.2}$$

(3a) Let $E(t)$ be the energy of the solution $u(x,t)$ to (3.1) at time $t > 0$, i.e.

$$E(t) := \frac{1}{2} \int_\Omega (u(x,t))^2 \,\mathrm{d}x. \tag{3.3}$$

Show that the solution $u(x,t)$ to (3.1) satisfies the energy inequality

$$E(t) \leq E(0) \quad \text{for all } t > 0. \tag{3.4}$$

HINT: Differentiate $E(t)$ with respect to time t and then use (3.1) and integration by parts

Next, we aim to discretize (3.1) using finite differences coupled with the a time stepping scheme.

To discretize the spatial domain $\Omega = (0,1)$, we subdivide the interval $[0,1]$ in $N+1$ subintervals using equispaced grid points $\{x_0 = 0, x_1, \dots, x_N, x_{N+1} = 1\}$. Let us denote by $h = |x_1 - x_0|$ the mesh size. To discretize the time domain $[0,T]$, consider $M \in \mathbb{N}$ equispaced time points

$$t_0 = 0, t_1, \dots, t_M = T,$$

and denote the time step size by $\Delta t := |t_{n+1} - t_n| = T/M$. Then, at a time point t with $n = 0, \dots, M-1$, the solution $u(x_j, t_n)$ at point x_j and at time t_n is denoted by U_j^n.

(3b) Write the fully discrete numerical scheme for (3.1) using central differences in space and the *explicit Euler* time stepping in time.

(3c) Suppose that we have implemented the numerical scheme deduced in subproblem (3b), and now we run it with $\Delta t = h$. Monitoring the energy E^n at each time step t_n, $n = 0, \dots, M-1$, we observe that $E^n \to \infty$ as $n \to \infty$. What is the reason for this blowup?

(3d) To make another test, we change the numerical scheme and discretize (3.1) using again central differences in space but now the *implicit Euler* time stepping. Again, we run our code with $\Delta t = h$ and this time we observe that the energy does not explode as time passes, and our scheme fulfills the discrete version of the energy inequality (3.3), i.e.

$$E^n \leq E^0 \tag{3.5}$$

with

$$E^n = \frac{1}{2} \sum_{j=1}^{N} (U_j^n)^2, \tag{3.6}$$

for every $n = 0, \dots, M-1$.

Explain why this time our numerical scheme is stable even choosing $\Delta t = h$.

Problem 4 Recasting Burgers' Equation [25 points]

Burgers' equation reads

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = 0 \quad \text{on } \mathbb{R} \times (0, T] \tag{4.1}$$

Assuming smoothness of u, an equivalent formulation for $w = u^2$ is

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial x}\left(\frac{2}{3}w^{3/2}\right) = 0 \quad \text{on } \mathbb{R} \times (0, T] \tag{4.2}$$

(4a) Show that if u is continuously differentiable (i.e. u is C^1) in both space and time, and solves (4.1) with initial condition $u(x, 0) = u_0(x) > 0$ for all $x \in \mathbb{R}$, then $w(x, t) := u(x, t)^2$ solves (4.2).

HINT: Since $u_0(x) > 0$ for all $x \in \mathbb{R}$, then, by the maximum principle, $u(x, t) > 0$ for all $(x, t) \in \mathbb{R} \times (0, T]$. Then the proof boils down to clever use of the chain rule.

(4b) What is the *flux function* for (4.2)? Show that it is convex.

(4c) Determine the *entropy solutions* of (4.1) and (4.2) for the Riemann problems

$$u_0(x) = \begin{cases} 2, & x < 0, \\ 1, & x > 0 \end{cases} \qquad w_0(x) = \begin{cases} 4, & x < 0 \\ 1, & x > 0. \end{cases}$$

(4d) Determine the *entropy solutions* of (4.1) and (4.2) for the Riemann problems

$$u_0(x) = \begin{cases} 1, & x < 0, \\ 2, & x > 0 \end{cases} \qquad w_0(x) = \begin{cases} 1, & x < 0 \\ 4, & x > 0. \end{cases}$$

(4e) *Compare the solutions of the two Riemann problems studied in subproblems (4c)–(4d). Comment on your findings in light of subproblem (4a) and try to find an explanation for the baffling mismatch.

We now want to discretize (4.1) (with some initial conditions) using a Finite Volume scheme in space and the *implicit Euler* scheme in time. To this aim, we solve the PDE (4.1) in the interval $[0, 1]$, considering the discrete points $x_j = \left(j + \frac{1}{2}\right)h$ for $j = 0, \ldots, N$, where $h = \frac{1}{N+1}$; we also define the midpoint values $x_{j-\frac{1}{2}} = x_j - \frac{h}{2} = jh$, for $j = 0, \ldots, N$ (the extension of the solution to the whole space $\mathbb{R}$ can be done using periodic boundary conditions, but you don't have to worry about this). The midpoint values define the control volumes

$$\mathcal{C}_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}),$$

with x_j the center of the control volume $\mathcal{C}_j$, for $j = 0, \ldots, N$.

To discretize the time domain $[0, T]$, consider $M \in \mathbb{N}$ equispaced time points

$$t_0 = 0, t_1, \ldots, t_M = T,$$

and denote the time step size by $\Delta t := |t_{n+1} - t_n| = T/M$.

We denote by U_j^n the approximate cell average of u in the control volume $\mathcal{C}_j$ and the time t_n, i.e.

$$U_j^n \approx \frac{1}{h} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x, t^n) \, \mathrm{d}x. \tag{4.3}$$

At the control volume interfaces, we use the Lax-Friedrichs numerical flux

$$F_{j+\frac{1}{2}}^n(U_j, U_{j+1}) = \frac{f(U_j^n) + f(U_{j+1}^n)}{2} - \frac{h}{2\Delta t}(U_{j+1}^n - U_j^n). \tag{4.4}$$

(4f) Using the indications above, write the fully discrete and stable numerical scheme to compute the generic cell average U_j^{n+1} in the control volume $\mathcal{C}_j$, $j = 1, \ldots, N$ (i.e. in a control volume not containing a boundary point), at time t^{n+1}, $n = 1, \ldots, M$ (so that you don't have to care about initial conditions).

Problem 5 Transport in One Dimension [15 points]

Consider the one-dimensional linear transport equation:

$$\begin{aligned} U_t + (a(x)U)_x &= 0, \qquad \forall (x,t) \in \mathbb{R} \times [0,1], \\ U(x,0) &= U_0(x), \qquad \forall x \in \mathbb{R}, \end{aligned} \tag{5.1}$$

with coefficient $a(x) \in C^1(\mathbb{R})$.

(5a) Write down the equation for characteristics of (5.1). Use it to derive an expression for the exact solution.

HINT: Assume that a is an increasing function of x.

(5b) Let $U(x,t)$ be a smooth solution of (5.1), that decays to zero at infinity. Then show that U satisfies the energy bound

$$\int_{\mathbb{R}} U^2(x,T)dx \leq e^{CT} \int_{\mathbb{R}} U_0^2(x)dx, \tag{5.2}$$

for all $T > 0$, with constant C depending on $\|a\|_{C^1}$.

HINT: The Gronwall's inequality may come into help: *Let $\beta(t)$ be continuous and $u(t)$ be differentiable on some interval $[a,b]$, and assume that*

$$u'(t) \leq \beta(t)u(t) \quad \forall t \in (a,b).$$

Then

$$u(t) \leq u(a) \exp\left(\int_a^t \beta(\tau) \, \mathrm{d}\tau \right) \quad \forall t \in [a,b].$$

(5c) Consider the equation (5.1) on the domain $D = (0,1)$ with periodic boundary conditions and $a = -1$. Denoting by U_j^n the numerical cell average for the generic cell j at a generic time t_n, write a stable numerical scheme to simulate (5.1) (given suitable initial conditions $u(x, T = 1) = u_0(x)$).