# Sample Size Calculations

## Practical Methods for Engineers and Scientists

(Software solutions to selected example problems, Version: 17 August 2010)

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(Software solutions to selected example problems, Version: 17 August 2010)

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Sample Size Calculations: Practical Methods for Engineers and Scientists
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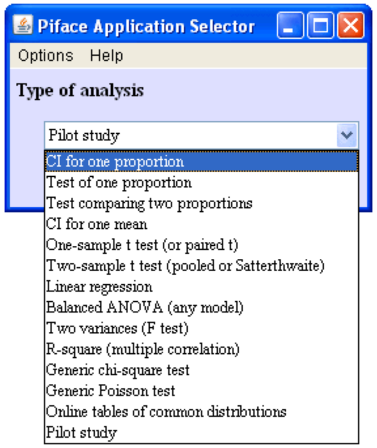
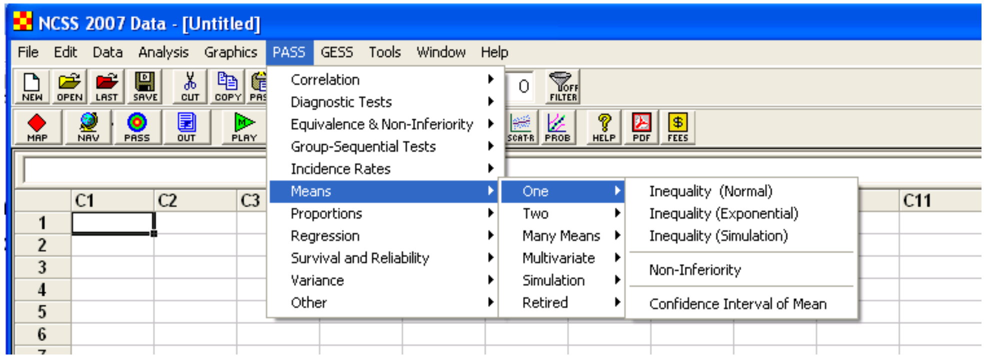
# Contents

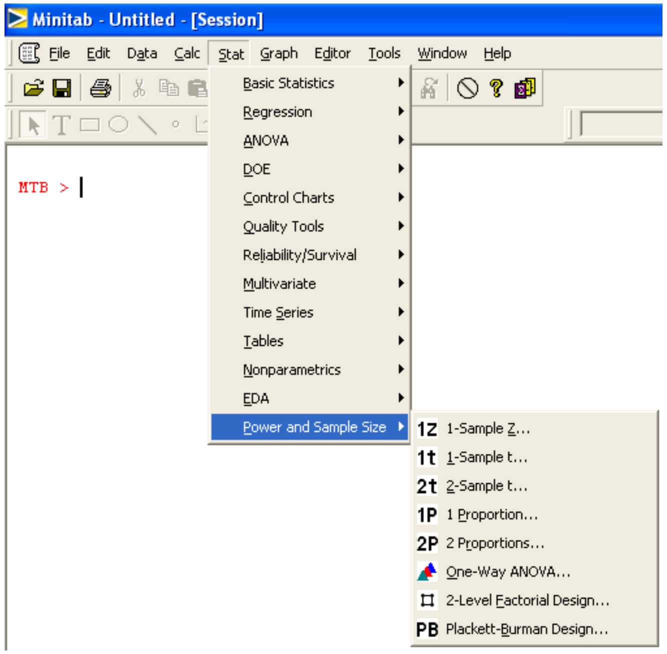
The purpose of this document is to present solutions to selected example problems from the book using PASS (2005), MINITAB (V15 and V16), Piface (V1.72), and R. (This version of the document, compiled on 17 August 2010, does not yet contain solutions using R.) All of the programs have more sample size and power calculation capabilities than what is included in the book. Some packages are particularly strong in certain areas. For example, PASS has the broadest scope, Piface offers an unmatched collection of ANOVA methods involving fixed, random, mixed, and nested designs and supports custom ANOVA models, and MINITAB has special methods for quality engineers including attribute and variables sampling plan design and reliability study design.

The following figures show screen captures of some of the methods available in Piface, PASS, and MINITAB.

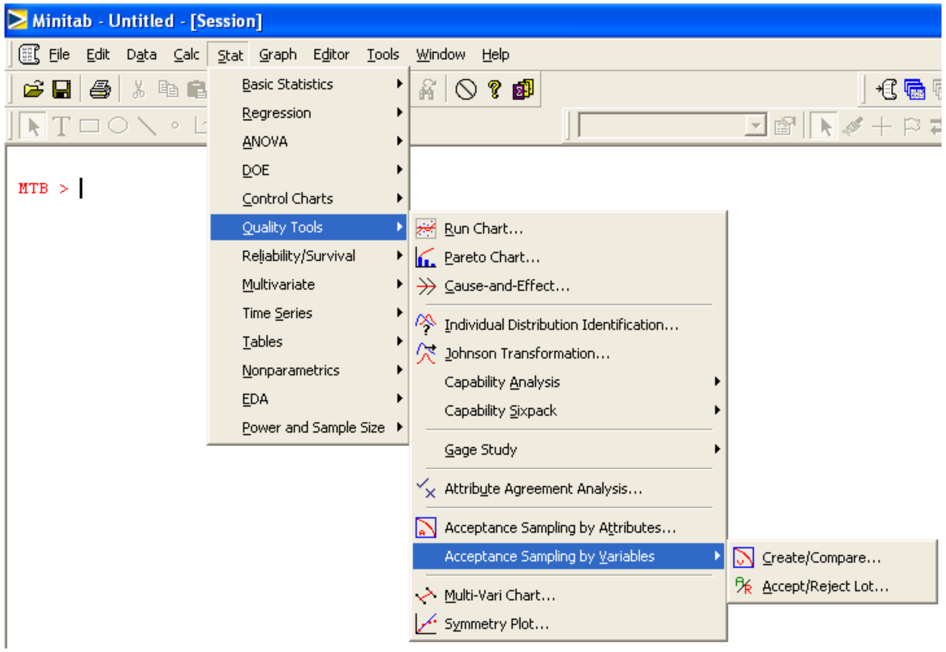


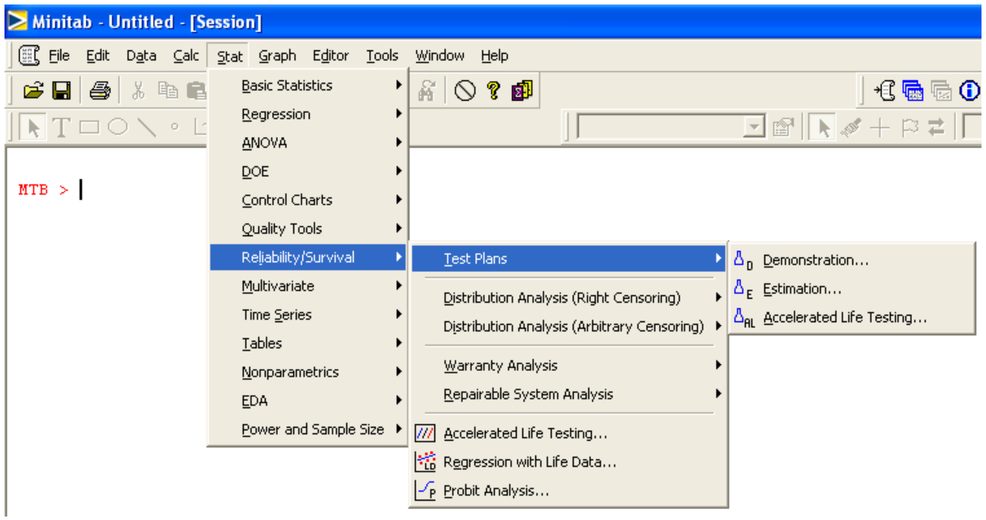



MINITAB V15:

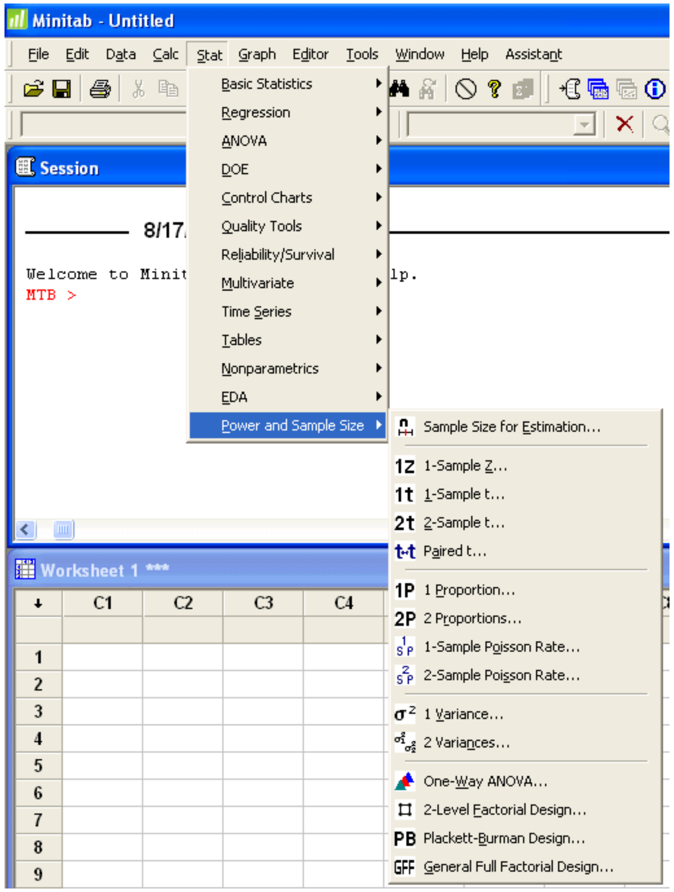

Minitab - Untitled - [Session]
File Edit Data Calc Stat Graph Editor Tools Window Help
MTB >
Basic Statistics
Regression
ANOVA
DOE
Control Charts
Quality Tools
Reliability/Survival
Multivariate
Time Series
Tables
Nonparametrics
EDA
Power and Sample Size
1Z 1-Sample Z...
1t 1-Sample t...
2t 2-Sample t...
1P 1 Proportion...
2P 2 Proportions...
One-Way ANOVA...
2-Level Factorial Design...
PB Plackett-Burman Design...

MINITAB V16:




Most of the programs emphasize sample size and power calculations for hypothesis tests but they can be tricked into performing approximate sample size calculations for


Minitab - Untitled
File Edit Data Calc Stat Graph Editor Tools Window Help Assistant
Session
Welcome to Minit
MTB >
Basic Statistics
Regression
ANOVA
DOE
Control Charts
Quality Tools
Reliability/Survival
Multivariate
Time Series
Tables
Nonparametrics
EDA
Power and Sample Size
Sample Size for Estimation...
1-Sample Z...
1-Sample t...
2-Sample t...
Paired t...
1 Proportion...
2 Proportions...
1-Sample Poisson Rate...
2-Sample Poisson Rate...
1 Variance...
2 Variances...
One-Way ANOVA...
2-Level Factorial Design...
Plackett-Burman Design...
General Full Factorial Design...
Worksheet 1 ***
C1
C2
C3
C4

confidence intervals by setting the hypothesis test power to $\pi = 0.50$. This trick is exact when the sampling distribution is normal because $z_{0.50} = 0$ and it is reasonably accurate when the sampling distribution is other than normal but symmetric. Be more careful when the sampling distribution is asymmetric.

The solutions in the book don't use the continuity correction when discrete distributions are approximated with continuous ones, however, the software solutions often include the continuity correction so answers to problems may differ slightly. If you have access to software that provides more accurate methods, then definitely use the software.

Some software provides several analysis methods for the same problem. For example, PASS offers six different methods for the significance test for one proportion expressed in terms of the proportion difference. The different methods usually give similar answers.

This document will be revised occasionally. The current version was compiled on 17 August 2010.

# Chapter 1

# Fundamentals

## 1.1 Motivation for Sample Size Calculations

## 1.2 Rationale for Sample Size and Power Calculations

**Example 1.1** Express the confidence interval $P(3.1 < \mu < 3.7) = 0.95$ in words.
**Solution:** The confidence interval indicates that we can be 95% confident that the true but unknown value of the population mean $\mu$ falls between $\mu = 3.1$ and $\mu = 3.7$. Apparently, the mean of the sample used to construct the confidence interval is $\bar{x} = 3.4$ and the confidence interval half-width is $\delta = 0.3$.

**Example 1.2** Data are to be collected for the purpose of estimating the mean of a mechanical measurement. Data from a similar process suggest that the standard deviation will be $\sigma_x = 0.003mm$. Determine the sample size required to estimate the value of the population mean with a 95% confidence interval of half-width $\delta = 0.002mm$.
**Solution:** With $z_{\alpha/2} = z_{0.025} = 1.96$ in Equation 1.4, the required sample size is

$$n = \left(\frac{1.96 \times 0.003}{0.002}\right)^2 = 8.64.$$

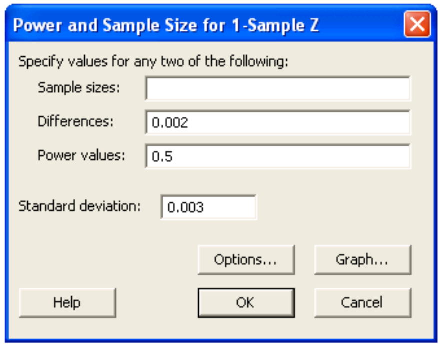
The sample size must be an integer; therefore, we round the calculated value of $n$ up to $n = 9$.

From **MINITAB**> **Stat**> **Power and Sample Size**> **1-Sample Z**:

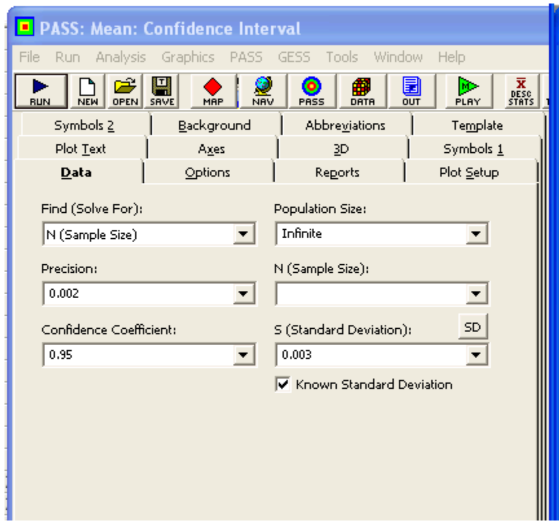
```
MTB > Power;
SUBC>   ZOne;
SUBC>     Difference 0.002;
SUBC>     Power 0.5;
SUBC>     Sigma 0.003;
SUBC>   GPCurve.
```

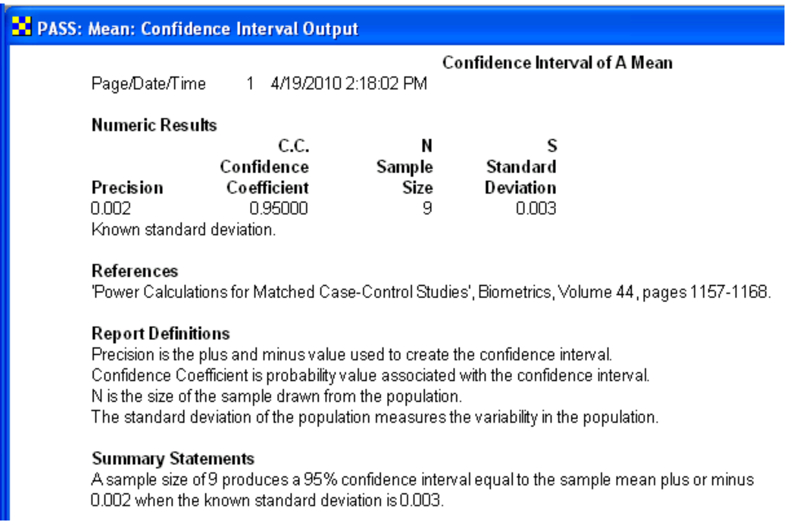
**Power and Sample Size**

```
1-Sample Z Test

Testing mean = null (versus not = null)
Calculating power for mean = null + difference
Alpha = 0.05  Assumed standard deviation = 0.003


            Sample  Target
Difference    Size   Power  Actual Power
     0.002       9     0.5      0.516005
```




From **PASS**> **Means**> **One**> **Confidence Interval of Mean**:



PASS: Mean: Confidence Interval Output

**Confidence Interval of A Mean**

Page/Date/Time 1 4/19/2010 2:18:02 PM

**Numeric Results**

| **Precision** | **C.C. Confidence Coefficient** | **N Sample Size** | **S Standard Deviation** |
|---|---|---|---|
| 0.002 | 0.95000 | 9 | 0.003 |

Known standard deviation.

**References**

'Power Calculations for Matched Case-Control Studies', Biometrics, Volume 44, pages 1157-1168.

**Report Definitions**

Precision is the plus and minus value used to create the confidence interval.
Confidence Coefficient is probability value associated with the confidence interval.
N is the size of the sample drawn from the population.
The standard deviation of the population measures the variability in the population.

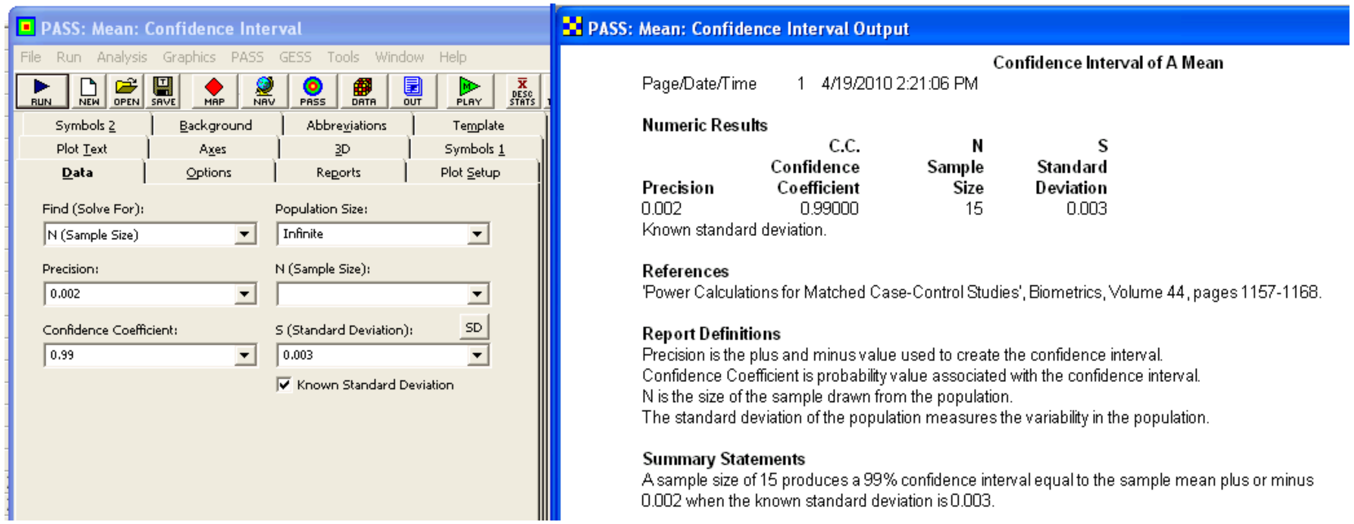
**Summary Statements**

A sample size of 9 produces a 95% confidence interval equal to the sample mean plus or minus 0.002 when the known standard deviation is 0.003.

**Example 1.3** What is the new sample size in Example 1.2 if the process owner prefers a 99% confidence interval?
**Solution:** With $z_{\alpha/2} = z_{0.005} = 2.575$ in Equation 1.4 the required sample size is

$$n = \left(\frac{2.575 \times 0.003}{0.002}\right)^2 = 15.$$

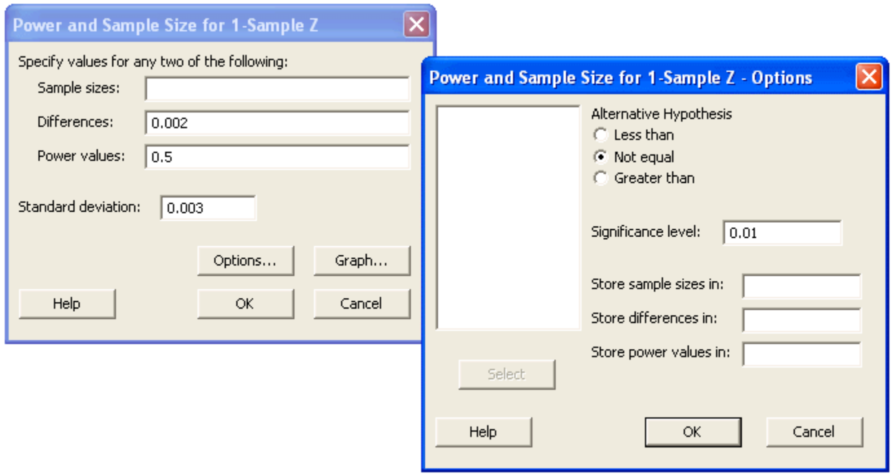
From **MINITAB**> **Stat**> **Power and Sample Size**> **1-Sample Z**:




```
MTB > Power;
SUBC>   ZOne;
SUBC>     Difference 0.002;
SUBC>     Power 0.5;
SUBC>     Sigma 0.003;
SUBC>     Alpha 0.01;
SUBC>   GPCurve.
```

**Power and Sample Size**

```
1-Sample Z Test

Testing mean = null (versus not = null)
Calculating power for mean = null + difference
Alpha = 0.01  Assumed standard deviation = 0.003


            Sample  Target
Difference    Size   Power  Actual Power
     0.002      15     0.5      0.502457
```




From **PASS**> **Means**> **One**> **Confidence Interval of Mean**:

**Example 1.4** What is the new sample size in Example 1.2 if the process owner prefers a 95% confidence level with $\delta = 0.001mm$ half-width?
**Solution:** With $z_{0.025} = 1.96$ and $\delta = 0.001mm$ in Equation 1.4 the required sample size is

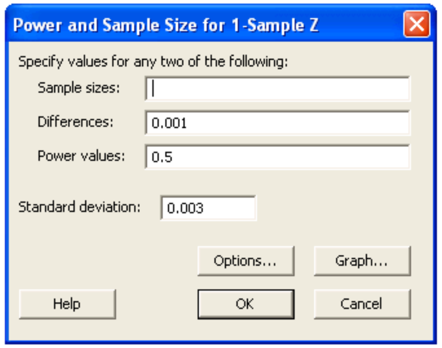
$$n = \left(\frac{1.96 \times 0.003}{0.001}\right)^2 = 35.$$

From **MINITAB**> **Stat**> **Power and Sample Size**> **1-Sample Z**:

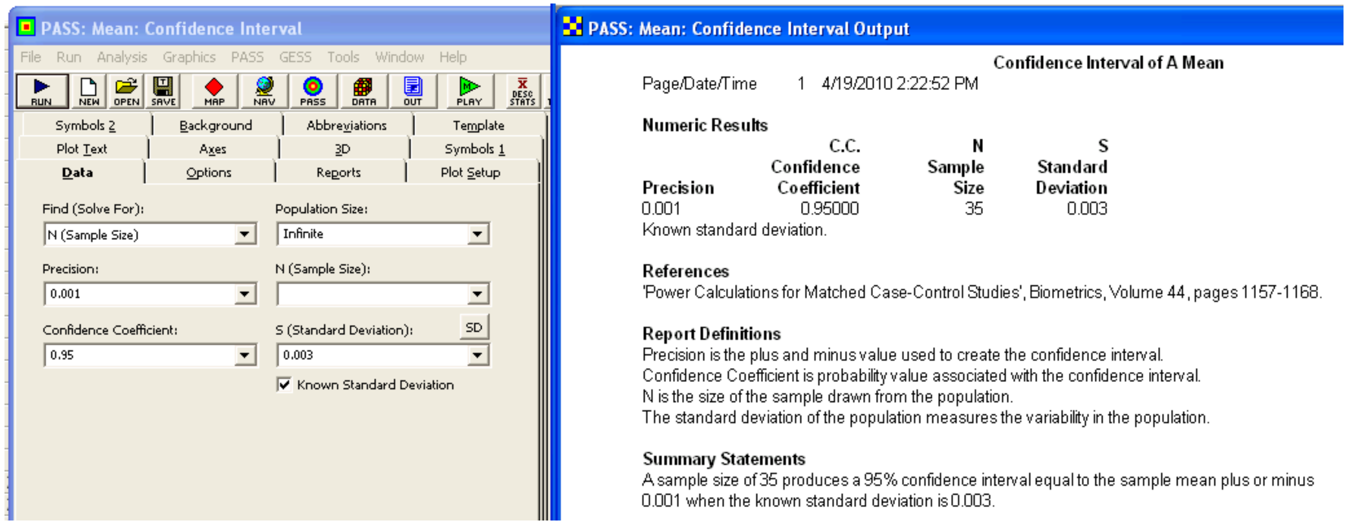
```
MTB > Power;
SUBC>   ZOne;
SUBC>     Difference 0.001;
SUBC>     Power 0.5;
SUBC>     Sigma 0.003;
SUBC>   GPCurve.
```

**Power and Sample Size**

```
1-Sample Z Test

Testing mean = null (versus not = null)
Calculating power for mean = null + difference
Alpha = 0.05  Assumed standard deviation = 0.003


            Sample  Target
Difference    Size   Power  Actual Power
     0.001      35     0.5      0.504854
```




From **PASS**> **Means**> **One**> **Confidence Interval of Mean**:




**Example 1.5** Determine the sample size required to estimate the mean of a population when $\sigma_x = 30$ is known and the population mean must not exceed the sample mean by more than $\delta = 10$ with 95% confidence.
**Solution:** A one-sided upper 95% confidence interval is required of the form

$$P\left(\mu < \bar{x} + \delta\right) = 0.95.$$

With $z_{0.05} = 1.645$ in Equation 1.8 the necessary sample size is

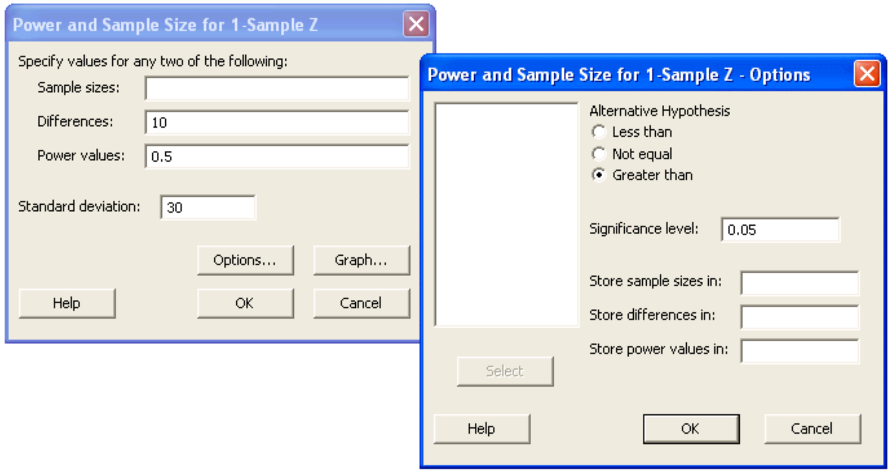
$$n = \left(\frac{1.645 \times 30}{10}\right)^2 = 25.$$

From **MINITAB**> **Stat**> **Power and Sample Size**> **1-Sample Z**:

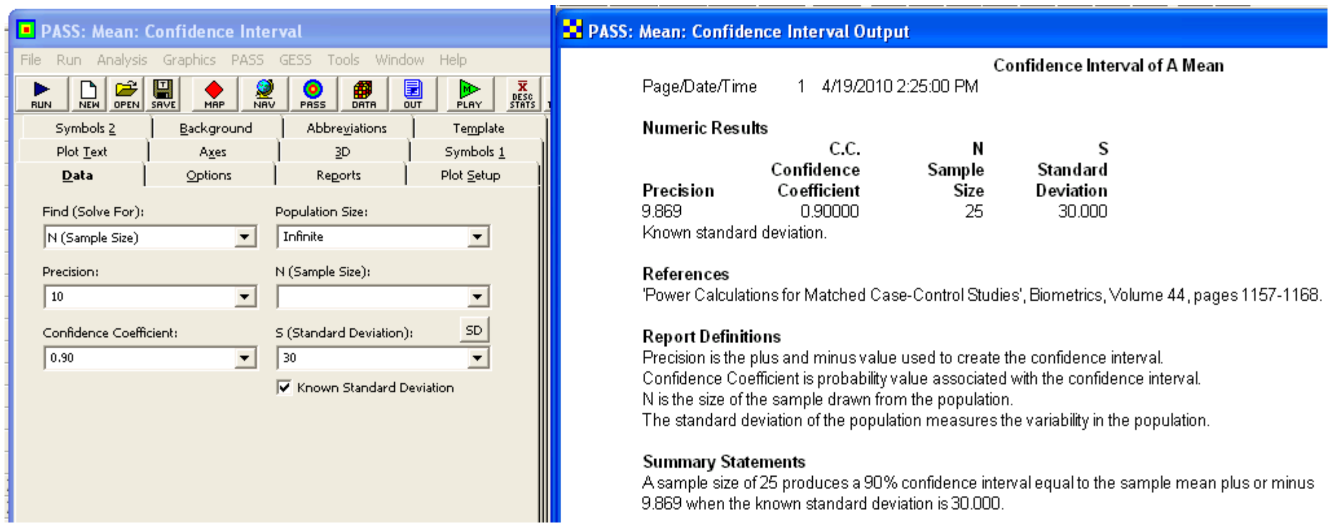
```
MTB > Power;
SUBC>   ZOne;
SUBC>     Difference 10;
SUBC>     Power 0.5;
SUBC>     Sigma 30;
SUBC>     Alternative 1;
SUBC>   GPCurve.
```

**Power and Sample Size**

```
1-Sample Z Test

Testing mean = null (versus > null)
Calculating power for mean = null + difference
Alpha = 0.05  Assumed standard deviation = 30


            Sample  Target
Difference    Size   Power  Actual Power
        10      25     0.5      0.508701
```




From **PASS**> **Means**> **One**> **Confidence Interval of Mean**:



## 1.3 Rationale for Hypothesis Tests

**Example 1.6** An experiment is planned to test the hypotheses $H_0 : \mu = 3200$ versus $H_A : \mu \neq 3200$. The process is known to be normally distributed with standard deviation $\sigma_x = 400$. What sample size is required to detect a practically significant shift in the process mean of $\delta = 300$ with power $\pi = 0.90$?
**Solution:** With $\beta = 1 - \pi = 0.10$ and assuming $\alpha = 0.05$ in Equation 1.12, the sample size required to detect a shift from $\mu = 3200$ to $\mu = 2900$ or $\mu = 3500$ with 90% power is

$$\begin{aligned} n &= \left( \frac{(z_{0.025} + z_{0.10})\, \sigma_x}{\delta} \right)^2 \\ &= \left( \frac{(1.96 + 1.282)\, 400}{300} \right)^2 \\ &= 19 \end{aligned}$$

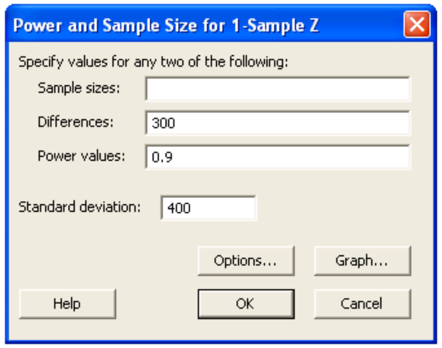
where the calculated value of $n$ was rounded up to the nearest integer value.

From **MINITAB**> **Stat**> **Power and Sample Size**> **1-Sample Z**:

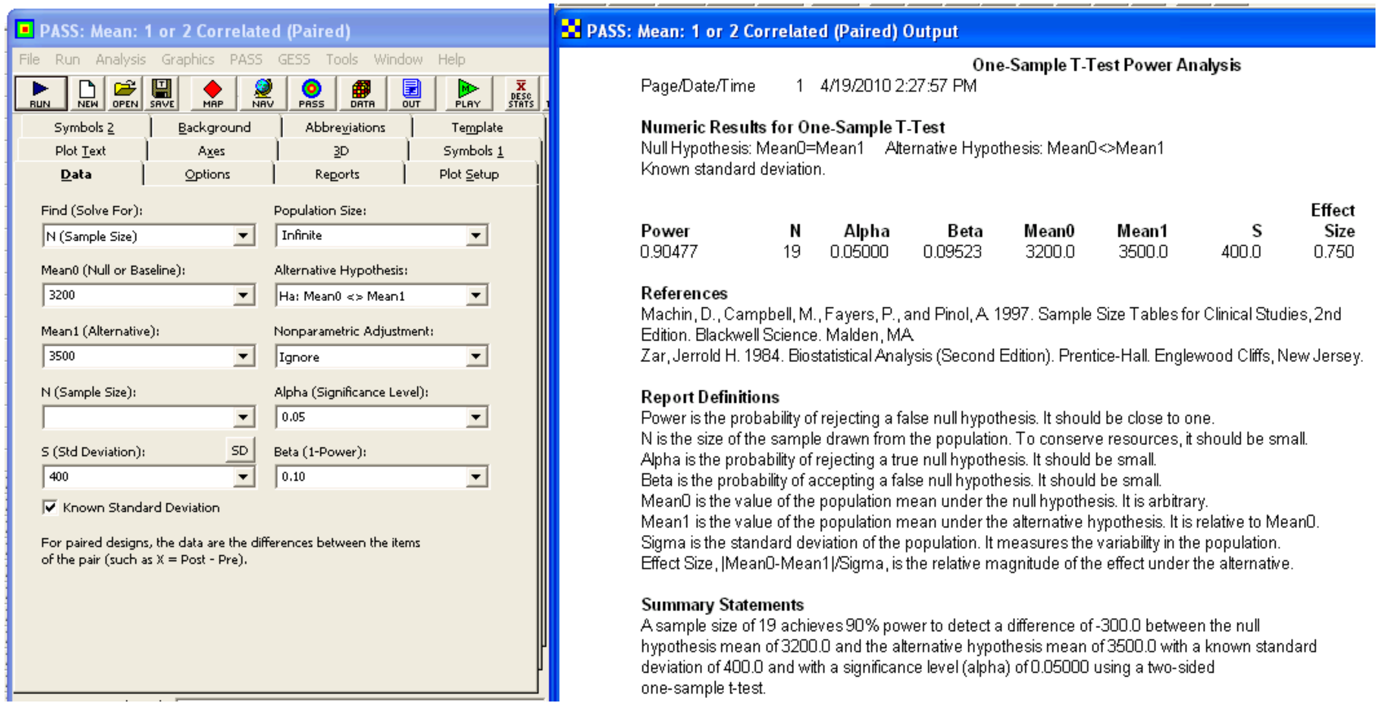
```
MTB > Power;
SUBC>   ZOne;
SUBC>     Difference 300;
SUBC>     Power 0.9;
SUBC>     Sigma 400;
SUBC>   GPCurve.
```

**Power and Sample Size**

```
1-Sample Z Test

Testing mean = null (versus not = null)
Calculating power for mean = null + difference
Alpha = 0.05  Assumed standard deviation = 400


            Sample  Target
Difference    Size   Power  Actual Power
       300      19     0.9      0.904769
```




From **PASS**> **Means**> **One**> **Inequality (Normal)**:




**Example 1.7** An experiment will be performed to test $H_0 : \mu = 8.0$ versus $H_A : \mu > 8.0$. What sample size is required to reject $H_0$ with 90% power when $\mu = 8.2$? The process is known to be normally distributed with $\sigma_x = 0.2$.
**Solution:** For the one-tailed hypothesis test with $\alpha = 0.05$, $\delta = 0.2$ and $\beta = 1 - \pi = 0.10$ the required sample size is

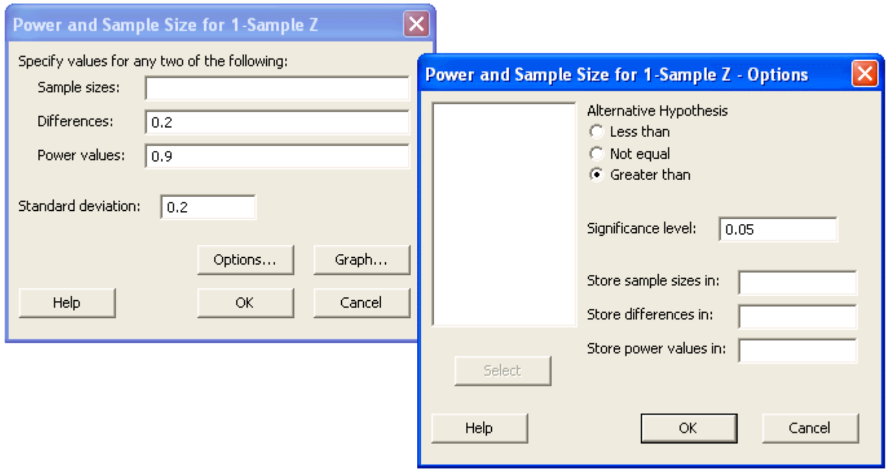
$$\begin{aligned} n &= \left(\frac{(z_\alpha + z_\beta)\,\sigma_x}{\delta}\right)^2 \\ &= \left(\frac{(z_{0.05} + z_{10})\,\sigma_x}{\delta}\right)^2 \\ &= \left(\frac{(1.645 + 1.282)\,0.2}{0.2}\right)^2 \\ &= 9. \end{aligned}$$

From **MINITAB**> **Stat**> **Power and Sample Size**> **1-Sample Z**:

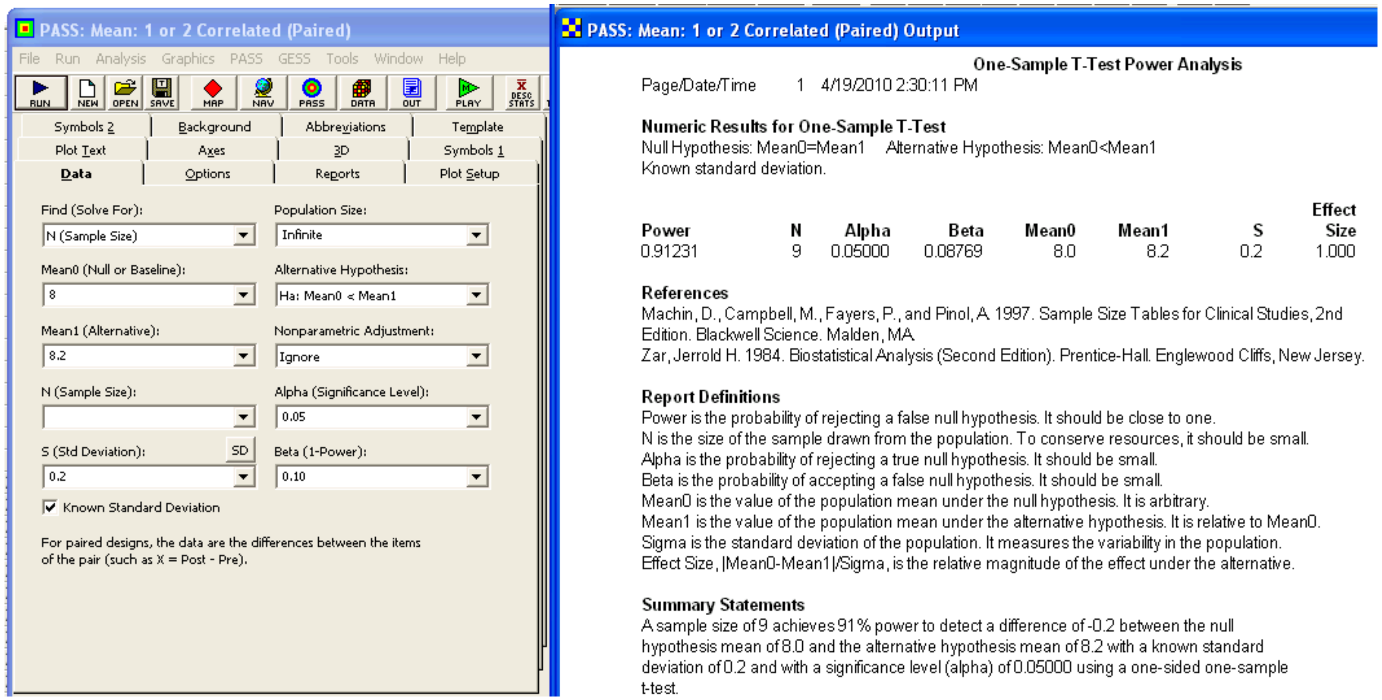
```
MTB > Power;
SUBC>   ZOne;
SUBC>     Difference 0.2;
SUBC>     Power 0.9;
SUBC>     Sigma 0.2;
SUBC>     Alternative 1;
SUBC>   GPCurve.
```

**Power and Sample Size**

```
1-Sample Z Test

Testing mean = null (versus > null)
Calculating power for mean = null + difference
Alpha = 0.05  Assumed standard deviation = 0.2


            Sample  Target
Difference    Size   Power  Actual Power
       0.2       9     0.9      0.912315
```




From **PASS**> **Means**> **One**> **Inequality (Normal)**:




**Example 1.8** Calculate the $p$ value for the test performed under the conditions of Example 1.6 if the sample mean was $\bar{x} = 3080$.

**Solution:** Figure 1.4 shows the contributions to the $p$ value from the two tails of the $\bar{x}$ distribution under $H_0$. The $z$ test statistic that corresponds to $\bar{x}$ is

$$\begin{aligned} z &= \frac{\bar{x}-\mu_0}{\sigma_{\bar{x}}} \\ &= \frac{\bar{x}-\mu_0}{\sigma_x/\sqrt{n}} \\ &= \frac{3080-3200}{400/\sqrt{19}} \\ &= -1.31, \end{aligned}$$

so the $p$ value is

$$\begin{aligned} p &= 1-\Phi\left(-1.31 < z < 1.31\right) \\ &= 0.19. \end{aligned}$$

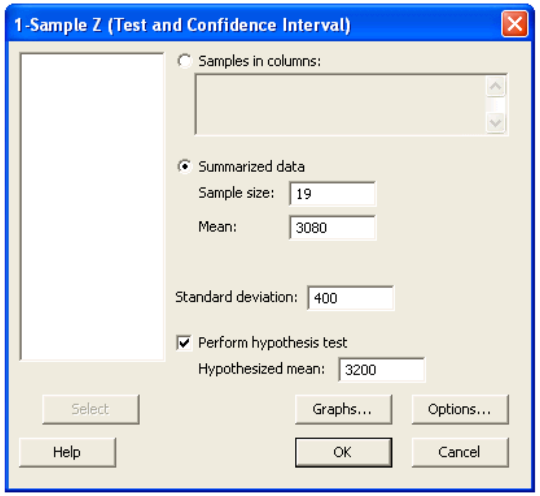
Because $(p = 0.19) > (\alpha = 0.05)$, the observed sample mean is statistically consistent with $H_0 : \mu = 3200$, so we can not reject $H_0$.

From **MINITAB**> **Stat**> **Power and Sample Size**> **1-Sample Z**:

```
MTB > OneZ 19 3080;
SUBC>   Sigma 400;
SUBC>   Test 3200.
```

**One-Sample Z**

```
Test of mu = 3200 vs not = 3200
The assumed standard deviation = 400


 N    Mean  SE Mean        95% CI            Z      P
19  3080.0     91.8  (2900.1, 3259.9)  -1.31  0.191
```




**Example 1.9** Calculate the $p$ value for the test performed under the conditions of Example 1.7 if the sample mean was $\bar{x} = 8.39$.

**Solution:** Figure 1.5 shows the single contribution to the $p$ value from the right tail of the $\bar{x}$ distribution under $H_0$. The $z$ test statistic that corresponds to $\bar{x}$ is

$$\begin{aligned} z &= \frac{8.39-8.2}{0.2/\sqrt{9}} \\ &= 2.85, \end{aligned}$$

so the $p$ value is

$$\begin{aligned} p &= \Phi(2.85 < z < \infty) \\ &= 0.0022. \end{aligned}$$

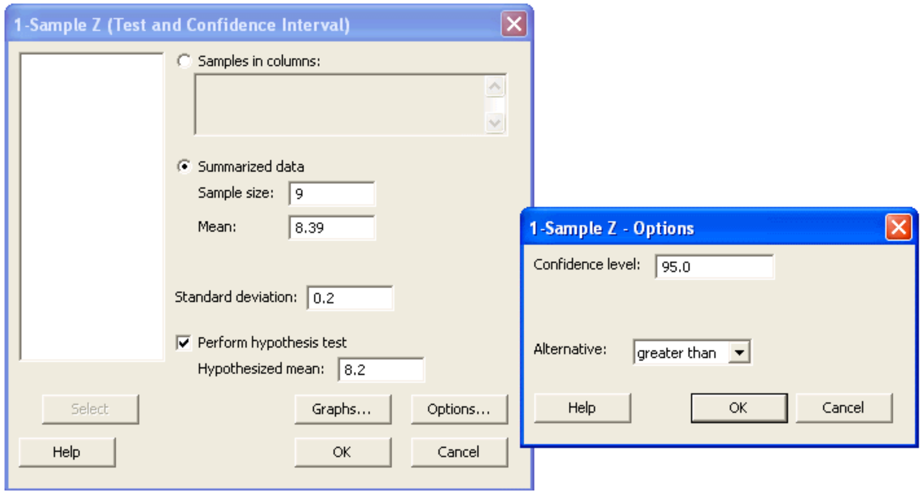
Because $(p = 0.0022) < (\alpha = 0.05)$, the observed sample mean is an improbable result under $H_0 : \mu = 8.2$, so we must reject $H_0$.

From **MINITAB**> **Stat**> **Power and Sample Size**> **1-Sample Z**:

```
MTB > OneZ 9 8.39;
SUBC>   Sigma 0.2;
SUBC>   Test 8.2;
SUBC>   Alternative 1.
```

**One-Sample Z**

```
Test of mu = 8.2 vs > 8.2
The assumed standard deviation = 0.2


                      95% Lower
N    Mean  SE Mean       Bound     Z      P
9  8.3900   0.0667      8.2803  2.85  0.002
```

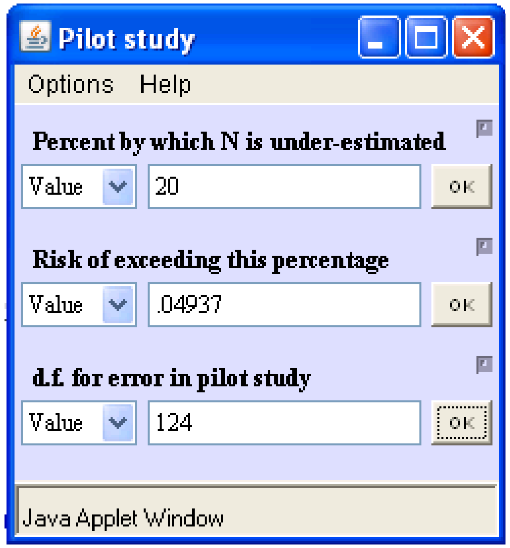



## 1.4 Practical Considerations

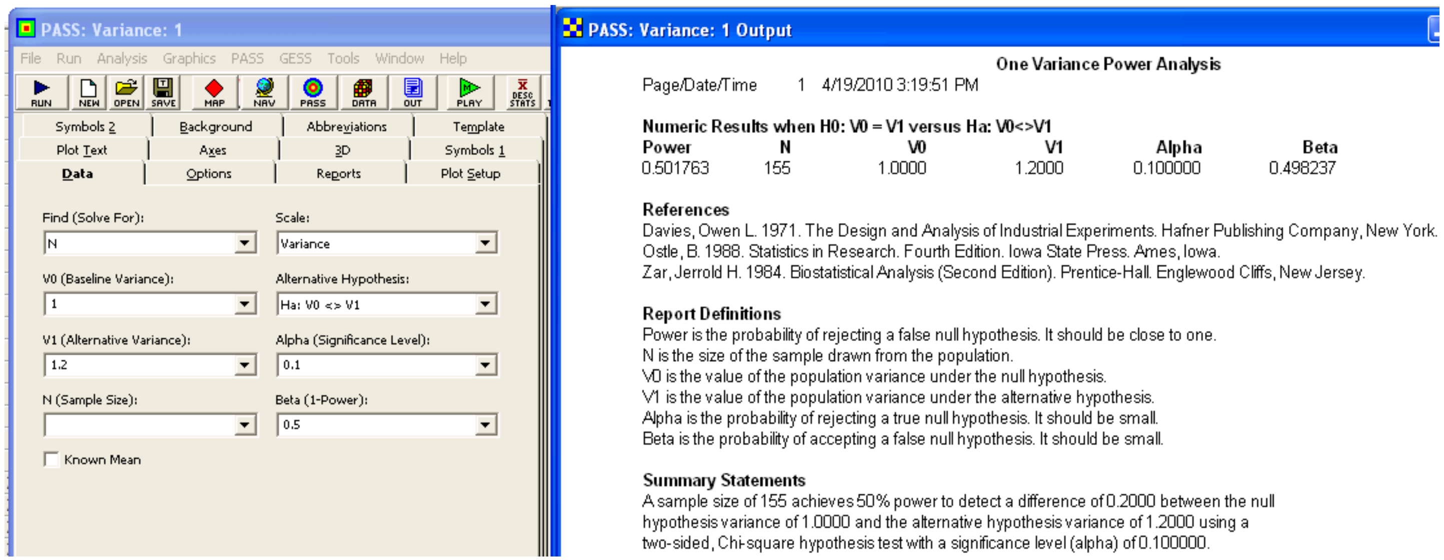
**Example 1.10** What sample size is required for a pilot study to estimate the standard deviation to be used in the sample size calculation for a primary experiment if the sample size for the primary experiment should be within 20% of the correct value with 90% confidence?
**Solution:** With $\delta = 0.20$ and $\alpha = 0.10$ in Equation 1.20, the required sample size for the preliminary experiment to estimate the standard deviation is

$$\begin{aligned} n &\simeq 2\left(\frac{1.645}{0.20}\right)^2 \\ &\simeq 136. \end{aligned}$$

From **Piface**> **Pilot Study**:




From **PASS**> **Variance**> **Variance: 1 Group**:




**Example 1.11** An engineer must obtain approval from his manager to test a certain number of units to determine the mean response for a validation study. The standard deviation of the response is $\sigma_x = 600$ and the smallest practically significant shift in the mean that the experiment should detect is understood to be $\delta = 400$. What graph should the engineer use to present his case?

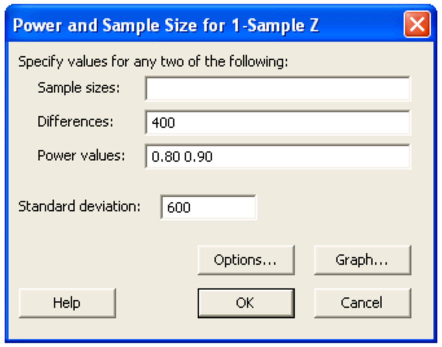
**Solution:** The value of the effect size of interest is firm at $\delta = 400$. The sample size is going to affect the power of the test, so an appropriate graph is power versus sample size. The sample size required to obtain a specified value of power for the test of $H_0 : \delta = 0$ versus $H_A : \delta \neq 0$ is given by Equation 1.12. Figure 1.6 shows the resulting power curve. The sample size required to obtain 80% power is $n = 18$ and the sample size required for 90% power is $n = 24$.

From **MINITAB**> **Stat**> **Power and Sample Size**> **1-Sample Z**:

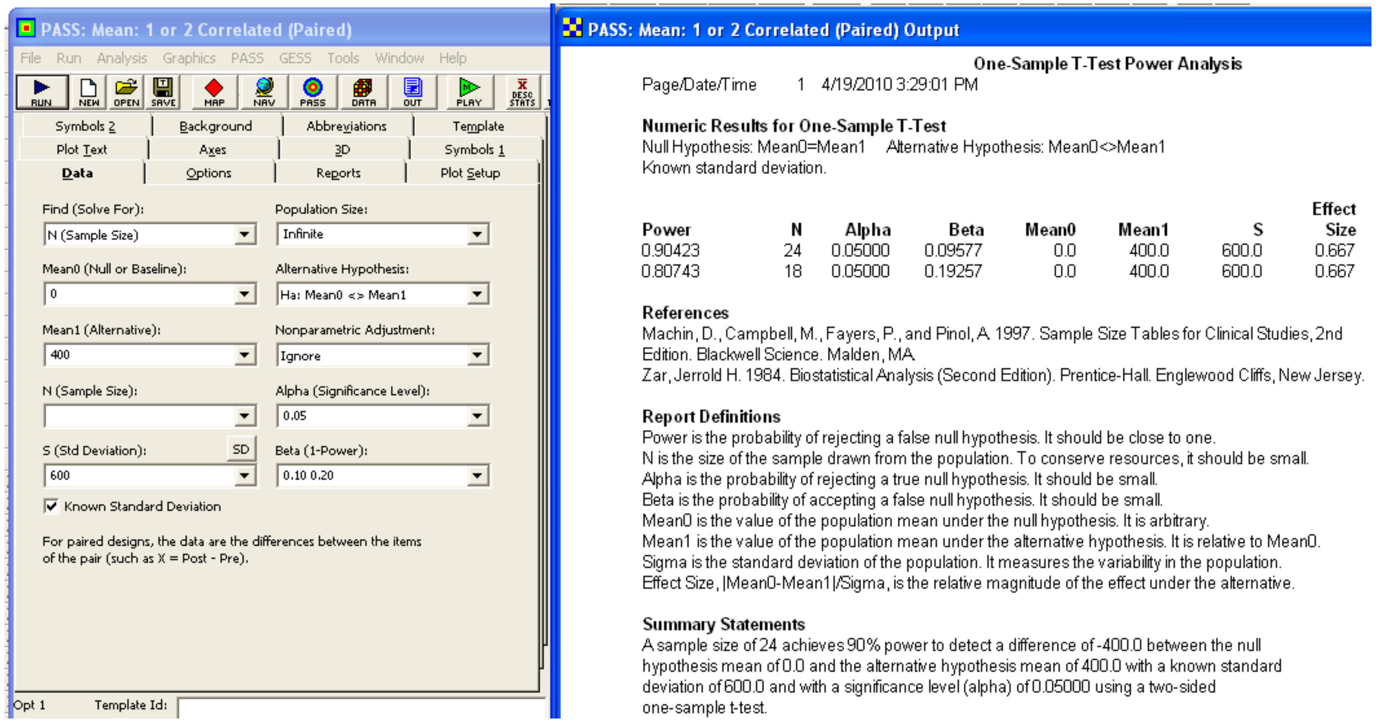
```
MTB > Power;
SUBC>   ZOne;
SUBC>     Difference 400;
SUBC>     Power 0.80 0.90;
SUBC>     Sigma 600;
SUBC>   GPCurve;
SUBC>     NSize 4:40.
```

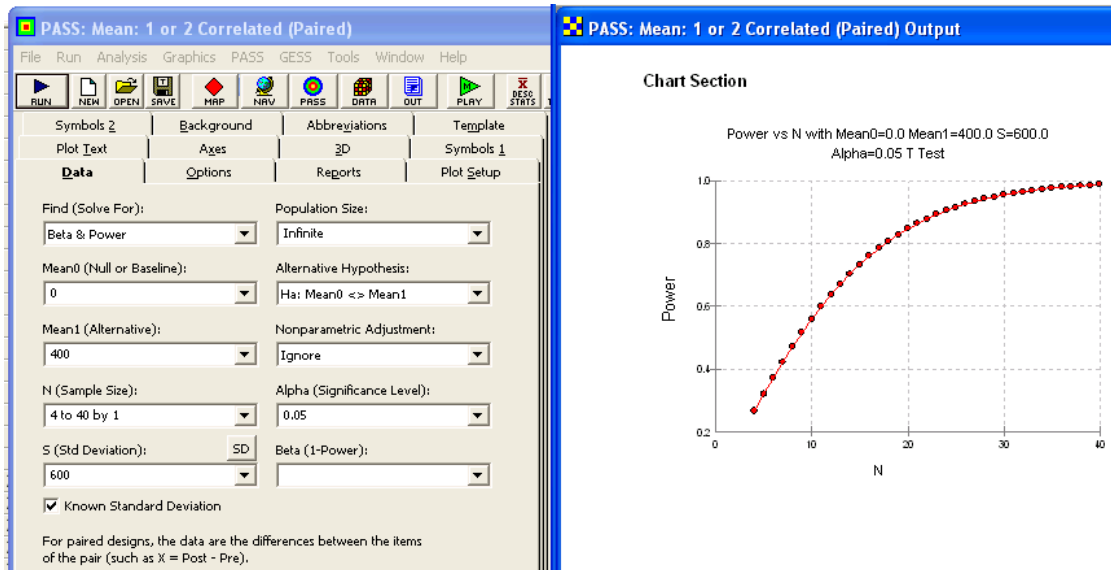
**Power and Sample Size**

```
1-Sample Z Test

Testing mean = null (versus not = null)
Calculating power for mean = null + difference
Alpha = 0.05  Assumed standard deviation = 600


            Sample  Target
Difference    Size   Power  Actual Power
       400      18     0.8      0.807430
       400      24     0.9      0.904228
```




From **PASS**> **Means**> **One**> **Inequality (Normal)**:


PASS: Mean: 1 or 2 Correlated (Paired)
File Run Analysis Graphics PASS GESS Tools Window Help
RUN NEW OPEN SAVE MAP NAV PASS DATA OUT PLAY DESC STATS
Symbols 2 Background Abbreviations Template
Plot Text Axes 3D Symbols 1
Data Options Reports Plot Setup
Find (Solve For):
N (Sample Size)
Population Size:
Infinite
Mean0 (Null or Baseline):
0
Alternative Hypothesis:
Ha: Mean0 <> Mean1
Mean1 (Alternative):
400
Nonparametric Adjustment:
Ignore
N (Sample Size):
Alpha (Significance Level):
0.05
S (Std Deviation): SD
600
Beta (1-Power):
0.10 0.20
Known Standard Deviation
For paired designs, the data are the differences between the items of the pair (such as X = Post - Pre).
Opt 1 Template Id:
PASS: Mean: 1 or 2 Correlated (Paired) Output
One-Sample T-Test Power Analysis
Page/Date/Time 1 4/19/2010 3:29:01 PM
Numeric Results for One-Sample T-Test
Null Hypothesis: Mean0=Mean1 Alternative Hypothesis: Mean0<>Mean1
Known standard deviation.
Power N Alpha Beta Mean0 Mean1 S Effect Size
0.90423 24 0.05000 0.09577 0.0 400.0 600.0 0.667
0.80743 18 0.05000 0.19257 0.0 400.0 600.0 0.667
References
Machin, D., Campbell, M., Fayers, P., and Pinol, A. 1997. Sample Size Tables for Clinical Studies, 2nd Edition. Blackwell Science. Malden, MA.
Zar, Jerrold H. 1984. Biostatistical Analysis (Second Edition). Prentice-Hall. Englewood Cliffs, New Jersey.
Report Definitions
Power is the probability of rejecting a false null hypothesis. It should be close to one.
N is the size of the sample drawn from the population. To conserve resources, it should be small.
Alpha is the probability of rejecting a true null hypothesis. It should be small.
Beta is the probability of accepting a false null hypothesis. It should be small.
Mean0 is the value of the population mean under the null hypothesis. It is arbitrary.
Mean1 is the value of the population mean under the alternative hypothesis. It is relative to Mean0.
Sigma is the standard deviation of the population. It measures the variability in the population.
Effect Size, |Mean0-Mean1|/Sigma, is the relative magnitude of the effect under the alternative.
Summary Statements
A sample size of 24 achieves 90% power to detect a difference of -400.0 between the null hypothesis mean of 0.0 and the alternative hypothesis mean of 400.0 with a known standard deviation of 600.0 and with a significance level (alpha) of 0.05000 using a two-sided one-sample t-test.




**Example 1.12** Suppose that the manager in Example 1.11 approves the use of $n = 24$ units in the validation study. What power does the study have to reject $H_0$ when the effect size is $\delta = 200$, 400, and 600?[7]
**Solution:** The power is given by

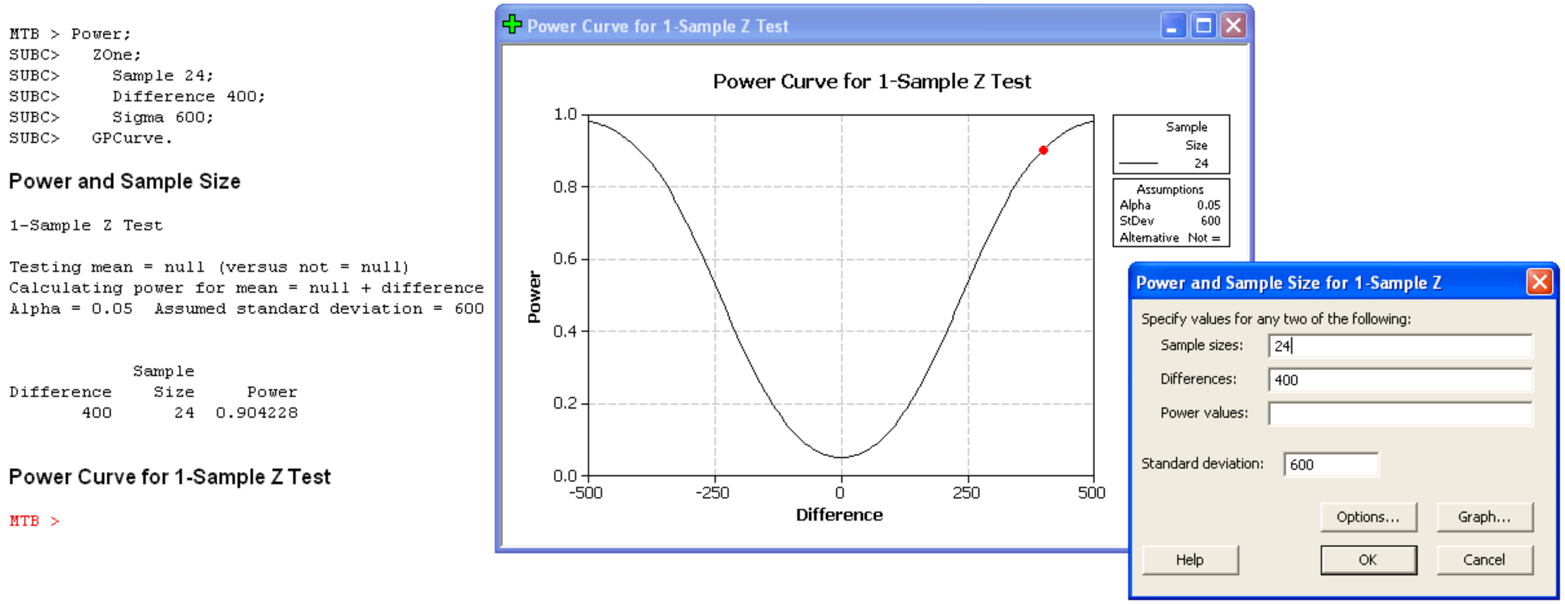
$$\pi = \Phi\left(-z_\beta < z < \infty\right)$$

where $z_\beta$ is determined from Equation 1.12:

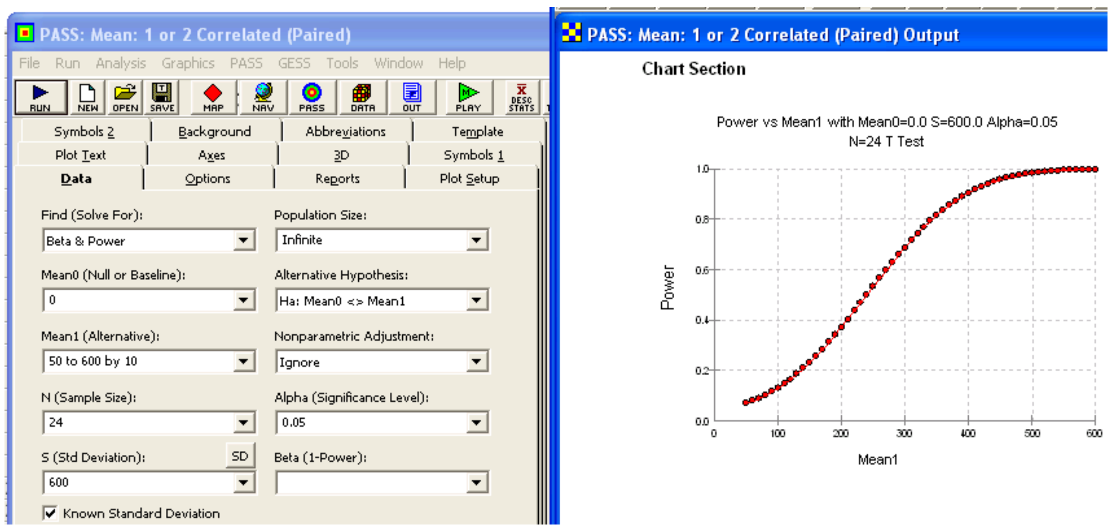
$$z_\beta = \sqrt{n}\frac{\delta}{\sigma_x} - z_{\alpha/2}.$$

Figure 1.7 shows the power as a function of effect size. The power to reject $H_0$ when $\delta = 200$ is $\pi \simeq 0.37$, when $\delta = 400$ is $\pi \simeq 0.90$, and when $\delta = 600$ is $\pi \simeq 1$.

From **MINITAB**> **Stat**> **Power and Sample Size**> **1-Sample Z**:




From **PASS**> **Means**> **One**> **Inequality (Normal)**:



## 1.5 Problems and Solutions

## 1.6 Software

# Chapter 2

# Means

## 2.1 Assumptions

## 2.2 One Mean

**Example 2.1** Find the sample size required to estimate the unknown mean of a population to within $\pm 3$ with 95% confidence if the population standard deviation is known to be $\sigma = 5$.
**Solution:** With $\alpha = 0.05$, $z_{0.025} = 1.96$, and $\delta = 3$ in Equation **??**, the required sample size is

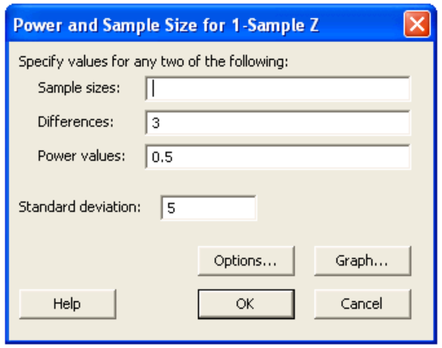
$$\begin{aligned} n &\geq \left(\frac{1.96 \times 5}{3}\right)^2 \\ &\geq 11. \end{aligned}$$

From **MINITAB**> **Stat**> **Power and Sample Size**> **1-Sample Z**:

```
MTB > Power;
SUBC>   ZOne;
SUBC>     Difference 3;
SUBC>     Power 0.5;
SUBC>     Sigma 5;
SUBC>   GPCurve.
```

**Power and Sample Size**

```
1-Sample Z Test

Testing mean = null (versus not = null)
Calculating power for mean = null + difference
Alpha = 0.05  Assumed standard deviation = 5


            Sample  Target
Difference    Size   Power  Actual Power
         3      11     0.5      0.512010
```




**Example 2.2** Find the sample size required to estimate the unknown mean of a population to within $\delta = 3$ measurement units with 95% confidence if the estimated population standard deviation is $\widehat{\sigma} = 5$.
**Solution:** From Equation 2.7 with $t_{0.025} \simeq (z_{0.025} = 1.96)$ in the first iteration,

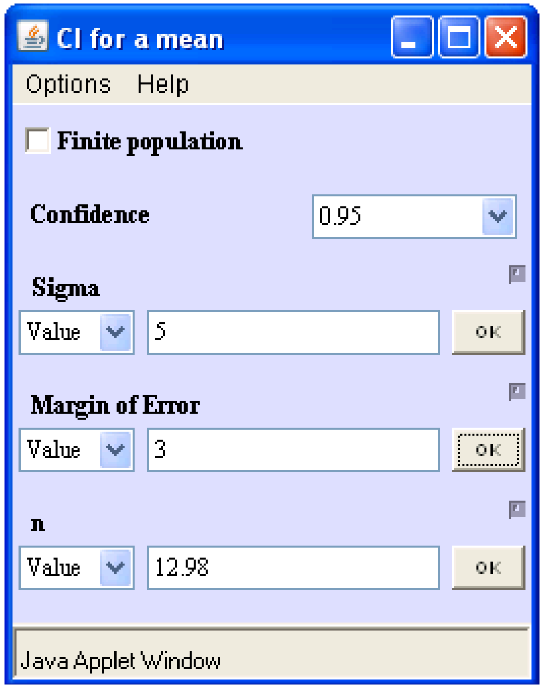
$$n \geq \left(\frac{1.96 \times 5}{3}\right)^2 = 11.$$

In the second iteration with $t_{0.025,10} = 2.228$,

$$n \geq \left(\frac{2.228 \times 5}{3}\right)^2 = 14.$$

Another iteration indicates that $n = 13$ is the smallest sample size that satisfies the sample size condition.

From **Piface**> **CI for one mean**:

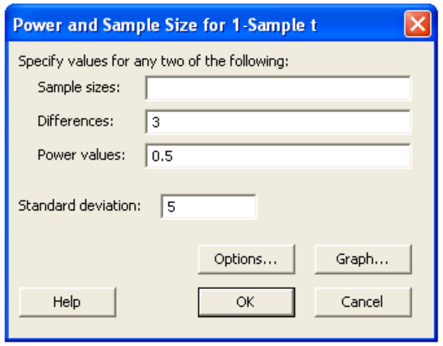



From **MINITAB**> **Stat**> **Power and Sample Size**> **1-Sample t**:

```
MTB > Power;
SUBC>   TOne;
SUBC>     Difference 3;
SUBC>     Power 0.5;
SUBC>     Sigma 5;
SUBC>   GPCurve.
```

**Power and Sample Size**

```
1-Sample t Test

Testing mean = null (versus not = null)
Calculating power for mean = null + difference
Alpha = 0.05  Assumed standard deviation = 5


            Sample  Target
Difference    Size   Power  Actual Power
         3      13     0.5      0.511701
```

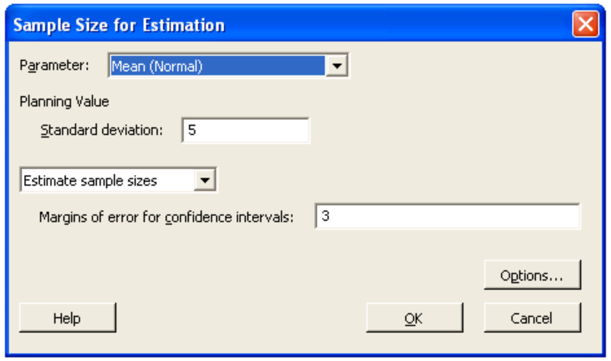



From **MINITAB (V16)**> **Stat**> **Power and Sample Size**> **Sample Size for Estimation**> **Mean (Normal)**:

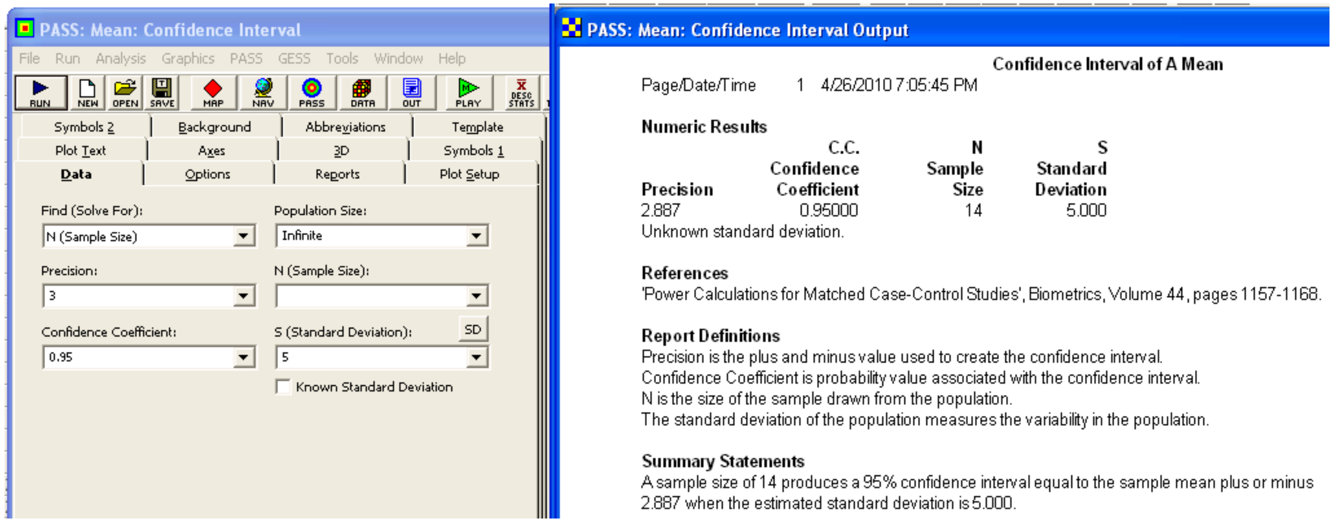
```
MTB > SSCI;
SUBC>   NMean;
SUBC>     Sigma 5;
SUBC>   Confidence 95.0;
SUBC>   IType 0;
SUBC>   MError 3.
```

**Sample Size for Estimation**

```
Method

Parameter             Mean
Distribution          Normal
Standard deviation    5 (estimate)
Confidence level      95%
Confidence interval   Two-sided


Results

  Margin  Sample
of Error    Size
       3      14
```



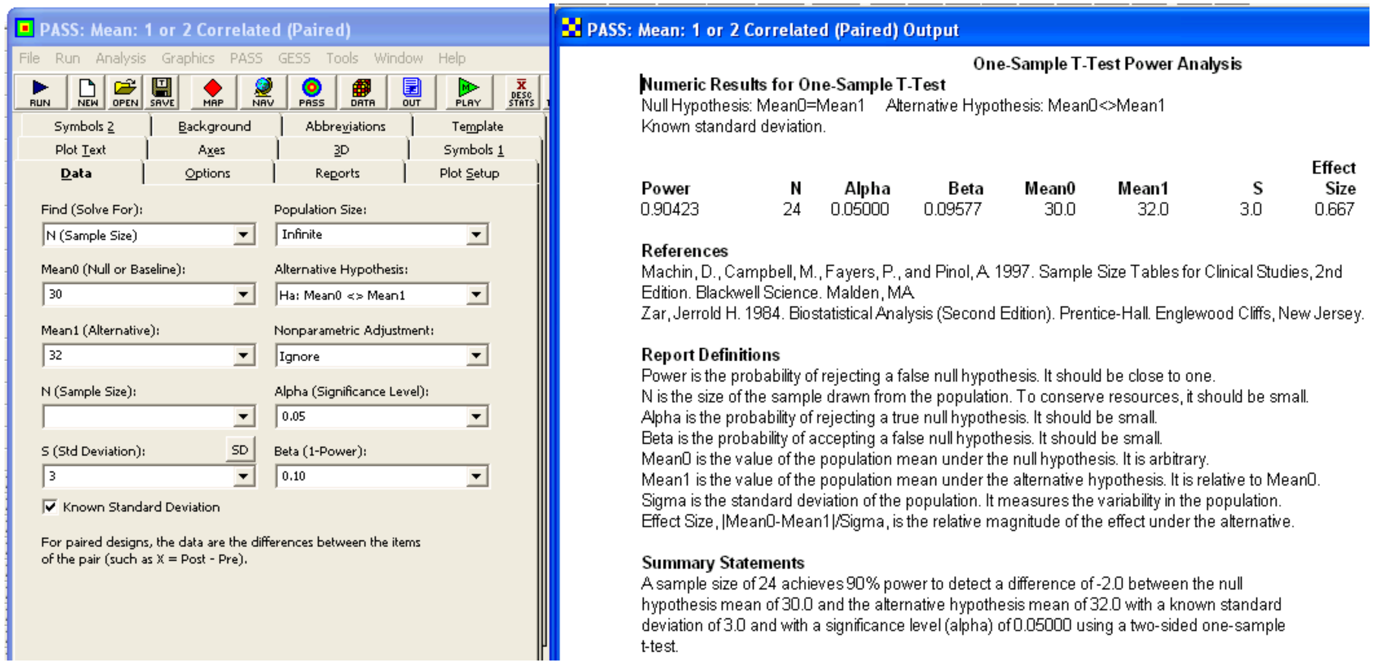

From **PASS> Means> One> Confidence Interval of Mean**:




**Example 2.3** For the one-sample test of $H_0 : \mu = 30$ versus $H_A : \mu \neq 30$ when the population is known to be normal with $\sigma = 3$, what sample size is required to detect a shift to $\mu = 32$ with 90% power?
**Solution:** By Equation 2.16 with $\delta = 2$, $z_{0.025} = 1.96$, and $z_{0.10} = 1.28$, the necessary sample size is

$$n \geq (1.96 + 1.28)^2 \left(\frac{3}{2}\right)^2 = 24.$$

From **PASS**> **Means**> **One**> **Inequality (Normal)**:




**Example 2.4** For the one-sample test of $H_0 : \mu = 30$ versus $H_A : \mu \neq 30$, what sample size is required to detect a shift to $\mu = 32$ with 90% power? The population standard deviation is unknown but expected to be $\sigma \simeq 1.5$.
**Solution:** The sample size condition given by Equation 2.21 is transcendental, so the correct value of $n$ must be determined iteratively. With $t \simeq z$ as a first guess, $z_{0.025} = 1.96$, $z_{0.10} = 1.282$, and

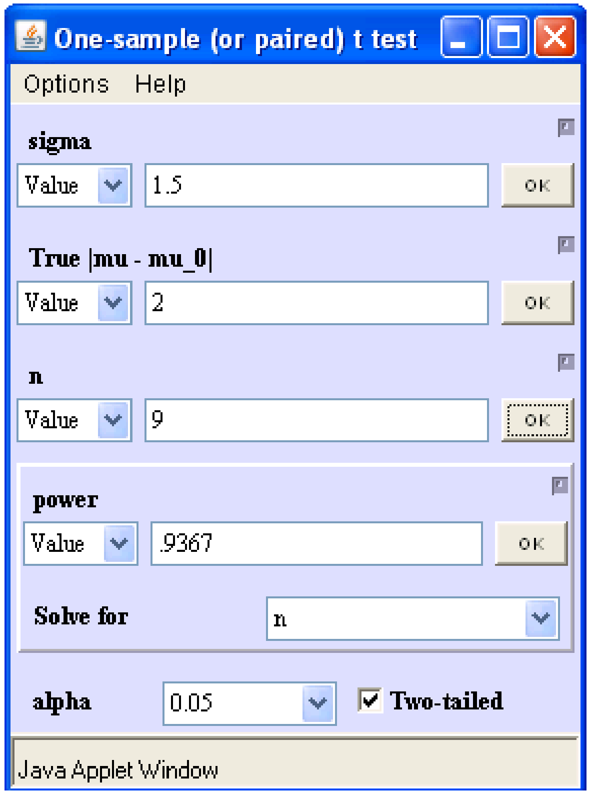
$$n = (1.96 + 1.282)^2 \left(\frac{1.5}{2}\right)^2 = 6.$$

Then with $df_\epsilon = 5$, $t_{0.025,5} = 2.571$, and $t_{0.10,5} = 1.476$ the new sample size estimate is

$$n \geq (2.571 + 1.476)^2 \left(\frac{1.5}{2}\right)^2 = 9.21.$$

Further iterations are required because $(n = 6) \ngeq 9.21$. Another iteration indicates that $n = 9$ delivers the desired power.

From **Piface**> **One-sample t test (or paired t)**:

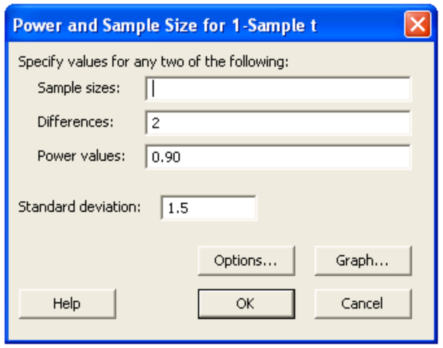



From **MINITAB**> **Stat**> **Power and Sample Size**> **1-Sample t**:

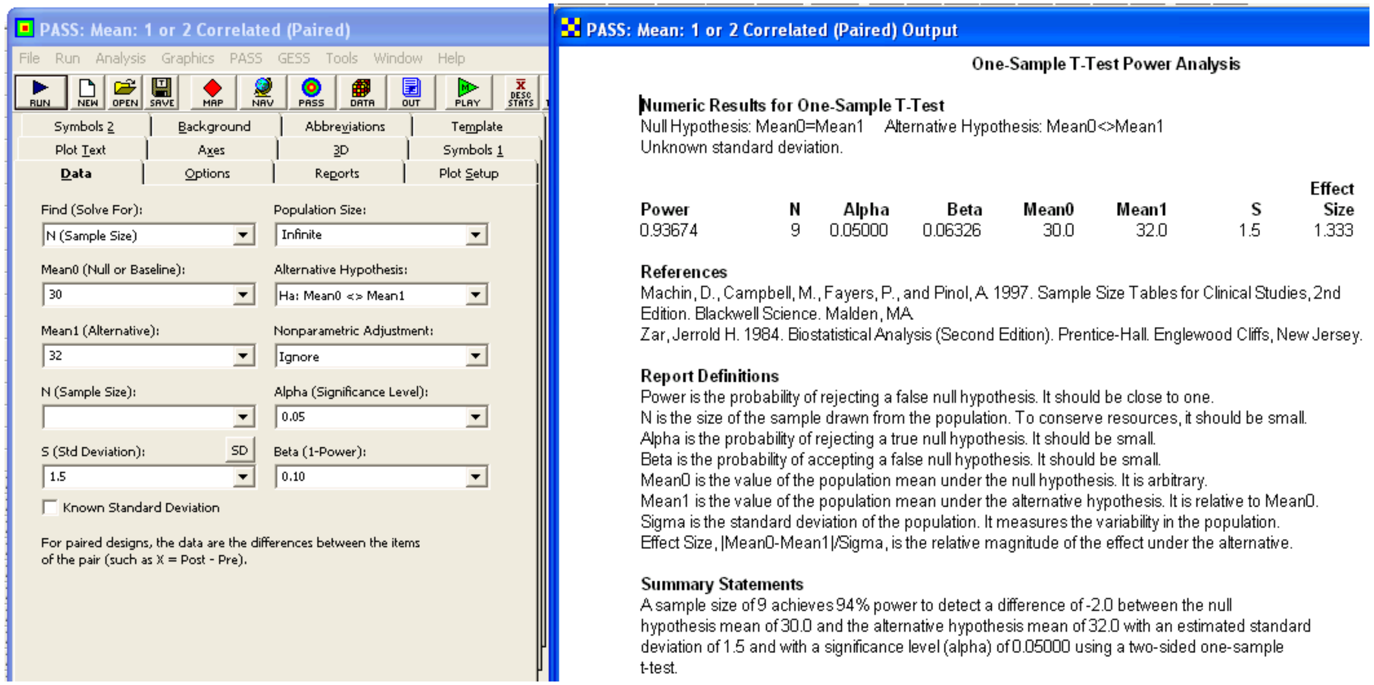
```
MTB > Power;
SUBC>   TOne;
SUBC>     Difference 2;
SUBC>     Power 0.90;
SUBC>     Sigma 1.5;
SUBC>   GPCurve.
```

**Power and Sample Size**

```
1-Sample t Test

Testing mean = null (versus not = null)
Calculating power for mean = null + difference
Alpha = 0.05  Assumed standard deviation = 1.5


            Sample  Target
Difference    Size   Power  Actual Power
         2       9     0.9      0.936743
```



From **PASS**> **Means**> **One**> **Inequality (Normal)**:




**Example 2.5** Find the approximate and exact power for the solution obtained for Example 2.4.
**Solution:** With $n = 9$ and $t_{0.025,8} = 2.306$ the approximate power by Equation 2.19 is

$$\begin{aligned}\pi &= P\left(-\infty < t < \frac{\delta}{\hat{\sigma}/\sqrt{n}} - t_{\alpha/2}\right) \\ &= P\left(-\infty < t < \frac{2}{1.5/\sqrt{9}} - 2.306\right) \\ &= P\left(-\infty < t < 1.694\right) \\ &= 0.9356.\end{aligned}$$

From Equation 2.23 the $t$ distribution noncentrality parameter is

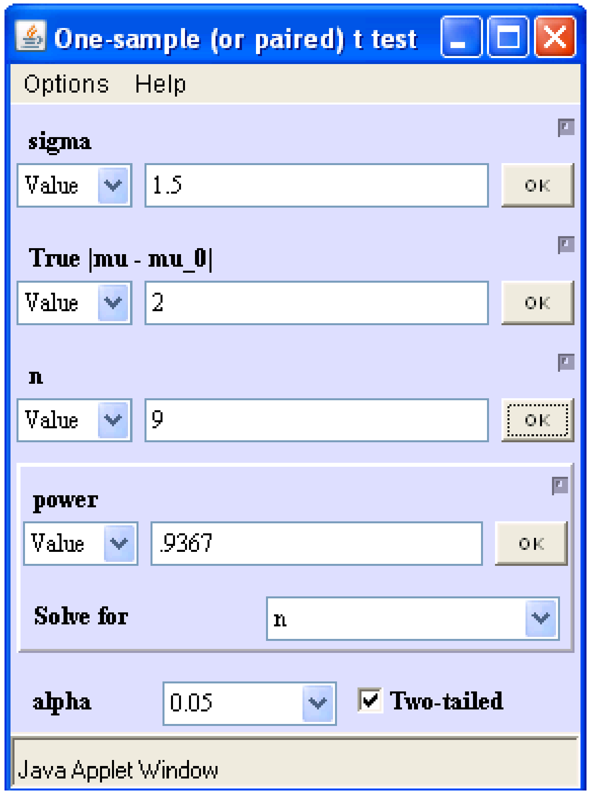
$$\phi = \frac{2}{1.5/\sqrt{9}} = 4.00,$$

so, from Equation 2.22,

$$t_{0.025} = 2.306 = t_{\beta,4.0},$$

which is satisfied by $\beta = 0.0633$ and power $\pi = 1 - \beta = 0.9367$. This value is in excellent agreement with the value obtained by the approximate method even though the sample size is relatively small.

From **Piface**> **One-sample t test (or paired t)**:



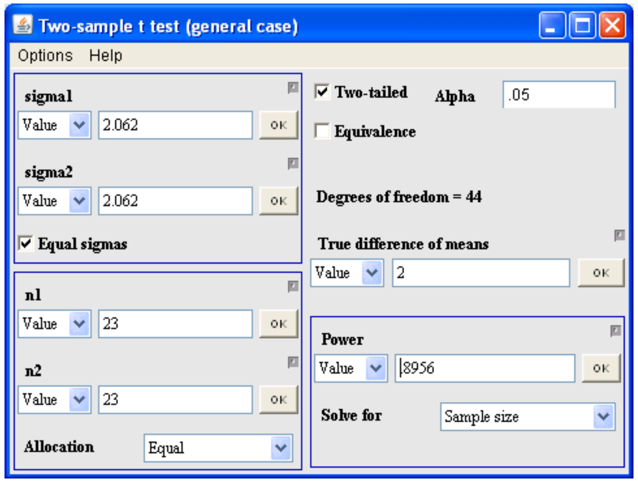

**Example 2.6** Compare the sample sizes for the two-independent-samples experiment and the paired-sample experiment if they must detect a bias between two treatments of $\Delta\mu = 2$ with 90% power when the standard deviation of individual units is $\widehat{\sigma}_x = 2$ and the measurement precision error is $\widehat{\sigma}_\epsilon = 0.5$.
**Solution:** For the two-independent-sample $t$ test the characteristic standard deviation for each treatment is (from Equation 2.27)

$$\widehat{\sigma}_{independent} = \sqrt{2^2 + 0.5^2} = 2.062.$$

Then, from Equation 2.62, the required sample size for each treatment is

$$\begin{aligned} n &\geq 2\left(t_{0.025}+t_{0.10}\right)^2\left(\frac{\widehat{\sigma}_{independent}}{\Delta\mu}\right)^2 \\ &\geq 2\left(t_{0.025}+t_{0.10}\right)^2\left(\frac{2.062}{2}\right)^2 \\ &\geq 24. \end{aligned}$$

From **Piface**> **Two-sample t test (pooled or Satterthwaite)**:

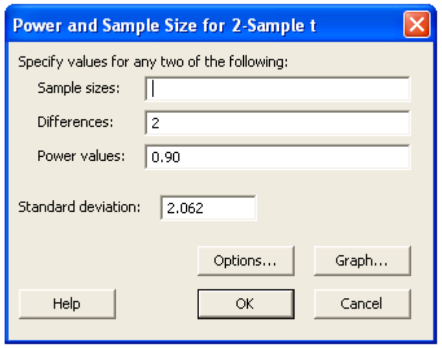



From **MINITAB**> **Stat**> **Power and Sample Size**> **2-Sample t**:

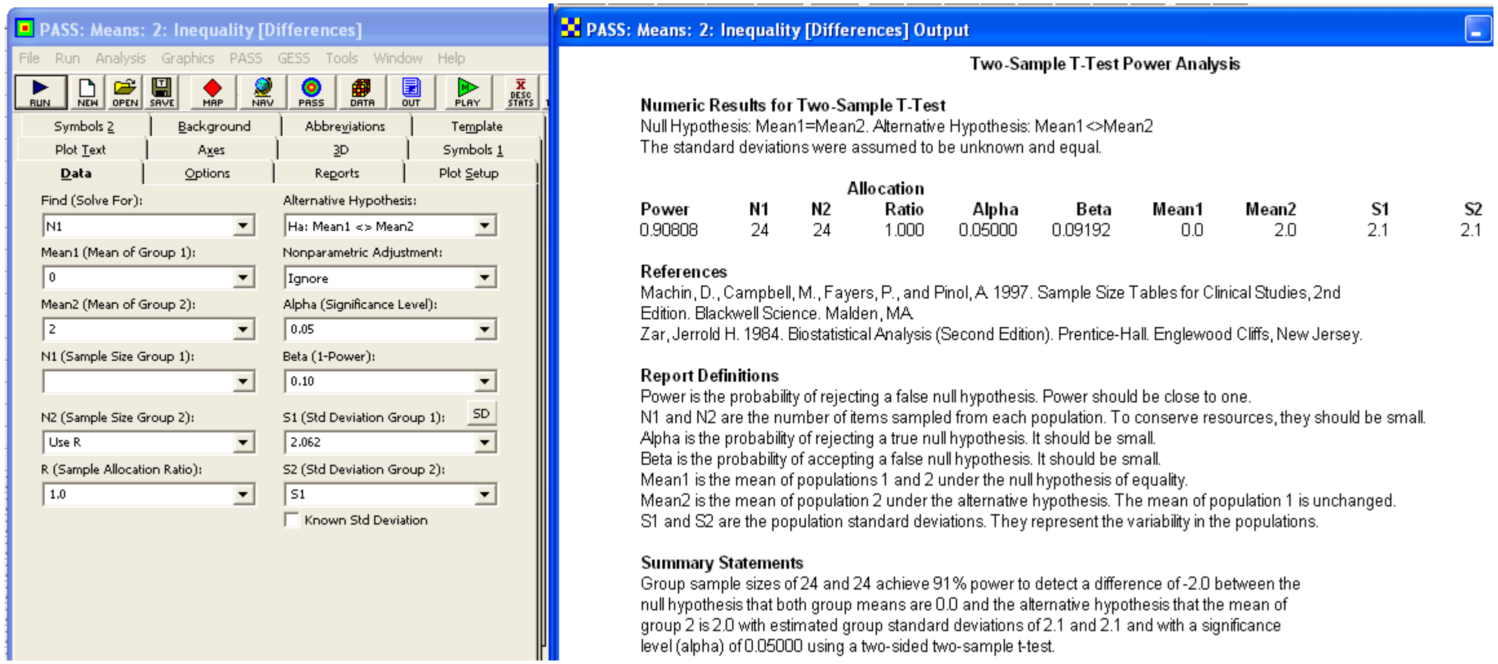
```
MTB > Power;
SUBC>   TTwo;
SUBC>     Difference 2;
SUBC>     Power 0.90;
SUBC>     Sigma 2.062;
SUBC>   GPCurve.
```

**Power and Sample Size**

```
2-Sample t Test

Testing mean 1 = mean 2 (versus not =)
Calculating power for mean 1 = mean 2 + difference
Alpha = 0.05  Assumed standard deviation = 2.062


            Sample  Target
Difference    Size   Power  Actual Power
         2      24     0.9      0.908083

The sample size is for each group.
```



From **PASS**> **Means**> **Two**> **Independent**> **Inequality (Normal) [Differences]**:




For the paired-sample $t$ test, the characteristic standard deviation for the $\Delta x_i$ can be estimated from Equation 2.28:

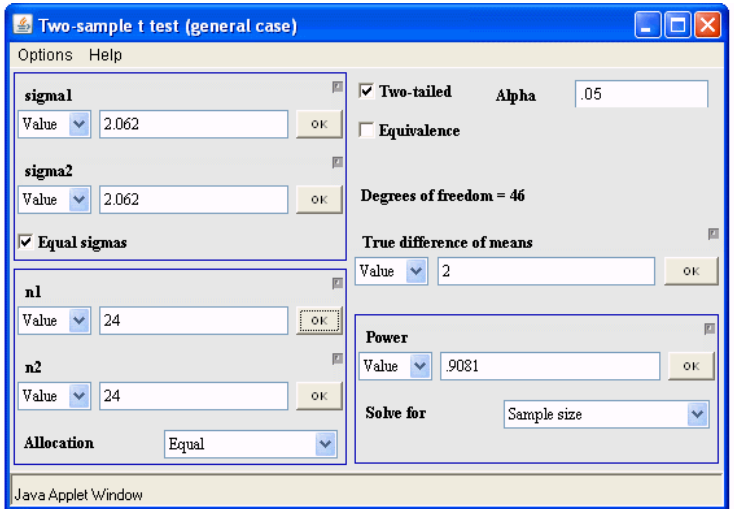
$$\widehat{\sigma}_{\Delta x} = \sqrt{2}\widehat{\sigma}_{\epsilon} = \sqrt{2} \times 0.5 = 0.707.$$

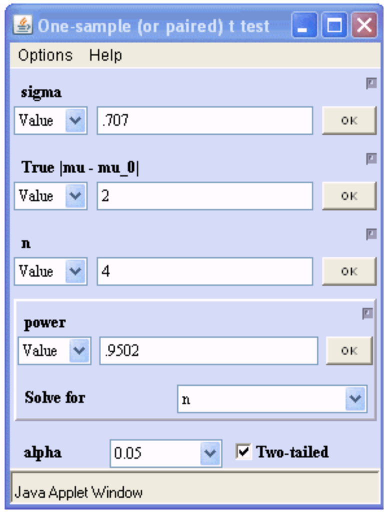
Then, from Equation 2.21, the required sample size is approximately

$$\begin{aligned} n &\geq (t_{0.025} + t_{0.10})^2 \left( \frac{\widehat{\sigma}_{\Delta x}}{\Delta \mu} \right)^2 \\ &\geq (t_{0.025} + t_{0.10})^2 \left( \frac{0.707}{2} \right)^2 \\ &\geq 4 \end{aligned}$$

and further iterations confirm that $n = 4$. When the independent-samples design requires two samples of size $n = 24$ units each, for a total of $48$ measurements, the paired-sample design requires only $n = 4$ units for a total of $8$ measurements!

From **Piface**> **One-sample t test (or paired t)**:



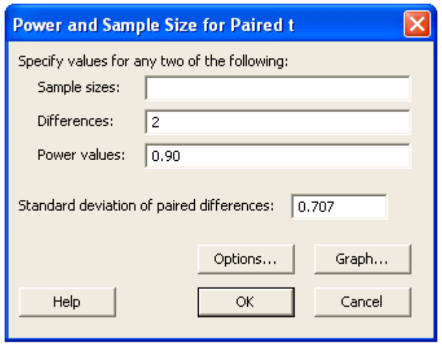




From **MINITAB (V16)**> **Stat**> **Power and Sample Size**> **Paired t**:

```
MTB > Power;
SUBC>   TPaired;
SUBC>     Difference 2;
SUBC>     Power 0.90;
SUBC>     Sigma 0.707;
SUBC>   GPCurve.
```

**Power and Sample Size**

```
Paired t Test

Testing mean paired difference = 0 (versus not = 0)
Calculating power for mean paired difference = difference
Alpha = 0.05  Assumed standard deviation of paired differences = 0.707


            Sample  Target
Difference    Size   Power  Actual Power
         2       4     0.9      0.950211
```

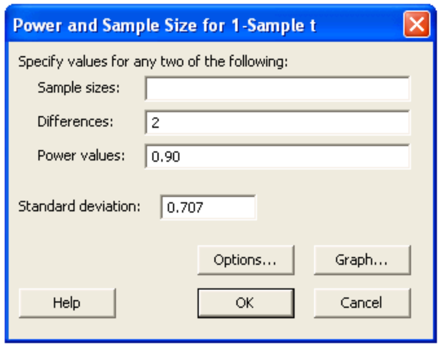



From **MINITAB**> **Stat**> **Power and Sample Size**> **1-Sample t**:

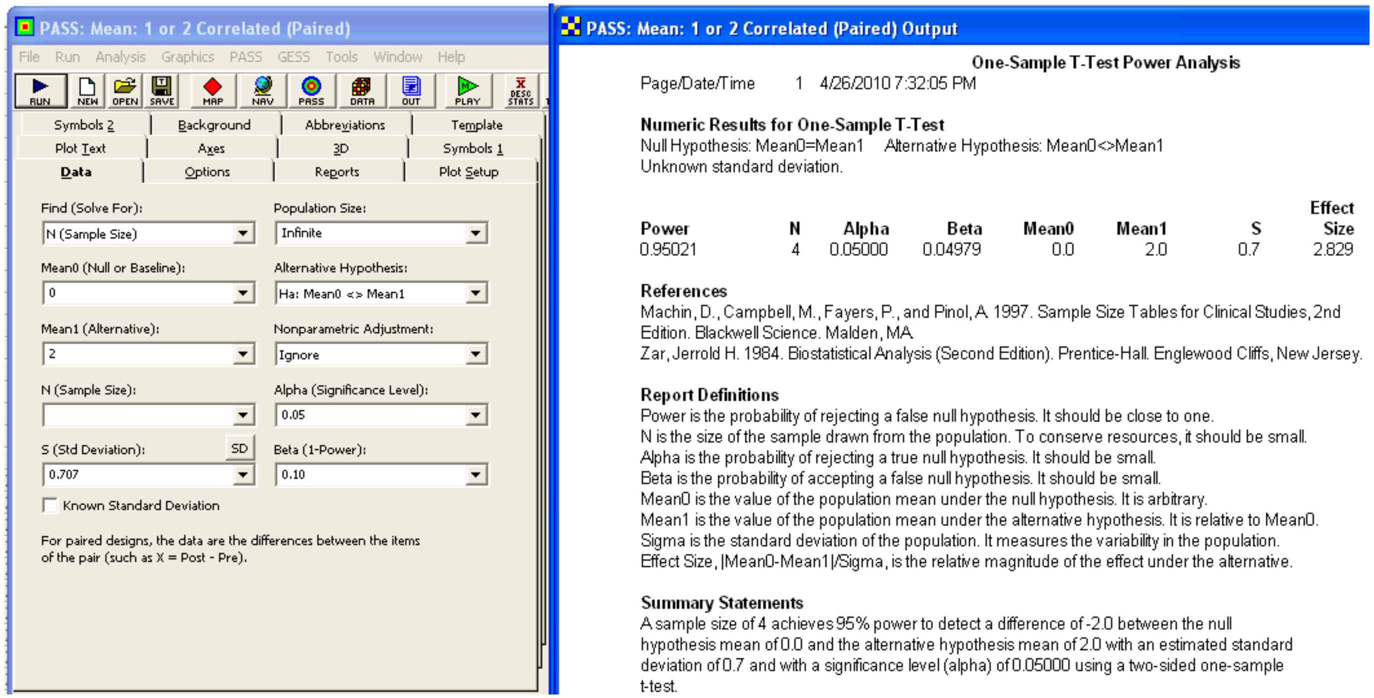
```
MTB > Power;
SUBC>   TOne;
SUBC>     Difference 2;
SUBC>     Power 0.90;
SUBC>     Sigma 0.707;
SUBC>   GPCurve.
```

**Power and Sample Size**

```
1-Sample t Test

Testing mean = null (versus not = null)
Calculating power for mean = null + difference
Alpha = 0.05  Assumed standard deviation = 0.707


            Sample  Target
Difference    Size   Power  Actual Power
         2       4     0.9      0.950211
```




From **PASS**> **Means**> **One**> **Inequality (Normal)**:



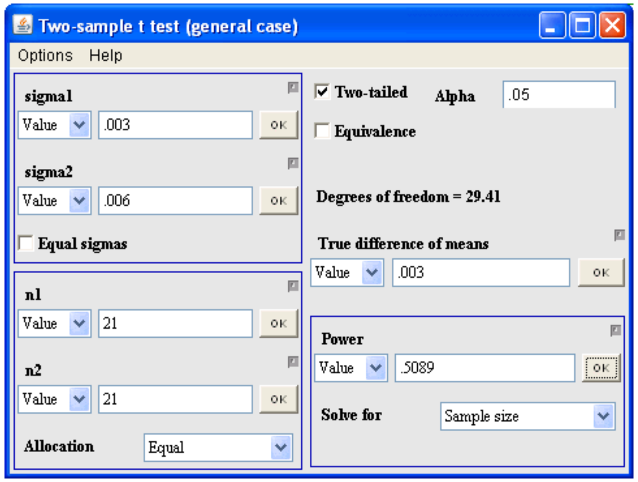
## 2.3 Two Independent Means

**Example 2.7** Find the sample sizes required for the a) equal-allocation and b) optimal-allocation conditions if the 95% two-sided confidence interval for $\Delta\mu$ must have half-width $\delta = 0.003$ when $\sigma_1 = 0.003$ and $\sigma_2 = 0.006$. Compare the total sample sizes required by the two methods.
**Solution:**

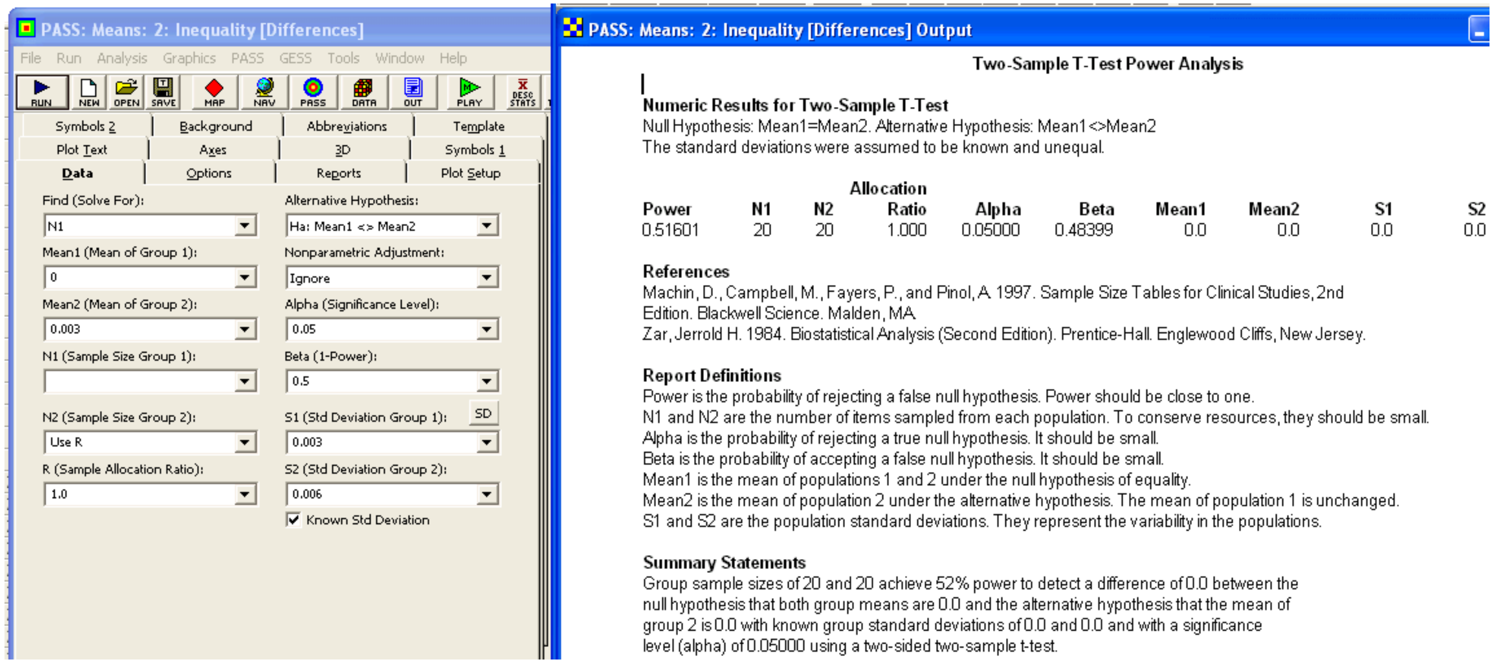
a) By Equation 2.32 the sample size required for equal allocation is

$$n = (1.96)^2 \frac{(0.003)^2 + (0.006)^2}{(0.003)^2} = 20.$$

From **Piface**> **Two-sample t test (pooled or Satterthwaite)**:




From **PASS**> **Means**> **Two**> **Independent**> **Inequality (Normal) [Differences]**:

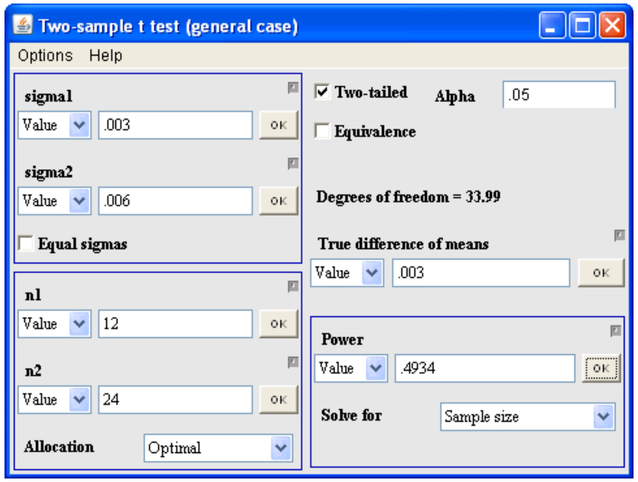



b) By Equations 2.33a and b, the sample sizes required for optimal allocation are

$$
\begin{aligned}
n_1 &= (1.96)^2 \frac{(0.003)\,(0.003 + 0.006)}{(0.003)^2} = 12 \\
n_2 &= 11.5 \left( \frac{0.006}{0.003} \right) = 24.
\end{aligned}
$$

For the equal-allocation method, the total sample size is $2n = 40$, and for the optimal-allocation method, the total sample size is $n_1 + n_2 = 36$ - a 10% savings in sample size.

From **Piface**> **Two-sample t test (pooled or Satterthwaite)**:




**Example 2.8** Determine the sample size required to obtain a confidence interval half-width $\delta = 50$ when $\widehat{\sigma}_1 = \widehat{\sigma}_2 = 80$.
**Solution:** With $t_{0.025} \simeq z_{0.025}$ for the first iteration, the sample size is

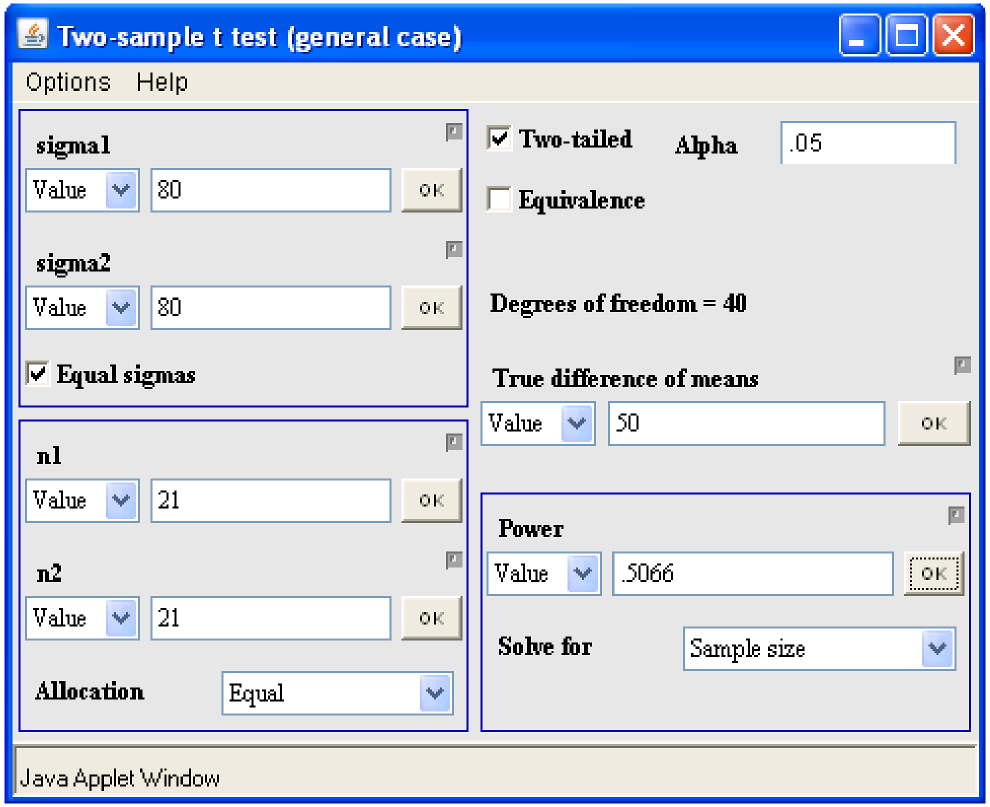
$$n = 2\left(\frac{1.96 \times 80}{50}\right)^2 = 20. \tag{2.1}$$

Another iteration with $t_{0.025,38} = 2.024$ gives

$$n = 2\left(\frac{2.024 \times 80}{50}\right)^2 = 21. \tag{2.2}$$

A third iteration (not shown) confirms that $n = 21$ is the necessary sample size.

From **Piface**> **Two-sample t test (pooled or Satterthwaite)**:



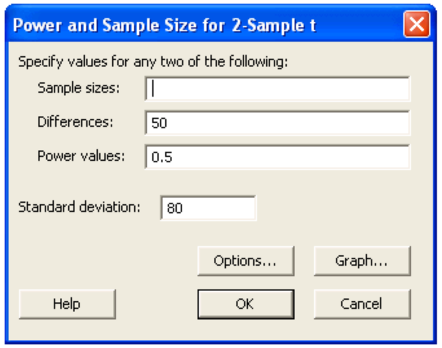

From **MINITAB**> **Stat**> **Power and Sample Size**> **2-Sample t**:

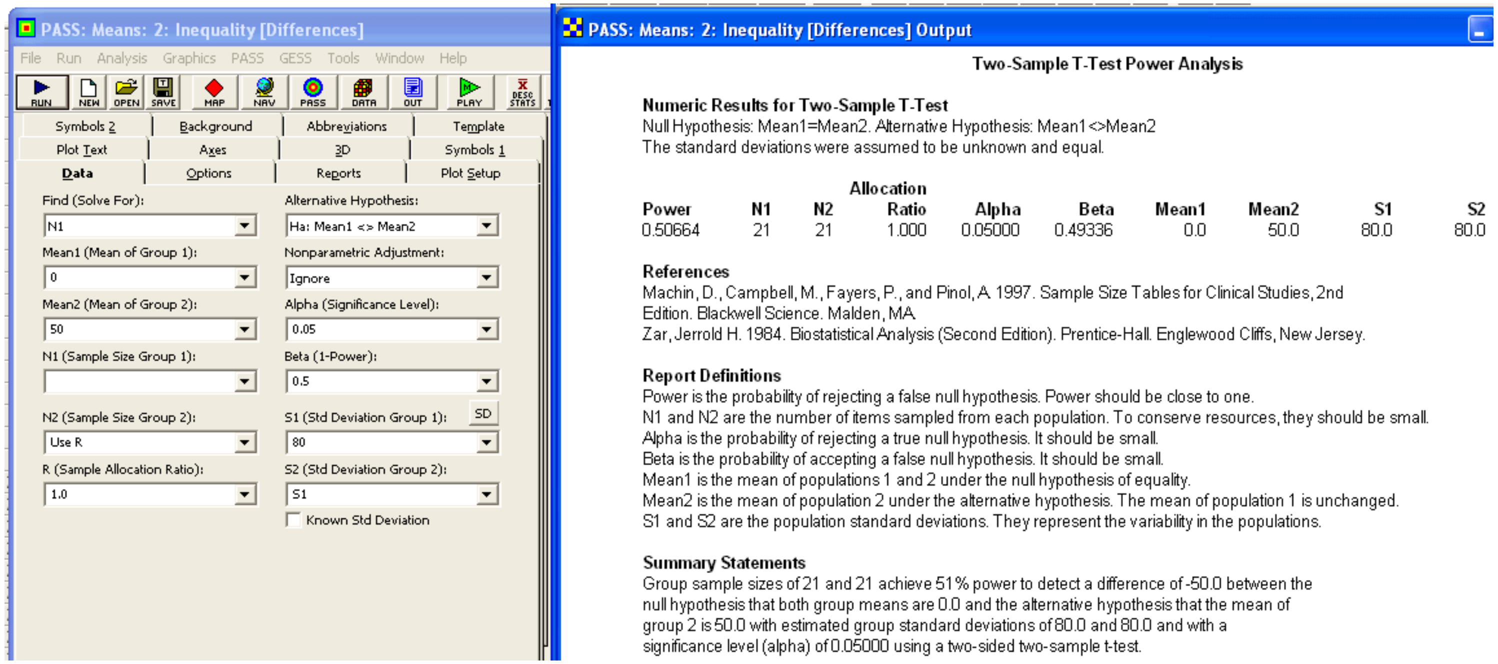
From **PASS**> **Means**> **Two**> **Independent**> **Inequality (Normal) [Differences]**:

```
MTB > Power;
SUBC>   TTwo;
SUBC>     Difference 50;
SUBC>     Power 0.5;
SUBC>     Sigma 80;
SUBC>   GPCurve.
```

**Power and Sample Size**

```
2-Sample t Test

Testing mean 1 = mean 2 (versus not =)
Calculating power for mean 1 = mean 2 + difference
Alpha = 0.05  Assumed standard deviation = 80


            Sample  Target
Difference    Size   Power  Actual Power
        50      21     0.5      0.506639
```




**Example 2.9** What optimal sample sizes are required to determine a confidence interval for the difference between two population means with confidence interval half-width $\delta = 15$ when $\widehat{\sigma}_1 = 24$ and $\widehat{\sigma}_2 = 8$?

**Solution:** From Equations 2.33a and b, initial guesses for the sample sizes are

$$n_1 \simeq (1.96)^2 \frac{24\,(24+8)}{15^2} \simeq 14$$

and

$$n_2 \simeq 14\left(\frac{8}{24}\right) \simeq 5.$$

To obtain optimal sample size allocation $n_1$ and $n_2$ must be in the ratio

$$(n_1 : n_2) = (\widehat{\sigma}_1 : \widehat{\sigma}_2) = (24 : 8) = (3 : 1)\,,$$

so reasonable choices for the sample sizes are $n_1 = 15$ and $n_2 = 5$. By Equation 2.41, the $t$ distribution degrees of freedom will be

$$df_\epsilon = \frac{\left(\frac{24^2}{15} + \frac{8^2}{5}\right)^2}{\frac{24^4}{15^2(15+1)} + \frac{8^4}{5^2(5+1)}} - 2 = 20.$$

With $t_{0.025,20} = 2.086$ the next iteration on the sample sizes gives

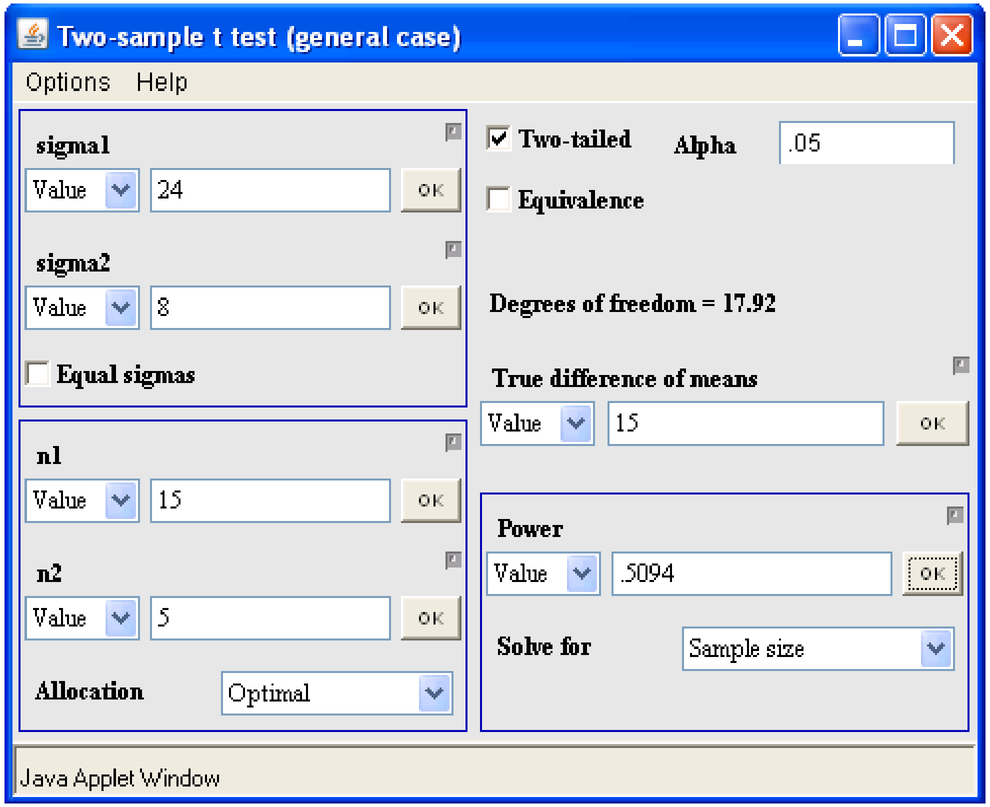
$$n_1 \simeq (2.086)^2 \frac{24\,(24+8)}{15^2} \simeq 15$$

and

$$n_2 \simeq 15\left(\frac{8}{24}\right) = 5$$

which must be the correct values.

From **Piface**> **Two-sample t test (pooled or Satterthwaite)**:




**Example 2.10** Calculate the sample size for the two-sample $t$ test to reject $H_0$ with 90% power when $|\mu_1 - \mu_2| = 5$. Assume that the sample sizes will be equal and that the two populations have equal standard deviations estimated to be $\widehat{\sigma}_\epsilon = 3$. Compare the approximate and exact powers.
**Solution:** With $\Delta\mu = 5$ and $\widehat{\sigma}_\epsilon = 3$ in Equation 2.62, the sample size predicted in the first iteration with $t \simeq z$ is

$$n = 2\left(\frac{(1.96 + 1.282)\,3}{5}\right)^2 = 8.$$

A second and third iteration indicate that the required sample size is $n = 9$.

With $n = 9$ for both samples, $df_\epsilon = 18 - 2 = 16$ and the approximate power is given by Equations 2.58 and 2.60:

$$\begin{aligned}
\pi &= P\left(-\infty < t < \sqrt{\frac{n}{2}}\frac{\Delta\mu}{\widehat{\sigma}_\epsilon} - t_{0.025,16}\right) \\
&= P\left(-\infty < t < \sqrt{\frac{9}{2}}\frac{5}{3} - 2.12\right) \\
&= P\left(-\infty < t < 1.416\right) \\
&= 0.912.
\end{aligned}$$

The $t$ distribution noncentrality parameter is given by Equation 2.64:

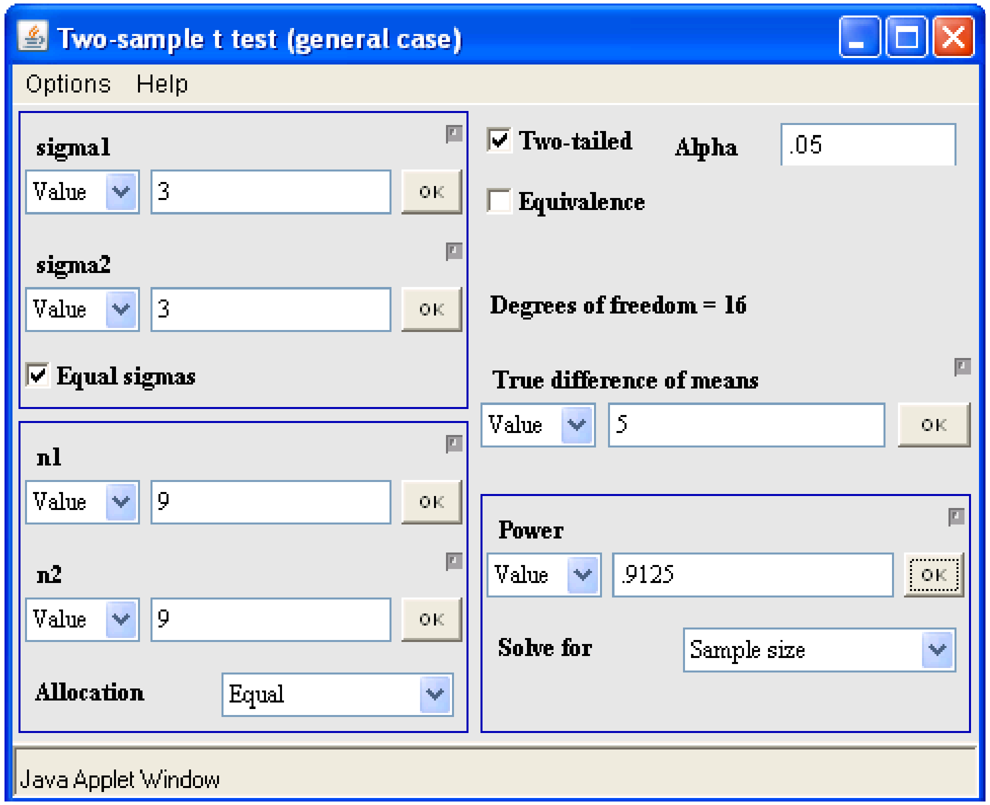
$$\phi = \sqrt{\frac{9}{2}}\frac{5}{3} = 3.536.$$

The exact power is determined by Equation 2.63 with $\alpha = 0.05$:

$$t_{0.025} = 2.120 = t_{\beta,3.536},$$

which is satisfied by $\beta = 0.087$, so the exact power is $\pi = 0.913$. The exact power is in excellent agreement with the approximate power despite the somewhat small sample size.

From **Piface**> **Two-sample t test (pooled or Satterthwaite)**:

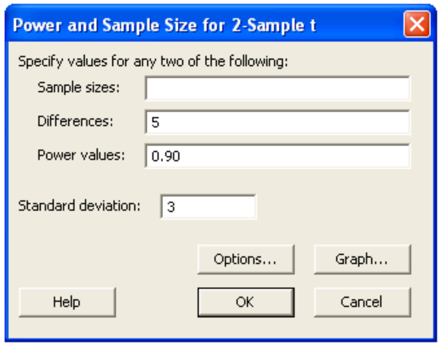



From **MINITAB**> **Stat**> **Power and Sample Size**> **2-Sample t**:

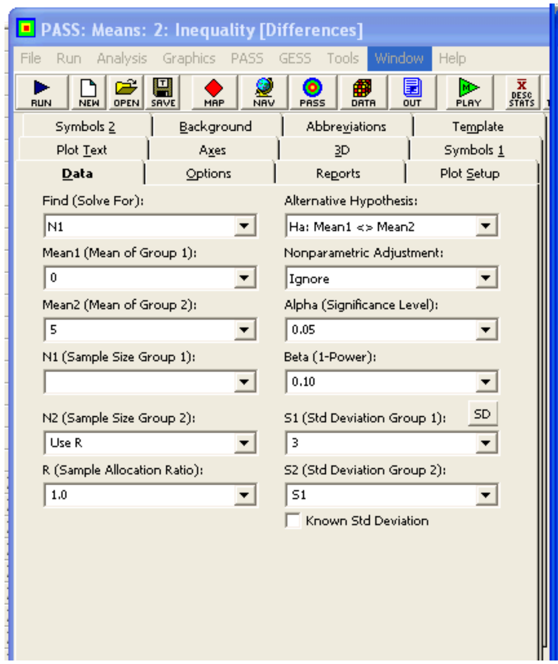
```
MTB > Power;
SUBC>   TTwo;
SUBC>     Difference 5;
SUBC>     Power 0.90;
SUBC>     Sigma 3;
SUBC>   GPCurve.
```

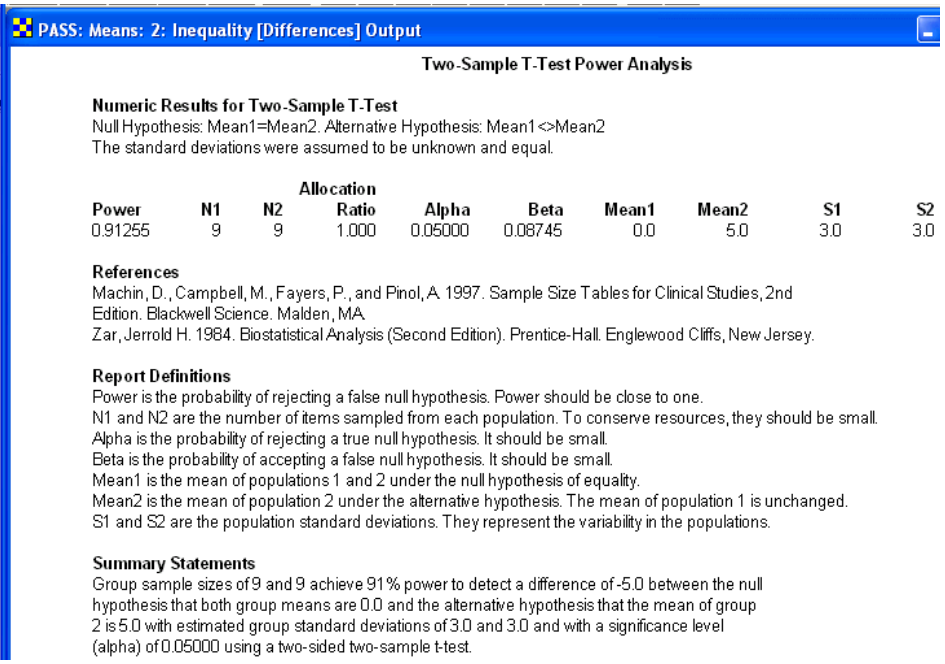
**Power and Sample Size**

```
2-Sample t Test

Testing mean 1 = mean 2 (versus not =)
Calculating power for mean 1 = mean 2 + difference
Alpha = 0.05  Assumed standard deviation = 3


             Sample  Target
Difference     Size   Power  Actual Power
         5        9     0.9      0.912548

The sample size is for each group.
```




From **PASS**> **Means**> **Two**> **Independent**> **Inequality (Normal) [Differences]**:





PASS: Means: 2: Inequality [Differences] Output

**Two-Sample T-Test Power Analysis**

**Numeric Results for Two-Sample T-Test**
Null Hypothesis: Mean1=Mean2. Alternative Hypothesis: Mean1<>Mean2
The standard deviations were assumed to be unknown and equal.

| Power | N1 | N2 | Allocation Ratio | Alpha | Beta | Mean1 | Mean2 | S1 | S2 |
|---|---|---|---|---|---|---|---|---|---|
| 0.91255 | 9 | 9 | 1.000 | 0.05000 | 0.08745 | 0.0 | 5.0 | 3.0 | 3.0 |

**References**
Machin, D., Campbell, M., Fayers, P., and Pinol, A. 1997. Sample Size Tables for Clinical Studies, 2nd Edition. Blackwell Science. Malden, MA.
Zar, Jerrold H. 1984. Biostatistical Analysis (Second Edition). Prentice-Hall. Englewood Cliffs, New Jersey.

**Report Definitions**
Power is the probability of rejecting a false null hypothesis. Power should be close to one.
N1 and N2 are the number of items sampled from each population. To conserve resources, they should be small.
Alpha is the probability of rejecting a true null hypothesis. It should be small.
Beta is the probability of accepting a false null hypothesis. It should be small.
Mean1 is the mean of populations 1 and 2 under the null hypothesis of equality.
Mean2 is the mean of population 2 under the alternative hypothesis. The mean of population 1 is unchanged.
S1 and S2 are the population standard deviations. They represent the variability in the populations.

**Summary Statements**
Group sample sizes of 9 and 9 achieve 91% power to detect a difference of -5.0 between the null hypothesis that both group means are 0.0 and the alternative hypothesis that the mean of group 2 is 5.0 with estimated group standard deviations of 3.0 and 3.0 and with a significance level (alpha) of 0.05000 using a two-sided two-sample t-test.

**Example 2.11** Determine the size of the second sample under the conditions described in Example 2.10 if the first sample size must be $n = 6$.

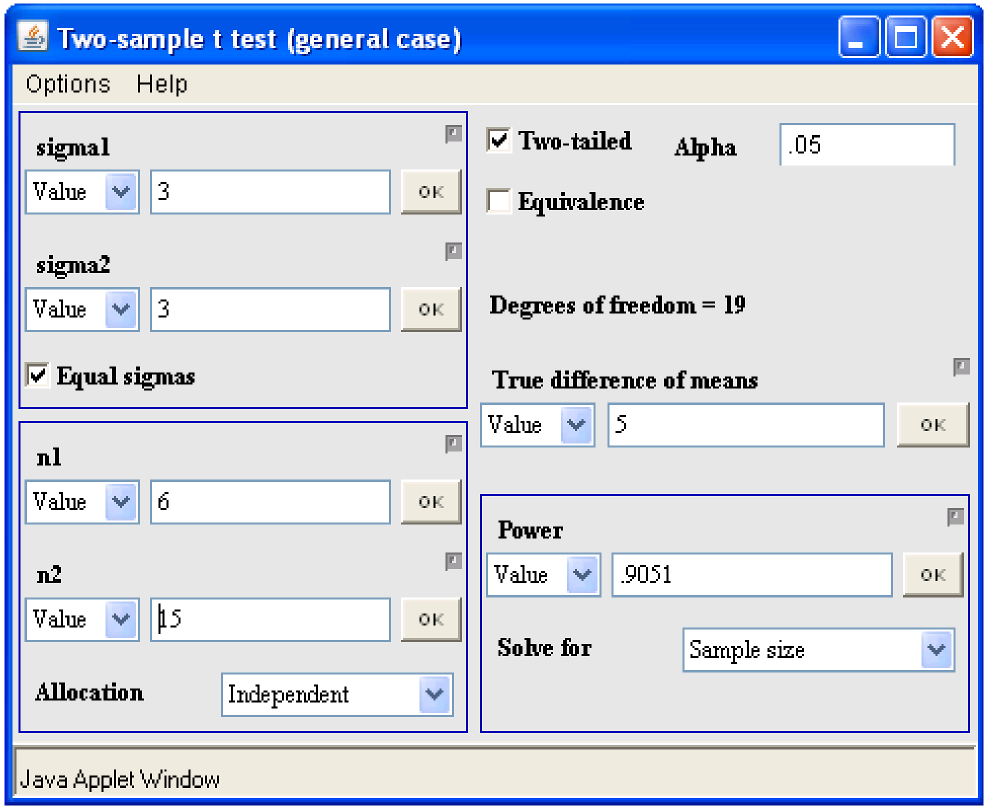
**Solution:** From Example 2.10, the optimal equal sample sizes are $n' = 9$. If $n_1 = 6$ is fixed, then, from Equation 2.68, the approximate value of the second sample size must be

$$n_2 = \frac{6 \times 9}{(2 \times 6) - 9} = 18.$$

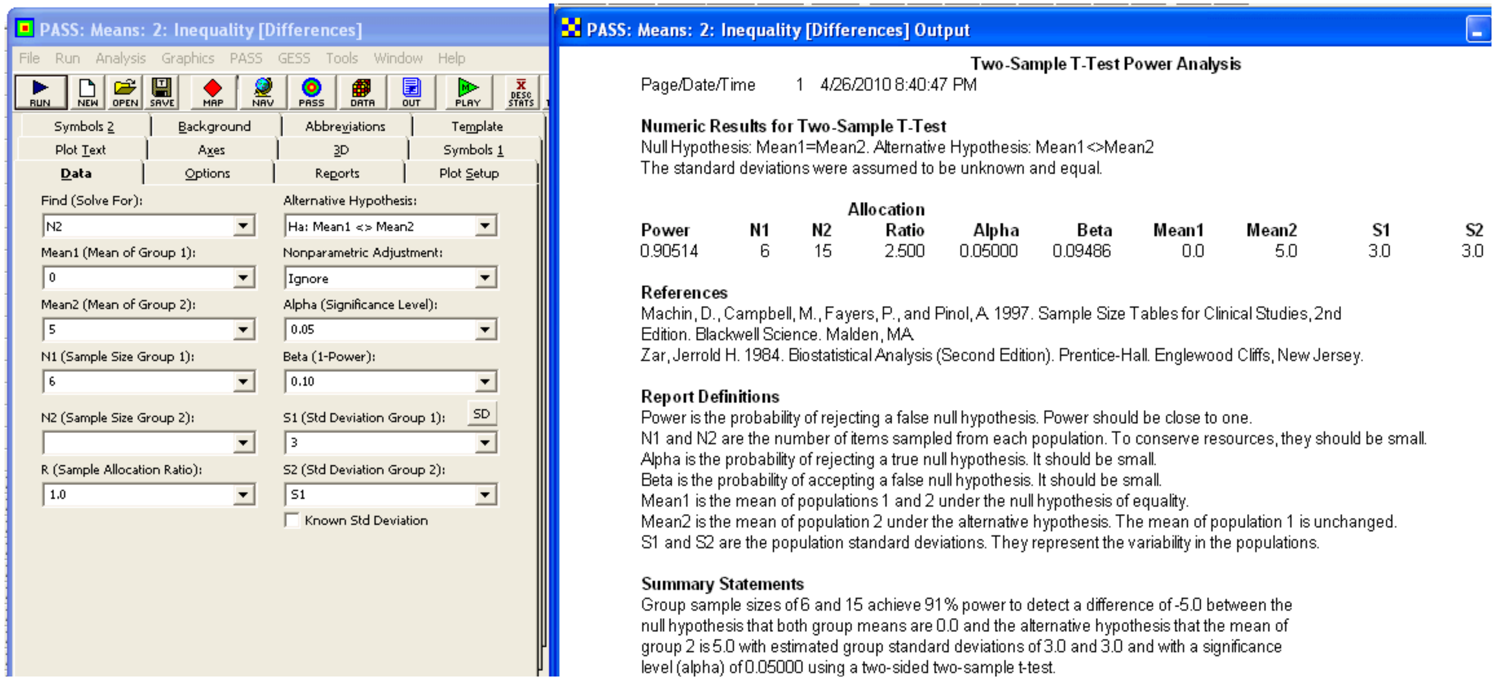
In the equal-$n$ solution, we had $n_1 + n_2 = 18$ and $df_\epsilon = 16$ with 91% power; therefore, we know that $n_1 + n_2 = 6 + 18 = 24$ and $df_\epsilon = 22$ will give a slightly larger power, so the next guess for $n_2$ can be a value less than $n_2 = 18$. By appropriate guesses and iterations, the required value of $n_2$ is determined to be $n_2 = 15$ with approximate power

$$\begin{aligned}
\pi &= P\left(-\infty < t < \frac{1}{\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}\left(\frac{\Delta\mu}{\widehat{\sigma}}\right) - t_{0.025,19}\right) \\
&= P\left(-\infty < t < \frac{1}{\sqrt{\frac{1}{6} + \frac{1}{15}}}\left(\frac{5}{3}\right) - 2.093\right) \\
&= P\left(-\infty < t < 1.357\right) \\
&= 0.905.
\end{aligned}$$

From **Piface**> **Two-sample t test (pooled or Satterthwaite)**:




From **PASS**> **Means**> **Two**> **Independent**> **Inequality (Normal) [Differences]**:




## 2.4 Equivalence Tests

**Example 2.12** Determine the sample size required for a one-sample equivalence test of the hypotheses $H_0 : \mu < 490$ *or* $\mu > 510$ versus $H_A : 490 < \mu < 510$ if the experiment must have 90% power to reject $H_0$ when $\mu = 505$ and $\sigma = 4$.
**Solution:** With $\mu_0 = 500$, $\mu = 505$, and $\delta = 10$, the sample size given by Equation 2.74 is

$$\begin{aligned} n &= \left( \frac{(z_{0.05} + z_{0.10})\,\sigma}{\delta - \Delta\mu} \right)^2 \\ &= \left( \frac{(1.645 + 1.282)\,4}{10 - 5} \right)^2 \\ &= 6. \end{aligned}$$

**Example 2.13** Determine the power of the two independent-sample equivalence test where $\mu_1$ and $\mu_2$ are considered to be practically equivalent if $|\Delta\mu| < 2$ when $\Delta\mu = 0.2$, $\sigma_1 = \sigma_2 = 2$, and $n_1 = n_2 = 20$.
**Solution:** With $\delta = 2$ as the limit of practical equivalence, the hypotheses to be tested are

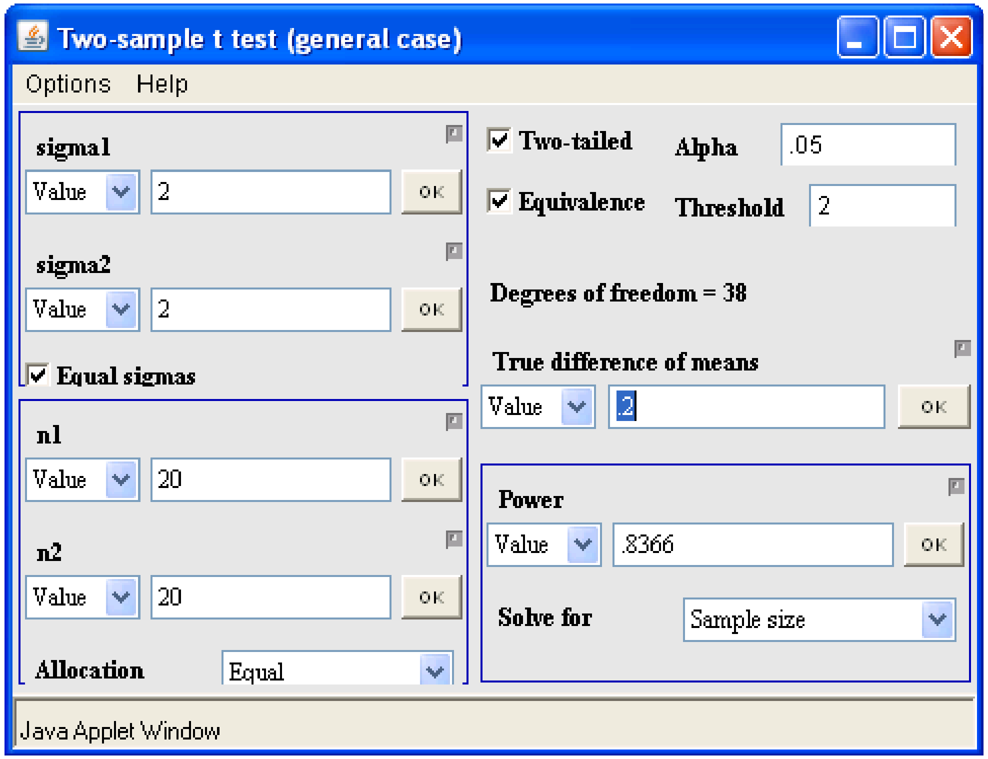
$$\begin{aligned} H_{01} &: \Delta\mu \leq -2 \text{ versus } H_{A1} : \Delta\mu > -2 \\ H_{02} &: \Delta\mu \geq 2 \text{ versus } H_{A2} : \Delta\mu < 2. \end{aligned}$$

From Equation 2.79 with $\Delta\mu = 0.2$, the power of the equivalence test is

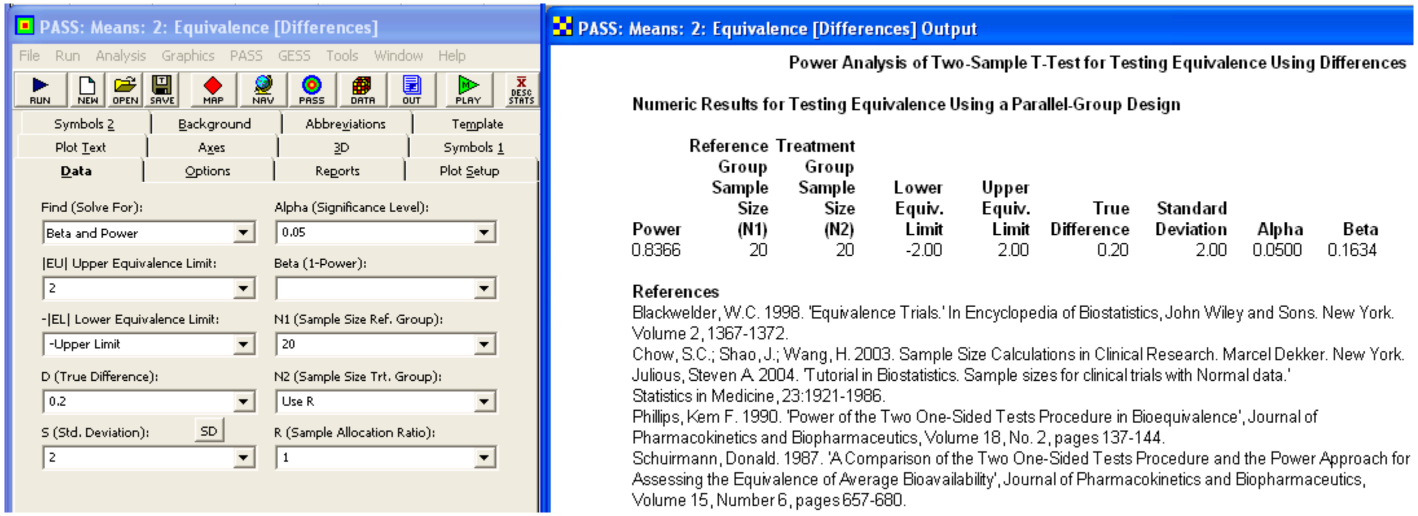
$$\begin{aligned}\pi &= \Phi\left(\frac{-2-0.2}{\sqrt{\frac{2}{20}2}} + 1.645 < z < \frac{2-0.2}{\sqrt{\frac{2}{20}2}} - 1.645\right)\\ &= \Phi\left(-1.83 < z < 1.20\right)\\ &= 0.85.\end{aligned}$$

PASS and Piface do the two-sample $t$ equivalence test which gives power comparable to that of the $z$ test for this example with relatively large error degrees of freedom ($df_\epsilon = 20 + 20 - 2 = 38$).

**Piface> Two-sample t test:**




**PASS> Means> Two> Independent> Equivalence [Difference]:**

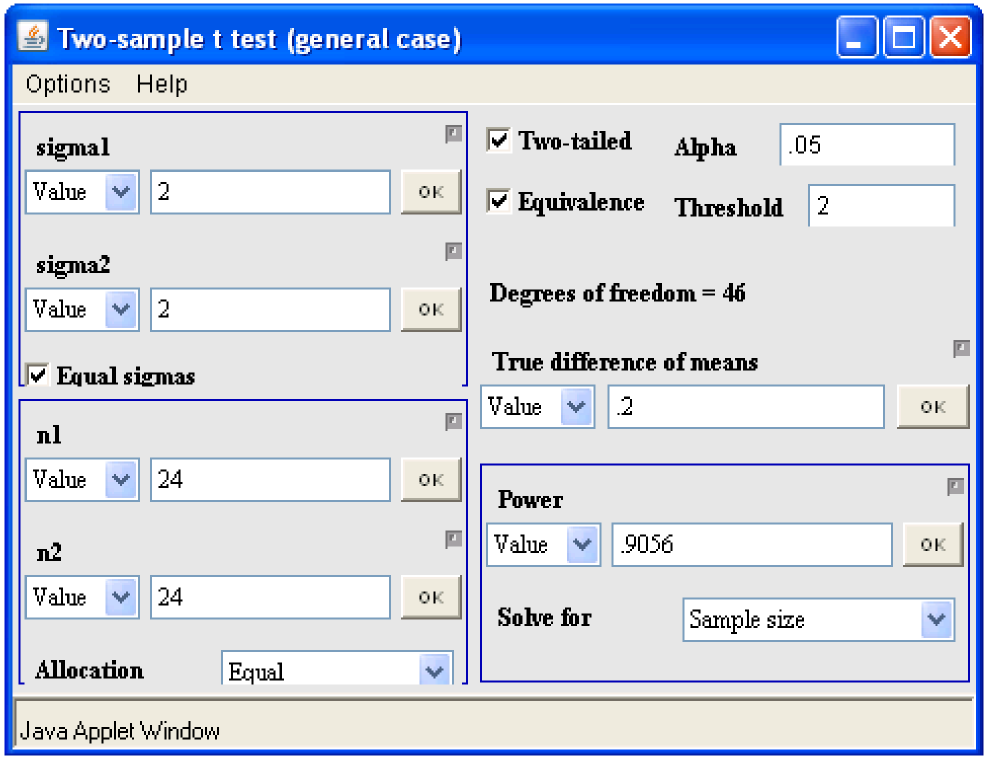



**Example 2.14** What sample size is required in Example 2.13 to obtain 90% power?
**Solution:** From Equation 2.80 with $\beta = 0.10$, the sample size is

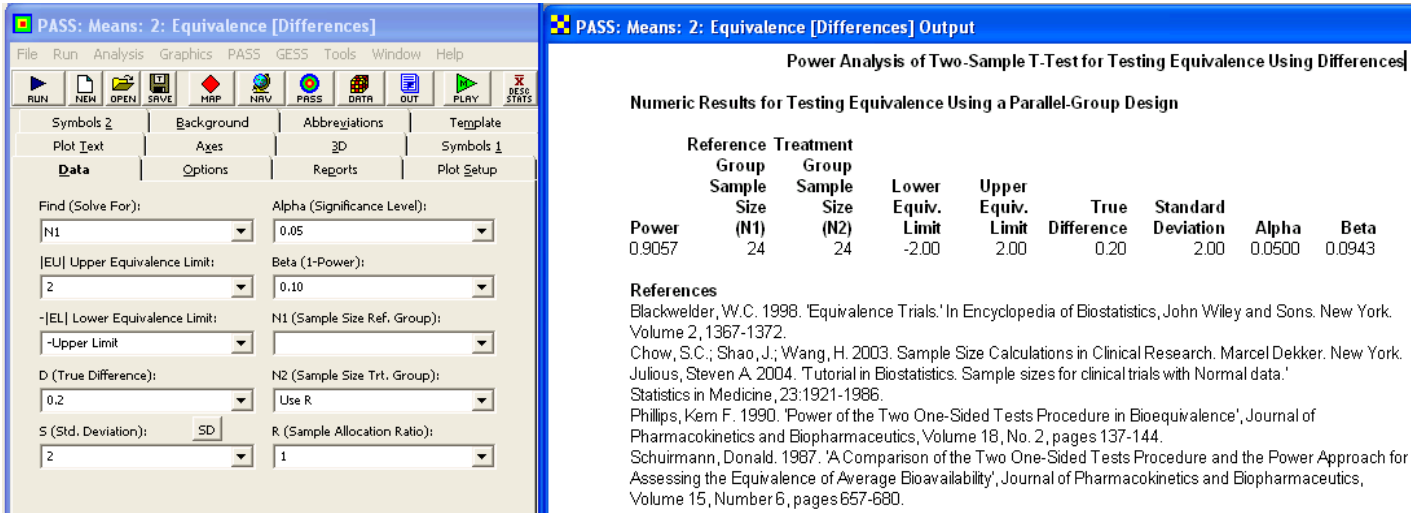
$$
\begin{aligned}
n &= 2\left(\frac{(1.645+1.282)\,2}{2-0.2}\right)^2 \\
&= 22.
\end{aligned}
$$

PASS and Piface do the two-sample $t$ equivalence test which gives comparable sample size to that of the $z$ test for this example with relatively large error degrees of freedom.

From **Piface**$>$ **Two-sample t test:**




From **PASS**> **Means**> **Two**> **Independent**> **Equivalence [Difference]**:




## 2.5 Contrasts

**Example 2.15** How many observations per treatment group are required to estimate the contrast

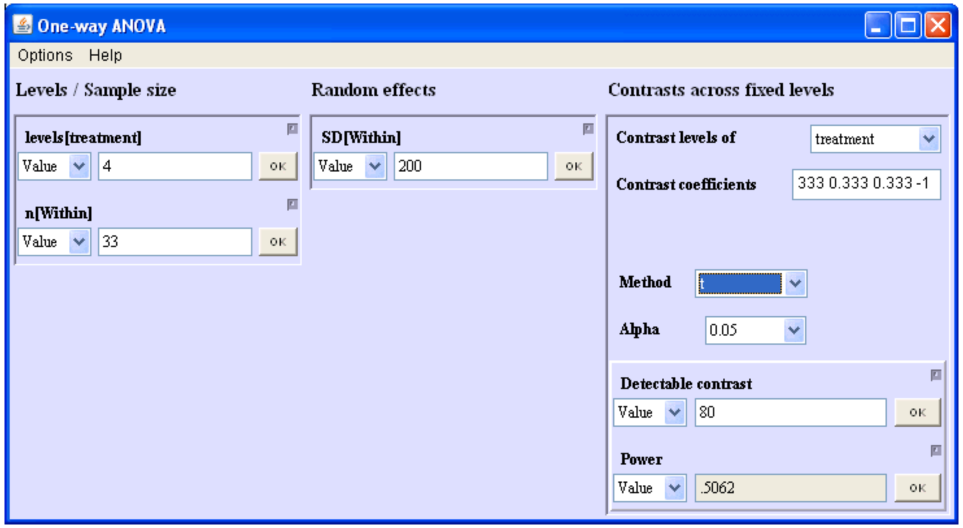
$$\mu_c = \left( \frac{\mu_1 + \mu_2 + \mu_3}{3} \right) - \mu_4$$

to within $\delta = 80$ measurement units with 95% confidence if the one-way ANOVA standard error is $s_\epsilon = 200$?
**Solution:** The goal is to obtain a 95% confidence interval for the contrast of the form given in Equation 2.85 with a confidence interval half-width of $\delta = 80$. The contrast coefficients are $c_i = \left\{ \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -1 \right\}$. If there are sufficient error degrees of freedom so that $t \simeq z$, then, from Equation 2.87, the approximate sample size is

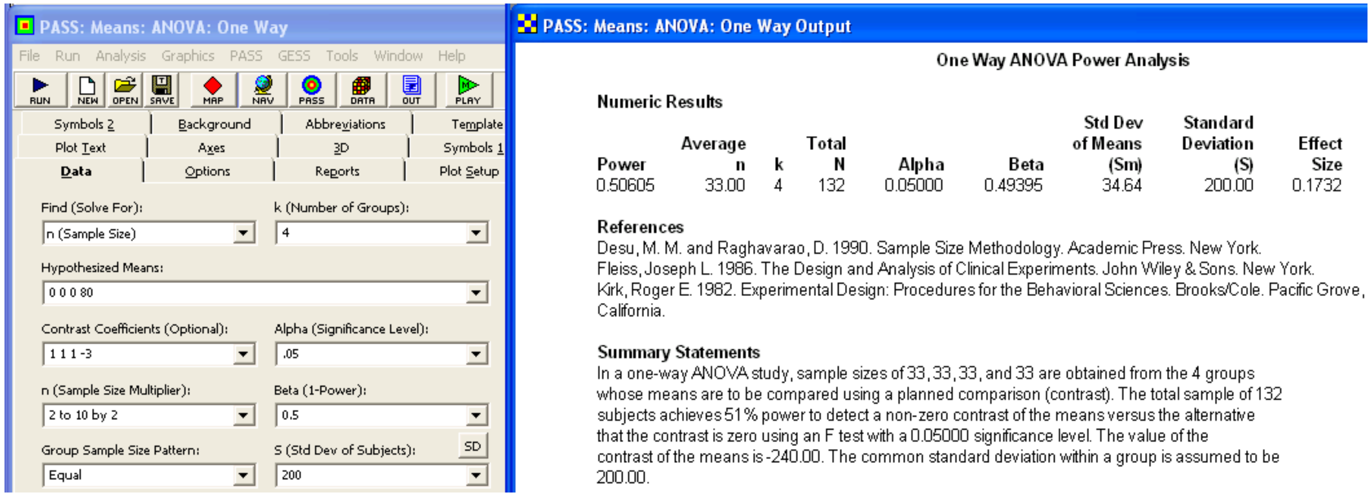
$$\begin{aligned} n &\simeq \left( \frac{1.96 \times 200}{80} \right)^2 \left( \left( \frac{1}{3} \right)^2 + \left( \frac{1}{3} \right)^2 + \left( \frac{1}{3} \right)^2 + (-1)^2 \right) \\ &\simeq 33. \end{aligned}$$

With $df_\epsilon = k(n-1) = 4(32) = 128$ error degrees of freedom the $t \simeq z$ approximation is satisfied, so the sample size is accurate.

Piface doesn't calculate the sample size, but it can be used to confirm the answer by showing that the sample size $n = 33$ produces about 50% power. From **Piface**> **Balanced ANOVA**> **One-way ANOVA**> **Differences/Contrasts**:




From **PASS**> **Means**> **Many Means**> **ANOVA: One-Way**:




## 2.6 Multiple Comparisons Tests

**Example 2.16** Determine the sample size required per treatment to detect a difference $\Delta\mu = 200$ between two treatment means using Bonferroni-corrected two-sample $t$ tests for all possible pairs of five treatments with 90% power. Assume that the five populations are normal and homoscedastic with $\widehat{\sigma}_\epsilon = 100$.

**Solution:** With $k = 5$ treatments there will be $K = \binom{5}{2} = 10$ two-sample $t$ tests to perform. To restrict the family error rate to $\alpha_{family} = 0.05$, the Bonferroni-corrected error rate for individual tests is

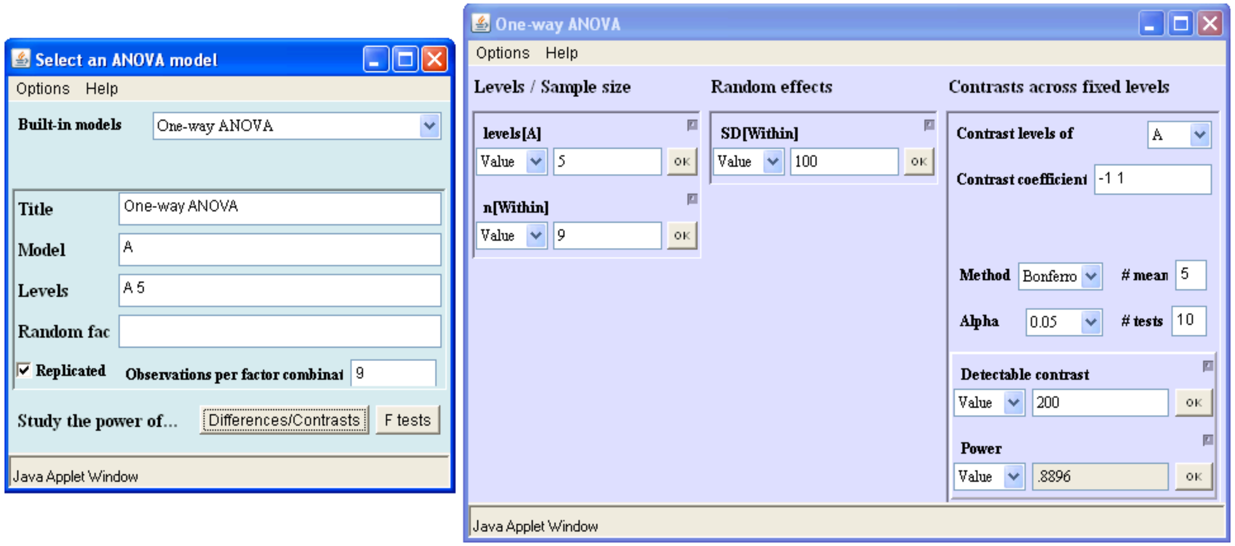
$$\alpha = \frac{0.05}{10} = 0.005.$$

By Equation 2.62 with $t \simeq z$, the sample size is

$$\begin{aligned} n &= 2\left(\frac{(z_{0.0025} + z_{0.10})\,\widehat{\sigma}_\epsilon}{\delta}\right)^2 \\ &= 2\left(\frac{(2.81 + 1.282)\,100}{200}\right)^2 = 9. \end{aligned}$$

There will be $df_\epsilon = df_{total} - df_{model} = (5 \times 9 - 1) - (4) = 40$ degrees of freedom to estimate $\widehat{\sigma}_\epsilon$ from the pooled treatment standard deviations, so the approximation $t \simeq z$ is justified.

From **Piface**> **Balanced ANOVA**> **One-way ANOVA**> **Differences/Contrasts**:




PASS uses a more conservative method for analyzing multiple comparisons which gives a larger sample size.

**Example 2.17** Determine the approximate power for the sample size calculated in Example 2.16.

**Solution:** The approximate power for the test is given by Equations 2.58 and 2.60 with $\alpha = 0.005$:

$$\begin{aligned}\pi &= P(-\infty < t < t_\beta) \\ &= P\left(-\infty < t < \left(\sqrt{\frac{n}{2}}\frac{\Delta\mu}{\widehat{\sigma}} - t_{\alpha/2}\right)\right) \\ &= P\left(-\infty < t < \left(\sqrt{\frac{9}{2}}\frac{200}{100} - t_{0.0025,40}\right)\right) \\ &= P(-\infty < t < 1.273) \\ &= 0.895.\end{aligned}$$

**Example 2.18** Bonferroni's method becomes very conservative when the number of tests gets very large. A less conservative method for determining $\alpha$ for individual tests is given by Sidak's method:

$$\alpha = 1 - (1 - \alpha_{family})^{1/K}. \tag{2.93}$$

Compare the sample sizes determined using Bonferroni's and Sidak's methods for multiple comparisons between all possible pairs of fifteen treatments when the tests must detect a difference of $\Delta\mu = 8$ with 90% power when $\widehat{\sigma}_\epsilon = 6$.
**Solution:** The number of multiple comparisons tests required is

$$\binom{15}{2} = \frac{15 \times 14}{2} = 105.$$

By Bonferroni's method with $\alpha_{family} = 0.05$, the $\alpha$ for individual tests is

$$\alpha = \frac{0.05}{105} = 0.000476,$$

so with $t \simeq z$ in Equation 2.62 the sample size is

$$\begin{aligned}n &= 2\left(\frac{(z_{0.000476/2} + z_{0.10})\,\widehat{\sigma}_\epsilon}{\delta}\right)^2 \\ &= 2\left(\frac{(3.494 + 1.282)\,6}{8}\right)^2 = 26.\end{aligned}$$

By Sidak's method (Equation 2.63), the $\alpha$ for individual tests is

$$\alpha = 1 - (1 - 0.05)^{1/105} = 0.000488,$$

so the sample size is

$$\begin{aligned}n &= 2\left(\frac{(z_{0.000488/2} + z_{0.10})\,\widehat{\sigma}_\epsilon}{\delta}\right)^2 \\ &= 2\left(\frac{(3.487 + 1.282)\,6}{8}\right)^2 = 26.\end{aligned}$$

Even with over 100 multiple comparisons, the sample sizes by the two calculation methods are still equal.

**Example 2.19** An experiment will be performed to compare four treatment groups to a control group. Determine the sample size required to detect a difference $\delta = 200$ between the treatments and the control using Bonferroni-corrected two-sample $t$ tests with 90% power. Use a balanced design with the same number of observations in each of the five groups and assume that the five populations are normal and homoscedastic with $\widehat{\sigma}_\epsilon = 100$.
**Solution:** To restrict the family error rate to $\alpha_{family} = 0.05$ with $K = 4$ tests, the Bonferroni-corrected error rate for individual tests is

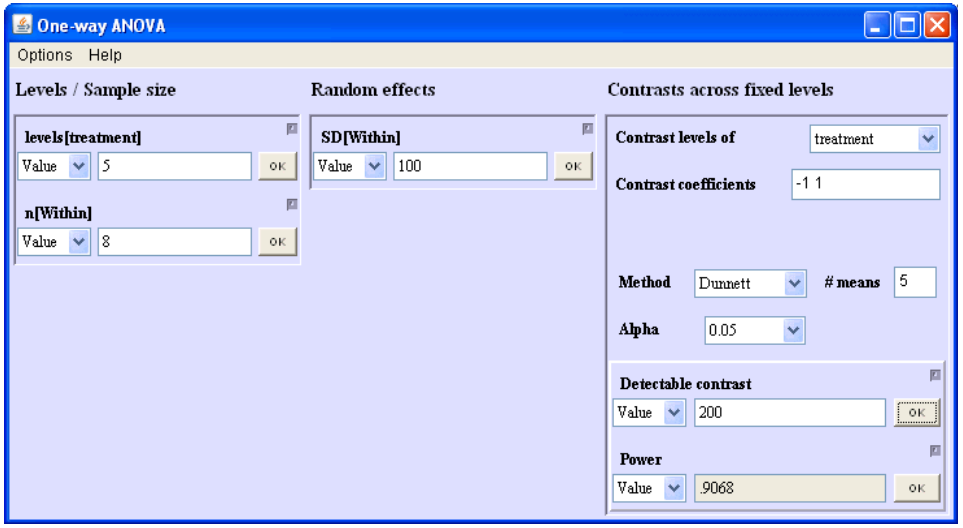
$$\alpha = \frac{0.05}{4} = 0.0125.$$

By Equation 2.62 with $t \simeq z$, the sample size is

$$\begin{aligned} n &= 2\left(\frac{\left(z_{0.0125/2} + z_{0.10}\right)\widehat{\sigma}_\epsilon}{\delta}\right)^2 \\ &= 2\left(\frac{(2.50 + 1.282)\,100}{200}\right)^2 = 8. \end{aligned}$$

Despite the small treatment-group sample size, the approximation $t \simeq z$ is justified because there will be $df_\epsilon = df_{total} - df_{model} = (5 \times 8 - 1) - (4) = 35$ degrees of freedom to estimate $\widehat{\sigma}_\epsilon$ from the five pooled treatment standard deviations.

Piface offers Dunnett's test, but it uses the Bonferroni correction to approximate Dunnett's method so it gives the same result. From **Piface**> **Balanced ANOVA**> **One-way ANOVA**> **Differences/Contrasts**:




**Example 2.20** Repeat Example 2.19 using the optimal allocation of units to treatments and controls.
**Solution:** From Equation 2.98 with $t \simeq z$ and $K = 4$,

$$n_i = \left(1 + \frac{1}{\sqrt{4}}\right)\left(\frac{(2.50 + 1.282)\,100}{200}\right)^2 = 6$$

and

$$n_0 = n_i\sqrt{K} = 6\sqrt{4} = 12.$$

The approximation $t \simeq z$ is still justified because the error degrees of freedom will be $df_\epsilon = (4 \times 6 + 12) - 4 = 32$. The original experiment required $5 \times 8 = 40$ units, but the optimal experiment requires only $4 \times 6 + 12 = 36$ units to obtain the same power.

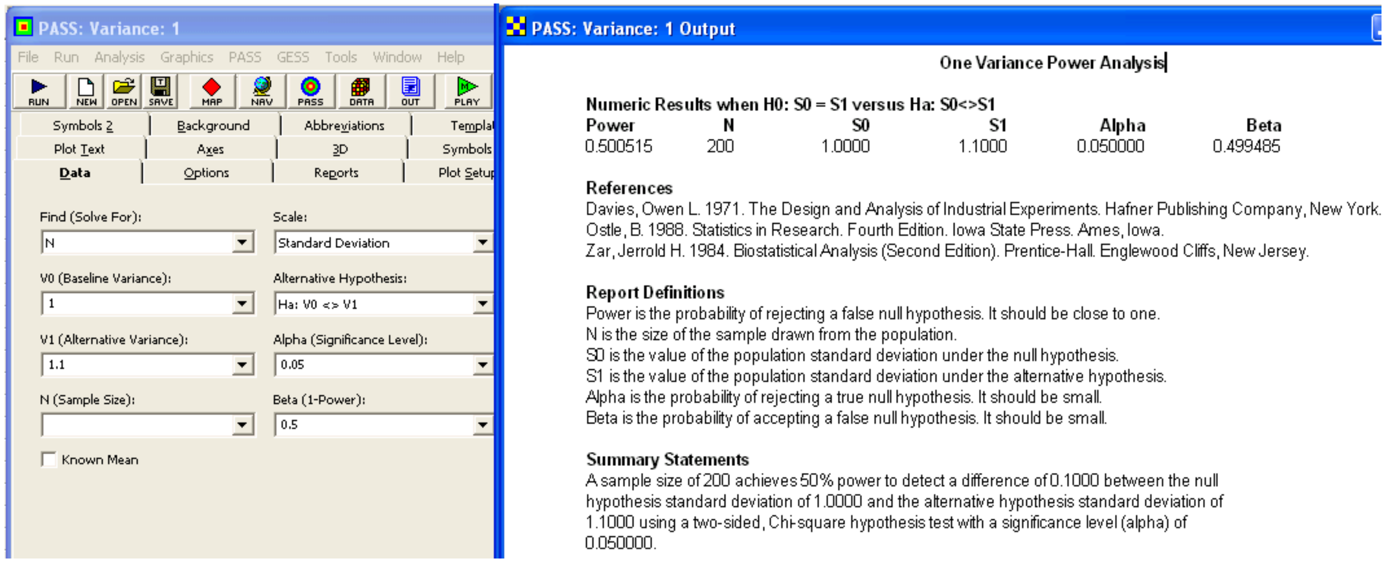
# Chapter 3

# Standard Deviations

## 3.1 One Standard Deviation

**Example 3.1** Determine the sample size required to construct the 95% confidence interval for $\sigma$ based on a random sample of size $n$ drawn from a normal population if the confidence interval half-width must be about 10% of the sample standard deviation.
**Solution:** From Table 3.1 the sample size must be about $n = 200$. The lower and upper confidence limits will fall at about -9% and +11% relative to the sample standard deviation, so the asymmetry for this relatively large sample size is not too severe.

From **PASS**> **Variance**> **Variance: 1 Group** (Note that when the **Scale** text box is set to *Standard Deviation*, other text boxes on the form with labels that refer to variances are interpreted as standard deviations.):




**Example 3.2** Use the large sample approximation method to determine the sample size for the situation in Example 3.1.
**Solution:** The required confidence interval has the form

$$P\left(s\left(1-0.10\right)<\sigma<s\left(1+0.10\right)\right)=0.95.$$

With $\alpha=0.05$ and $\delta=0.10$ in Equation 3.8, the sample size required to obtain a confidence interval of the desired half-width is

$$n=\frac{1}{2}\left(\frac{1.96}{0.10}\right)^{2}=193,$$

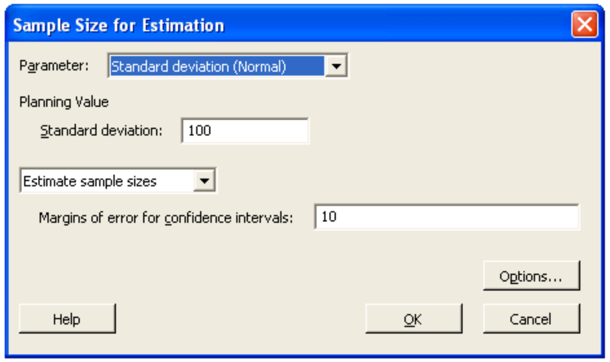
which is in excellent agreement with the original solution.

From **MINITAB (V16)**> **Stat**> **Power and Sample Size**> **Sample Size for Estimation**> **Standard deviation (Normal)**:

```
MTB > SSCI;
SUBC>   NStDev 100;
SUBC>   Confidence 95.0;
SUBC>   IType 0;
SUBC>   MError 10.
```

**Sample Size for Estimation**

```
Method

Parameter            Standard deviation
Distribution         Normal
Standard deviation   100
Confidence level     95%
Confidence interval  Two-sided


Results

  Margin  Sample
of Error    Size
      10     234
```




**Example 3.3** For the test of $H_0 : \sigma^2 = 10$ versus $H_A : \sigma^2 > 10$, find the power associated with $\sigma^2 = 20$ when the sample size is $n = 20$ using $\alpha = 0.05$.
**Solution:** From Equation 3.14 the power is given by

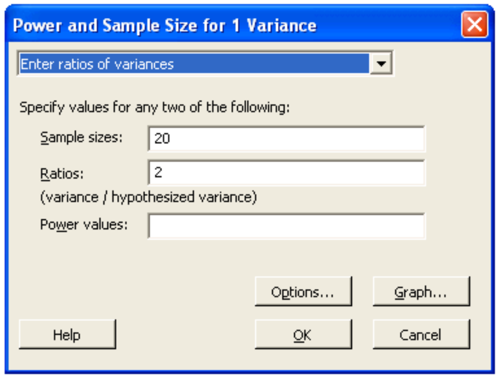
$$\begin{aligned} \pi &= P\left(\chi^2_{0.95}\left(\frac{10}{20}\right) < \chi^2 < \infty\right) \\ &= P\left(15.1 < \chi^2 < \infty\right) \\ &= 0.72. \end{aligned}$$

From **MINITAB (V16)**> **Stat**> **Power and Sample Size**> **1 Variance**:

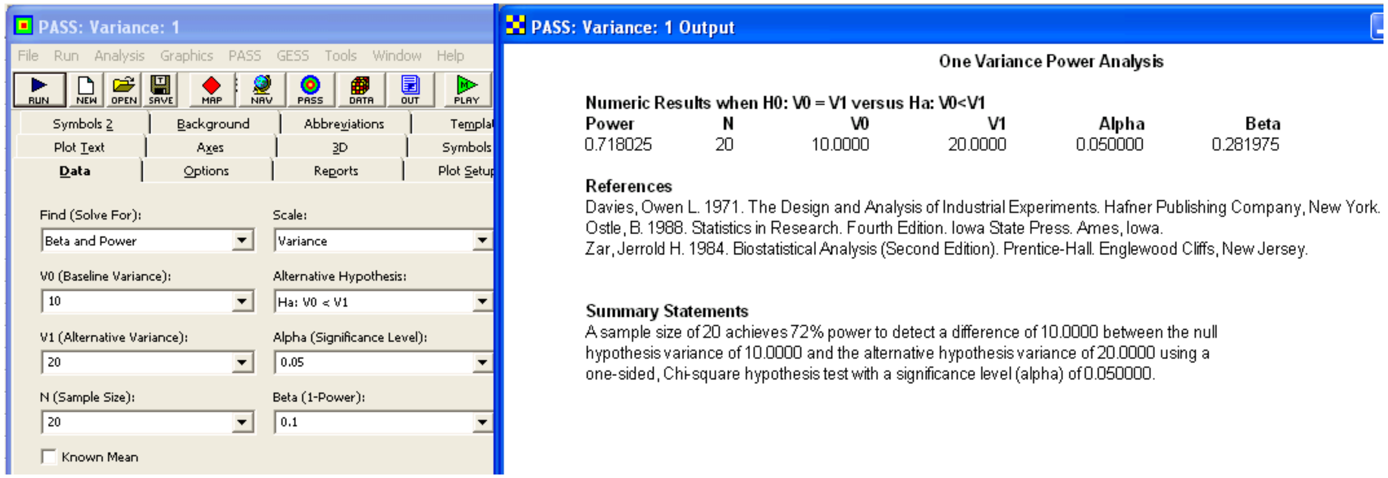
```
MTB > Power;
SUBC>   OneVariance;
SUBC>   Sample 20;
SUBC>   Ratio 2;
SUBC>   Alternative 1;
SUBC>   Alpha 0.05;
SUBC>   GPCurve.
```

**Power and Sample Size**

```
Test for One Variance

Testing variance = null (versus > null)
Calculating power for (variance / null) = ratio
Alpha = 0.05


         Sample
Ratio      Size     Power
    2        20  0.718025
```




From **PASS**> **Variance**> **Variance: 1 Group:**




**Example 3.4** Find the sample size required to reject $H_0 : \sigma^2 = 40$ with 90% power when $\sigma^2 = 100$ using $H_A : \sigma^2 > 40$ with $\alpha = 0.05$.
**Solution:** From Equation 3.15 with $\sigma_0^2 = 40$ and $\sigma_1^2 = 100$, the necessary sample size is the smallest value of $n$ that meets the requirement

$$\begin{aligned}\frac{\chi^2_{0.95}}{\chi^2_{0.10}} &\leq \frac{100}{40}\\ &\leq 2.5.\end{aligned}$$

By inspecting Table 3.2 and a table of $\chi^2$ values, the required sample size is $n = 22$ for which

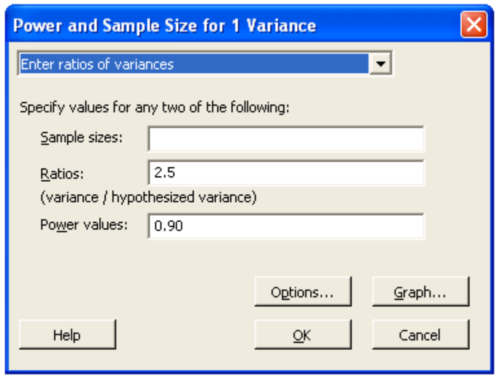
$$\left(\frac{\chi^2_{0.95}}{\chi^2_{0.10}} = \frac{32.67}{13.24} = 2.469\right) \leq 2.5.$$

From **MINITAB (V16)**> **Stat**> **Power and Sample Size**> **1 Variance**:

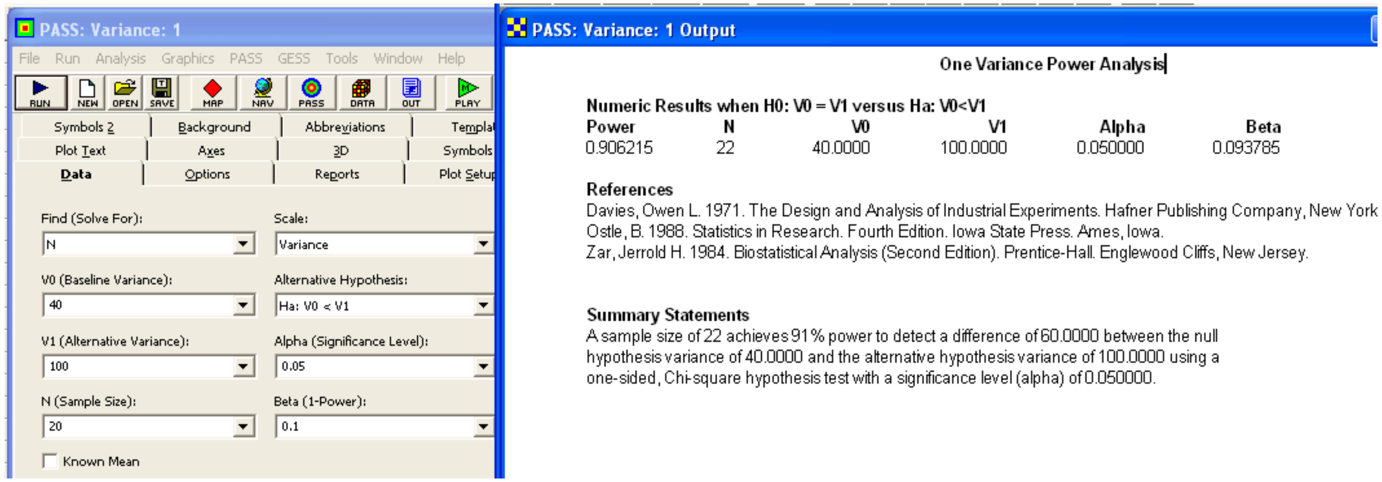
```
MTB > Power;
SUBC>   OneVariance;
SUBC>   Ratio 2.5;
SUBC>   Power 0.90;
SUBC>   Alternative 1;
SUBC>   Alpha 0.05;
SUBC>   GPCurve.
```

**Power and Sample Size**

```
Test for One Variance

Testing variance = null (versus > null)
Calculating power for (variance / null) = ratio
Alpha = 0.05


       Sample  Target
Ratio    Size   Power  Actual Power
  2.5      22     0.9      0.906215
```



From **PASS**> **Variance**> **Variance: 1 Group:**




**Example 3.5** Find the sample size required to reject $H_0 : \sigma = 0.003$ in favor of $H_A : \sigma < 0.003$ with 90% power when in fact $\sigma = 0.001$.
**Solution:** With $\alpha = 0.05$ and $\beta = 1 - \pi = 0.10$, the sample size condition given by Equation 3.17 is

$$\begin{aligned} \frac{\chi^2_{0.90}}{\chi^2_{0.05}} &\geq \left(\frac{0.003}{0.001}\right)^2 \\ &\geq 9.0, \end{aligned}$$

which, from Table 3.2, is satisfied by $n = 5$.

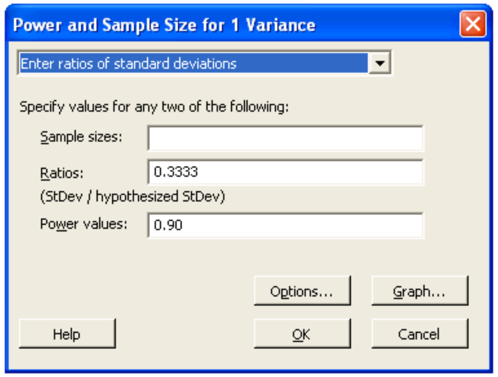
From **MINITAB (V16)**> **Stat**> **Power and Sample Size**> **1 Variance**:

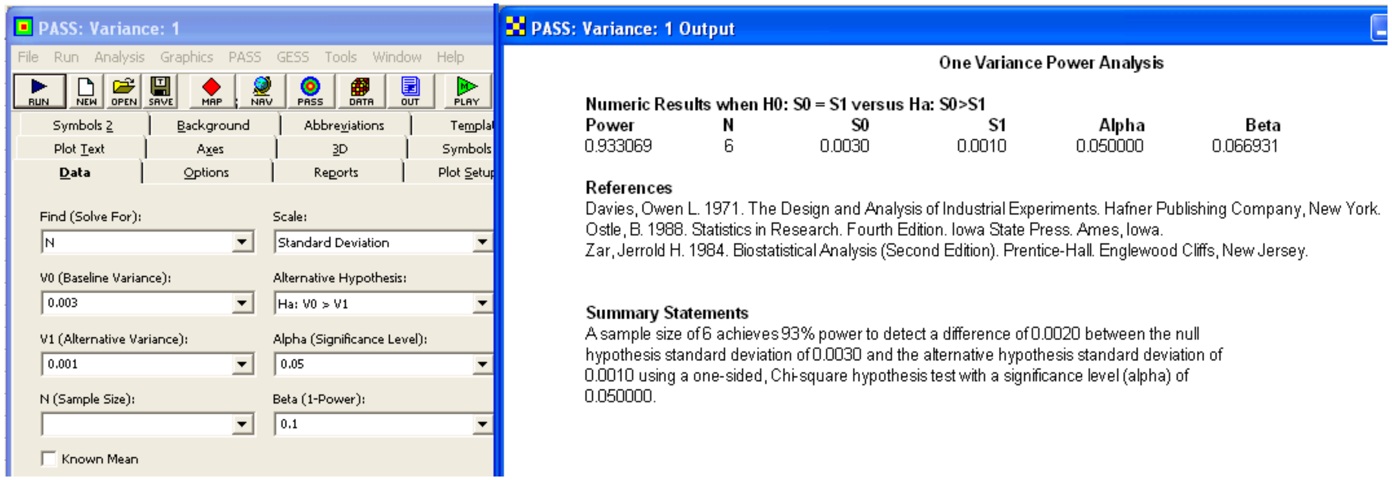
From **PASS**> **Variance**> **Variance: 1 Group:**

```
MTB > Power;
SUBC>   OneVariance;
SUBC>   StDeviation;
SUBC>   Ratio 0.3333;
SUBC>   Power 0.90;
SUBC>   Alternative -1;
SUBC>   Alpha 0.05;
SUBC>   GPCurve.
```

**Power and Sample Size**

```
Test for One Standard Deviation

Testing StDev = null (versus < null)
Calculating power for (StDev / null) = ratio
Alpha = 0.05


         Sample  Target
 Ratio     Size   Power  Actual Power
0.3333        6     0.9      0.933121
```




**Example 3.6** Compare the power determined by the large-sample approximation method to the exact power determined in Example 3.3.

**Solution:** The null hypothesis may be written as $H_0 : \ln(\sigma) = \ln\left(\sqrt{10}\right)$ and we wish to find the power to reject $H_0$ when $\ln(\sigma) = \ln\left(\sqrt{20}\right)$ with $n = 20$. From Equation 3.20 we have

$$z_\beta = \sqrt{2 \times 20} \ln\left(\sqrt{\frac{20}{10}}\right) - z_{0.05} = 0.547$$

and by Equation 3.19 the approximate power is

$$\begin{aligned}\pi &= \Phi\left(-0.547 < z < \infty\right) \\ &= 0.71.\end{aligned}$$

This result is still in good agreement with the exact power of 72% despite the rather small sample size.

**Example 3.7** Compare the sample size determined by the large-sample approximation method to the exact sample size determined in Example 3.4.
**Solution:** The problem is to find the sample size to reject $H_0 : \ln(\sigma) = \ln\left(\sqrt{40}\right)$ with 90% power when $\ln(\sigma) = \ln\left(\sqrt{100}\right)$. With $\alpha = 0.05$ and $\beta = 0.10$ in Equation **??** the approximate sample size required is

$$n = \frac{1}{2}\left(\frac{1.645 + 1.282}{\ln\left(\sqrt{\frac{100}{40}}\right)}\right)^2 = 21,$$

which is in good agreement with the exact sample size of $n = 22$.

## 3.2 Two Standard Deviations

**Example 3.8** What equal-$n$ sample size is required by an experiment to deliver a confidence interval for the ratio of two independent population standard deviations if the true ratio should fall within 20% of the experimental ratio with 95% confidence?
**Solution:** The goal of the experiment is to determine an interval of the form

$$P\left(\frac{s_1}{s_2}(1-0.2) < \frac{\sigma_1}{\sigma_2} < \frac{s_1}{s_2}(1+0.2)\right) = 1-\alpha.$$

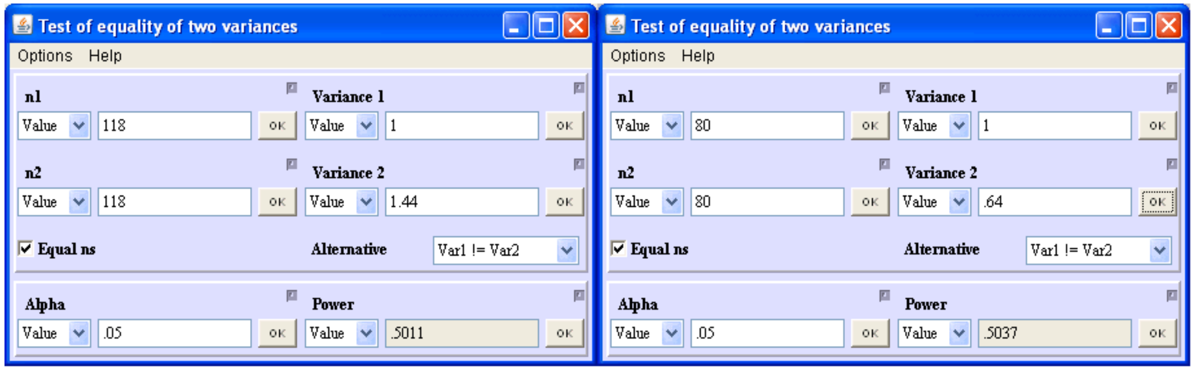
Then, from Equation **??** with $\delta = 0.2$, the required sample sizes are

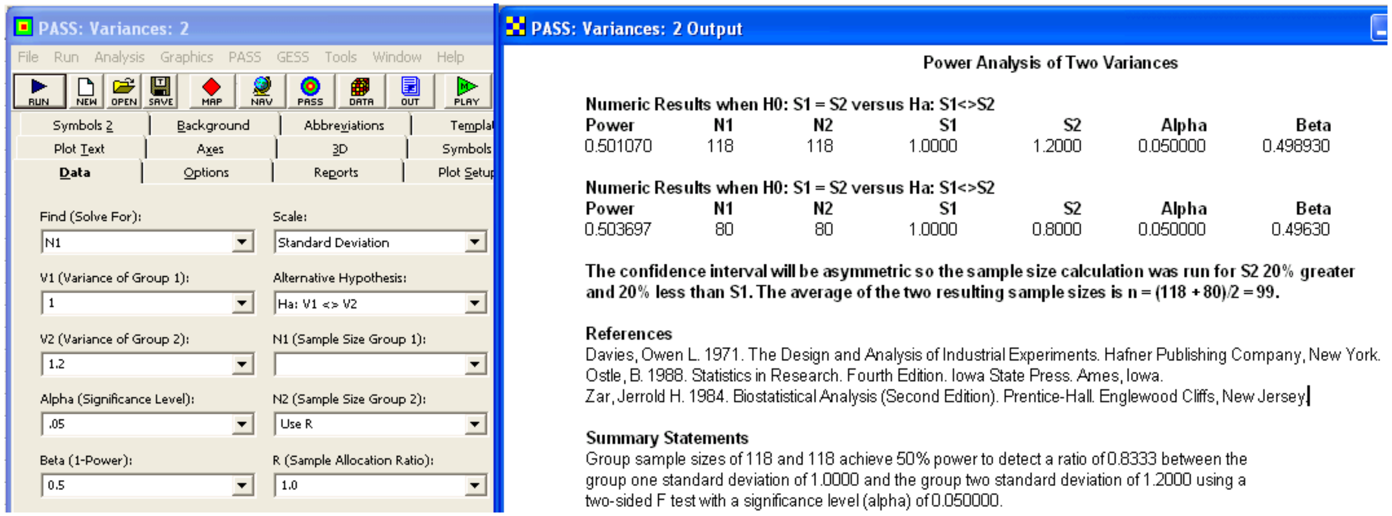
$$n_1 = n_2 = \left(\frac{z_{0.025}}{\delta}\right)^2 = \left(\frac{1.96}{0.20}\right)^2 = 97.$$

Piface and PASS support the $F$ test for two variances which can be tricked into confirming the sample size for the confidence interval. The confidence interval is asymmetric so the sample size is taken as the average of the sample sizes required for $\sigma_2/\sigma_1 = 1.2$ and $\sigma_2/\sigma_1 = 0.8$.

From **Piface**> **Two Variances (F Test)**:

From **PASS**> **Variance**> **Variance: 2 Groups:**




The sample sizes are $n = 118$ and $n = 80$, respectively, so the average sample size is $n = (118 + 80)/2 = 99$ which is in excellent agreement with the normal approximation.

**Example 3.9** Find the power to reject $H_0 : \sigma_1^2 = \sigma_2^2$ in favor of $H_A : \sigma_1^2 > \sigma_2^2$ if $n_1 = n_2 = 26$, $\sigma_1^2 = 15$, and $\sigma_2^2 = 5$ using $\alpha = 0.05$.

**Solution:** From Equation 3.34 the power is

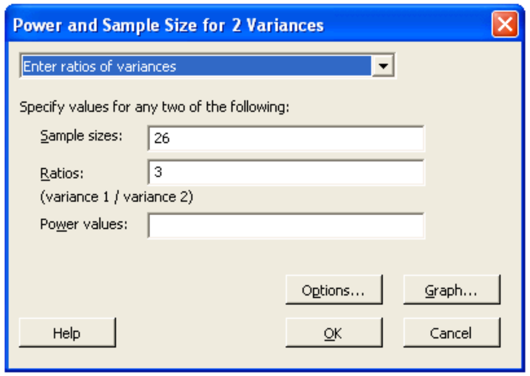
$$\begin{aligned}
\pi &= P\left(\left(\frac{\sigma_2}{\sigma_1}\right)^2 F_{1-\alpha} < F < \infty\right) \\
&= P\left(\left(\frac{5}{15}\right) F_{0.95,25,25} < F < \infty\right) \\
&= P\left(0.652 < F < \infty\right) \\
&= 0.854.
\end{aligned}$$

From **MINITAB (V16)**> **Stat**> **Power and Sample Size**> **2 Variances**:

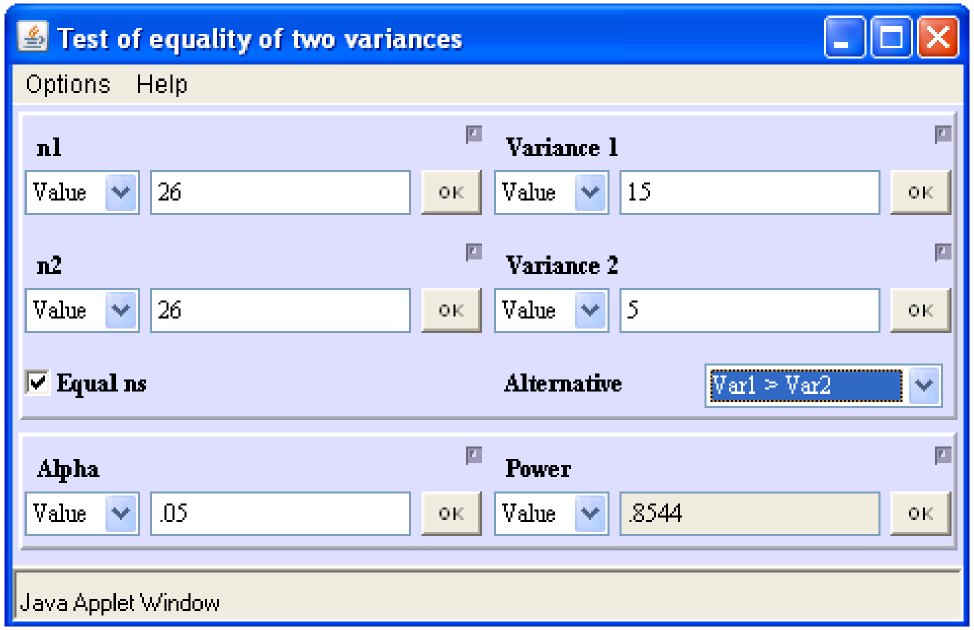
```
MTB > Power;
SUBC>   TwoVariance;
SUBC>   Sample 26;
SUBC>   Ratio 3;
SUBC>   Alternative 1;
SUBC>   Alpha 0.05;
SUBC>   FTest;
SUBC>   GPCurve.
```

**Power and Sample Size**

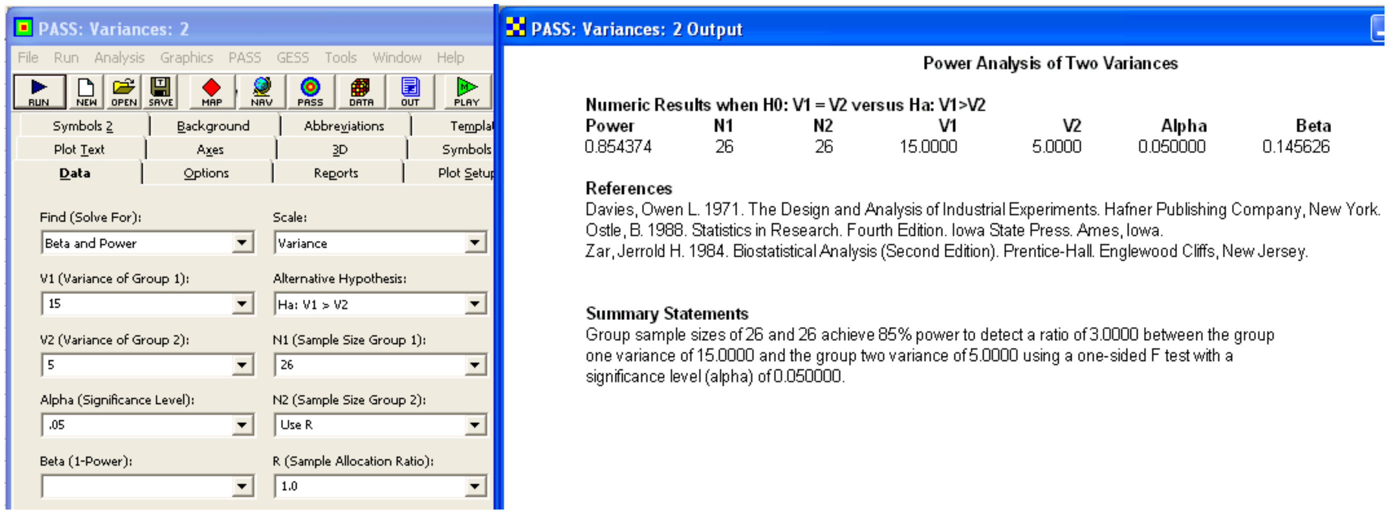
```
Test for Two Variances

Testing (variance 1 / variance 2) = 1 (versus >)
Calculating power for (variance 1 / variance 2) = ratio
Alpha = 0.05
Method:  F Test


        Sample
Ratio     Size     Power
    3       26  0.854374

The sample size is for each group.
```



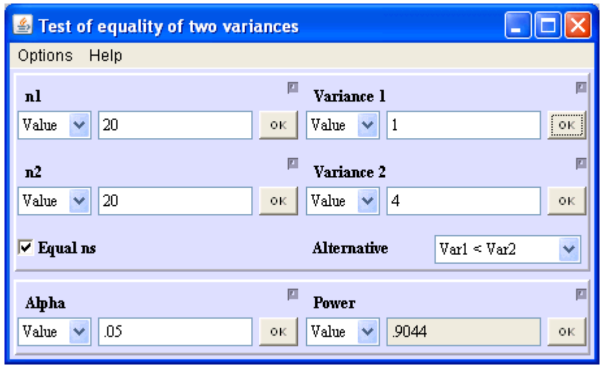

From **Piface**> **Two Variances (F Test)**:




From **PASS**> **Variance**> **Variance: 2 Groups:**




**Example 3.10** What equal sample size is required to detect a factor of two difference between two population standard deviations with 90% power and $\alpha = 0.05$?




**Solution:** A factor of two difference in population standard deviation corresponds to a factor of four difference in variance, so we need to determine the sample size such that

$$P\left(\frac{1}{4}F_{1-\alpha} < F < \infty\right) = 0.90.$$

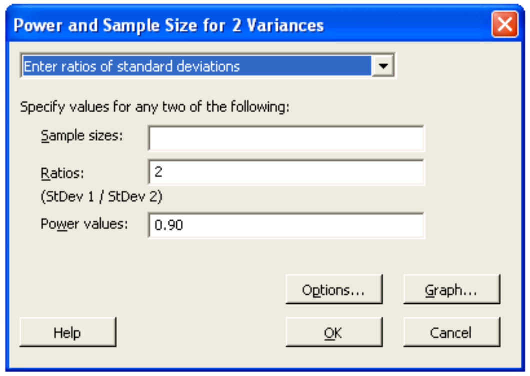
By iterating through several values of sample size, we find that when $n_1 = n_2 = 20$, $F_{0.95,19,19} = 2.168$ and $F_{0.096} = 2.168/4 = 0.542$, which satisfies the problem statement.

From **MINITAB (V16)**> **Stat**> **Power and Sample Size**> **2 Variances**:

```
MTB > Power;
SUBC>   TwoVariance;
SUBC>   StDeviation;
SUBC>   Ratio 2;
SUBC>   Power 0.90;
SUBC>   Alternative 1;
SUBC>   Alpha 0.05;
SUBC>   FTest;
SUBC>   GPCurve.
```

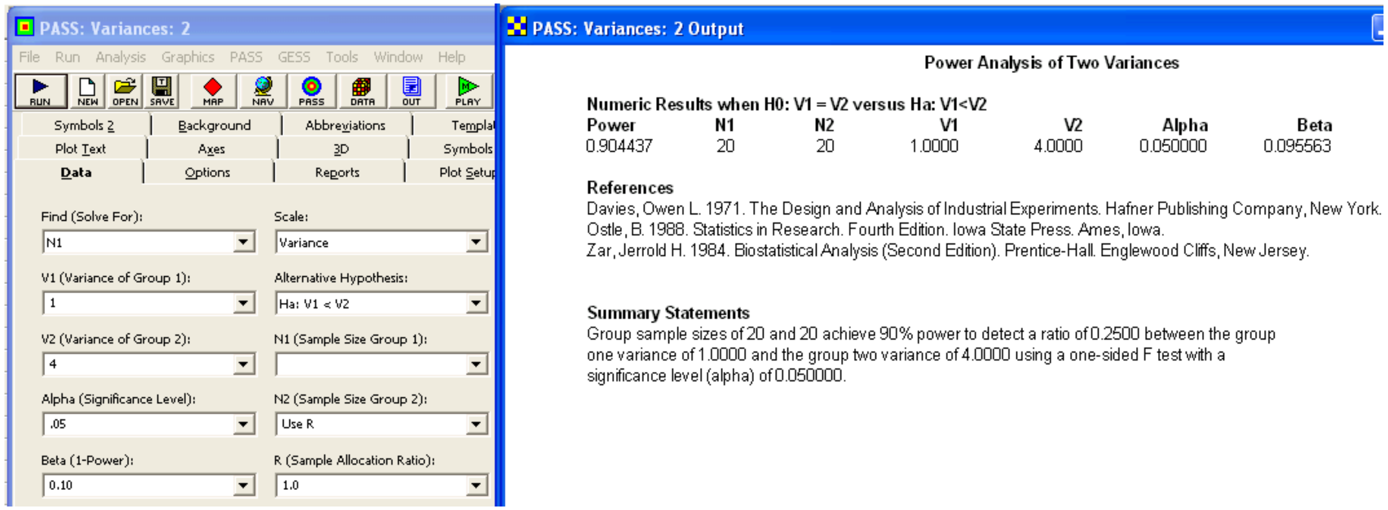
**Power and Sample Size**

```
Test for Two Standard Deviations

Testing (StDev 1 / StDev 2) = 1 (versus >)
Calculating power for (StDev 1 / StDev 2) = ratio
Alpha = 0.05
Method:  F Test


        Sample  Target
Ratio     Size   Power  Actual Power
    2       20     0.9      0.904437

The sample size is for each group.
```




From **Piface**> **Two Variances (F Test)**:

From **PASS**> **Variance**> **Variance: 2 Groups:**




**Example 3.11** Repeat Example 3.9 using the large-sample approximation method.
**Solution:** From the information given in the example problem statement

$$z_\beta = \frac{\ln\left(\sqrt{\frac{15}{5}}\right)}{\sqrt{\frac{1}{2}\left(\frac{1}{26}+\frac{1}{26}\right)}} - 1.645 = 1.16$$

so the power is

$$\pi = \Phi\left(-1.16 < z < \infty\right) = 0.877,$$

which is in good agreement with the exact solution of $\pi = 0.854$.

**Example 3.12** Repeat Example 3.10 using the large-sample approximation method.
**Solution:** From the information given in the example problem statement

$$n_1 = n_2 = \left(\frac{1.645 + 1.282}{\ln\left(2\right)}\right)^2 = 18,$$

which slightly underestimates the exact solution $n = 20$.

## 3.3 Coefficient of Variation

**Example 3.13** Determine the sample size required to estimate the population coefficient of variation to within ±25% with 95% confidence if the coefficient of variation is expected to be about 30%.

**Solution:** With $\alpha = 0.05$, $\delta = 0.25$, and $\widehat{CV} = 0.3$ in Equation 3.45, the sample size must be

$$n = \left(\frac{1.96}{0.25}\right)^2 \left((0.3)^2 + \frac{1}{2}\right) = 37.$$

**Example 3.14** Determine the sample size required to reject $H_0 : CV = 0.5$ with 90% power when $CV = 0.8$.
**Solution:** With $CV_0 = 0.5$, $CV_1 = 0.8$, $\alpha = 0.05$, and $\beta = 0.10$ in Equation 3.49, the required sample size is

$$n = \left(\frac{1.96 \times 0.5\sqrt{(0.5)^2 + \frac{1}{2}} + 1.282 \times 0.8\sqrt{(0.8)^2 + \frac{1}{2}}}{0.8 - 0.5}\right)^2 = 42.$$

**Example 3.15** Determine the sample size required to reject $H_0 : CV_1 = CV_2$ in favor of $H_A : CV_1 \neq CV_2$ with 90% power when $CV_1 = 0.3$ and $CV_2 = 0.5$.
**Solution:** With $CV_1 = 0.3$, $CV_2 = 0.5$, $\alpha = 0.05$, and $\beta = 0.10$ in Equation 3.57, the required sample size is

$$n = \left(\frac{1.96 \times 0.3\sqrt{(0.3)^2 + \frac{1}{2}} + 1.282 \times 0.5\sqrt{(0.5)^2 + \frac{1}{2}}}{0.3 - 0.5}\right)^2 = 26.$$

# Chapter 4

# Proportions

## 4.1 One Proportion (Large Population)

**Example 4.1** How large a random sample is required to demonstrate that the fraction defective of a process is less than 1% with 95% confidence?
**Solution:** The required confidence interval has the form

$$P(0 < p < 0.01) = 0.95$$

so $p_U = 0.01$ and $\alpha = 0.05$. If we assume that the sample size is small compared to the lot size, then Equation 4.4 can be used to approximate the sample size. However, because the number of defectives allowed in the sample was not specified, we must consider the possibility of different $X$ values. For $X = 0$, by the rule of three (Equation 4.5), the sample size is

$$\begin{aligned} n &\simeq \frac{3}{0.01} \\ &\simeq 300. \end{aligned}$$

For $X = 1$, by Equation 4.4

$$\begin{aligned} n &\simeq \frac{\chi^2_{0.95,4}}{2(0.01)} \\ &\simeq \frac{9.49}{2(0.01)} \\ &\simeq 475. \end{aligned}$$

The values of $n$ can be found for other choices of $X$ in a similar manner.

**Piface**> **CI for one proportion, PASS**> **Proportions**> **One Group**> **Confidence Interval - Proportion**, and **MINITAB**> **Stat**> **Power and Sample Size**> **1 Proportion** use the normal approximation to the binomial distribution to calculate the sample size for a symmetric two-tailed confidence interval but the normal approximation isn't valid for this problem.

**Example 4.2** What fraction of a large population must be inspected and found to be free of defectives to be 95% confident that the population contains no more than ten defectives?
**Solution:** The goal of the experiment is to demonstrate that the population defective count satisfies the confidence interval $P(0 < S \leq 10) = 0.95$. With $X = 0$ and $\alpha = 0.05$ in Equation 4.7, the fraction of the population that will need to be inspected is

$$\frac{n}{N} \simeq \frac{\chi^2_{0.95}}{2S_U} \quad (4.1)$$

$$\simeq \frac{3}{10} \quad (4.2)$$

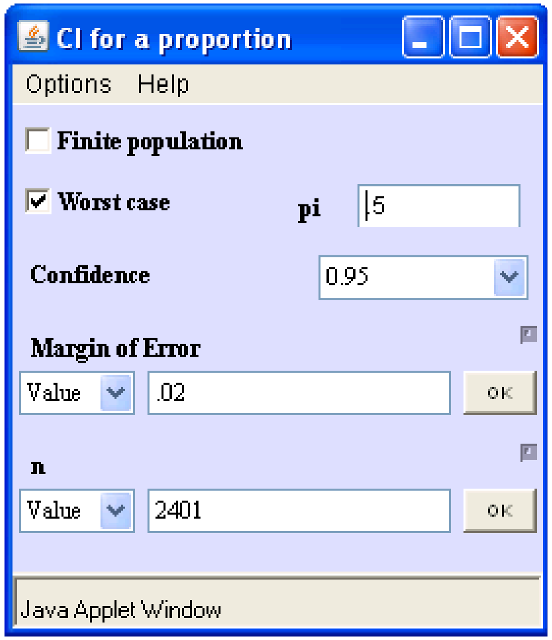
$$\simeq 0.30.$$

This result violates the small-sample approximation requirement that $n \ll N$, but it provides a good starting point for iterations toward a more accurate result. When $n$ becomes a substantial fraction of $N$, use the method shown in Section 10.4.1.2 instead. (This example is re-solved using that method in Example 10.21.)

**Example 4.3** How many people should be polled to estimate voter preference for two candidates in a close election if the poll result must be within 2% of the truth with 95% confidence?
**Solution:** From Equation 4.15 with confidence interval half-width $\delta = 0.02$ the required sample size is

$$n = \frac{1}{(0.02)^2} = 2500.$$

From **Piface**> **CI for one proportion**:

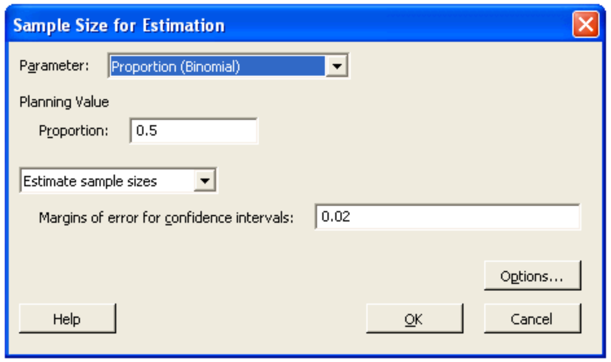


From **MINITAB (V16)**> **Stat**> **Power and Sample Size**> **Sample Size for Estimation**> **Proportion (Binomial)**:

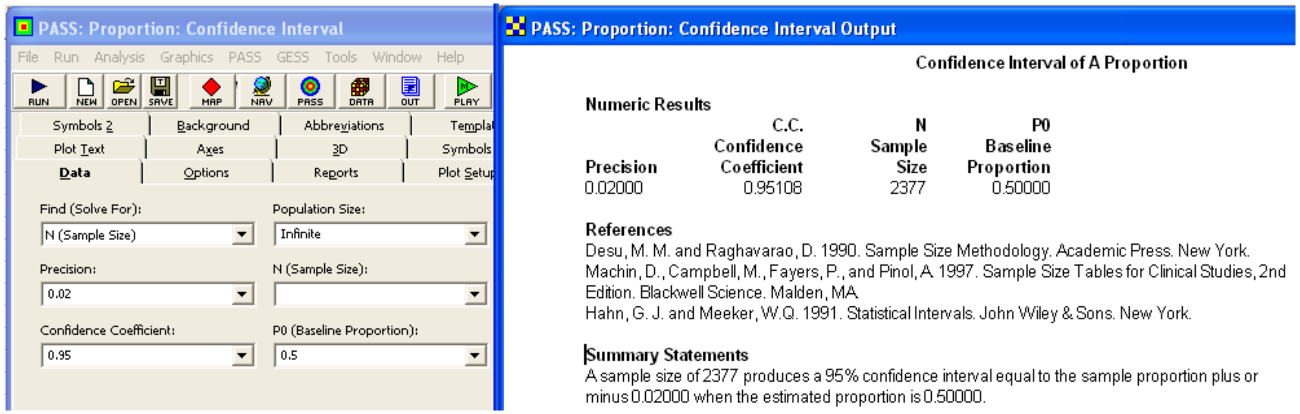
```
MTB > SSCI;
SUBC>   BProportion 0.5;
SUBC>   Confidence 95.0;
SUBC>   IType 0;
SUBC>   MError 0.02.
```

**Sample Size for Estimation**

```
Method

Parameter            Proportion
Distribution         Binomial
Proportion           0.5
Confidence level     95%
Confidence interval  Two-sided


Results

  Margin  Sample
of Error    Size
    0.02    2449
```




From **PASS**> **Proportions**> **One Group**> **Confidence Interval - Proportion**:

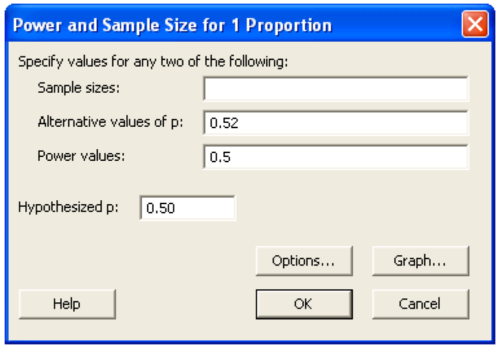



From **MINITAB**> **Stat**> **Power and Sample Size**> **1 Proportion**

```
MTB > Power;
SUBC>   POne;
SUBC>     PAlternative 0.52;
SUBC>     Power 0.5;
SUBC>     PNull 0.50;
SUBC>   GPCurve.
```

**Power and Sample Size**

```
Test for One Proportion

Testing proportion = 0.5 (versus not = 0.5)
Alpha = 0.05


Alternative  Sample  Target
 Proportion    Size   Power  Actual Power
       0.52    2401     0.5      0.500058
```



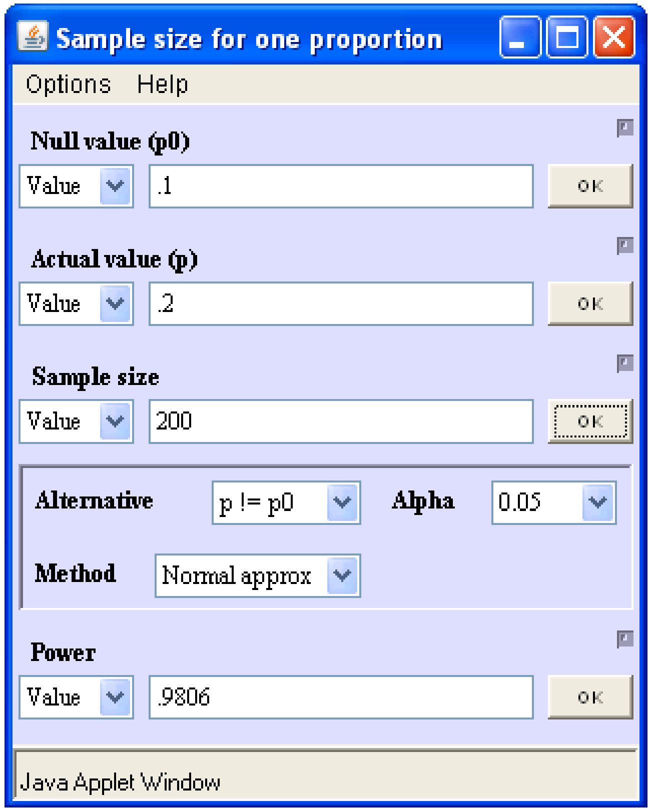

**Example 4.4** Find the power to reject $H_0 : p = 0.1$ when in fact $p = 0.2$ and the sample will be of size $n = 200$.
**Solution:** Under both $H_0$ and $H_A$ the sample size is sufficiently large to justify the use of normal approximations to the binomial distributions. From Equation 4.21 with $\alpha = 0.05$ we have

$$z_\beta = \frac{\sqrt{200}\,|0.2 - 0.1| - z_{0.025}\sqrt{(0.1)\,(1-0.1)}}{\sqrt{(0.2)\,(1-0.2)}} = 2.066,$$

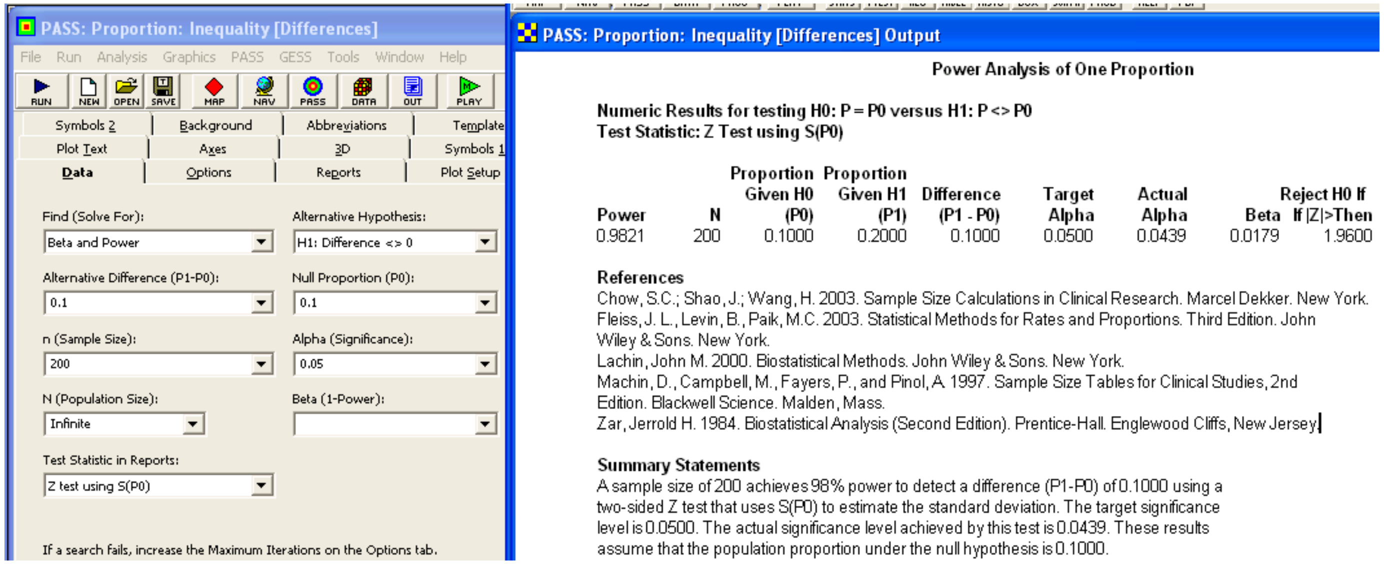
so the power is

$$\begin{aligned}\pi &= 1 - \Phi\left(-\infty < z < 2.066\right) \\ &= 0.981.\end{aligned}$$

From **Piface**> **Test of one proportion**:




From **PASS**> **Proportions**> **One Group**> **Inequality [Differences]:**

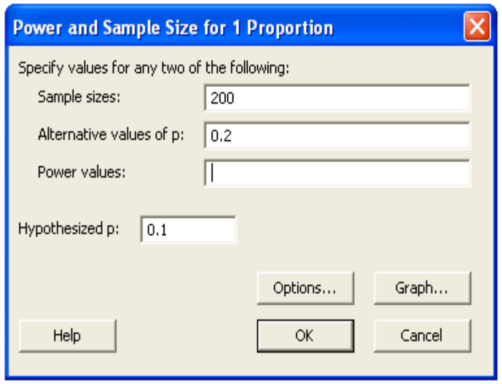



From **MINITAB**> **Stat**> **Power and Sample Size**> **1 Proportion:**

```
MTB > Power;
SUBC>   POne;
SUBC>     Sample 200;
SUBC>     PAlternative 0.2;
SUBC>     PNull 0.1;
SUBC>   GPCurve.
```

**Power and Sample Size**

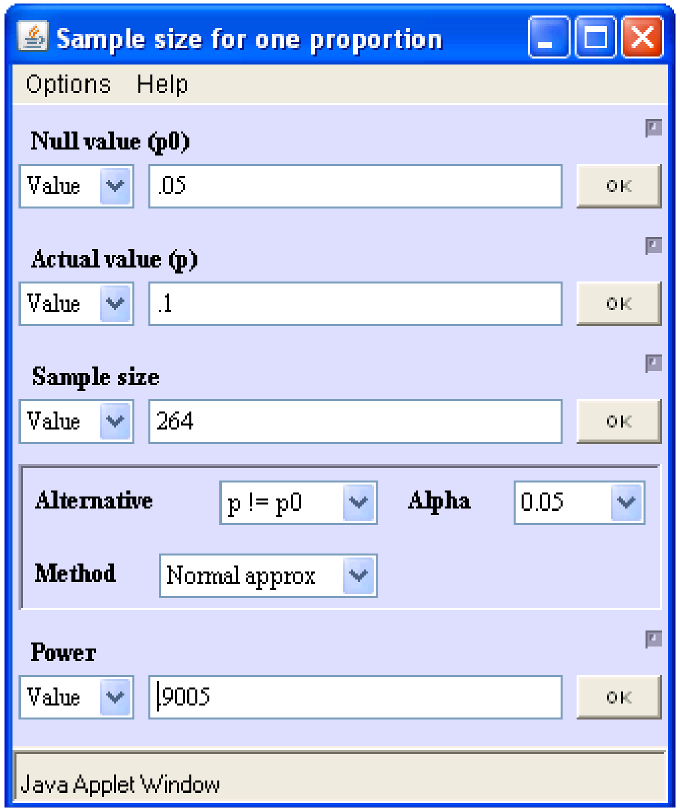
```
Test for One Proportion

Testing proportion = 0.1 (versus not = 0.1)
Alpha = 0.05


Alternative  Sample
 Proportion    Size     Power
        0.2     200  0.980565
```



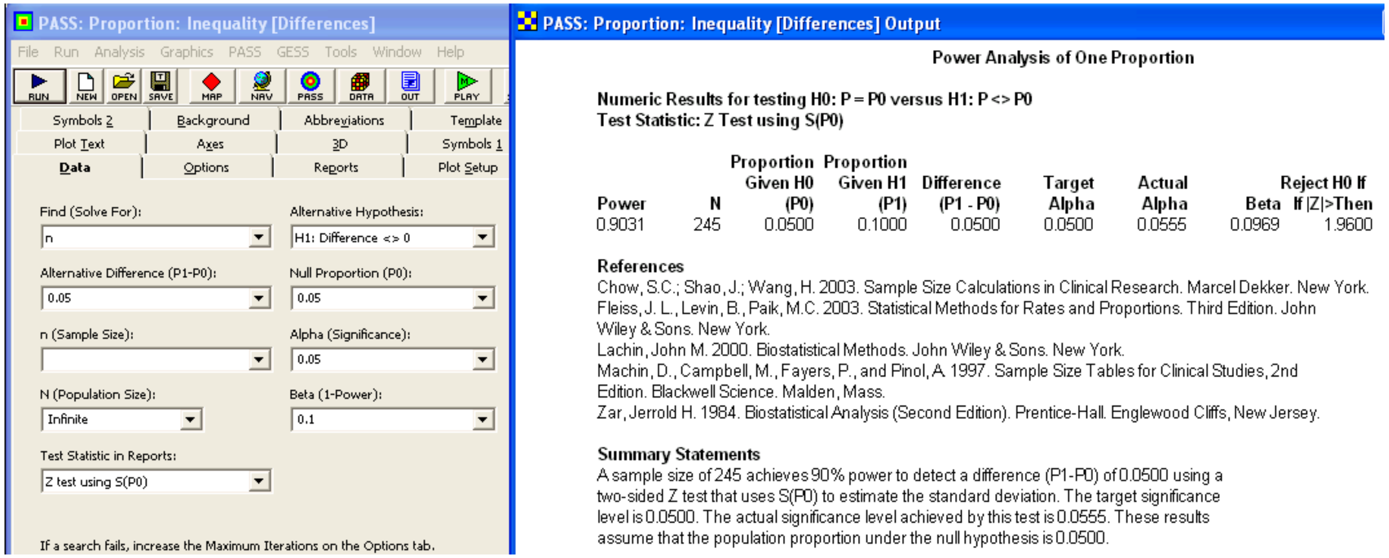

**Example 4.5** What sample size is required to reject $H_0 : p = 0.05$ when in fact $p = 0.10$ using a two-sided test with 90% power?
**Solution:** Assuming that the sample size will be sufficiently large to justify the normal approximation method, from Equation 4.22 the required sample size is

$$\begin{aligned} n &= \left( \frac{1.96\sqrt{(0.05)(1-0.05)} + 1.282\sqrt{(0.10)(1-0.10)}}{0.10 - 0.05} \right)^2 \\ &= 264. \end{aligned}$$

From **Piface**> **Test of one proportion**:




From **PASS**> **Proportions**> **One Group**> **Inequality [Differences]:**

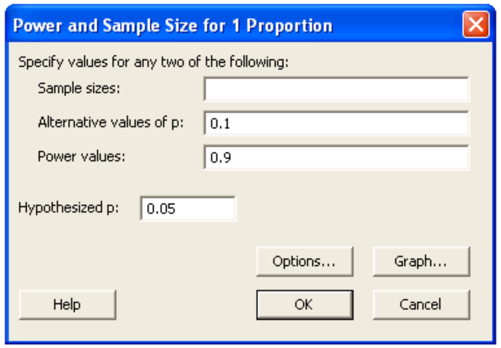



From **MINITAB**> **Power and Sample Size**> **1 Proportion:**

```
MTB > Power;
SUBC>   POne;
SUBC>     PAlternative 0.1;
SUBC>     Power 0.9;
SUBC>     PNull 0.05;
SUBC>   GPCurve.
```

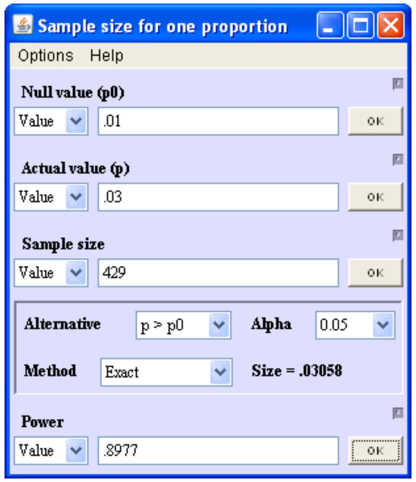
**Power and Sample Size**

```
Test for One Proportion

Testing proportion = 0.05 (versus not = 0.05)
Alpha = 0.05


Alternative  Sample  Target
 Proportion    Size   Power  Actual Power
        0.1     264     0.9      0.900470
```

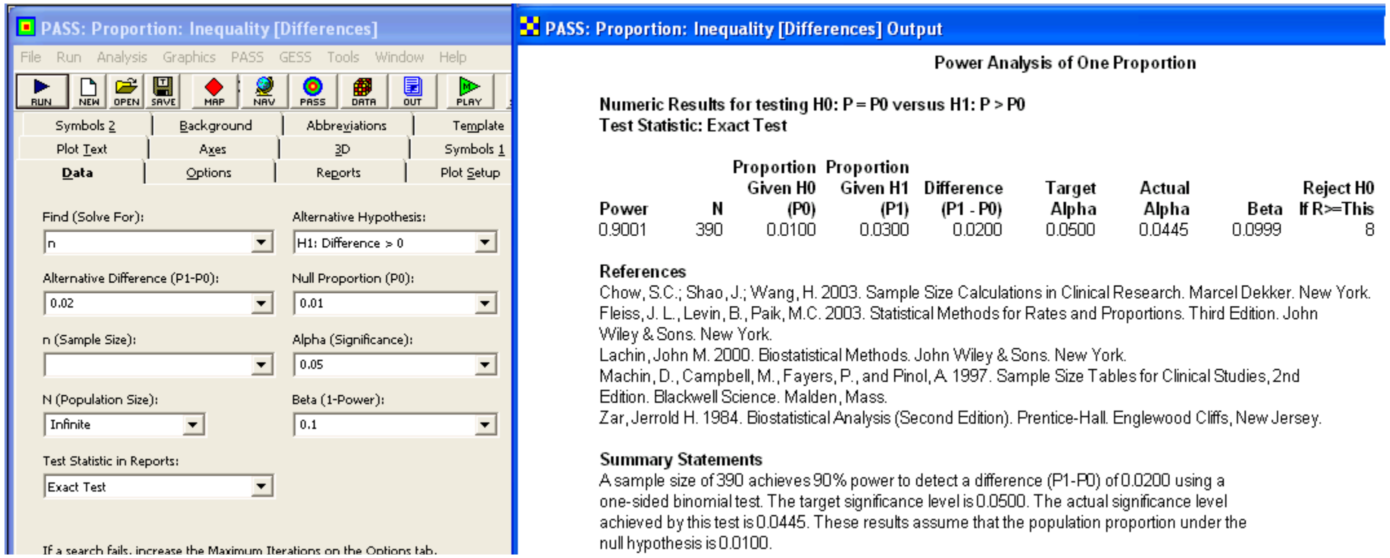



**Example 4.6** What sample size is required to reject $H_0 : p = 0.01$ with 90% power when in fact $p = 0.03$?
**Solution:** The hypotheses to be tested are $H_0 : p = 0.01$ versus $H_A : p > 0.01$ and the two points on the OC curve are $(p_0, 1 - \alpha) = (0.01, 0.95)$ and $(p_1, \beta) = (0.03, 0.10)$. The exact simultaneous solution to Equations 4.24 and 4.25, obtained using Larson's nomogram and then iterating to the exact solution using a binomial calculator, is $(n, c) = (390, 7)$. The distributions of the success counts under $H_0$ and $H_A$ are shown in Figure 4.2.

From **Piface**> **Test of one proportion**:




From **PASS**> **Proportions**> **One Group**> **Inequality [Differences]:**

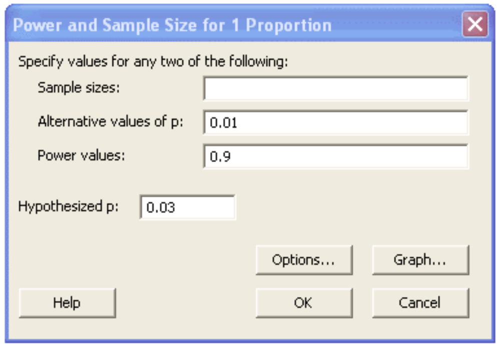


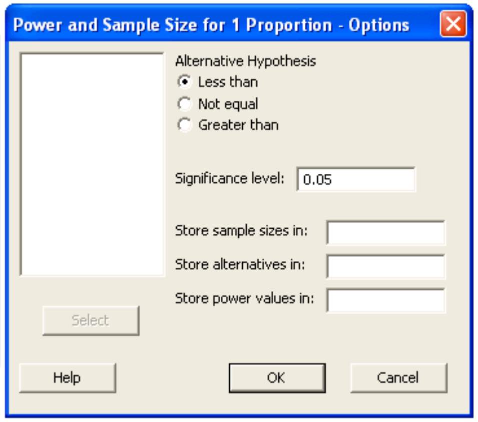

From **MINITAB**> **Power and Sample Size**> **1 Proportion:**

```
MTB > Power;
SUBC>   POne;
SUBC>     PAlternative 0.01;
SUBC>     Power 0.9;
SUBC>     PNull 0.03;
SUBC>     Alternative -1;
SUBC>   GPCurve.
```

**Power and Sample Size**

```
Test for One Proportion

Testing proportion = 0.03 (versus < 0.03)
Alpha = 0.05


Alternative  Sample  Target
 Proportion    Size   Power  Actual Power
       0.01     417     0.9      0.900542
```

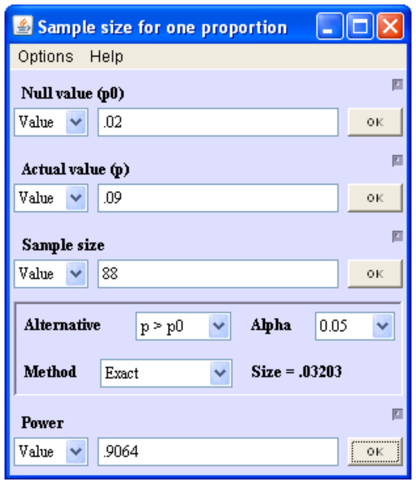


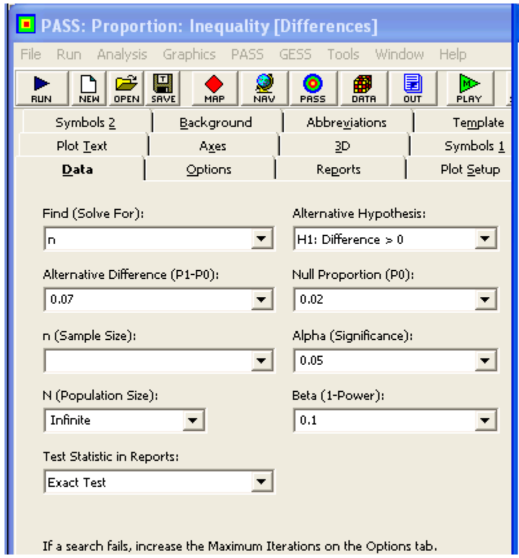

**Example 4.7** Use Larson's nomogram to find $n$ and $c$ for the sampling plan for defectives that will accept 95% of lots with 2% defectives and 10% of lots with 8% defectives. Draw the OC curve.

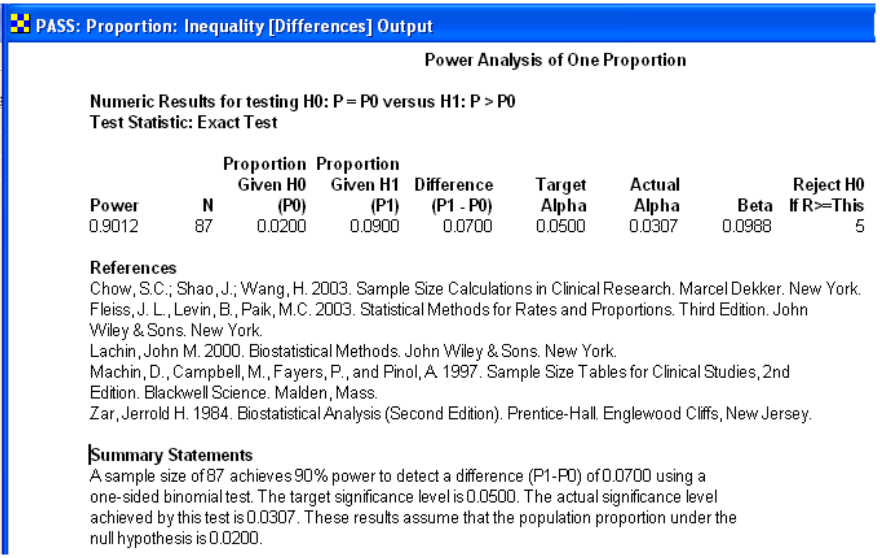
**Solution:** Figure 4.3 shows the solution using Larson's nomogram with the two specified points on the OC curve at $(p, P_A(H_0)) = (0.02, 0.95)$ and $(0.09, 0.10)$. The required sampling plan is $n = 100$ and $c = 4$. The OC curve is shown in Figure 4.4. Points on the OC were obtained by rocking a line about the point at $n = 100$ and $c = 4$ in the nomogram and reading off $p$ and $P_A$ values.

From **Piface**> **Test of one proportion**:




From **PASS**> **Proportions**> **One Group**> **Inequality [Differences]:**



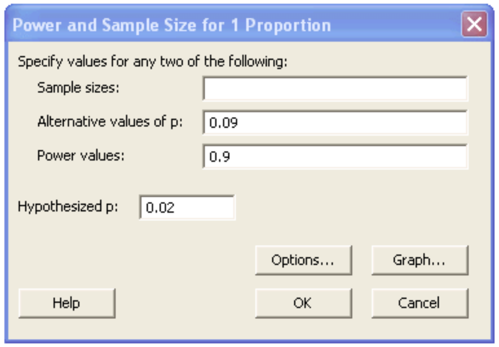



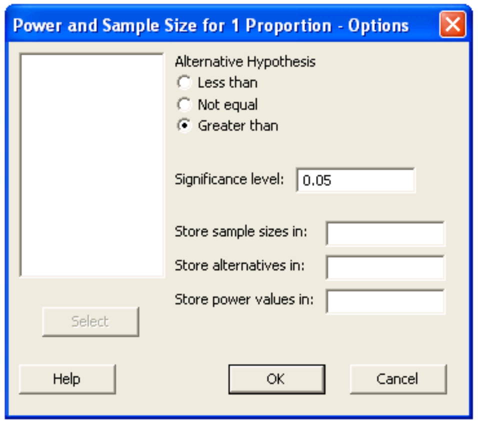

From **MINITAB**> **Stat**> **Power and Sample Size**> **1 Proportion:**

```
MTB > Power;
SUBC>   POne;
SUBC>     PAlternative 0.09;
SUBC>     Power 0.9;
SUBC>     PNull 0.02;
SUBC>     Alternative 1;
SUBC>   GPCurve.
```

**Power and Sample Size**

```
Test for One Proportion

Testing proportion = 0.02 (versus > 0.02)
Alpha = 0.05


Alternative  Sample  Target
 Proportion    Size   Power  Actual Power
       0.09      73     0.9      0.900639
```

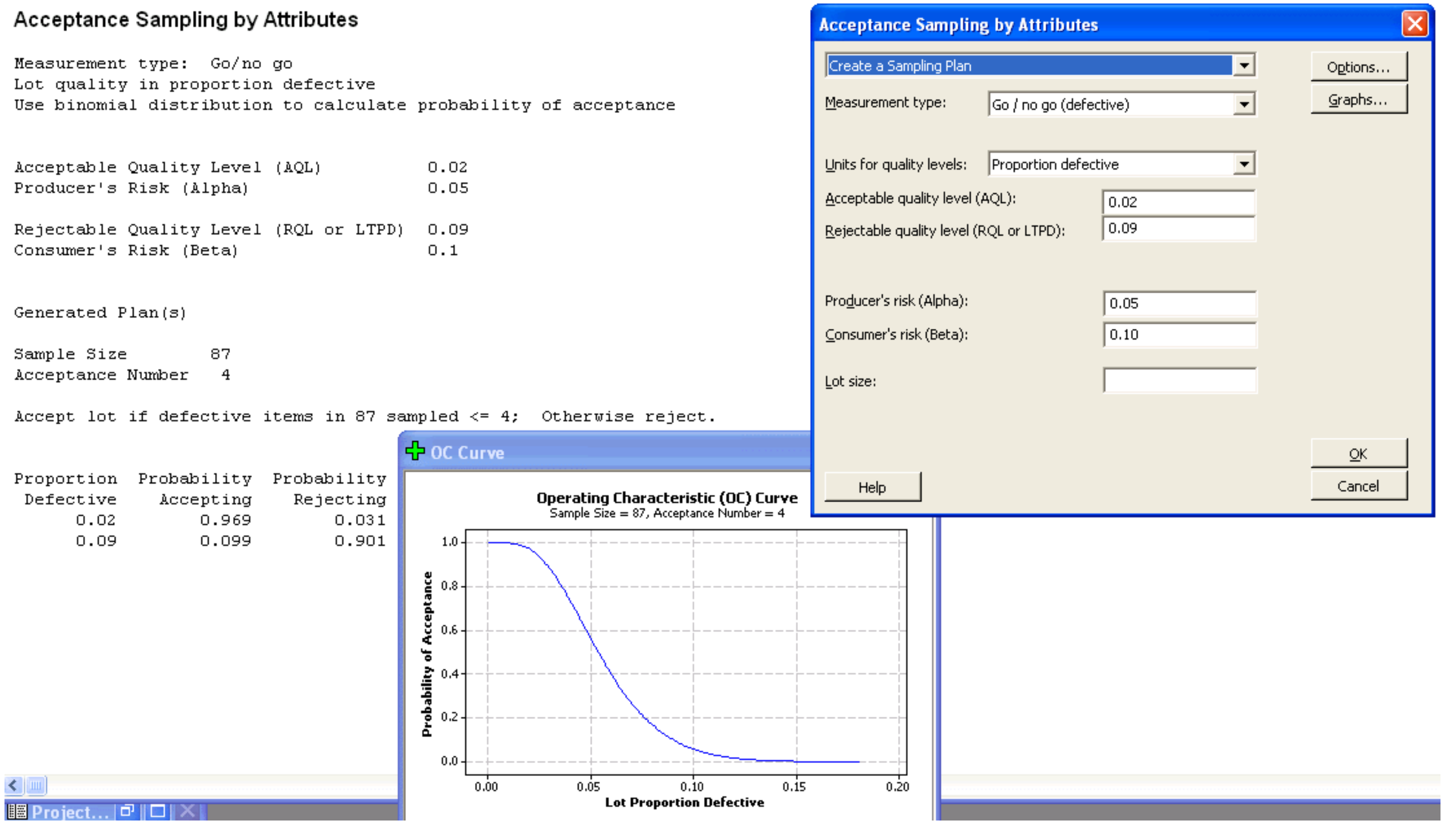


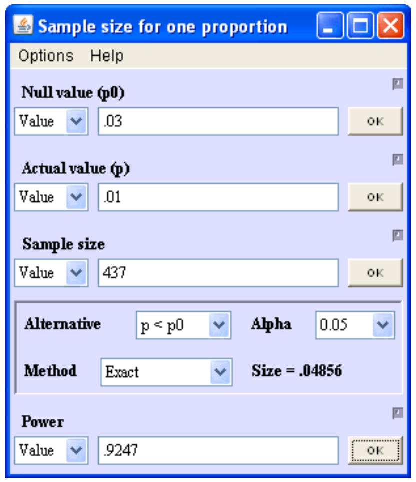




From **MINITAB**> **Stat**> **Quality Tools**> **Acceptance Sampling by Attributes:**

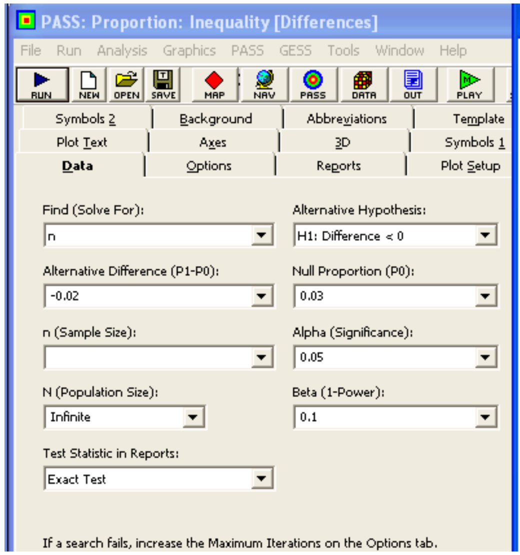
**Example 4.8** What sample size is required to reject $H_0 : p = 0.03$ with 90% power when in fact $p = 0.01$?
**Solution:** The hypotheses to be tested are $H_0 : p = p_0$ versus $H_A : p < p_0$ and the two points on the OC curve are $(p_0, 1 - \alpha) = (0.03, 0.95)$ and $(p_1, \beta) = (0.01, 0.10)$. The exact simultaneous solution to Equations 4.26 and 4.27, determined using Larson's nomogram followed by manual iterations with a binomial calculator, is $(n, r) = (436, 7)$.

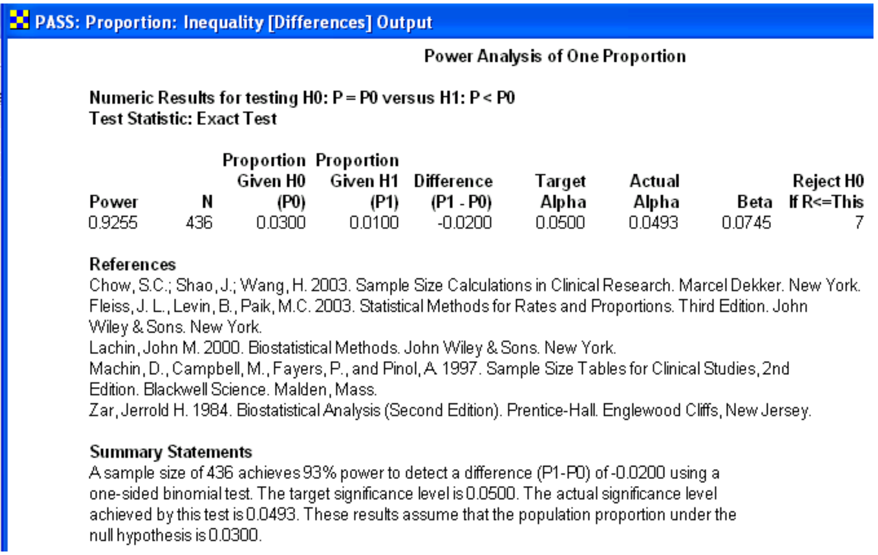
From **Piface**> **Test of one proportion**:


Acceptance Sampling by Attributes
Measurement type: Go/no go
Lot quality in proportion defective
Use binomial distribution to calculate probability of acceptance
Acceptable Quality Level (AQL) 0.02
Producer's Risk (Alpha) 0.05
Rejectable Quality Level (RQL or LTPD) 0.09
Consumer's Risk (Beta) 0.1
Generated Plan(s)
Sample Size 87
Acceptance Number 4
Accept lot if defective items in 87 sampled <= 4; Otherwise reject.
Proportion Defective Probability Accepting Probability Rejecting
0.02 0.969 0.031
0.09 0.099 0.901
OC Curve
Operating Characteristic (OC) Curve
Sample Size = 87, Acceptance Number = 4
Probability of Acceptance
Lot Proportion Defective
Acceptance Sampling by Attributes
Create a Sampling Plan
Measurement type: Go / no go (defective)
Units for quality levels: Proportion defective
Acceptable quality level (AQL): 0.02
Rejectable quality level (RQL or LTPD): 0.09
Producer's risk (Alpha): 0.05
Consumer's risk (Beta): 0.10
Lot size:
Options...
Graphs...
OK
Cancel
Help
Project...




From **PASS> Proportions> One Group> Inequality [Differences]:**





PASS: Proportion: Inequality [Differences] Output

Power Analysis of One Proportion

Numeric Results for testing H0: P = P0 versus H1: P < P0
Test Statistic: Exact Test

| Power | N | Proportion Given H0 (P0) | Proportion Given H1 (P1) | Difference (P1 - P0) | Target Alpha | Actual Alpha | Beta | Reject H0 If R<=This |
|---|---|---|---|---|---|---|---|---|
| 0.9255 | 436 | 0.0300 | 0.0100 | -0.0200 | 0.0500 | 0.0493 | 0.0745 | 7 |

References
Chow, S.C.; Shao, J.; Wang, H. 2003. Sample Size Calculations in Clinical Research. Marcel Dekker. New York.
Fleiss, J. L., Levin, B., Paik, M.C. 2003. Statistical Methods for Rates and Proportions. Third Edition. John Wiley & Sons. New York.
Lachin, John M. 2000. Biostatistical Methods. John Wiley & Sons. New York.
Machin, D., Campbell, M., Fayers, P., and Pinol, A. 1997. Sample Size Tables for Clinical Studies, 2nd Edition. Blackwell Science. Malden, Mass.
Zar, Jerrold H. 1984. Biostatistical Analysis (Second Edition). Prentice-Hall. Englewood Cliffs, New Jersey.

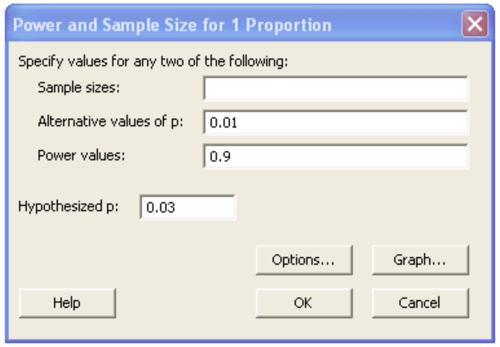
Summary Statements
A sample size of 436 achieves 93% power to detect a difference (P1-P0) of -0.0200 using a one-sided binomial test. The target significance level is 0.0500. The actual significance level achieved by this test is 0.0493. These results assume that the population proportion under the null hypothesis is 0.0300.

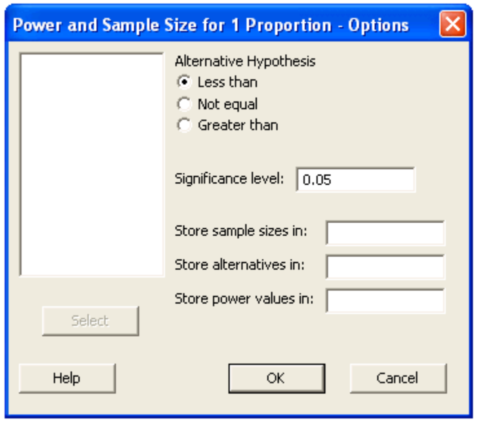
From **MINITAB> Stat> Power and Sample Size> 1 Proportion:**

```
MTB > Power;
SUBC>   POne;
SUBC>     PAlternative 0.01;
SUBC>     Power 0.9;
SUBC>     PNull 0.03;
SUBC>     Alternative -1;
SUBC>   GPCurve.
```

### Power and Sample Size

```
Test for One Proportion

Testing proportion = 0.03 (versus < 0.03)
Alpha = 0.05


Alternative  Sample  Target
 Proportion    Size   Power  Actual Power
       0.01     417     0.9      0.900542
```

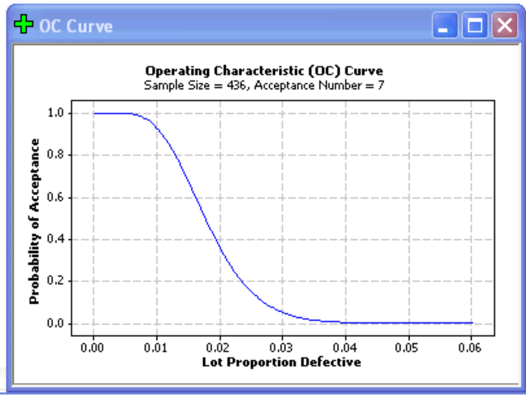


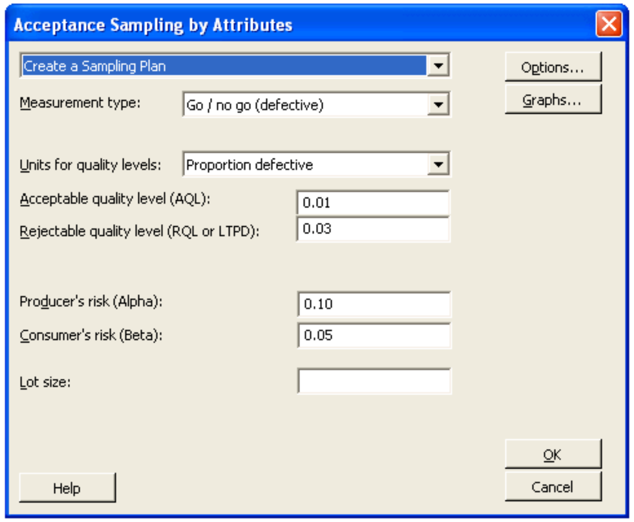




From **MINITAB**> **Stat**> **Quality Tools**> **Acceptance Sampling by Attributes:**

**Acceptance Sampling by Attributes**

```
Measurement type:  Go/no go
Lot quality in proportion defective
Use binomial distribution to calculate probability of acceptance


Acceptable Quality Level (AQL)          0.01
Producer's Risk (Alpha)                 0.1

Rejectable Quality Level (RQL or LTPD)  0.03
Consumer's Risk (Beta)                  0.05


Generated Plan(s)

Sample Size        436
Acceptance Number    7

Accept lot if defective items in 436 sampled <= 7;  Otherwise reject.


Proportion  Probability  Probability
 Defective    Accepting    Rejecting
      0.01        0.926        0.074
      0.03        0.049        0.951
```




## 4.2 One Proportion (Small Population)

**Example 4.9** Suppose that a sample of size $n = 20$ drawn from a population of $N = 100$ units was found to have $X = 2$ defective units. Determine the one-sided upper confidence limit for the population fraction defective.
**Solution:** From the following hypergeometric probabilities:

$$h(0 \le x \le 2; S = 26, N = 100, n = 20) = 0.0555$$
$$h(0 \le x \le 2; S = 27, N = 100, n = 20) = 0.0448$$

the smallest value of $S$ that satisfies the inequality in Equation 4.34 is $S = 27$, so the 95% one-sided upper confidence limit for $S$ is $S_U = 27$ or

$$P(S \le 27) \ge 0.95.$$

**Example 4.10** A hospital is asked by an auditor to confirm that its billing error rate is less than 10% for a day chosen randomly by the auditor. However, it is impractical to inspect all 120 bills issued on that day. How many of the bills must be inspected to demonstrate, with 95% confidence, that the billing error rate is less than 10%?
**Solution:** The goal of the analysis is to demonstrate that the one-sided upper 95% confidence limit on the billing error rate $p$ is 10% or

$$P\left(p \leq 0.10\right) = 0.95.$$

Under the assumption that the auditor will accept a zero defectives sampling plan, by the rule of three (Equation 4.5) the approximate sample size must be

$$n \simeq \frac{3}{p} = \frac{3}{0.10} = 30.$$

Because $n = 30$ is large compared to $N = 120$, the finite population correction factor (Equation 4.16) should be used and gives

$$\begin{aligned} n' &= \frac{30}{1 + \frac{30-1}{120}} \\ &= 25. \end{aligned}$$

Iterations with a hypergeometric probability calculator show that $n = 26$ is the smallest sample size that gives 95% confidence that the billing error rate is less than 10%.

**Example 4.11** What sample size $n$ must be drawn from a population of size $N = 200$ and found to be free of defectives if we need to demonstrate,with 95% confidence, that there are no more than four defectives in the population?
**Solution:** The goal of the experiment is to demonstrate the confidence interval

$$P\left(0 \leq S \leq 4\right) \geq 0.95$$

using a zero-successes $(X = 0)$ sampling plan. By the small-sample binomial approximation with $S_U = 4$ and $\alpha = 0.05$, the required sample size by Equation 4.40 is given by

$$n = \frac{\ln\left(0.05\right)}{\ln\left(1 - \frac{4}{200}\right)} = 149,$$

which violates the small-sample assumption. By Equation 4.42, the rare-event binomial approximation gives

$$\begin{aligned} n &\geq N\left(1 - \alpha^{1/S_U}\right) \\ &\geq 200\left(1 - 0.05^{1/4}\right) \\ &\geq 106. \end{aligned}$$

This solution meets the requirements of the rare-event approximation method, but just to check this result, the corresponding exact hypergeometric probability is $h\left(0; 4, 200, 106\right) = 0.047$ which is less than $\alpha = 0.05$ as required, however, because $h\left(0, 4, 200, 105\right) = 0.049$, the sample size $n = 105$ is the exact solution to the problem.

**Example 4.12** A biologist needs to test the fraction of female frogs in a single brood, but the sex of the frog tadpoles is difficult to determine. The hypotheses to be tested are $H_0 : p = 0.5$ versus $H_A : p > 0.5$ where $p$ is the fraction of the frogs that are female. If there are $N = 212$ viable frogs in the brood, how many of them must she sample to reject $H_0$ with 90% power when $p = 0.65$?
**Solution:** The exact sample size $(n)$ and acceptance number $(c)$ have to be determined by iteration. The approximate sample size given by the large-sample binomial approximation method in Equation 4.22 with $p_0 = 0.5$, $\alpha = 0.05$, $p_1 = 0.65$, and $\beta = 0.10$ is

$$n = \left( \frac{1.645\sqrt{0.5\,(1-0.5)} + 1.282\sqrt{0.65\,(1-0.65)}}{0.65 - 0.5} \right)^2 = 92.$$

However, this sample size is large compared to the population size, so the finite population correction factor (Equation 4.16) must be used, which gives

$$\begin{aligned} n' &= \frac{92}{1 + \frac{92-1}{212}} \\ &= 65. \end{aligned}$$

The exact values of $n$ and $c$ are determined from the simultaneous solution of Equations 4.43 and Equation 4.44 with $S_0 = Np_0 = 106$ and $S_1 = Np_1 = 138$, which gives

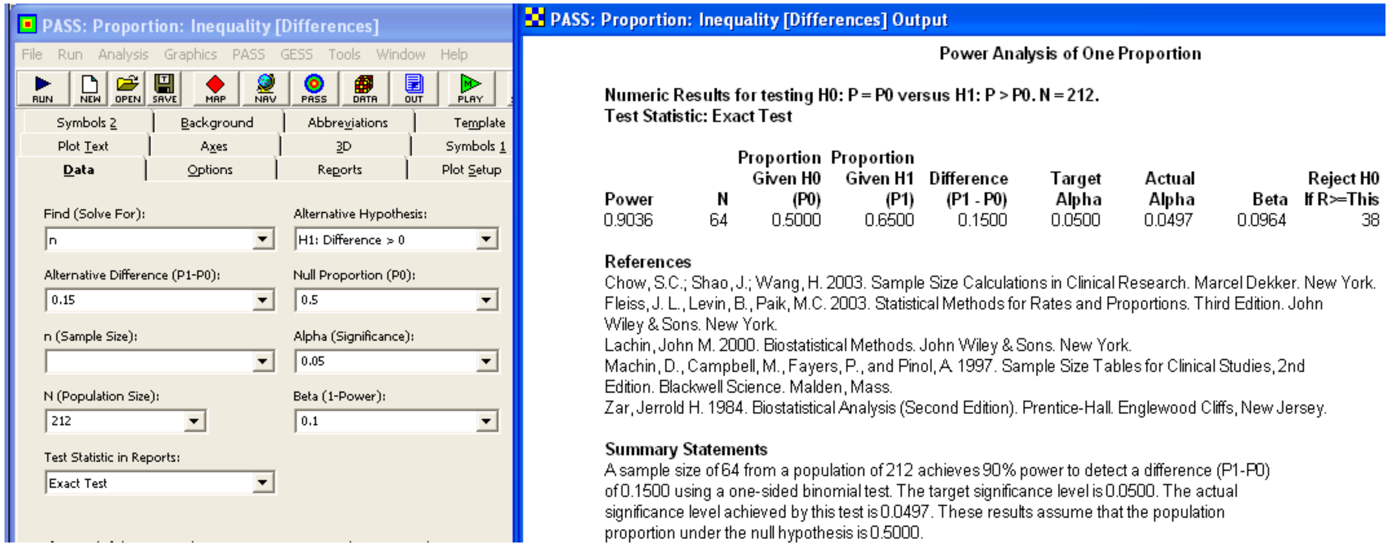
$$\begin{aligned} \sum_{x=0}^{c} h\,(x; S = 106, N = 212, n) &\geq 0.95 \\ \sum_{x=0}^{c} h\,(x; S = 138, N = 212, n) &\leq 0.10. \end{aligned}$$

Using a hypergeometric calculator with $n = 65$ we find

$$\begin{aligned} \sum_{x=0}^{38} h\,(x; S = 106, N = 212, n = 65) &= 0.963 \\ \sum_{x=0}^{38} h\,(x; S = 138, N = 212, n = 65) &= 0.117, \end{aligned}$$

which satisfies the $1 - \alpha \geq 0.95$ requirement but does not satisfy the $\beta \leq 0.10$ requirement. A few more iterations determine that $n = 69$ and $c = 40$ gives $\alpha = 0.039$ and $\pi = 0.912$ which meets both requirements. This means that the biologist must sample $n = 69$ frogs and can reject $H_0$ if $S > 40$.

The one proportion methods in Piface and MINITAB can be used to find the first step in the solution, $n = 92$, but they don't provide the opportunity to apply a small population correction. PASS does support finite populations using the binomial method instead of the normal approximation. From **PASS**> **Proportions**> **One Group**> **Inequality [Differences]**:




## 4.3 Two Proportions

**Example 4.13** Determine the sample size required to estimate the difference between two proportions to within 0.03 with 95% confidence if both proportions are expected to be about 0.45. Assume that the two sample sizes will be equal.
**Solution:** From Equation 4.54 with $\delta = 0.03$, $\bar{p} = 0.45$, $n_1/n_2 = 1$, and $\alpha = 0.05$, the required sample size is

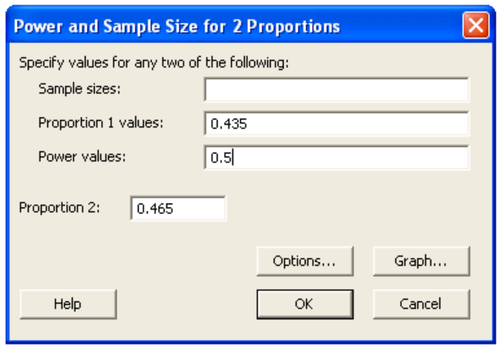
$$\begin{aligned} n_1 &= n_2 = \left(\frac{1.96}{0.03}\right)^2 (2 \times 0.45 \times (1 - 0.45)) \\ &= 2113. \end{aligned}$$

From **Piface**> **Test comparing two proportions** without and with the continuity correction:

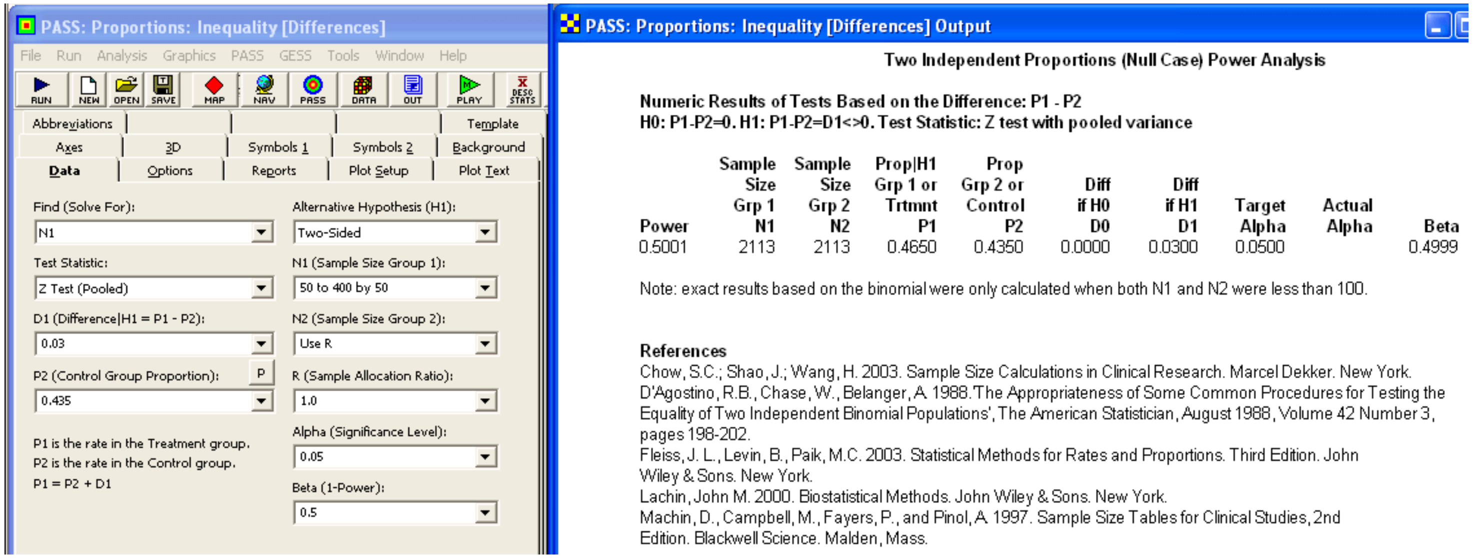


From **MINITAB**> **Stat**> **Power and Sample Size**> **Two Proportions**:

```
MTB > Power;
SUBC>   PTwo;
SUBC>     PrOne 0.435;
SUBC>     Power 0.5;
SUBC>     PrTwo 0.465;
SUBC>   GPCurve.

Power and Sample Size

Test for Two Proportions

Testing proportion 1 = proportion 2 (versus not =)
Calculating power for proportion 2 = 0.465
Alpha = 0.05


              Sample  Target
Proportion 1    Size   Power  Actual Power
       0.435    2113     0.5      0.500081

The sample size is for each group.
```




From **PASS**> **Proportions**> **Two Groups: Independent**> **Inequality [Differences]**:




**Example 4.14** An experiment is planned to estimate the risk ratio. The two proportions are expected to be $p_1 \simeq 0.2$ and $p_2 \simeq 0.05$. Determine the optimal allocation ratio and the sample size required to determine the risk ratio to within 20% of its true value with 95% confidence?

**Solution:** A 95% confidence interval for the risk ratio is required of the form in Equation 4.56. With $p_1 = 0.2$ and $p_2 = 0.05$, the anticipated value of the risk ratio is $RR \simeq 0.2/0.05 = 4$ and from Equation 4.62 the optimal sample size allocation ratio is

$$\frac{n_1}{n_2} = \sqrt{\frac{0.05/0.95}{0.2/0.8}} = 0.4588.$$

Then with $\delta = 0.2$ and $\alpha = 0.05$ in Equation 4.61, the required sample size $n_1$ is

$$\begin{aligned} n_1 &= \left(\frac{1.96}{0.2}\right)^2 \left(\frac{1-0.2}{0.2} + \frac{1-0.05}{0.05}(0.4588)\right) \\ &= 1222 \end{aligned}$$

and the sample size $n_2$ is

$$n_2 = \frac{n_1}{\left(\frac{n_1}{n_2}\right)} = \frac{1222}{0.4588} = 2664.$$

These sample sizes minimize the total number of samples required for the experiment.

**Example 4.15** An experiment is planned to estimate the odds ratio. The two proportions are expected to be $p_1 \simeq 0.5$ and $p_2 \simeq 0.25$. Determine the optimal allocation ratio and the sample size required to determine, with 90% confidence, the odds ratio to within 20% of its true value?

**Solution:** The desired confidence interval has the form given by Equation 4.64 with $\delta = 0.2$. With $p_1 = 0.5$ and $p_2 = 0.25$, the anticipated value of the odds ratio is $OR = \frac{0.5/0.5}{0.25/0.75} = 3$ and from Equation 4.70 the optimal sample size allocation ratio is

$$\frac{n_1}{n_2} = \sqrt{\frac{0.25 \times 0.75}{0.5 \times 0.5}} = 0.866.$$

Then with $\delta = 0.2$ and $\alpha = 0.10$ in Equation 4.69, the required sample size $n_1$ is

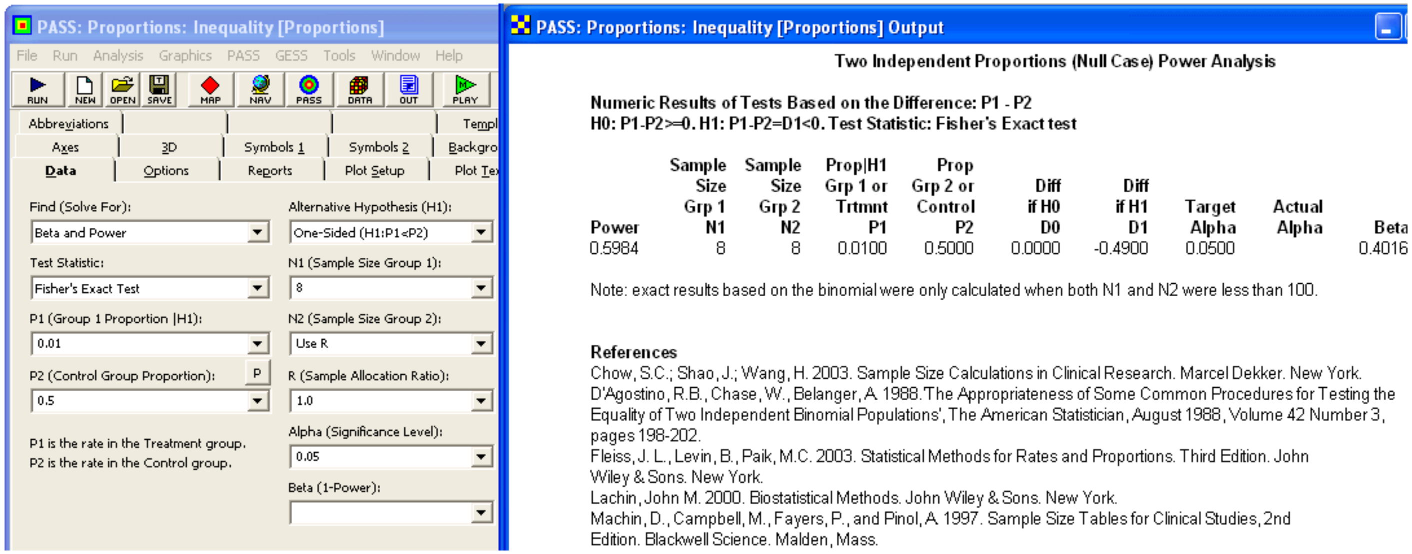
$$\begin{aligned} n_1 &= \left(\frac{1.645}{0.2}\right)^2 \left(\frac{1}{0.5 \times 0.5} + \frac{1}{0.25 \times 0.75}(0.866)\right) \\ &= 584 \end{aligned}$$

and the sample size $n_2$ is

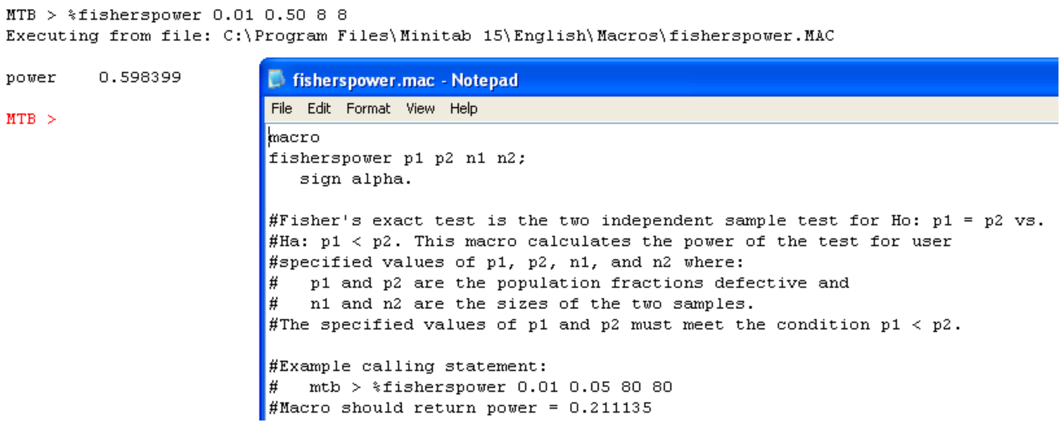
$$n_2 = \frac{n_1}{\left(\frac{n_1}{n_2}\right)} = \frac{584}{0.866} = 675.$$

**Example 4.16** Determine the power for Fisher's test to reject $H_0 : p_1 = p_2$ in favor of $H_A : p_1 < p_2$ when $p_1 = 0.01$, $p_2 = 0.50$, and $n_1 = n_2 = 8$.
**Solution:** The Fisher's test $p$ values for all possible combinations of $x_1$ and $x_2$ were calculated using Equation 4.71 and are shown in Table 4.3. The few cases that are statistically significant, where $p \leq 0.05$, are shown in a bold font in the upper right corner of the table. Table 4.4 shows the contributions to the power given by the product of the two binomial probabilities in Equation 4.74. The sum of the individual contributions, that is, the power of Fisher's test, is $\pi = 0.60$.

From **PASS**> **Proportions**> **Two Groups: Independent**> **Inequality [Differences]**:



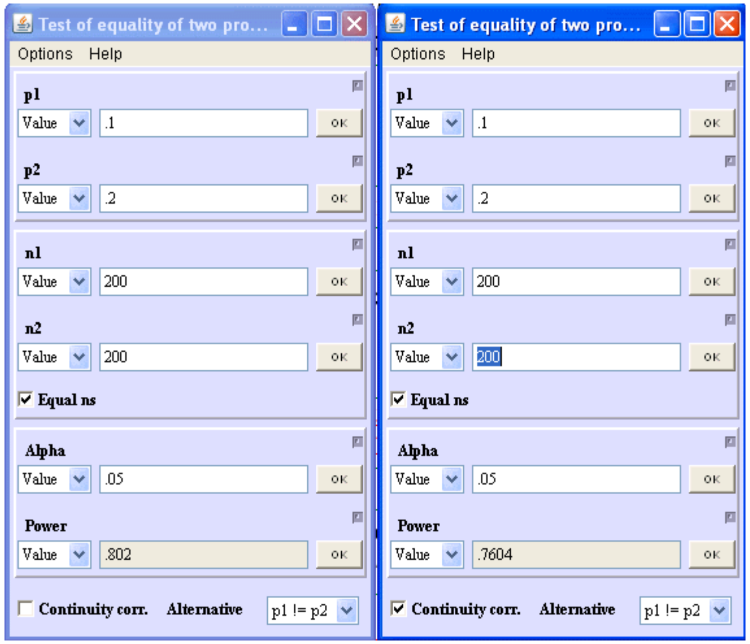
MINITAB's default sample size and power calculator uses the normal approximation but the Fisher's test power can be calculated using the custom MINITAB macro *fisherspower.mac* that is posted on www.mmbstatistical.com.




**Example 4.17** Determine the power for the test of $H_0 : p_1 = p_2$ versus $H_A : p_1 \neq p_2$ when $n_1 = n_2 = 200$, $p_1 = 0.10$, and $p_2 = 0.20$. Use a two-tailed test with $\alpha = 0.05$
**Solution:** The normal approximation to the binomial distribution is justified for both samples, so with $\widehat{p} = 0.15$ and $\Delta\widehat{p} = 0.10$ in Equations 4.78 and 4.79, the power is

$$\begin{aligned}\pi &= \Phi\left(-\infty < z < \frac{0.10}{\sqrt{\frac{2(0.15)(1-0.15)}{200}}} - 1.96\right) \\ &= \Phi\left(-\infty < z < 0.84\right) \\ &= 0.80.\end{aligned}$$

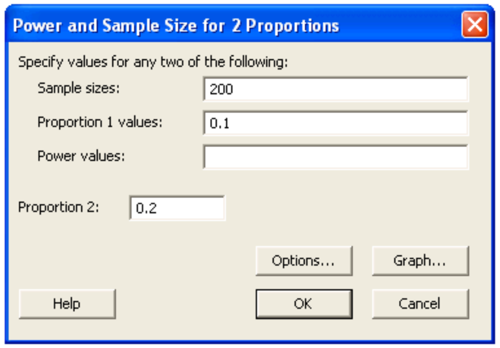
From **Piface> Test comparing two proportions** without and with the continuity correction:



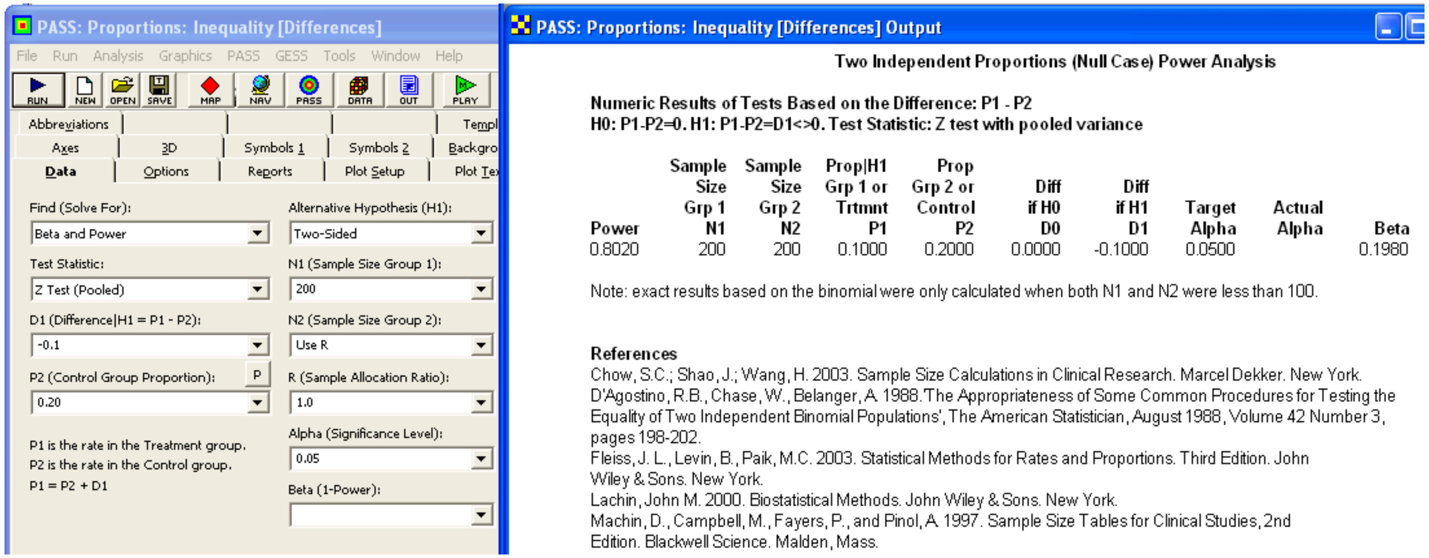

From **MINITAB**> **Stat**> **Power and Sample Size**> **Two Proportions**:

**Power and Sample Size**

```
Test for Two Proportions

Testing proportion 1 = proportion 2 (versus not =)
Calculating power for proportion 2 = 0.2
Alpha = 0.05


              Sample
Proportion 1    Size     Power
         0.1     200  0.802049

The sample size is for each group.
```



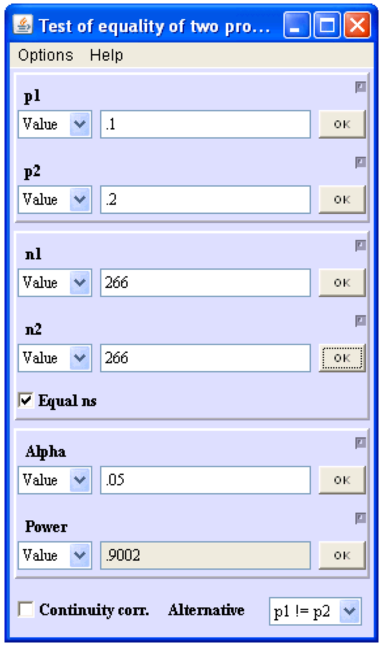

From **PASS**> **Proportions**> **Two Groups: Independent**> **Inequality [Differences]**:




**Example 4.18** What common sample size is required to resolve the difference between two proportions with 90% power using a two-sided test when $p_1 = 0.10$ and $p_2 = 0.20$ is expected?
**Solution:** From Equation 4.80 with $\widehat{p} = 0.15$ and $\Delta\widehat{p} = 0.10$ the required sample size is

$$\begin{aligned} n &= \frac{2 \times 0.15 \times 0.85}{(0.10)^2}(1.28 + 1.96)^2 \\ &= 268. \end{aligned}$$

From **Piface**> **Test comparing two proportions** without the continuity correction:

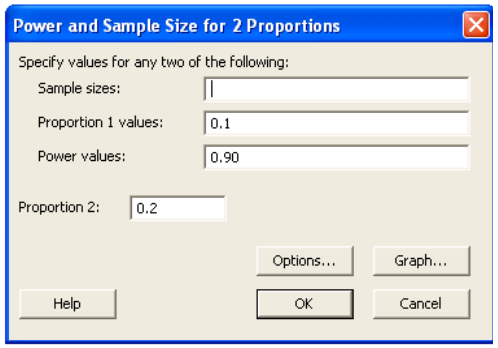



From **MINITAB**> **Stat**> **Power and Sample Size**> **Two Proportions**:

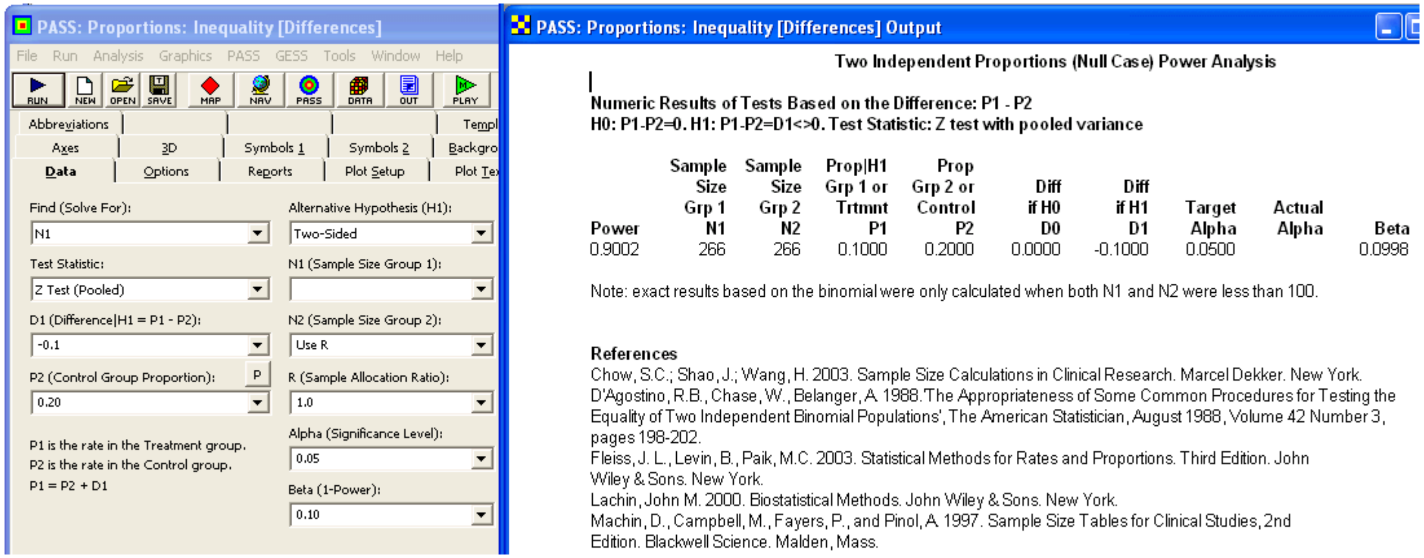
```
MTB > Power;
SUBC>   PTwo;
SUBC>     PrOne 0.1;
SUBC>     Power 0.90;
SUBC>     PrTwo 0.2;
SUBC>   GPCurve.
```

**Power and Sample Size**

```
Test for Two Proportions

Testing proportion 1 = proportion 2 (versus not =)
Calculating power for proportion 2 = 0.2
Alpha = 0.05


              Sample  Target
Proportion 1    Size   Power  Actual Power
         0.1     266     0.9      0.900155

The sample size is for each group.
```




From **PASS**> **Proportions**> **Two Groups: Independent**> **Inequality [Differences]**:




**Example 4.19** Repeat the calculation of the sample size for Example 4.18.
**Solution:** From Equation 4.86 the required sample size is

$$n = \frac{1}{2}\left(\frac{1.28 + 1.96}{arcsin\sqrt{0.10} - arcsin\sqrt{0.20}}\right)^2 = 261,$$

which is in excellent agreement with the sample size determined by the normal approximation method.

**Example 4.20** Repeat the calculation of the sample size for Example 4.18 using the log risk ratio method.
**Solution:** With $RR = 0.1/0.2 = 0.5$, $\alpha = 0.05$, $\beta = 0.10$, and $n_1/n_2 = 1$ in Equation 4.90, the required sample size is

$$n_1 = n_2 = \left(\frac{1.96 + 1.282}{\ln(0.5)}\right)^2 \left(\frac{1 - 0.1}{0.1} + \frac{1 - 0.2}{0.2}\right) = 285,$$

which is in excellent agreement with the sample size obtained by the normal approximation method.
See **PASS**> **Proportions**> **Two Groups: Independent**> **Inequality [Ratios]**.

**Example 4.21** Repeat the calculation of the sample size for Example 4.18 using the log odds ratio method.
**Solution:** With $n_1/n_2 = 1$ in Equation 4.97, the required common sample size is

$$n = \left(\frac{1.96 + 1.282}{\ln\left(\frac{0.10/0.90}{0.20/0.80}\right)}\right)^2 \left(\frac{1}{0.10\,(0.90)} + \frac{1}{0.20\,(0.80)}\right) = 278,$$

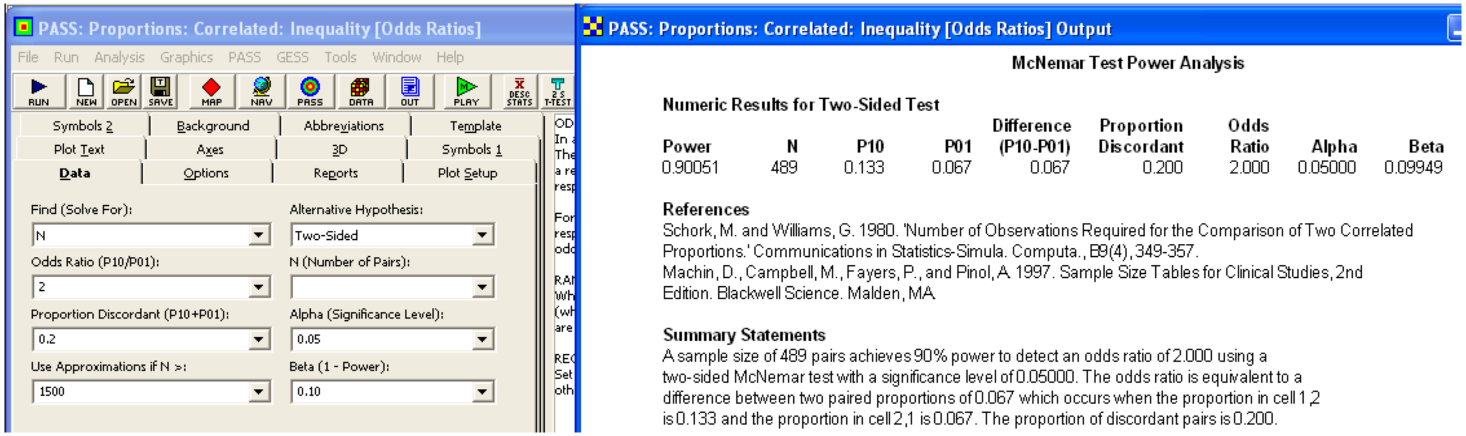
which is in excellent agreement with the sample size obtained by the normal approximation method.

See **PASS> Proportions> Two Groups: Independent> Inequality [Odds Ratios]**.

**Example 4.22** Determine the number of subjects required for McNemar's test to reject $H_0 : RR = 1$ with 90% power when $RR = 2$ and the rate of discordant observations is estimated to be $p_D = 0.2$ from a preliminary study.
**Solution:** With $\beta = 0.10$ and $\alpha = 0.05$ in Equation 4.105, the approximate number of subjects required for the study is

$$\begin{aligned}\sum_i \sum_j \widehat{f}_{ij} &\simeq \frac{(1.282 + 1.96)^2}{0.20}\left(\frac{2+1}{2-1}\right)^2 \\ &\simeq 473.\end{aligned}$$

From **PASS> Proportions> Two Groups: Paired or Correlated> Inequality (McNemar) [Odds Ratios]:**




**Example 4.23** Determine the McNemar's test power to reject $H_0 : RR = 1$ in favor of $H_A : RR \neq 1$ for a study with 200 subjects when in fact $RR = 3$ using $p_D = 0.3$.
**Solution:** With 200 subjects in the study, the expected number of discordant pairs is

$$\widehat{f}_{12} + \widehat{f}_{21} = p_D \sum_i \sum_j \widehat{f}_{ij} = 0.3 \times 200 = 60.$$

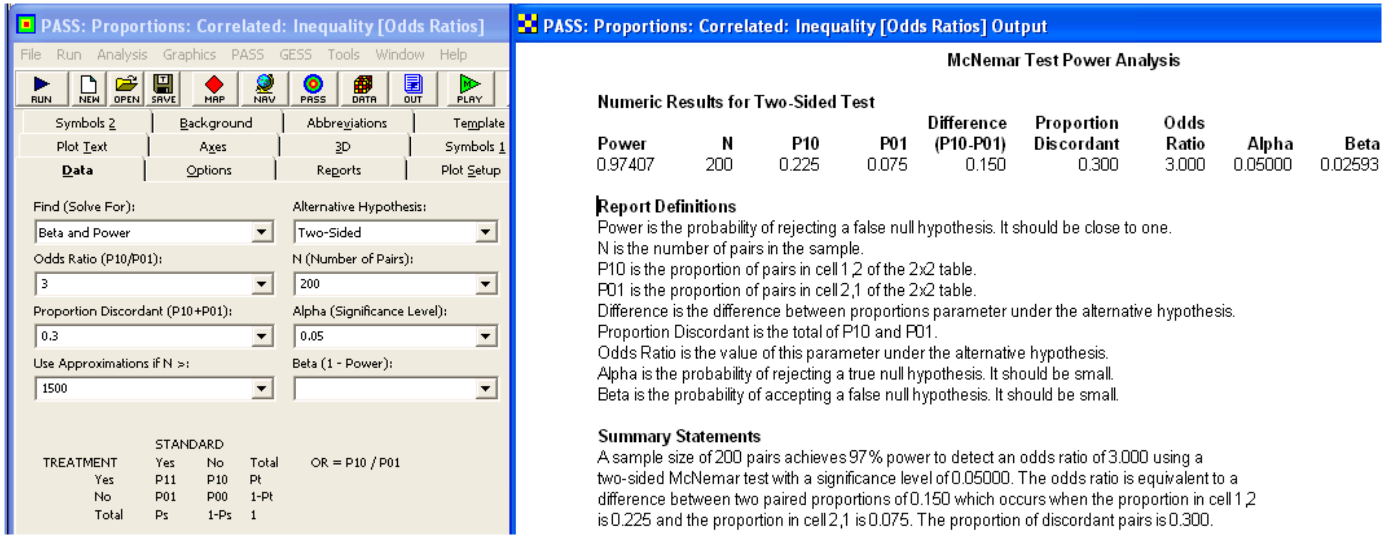
Under $H_A$ with $RR = 3$, we have $\widehat{f}_{21} = 15$ and $\widehat{f}_{12} = 45$, so the expected value of the McNemar's $z_M$ statistic is

$$z_M = \frac{|45 - 15|}{\sqrt{45 + 15}} = 3.87$$

and the approximate power is

$$\begin{aligned}\pi &= P\left(-\infty < z < z_{\beta}\right)\\ &= P\left(-\infty < z < \left(z_M - z_{\alpha/2}\right)\right)\\ &= P\left(-\infty < z < (3.87 - 1.96)\right)\\ &= P\left(-\infty < z < 1.91\right)\\ &= 0.972.\end{aligned}$$

From **PASS**> **Proportions**> **Two Groups: Paired or Correlated**> **Inequality (McNemar) [Odds Ratios]:**




## 4.4 Equivalence Tests

**Example 4.24** For the test of $H_0 : p < 0.45$ or $p > 0.55$ versus $H_A : 0.45 < p < 0.55$, calculate the exact and approximate power when $p = 0.5$ assuming that the sample size is $n = 800$ and $\alpha = 0.05$.
**Solution:** The value of $x_1$,determined from Equation 4.106, is $x_1 = 384$ because

$$\sum_{x=0}^{383} b\left(x; n = 800, p_1 = 0.45\right) = 0.952.$$

The value of $x_2$, determined from Equation 4.107, is $x_2 = 416$ because

$$\sum_{x=0}^{416} b\left(x; n = 800, p_2 = 0.55\right) = 0.048.$$

Then the power when $p = 0.5$ is given by Equation 4.108:

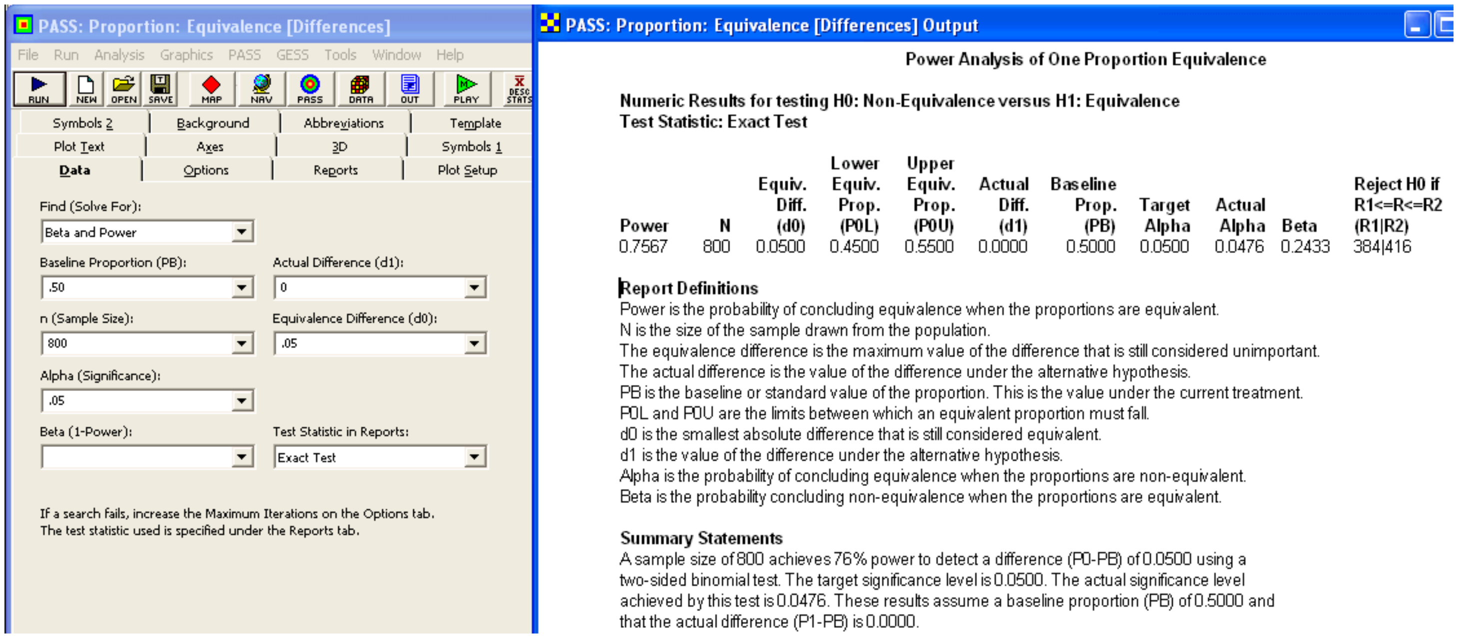
$$\begin{aligned}\pi &= \sum_{x=383}^{416} b\left(x; n = 800, p_2 = 0.50\right) \\ &= 0.757.\end{aligned}$$

The approximate power by the normal approximation method, given by Equation 4.111, is

$$\begin{aligned}\pi &= \Phi\left(\frac{0.45 - 0.5}{\sqrt{\frac{0.5(1-0.5)}{800}}} + 1.645 < z < \frac{0.55 - 0.5}{\sqrt{\frac{0.5(1-0.5)}{800}}} - 1.645\right) \\ &= \Phi\left(-1.183 < z < 1.183\right) \\ &= 0.763,\end{aligned}$$

which is in good agreement with the exact solution.

From **PASS**> **Proportions**> **One Group: Equivalence [Differences]**:

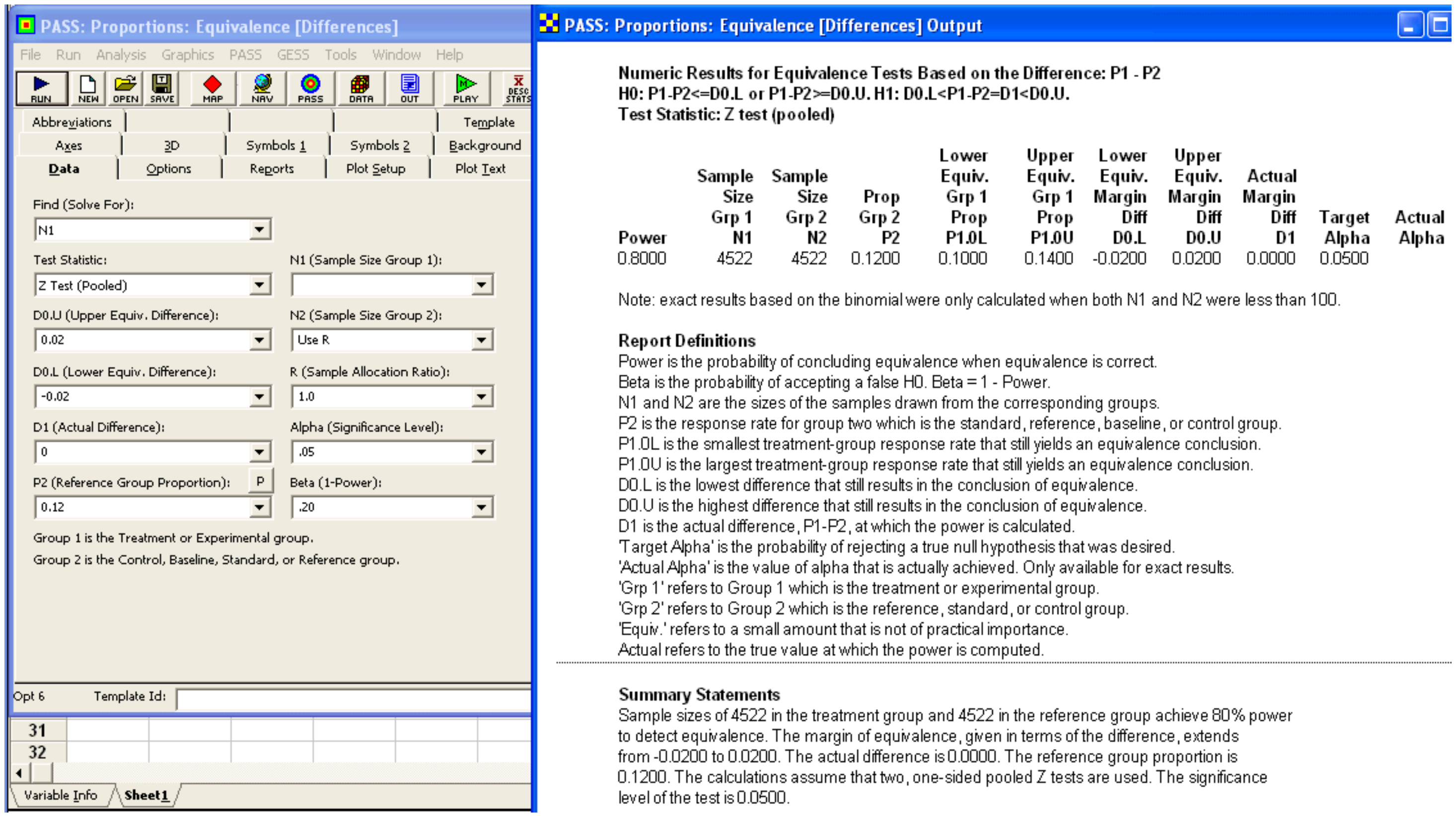



**Example 4.25** An experiment is to be performed to test the hypotheses $H_0 : p_1 \neq p_2$ versus $H_A : p_1 = p_2$. The two proportions are expected to be $p \simeq 0.12$ and the limit of practical equivalence is $\delta = 0.02$. What sample size is required to reject $H_0$ when $p_1 = p_2$ with 80% power?

**Solution:** With $\alpha = 0.05$ and $n_1/n_2 = 1$ in Equation 4.118, the sample size $n = n_1 = n_2$ is

$$
\begin{aligned}
n &= 2\left(z_{0.05} + z_{0.10}\right)^2 \frac{0.12\,(1-0.12)}{(0.02)^2} \\
&= 2\,(1.645 + 1.282)^2 \frac{0.12\,(1-0.12)}{(0.02)^2} \\
&= 4524.
\end{aligned}
$$

From **PASS**> **Proportions**> **Two Groups: Independent**> **Equivalence [Differences]**:

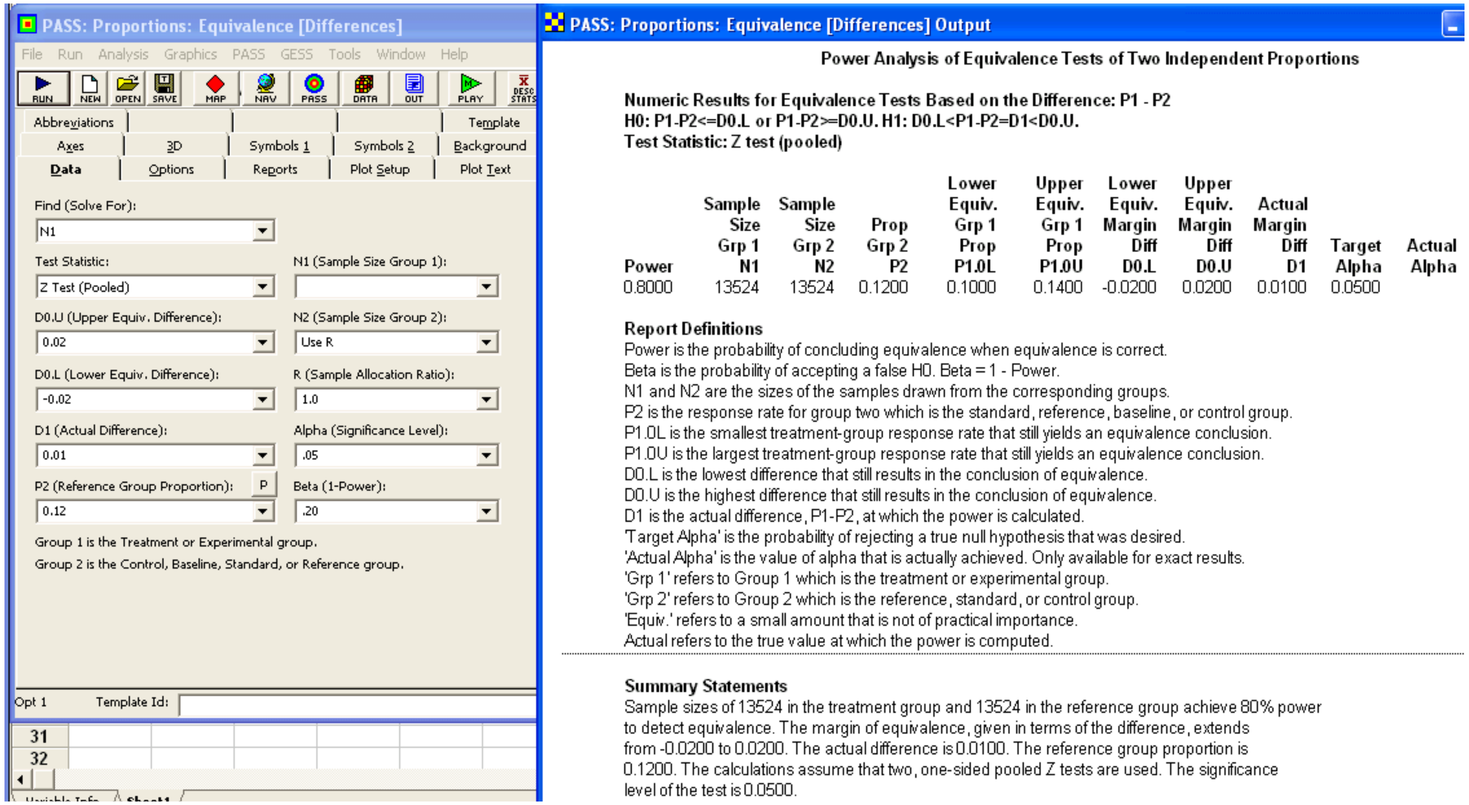



**Example 4.26** What sample size is required if the true difference between the two proportions in Example4.254.25 is $\Delta p = 0.01$?

**Solution:** $p_1$ and $p_2$ are not specified, but they are both approximately $p = 0.12$, so from Equation 4.119 the sample size must be

$$
\begin{aligned}
n_1 &\simeq 2\left(z_\alpha + z_\beta\right)^2 \frac{p\left(1-p\right)}{\left(\delta - |\Delta p|\right)^2} \\
&\simeq 2\left(1.645 + 0.842\right)^2 \frac{0.12\left(1-0.12\right)}{\left(0.02-0.01\right)^2} \\
&\simeq 13063.
\end{aligned}
$$

From **PASS**> **Proportions**> **Two Groups: Independent**> **Equivalence [Differences]**:



## 4.5 Chi-square Tests

**Example 4.27** Confirm the sample size for Example 4.18 using the $\chi^2$ test method for a $2 \times 2$ table.
**Solution:** Under $H_A$ with $p_1 = 0.10$ and $p_2 = 0.20$ the expected proportion of observations in each cell of the $2 \times 2$ table is

$$(p_{ij})_A = \frac{1}{2}\left\{\begin{array}{cc} 0.1 & 0.9 \\ 0.2 & 0.8 \end{array}\right\} = \left\{\begin{array}{cc} 0.05 & 0.45 \\ 0.1 & 0.4 \end{array}\right\}.$$

Under $H_0$ with $p_1 = p_2 = (0.1 + 0.2)/2 = 0.15$ the expected distribution of observations is

$$(p_{ij})_0 = \frac{1}{2}\left\{\begin{array}{cc} 0.15 & 0.85 \\ 0.15 & 0.85 \end{array}\right\} = \left\{\begin{array}{cc} 0.075 & 0.425 \\ 0.075 & 0.425 \end{array}\right\}.$$

From Equation 4.121 with a total of $2 \times 268 = 536$ observations the noncentrality parameter is

$$\begin{aligned} \phi &= 536\left(\frac{(0.05-0.075)^2}{0.075} + \frac{(0.1-0.075)^2}{0.075} + \frac{(0.45-0.425)^2}{0.425}\right. \\ &\quad \left. + \frac{(0.4-0.425)^2}{0.425}\right) \\ &= 10.51. \end{aligned}$$

Then, with $\alpha = 0.05$ and $df = 1$ degree of freedom in Equation 4.122,

$$\chi^2_{0.95} = 3.8415 = \chi^2_{\beta,10.51}$$

which is satisfied by $\beta = 0.10$, so the power is $\pi = 1 - \beta = 0.90$ and is consistent with the original example problem solution.

**Example 4.28** A large school district intends to perform pass/fail testing of students from four large schools to test for performance differences among schools. If 50 students are chosen randomly from each school, what is the power of the $\chi^2$ test to reject the null hypothesis of homogeneity when the student failure rates at the four schools are in fact 10%, 10%, 10%, and 30%?
**Solution:** To calculate the power of the $\chi^2$ test we must specify the two $2 \times 4$ tables (result by school) associated with $(p_{ij})_0$ and $(p_{ij})_A$. From the problem statement, under $H_A$ with $(p_{1j})_A = \{0.1, 0.1, 0.1, 0.3\}$, the table of $(p_{ij})_A$ is

$$\begin{aligned} (p_{ij})_A &= \frac{1}{4}\left\{\begin{array}{cccc} 0.1 & 0.1 & 0.1 & 0.3 \\ 0.9 & 0.9 & 0.9 & 0.7 \end{array}\right\} \\ &= \left\{\begin{array}{cccc} 0.025 & 0.025 & 0.025 & 0.075 \\ 0.225 & 0.225 & 0.225 & 0.175 \end{array}\right\}. \end{aligned}$$

The mean failure rate of all four schools is $(3(0.1) + 0.3)/4 = 0.15$ under $H_0$, so the corresponding table of $(p_{ij})_0$ is

$$\begin{aligned} (p_{ij})_0 &= \frac{1}{4}\left\{\begin{array}{cccc} 0.15 & 0.15 & 0.15 & 0.15 \\ 0.85 & 0.85 & 0.85 & 0.85 \end{array}\right\} \\ &= \left\{\begin{array}{cccc} 0.0375 & 0.0375 & 0.0375 & 0.0375 \\ 0.2125 & 0.2125 & 0.2125 & 0.2125 \end{array}\right\}. \end{aligned}$$

Under these definitions, the $\chi^2$ distribution noncentrality parameter is

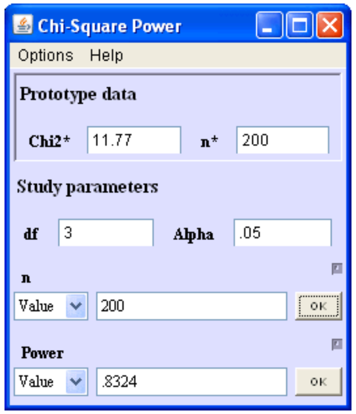
$$\begin{aligned}\phi &= 200\left[3\left(\frac{(0.025-0.0375)^2}{0.0375}\right)+\frac{(0.075-0.0375)^2}{0.0375}\right.\\ &\left.+3\left(\frac{(0.225-0.2125)^2}{0.2125}\right)+\frac{(0.175-0.2125)^2}{0.2125}\right]\\ &= 11.77.\end{aligned}$$

The $\chi^2$ test statistic will have $df = (2-1)(4-1) = 3$ degrees of freedom, so the critical value of the test statistic is $\chi^2_{0.95,3} = 7.81$. The power of the test determined from the condition

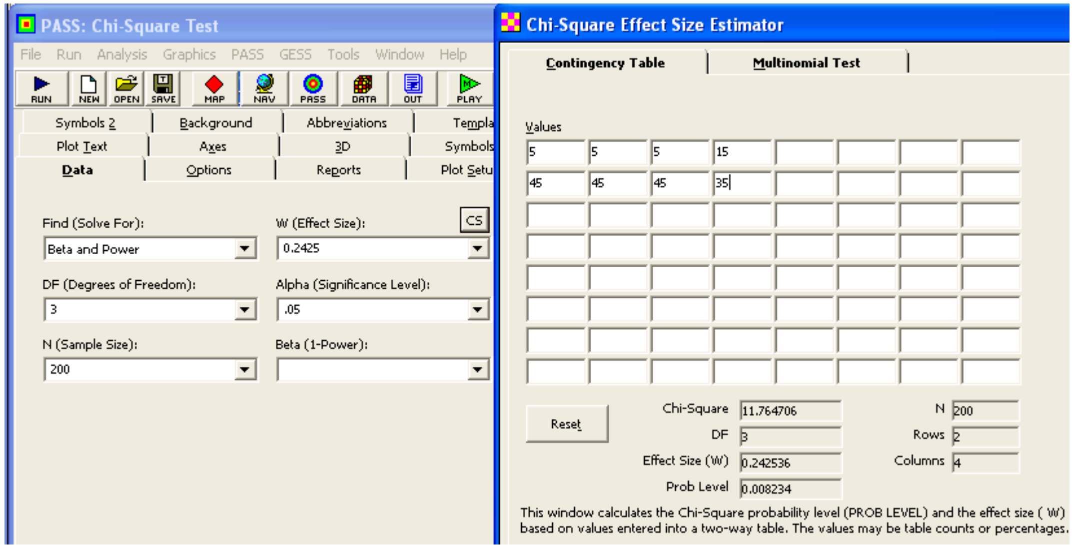
$$\chi^2_{0.95} = 7.81 = \chi^2_{1-\pi,11.77}$$

is $\pi = 0.833$.

From **Piface**> **Generic chi-square test**:




From **PASS**> **Proportions**> **Multi-Group: Chi-Square Test** with effect size $W = \sqrt{\phi/N} = \sqrt{11.77/200} = 0.2425$**:**




**Chi-Square Test Power Analysis**

**Numeric Results for Chi-Square Test**

| Power | N | W | Chi-Square | DF | Alpha | Beta |
|---|---|---|---|---|---|---|
| 0.83210 | 200 | 0.2425 | 11.7613 | 3 | 0.05000 | 0.16790 |

**References**

Cohen, Jacob. 1988. Statistical Power Analysis for the Behavioral Sciences, Lawrence Erlbaum Associates, Hillsdale, New Jersey.

**Report Definitions**

Power is the probability of rejecting a false null hypothesis. It should be close to one.
N is the size of the sample drawn from the population. To conserve resources, it should be small.
W is the effect size--a measure of the magnitude of the Chi-Square that is to be detected.
DF is the degrees of freedom of the Chi-Square distribution.
Alpha is the probability of rejecting a true null hypothesis.
Beta is the probability of accepting a false null hypothesis.

**Summary Statements**

A sample size of 200 achieves 83% power to detect an effect size (W) of 0.2425 using a 3 degrees of freedom Chi-Square Test with a significance level (alpha) of 0.05000.

**Example 4.29** What is the power to reject the claim that a die is balanced ($H_0 : \theta_i = \frac{1}{6}$ for $i = 1$ to 6) when it is in fact slightly biased toward one die face ($H_A : \theta_i = \{0.16, 0.16, 0.16, 0.16, 0.16, 0.20\}$) based on 100 rolls of the die?
**Solution:** The table of observations will have six cells and there will be no parameters estimated from the sample data, so the $\chi^2$ test will have $df = 6 - 1 = 5$ degrees of freedom. From Equation 4.121 the noncentrality parameter will be

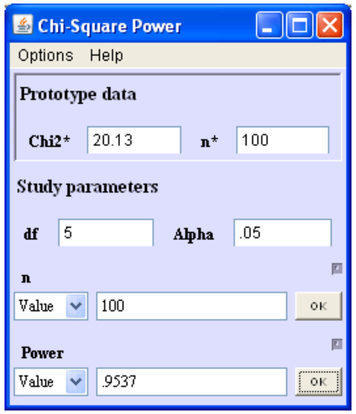
$$\phi = 100\left[5\left(\frac{\left(0.16 - \frac{1}{6}\right)^2}{\frac{1}{6}}\right) + \frac{\left(0.20 - \frac{1}{6}\right)}{\frac{1}{6}}\right] = 20.13.$$

With $\alpha = 0.05$ we have $\chi^2_{0.95} = 11.07$, so the power to reject $H_0$ is determined from the condition

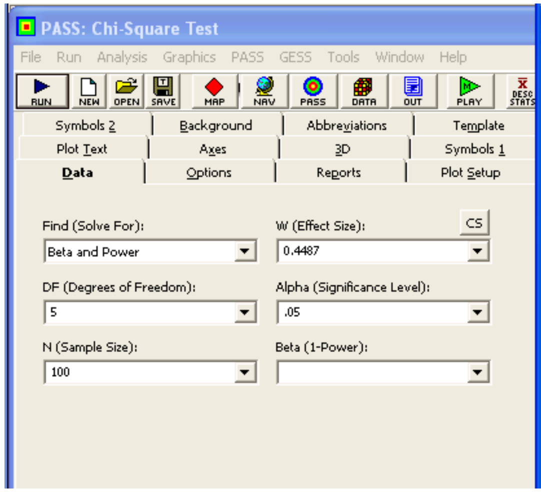
$$\chi^2_{0.95} = 11.07 = \chi^2_{1-\pi,20.13},$$

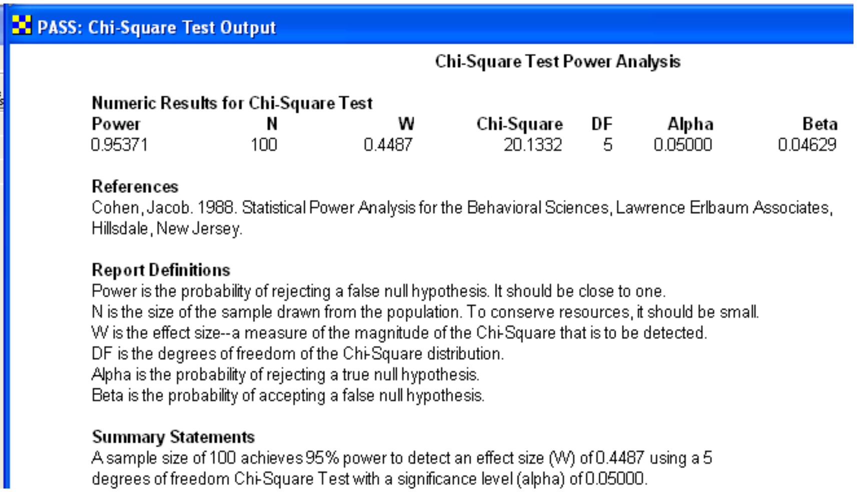
which is satisfied by $\pi = 0.954$.

From **Piface**> **Generic chi-square test**:




From **PASS**> **Proportions**> **Multi-Group: Chi-Square Test** with effect size $W = \sqrt{\phi/N} = \sqrt{20.13/100} = 0.4487$**:**



# Chapter 5

# Poisson Counts

## 5.1 One Poisson Count

**Example 5.1** How many Poisson events must be observed if the relative error of the estimate for $\lambda$ must be no larger than $\pm 10\%$ with 95% confidence?
**Solution:** The desired confidence interval for $\lambda$ has the form

$$P\left(\frac{x}{n}(1-0.10) < \lambda < \frac{x}{n}(1+0.10)\right) = 0.95,$$

so $\delta = 0.10$ and from Equation 5.10

$$x = \left(\frac{1.96}{0.10}\right)^2 = 385.$$

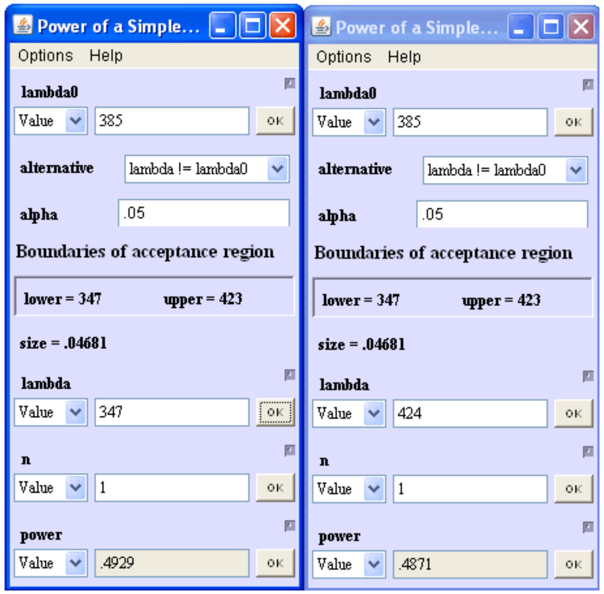
That is, if the Poisson process is sampled until $x = 385$ counts are obtained, then the 95% confidence limits for $\lambda$ will be

$$UCL/LCL = \left(\frac{385}{n}\right)(1 \pm 0.10)$$

or

$$P\left(\frac{346}{n} < \lambda < \frac{424}{n}\right) = 0.95.$$

From **Piface**> **Generic Poisson test**:

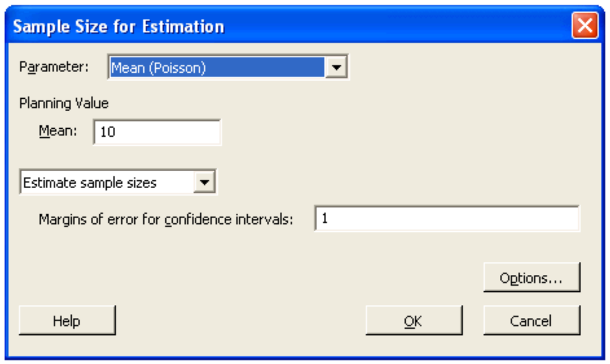



From **MINITAB (V16)**> **Stat**> **Power and Sample Size**> **Sample Size for Estimation**> **Mean (Poisson)**:

```
MTB > SSCI;
SUBC>   PMean 10;
SUBC>   Confidence 95.0;
SUBC>   IType 0;
SUBC>   MError 1.
```

**Sample Size for Estimation**

```
Method

Parameter            Mean
Distribution         Poisson
Mean                 10
Confidence level     95%
Confidence interval  Two-sided


Results

  Margin  Sample
of Error    Size
       1      43
```

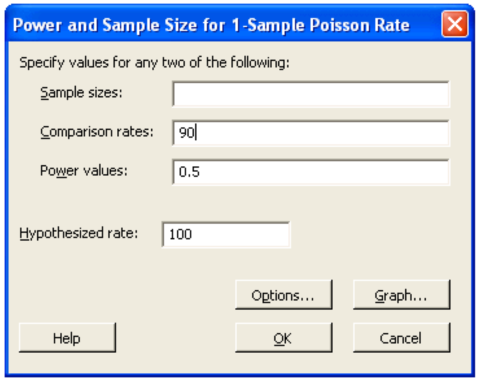



From **MINITAB (V16)**> **Stat**> **Power and Sample Size**> **1-Sample Poisson Rate**:

```
MTB > Power;
SUBC>   OneRate;
SUBC>   RCompare 90;
SUBC>   Power 0.5;
SUBC>   RNull 100;
SUBC>   Alternative 0;
SUBC>   Alpha 0.05;
SUBC>   Length 1.0;
SUBC>   GPCurve.
```

**Power and Sample Size**

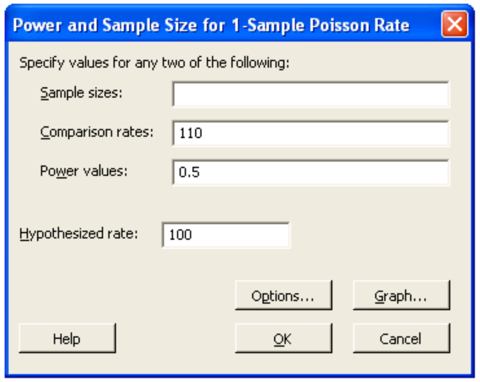
```
Test for 1-Sample Poisson Rate

Testing rate = 100 (versus not = 100)
Alpha = 0.05
"Length" of observation = 1


Comparison  Sample  Target
      Rate    Size   Power  Actual Power
        90       4     0.5      0.516846
```




```
MTB > Power;
SUBC>   OneRate;
SUBC>   RCompare 110;
SUBC>   Power 0.5;
SUBC>   RNull 100;
SUBC>   Alternative 0;
SUBC>   Alpha 0.05;
SUBC>   Length 1.0;
SUBC>   GPCurve.
```

**Power and Sample Size**

```
Test for 1-Sample Poisson Rate

Testing rate = 100 (versus not = 100)
Alpha = 0.05
"Length" of observation = 1


Comparison  Sample  Target
      Rate    Size   Power  Actual Power
       110       4     0.5      0.515305
```




**Example 5.2** For the hypothesis test of $H_0 : \lambda = 4$ versus $H_A : \lambda > 4$ based on a sample of size $n = 2$ units using $\alpha \leq 0.05$, determine the power to reject $H_0$ when $\lambda = 8$.
**Solution:** Under $H_0$, the distribution of the observed number of counts $x$ will be Poisson with $\mu_x = n\lambda_0 = 2 \times 4 = 8$. The acceptance interval for $H_0$ will be $0 \leq x \leq 13$ because

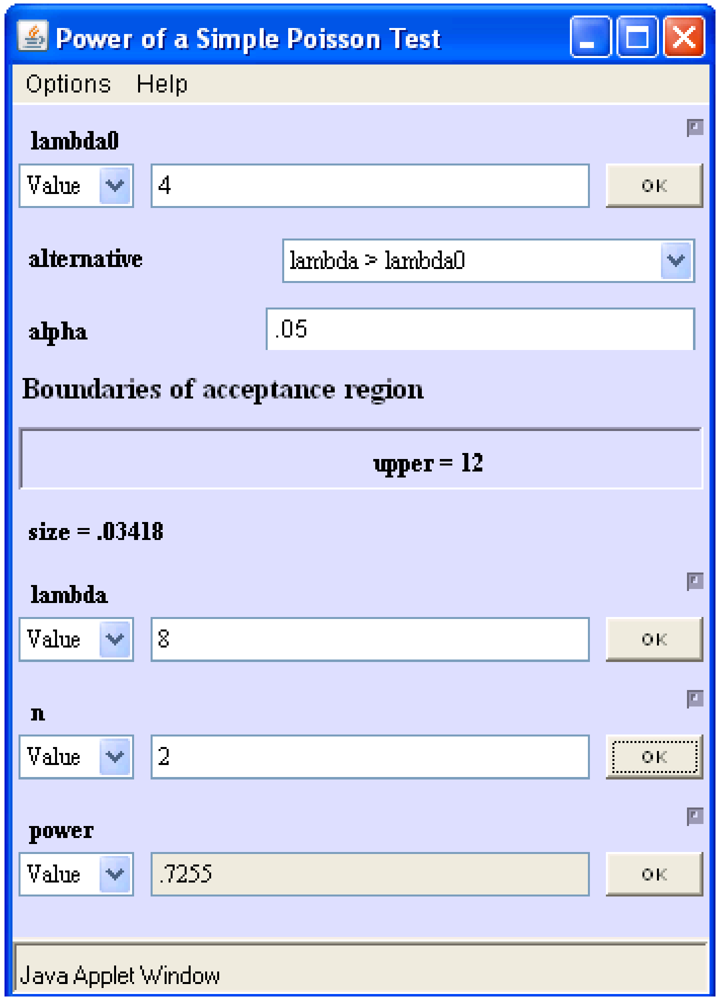
$$\begin{aligned}(1 - Poisson\,(0 \leq x \leq 12; 8) = 0.064) &> 0.05 \\ (1 - Poisson\,(0 \leq x \leq 13; 8) = 0.034) &< 0.05,\end{aligned}$$

so the exact significance level for the test will be $\alpha = 0.034$. With $\lambda = 8$, the power to reject $H_0$ is

$$\pi = 1 - Poisson\,(0 \leq x \leq 13; n\lambda = 16) = 0.725.$$

The count distributions under $H_0$ and $H_A$ are shown in Figure 5.1.

From **Piface**> **Generic Poisson test**:

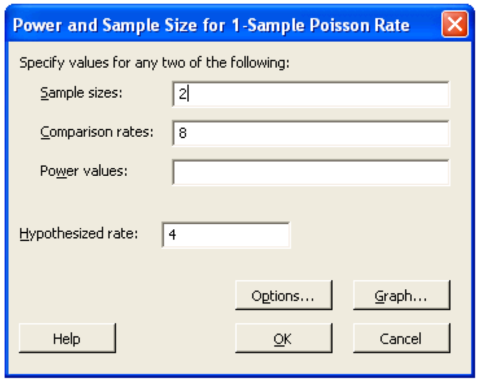



MINITAB V16 uses the normal approximation to the Poisson distribution so its answers are different from the exact answers. From **MINITAB (V16)**> **Stat**> **Power and Sample Size**> **1-Sample Poisson Rate**:

```
MTB > Power;
SUBC>   OneRate;
SUBC>   Sample 2;
SUBC>   RCompare 8;
SUBC>   RNull 4;
SUBC>   Alternative 1;
SUBC>   Alpha 0.05;
SUBC>   Length 1.0;
SUBC>   GPCurve.
```

**Power and Sample Size**

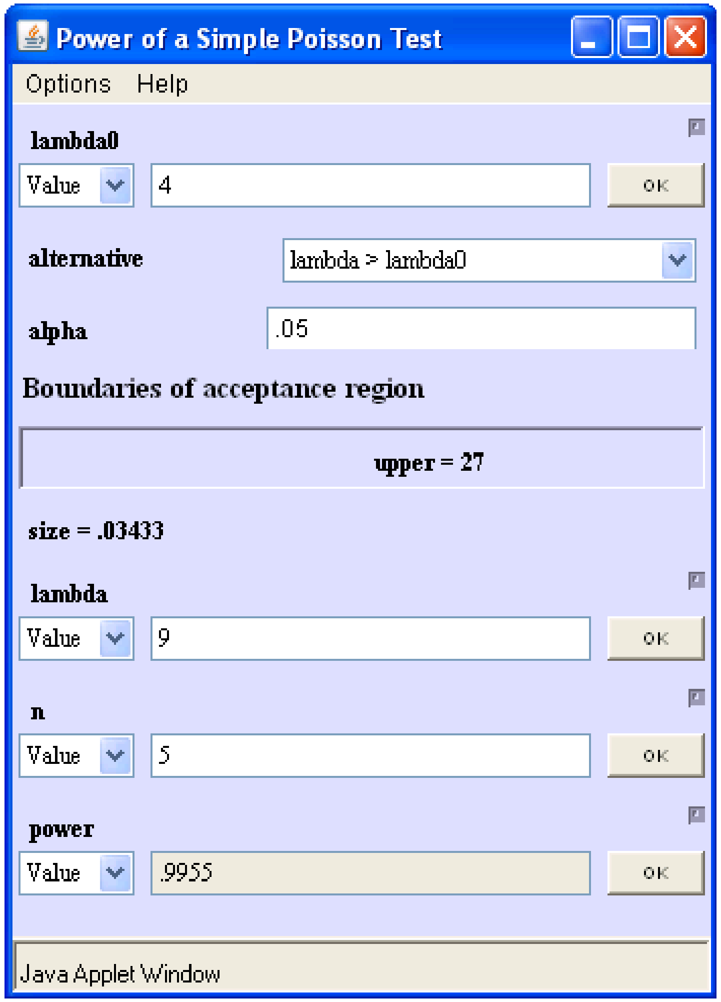
```
Test for 1-Sample Poisson Rate

Testing rate = 4 (versus > 4)
Alpha = 0.05
"Length" of observation = 1


Comparison  Sample
      Rate    Size     Power
         8       2  0.798679
```




**Example 5.3** For the hypothesis test of $H_0 : \lambda = 4$ versus $H_A : \lambda > 4$ based on a sample of size $n = 5$ units, determine the power to reject $H_0$ when $\lambda = 9$. Use the square root transformation method with $\alpha = 0.05$.
**Solution:** By Equation 5.16, the power to reject $H_0 : \lambda = 4$ when $\lambda = 9$ is

$$\begin{aligned}\pi &= \Phi\left(-2\sqrt{5}\left(\sqrt{9}-\sqrt{4}\right)+z_{0.05}<z<\infty\right)\\ &= \Phi\left(-2.83<z<\infty\right)\\ &= 0.9977.\end{aligned}$$

From **Piface**> **Generic Poisson test**:

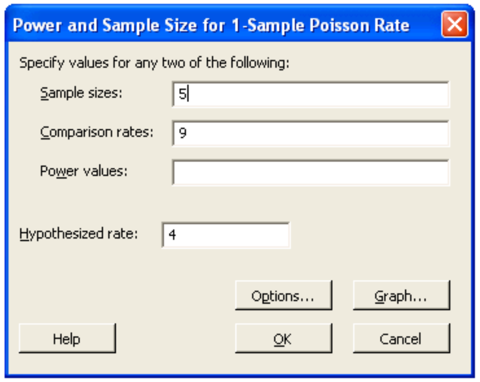



From **MINITAB (V16)**> **Stat**> **Power and Sample Size**> **1-Sample Poisson Rate**:

```
MTB > Power;
SUBC>   OneRate;
SUBC>   Sample 5;
SUBC>   RCompare 9;
SUBC>   RNull 4;
SUBC>   Alternative 1;
SUBC>   Alpha 0.05;
SUBC>   Length 1.0;
SUBC>   GPCurve.
```

**Power and Sample Size**

```
Test for 1-Sample Poisson Rate

Testing rate = 4 (versus > 4)
Alpha = 0.05
"Length" of observation = 1


Comparison  Sample
      Rate    Size     Power
         9       5  0.995733
```

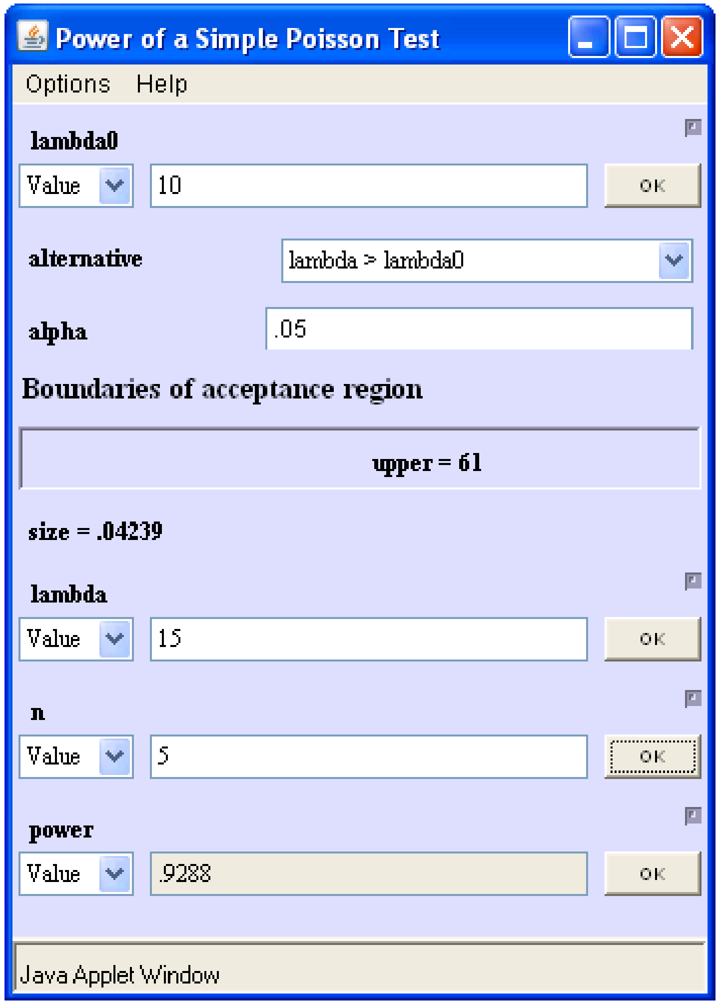



**Example 5.4** How many sampling units must be inspected to reject $H_0 : \lambda = 10$ with 90% power in favor of $H_A : \lambda > 10$ when in fact $\lambda = 15$?
**Solution:** By Equation 5.17 the necessary sample size is

$$n = \frac{1}{4}\left(\frac{1.645 + 1.282}{\sqrt{15} - \sqrt{10}}\right)^2 = 4.2,$$

which rounds up to $n = 5$ sampling units.

From **Piface**> **Generic Poisson test**:

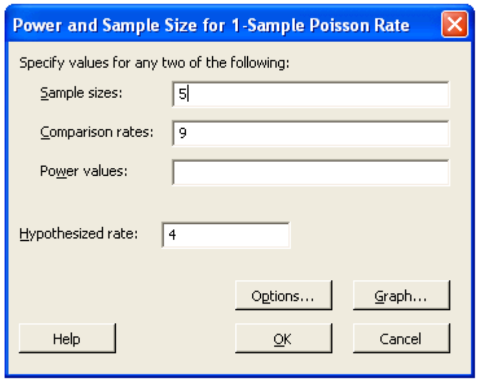



From **MINITAB (V16)**> **Stat**> **Power and Sample Size**> **1-Sample Poisson Rate**:

```
MTB > Power;
SUBC>   OneRate;
SUBC>   Sample 5;
SUBC>   RCompare 9;
SUBC>   RNull 4;
SUBC>   Alternative 1;
SUBC>   Alpha 0.05;
SUBC>   Length 1.0;
SUBC>   GPCurve.
```

**Power and Sample Size**

```
Test for 1-Sample Poisson Rate

Testing rate = 4 (versus > 4)
Alpha = 0.05
"Length" of observation = 1


Comparison  Sample
      Rate    Size     Power
         9       5  0.995733
```




## 5.2 Two Poisson Counts

**Example 5.5** What optimal sample sizes are required to estimate the difference between two Poisson means with 30% precision if the means are expected to be $\lambda_1 = 25$ and $\lambda_2 = 16$?
**Solution:** The difference between the means is expected to be $\Delta\lambda = 9$, so the confidence interval half-width must be 30% of that, or

$$\delta = 0.3 \times 9 = 2.7.$$

From Equation 5.24, the optimal sample size ratio is

$$\frac{n_1}{n_2} = \sqrt{\frac{\lambda_1}{\lambda_2}} = \sqrt{\frac{25}{16}} = 1.25.$$

From Equation 5.22, with $\alpha = 0.05$, the sample size $n_1$ must be

$$n_1 = \left(\frac{1.96}{2.7}\right)^2 (25 + 1.25 \times 16) = 23.7$$

and the sample size $n_2$ must be

$$n_2 = \frac{n_1}{\left(\frac{n_1}{n_2}\right)} = \frac{23.7}{1.25} = 18.96,$$

which round up to $n_1 = 24$ and $n_2 = 19$.

**Example 5.6** How many Poisson counts are required to estimate the ratio of the means of two independent Poisson distributions to within 20% of the true ratio with 95% confidence if the sample sizes will be the same and the ratio of the means is expected to be $\lambda_1/\lambda_2 \simeq 2$?

**Solution:** With $n_1/n_2 = 1$, $\lambda_1/\lambda_2 = 2$, $z_{0.025} = 1.96$, and $\delta = 0.02$ in Equation 5.30, the number of Poisson counts required in the first sample is

$$\begin{aligned} x_1 &= (1 + 1 \times 2)\left(\frac{1.96}{0.2}\right)^2 \\ &= 289. \end{aligned}$$

The corresponding required counts in the second sample are about half of the counts in the first: $x_2 = 289/2 = 145$.

**Example 5.7** Determine the power to reject $H_0 : \lambda_1 = \lambda_2$ in favor of $H_A : \lambda_1 < \lambda_2$ when $\lambda_1 = 10$, $n_1 = 8$ and $\lambda_2 = 15$, $n_2 = 6$. Use the large-sample normal approximation, square root transform, and $F$ test methods with $\alpha = 0.05$.
**Solution:** The expected number of counts from the first $(x_1)$ and second $(x_2)$ populations are both large enough to justify the large sample approximation method. By this method the power is

$$\begin{aligned} \pi &= \Phi\left(-\infty < z < \frac{15 - 10}{\sqrt{\frac{15}{6} + \frac{10}{8}}} - 1.645\right) \\ &= \Phi\left(-\infty < z < 0.937\right) \\ &= 0.826. \end{aligned}$$

By the log-transformation method the power is

$$\begin{aligned} \pi &= \Phi\left(-\infty < z < \frac{\log\left(15/10\right)}{\sqrt{\frac{1}{6\times 15} + \frac{1}{8\times 10}}} - 1.645\right) \\ &= \Phi\left(-\infty < z < 0.994\right) \\ &= 0.840. \end{aligned}$$

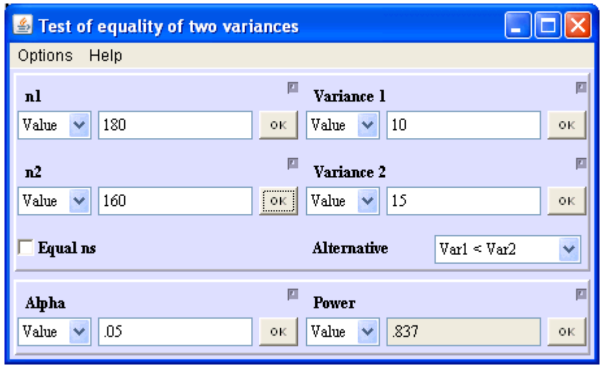
By the square-root transform method the power is

$$\begin{aligned} \pi &= \Phi\left(-\infty < z < \frac{\sqrt{15} - \sqrt{10}}{\frac{1}{2}\sqrt{\frac{1}{8} + \frac{1}{6}}} - 1.645\right) \\ &= \Phi\left(-\infty < z < 1.01\right) \\ &= 0.838. \end{aligned}$$

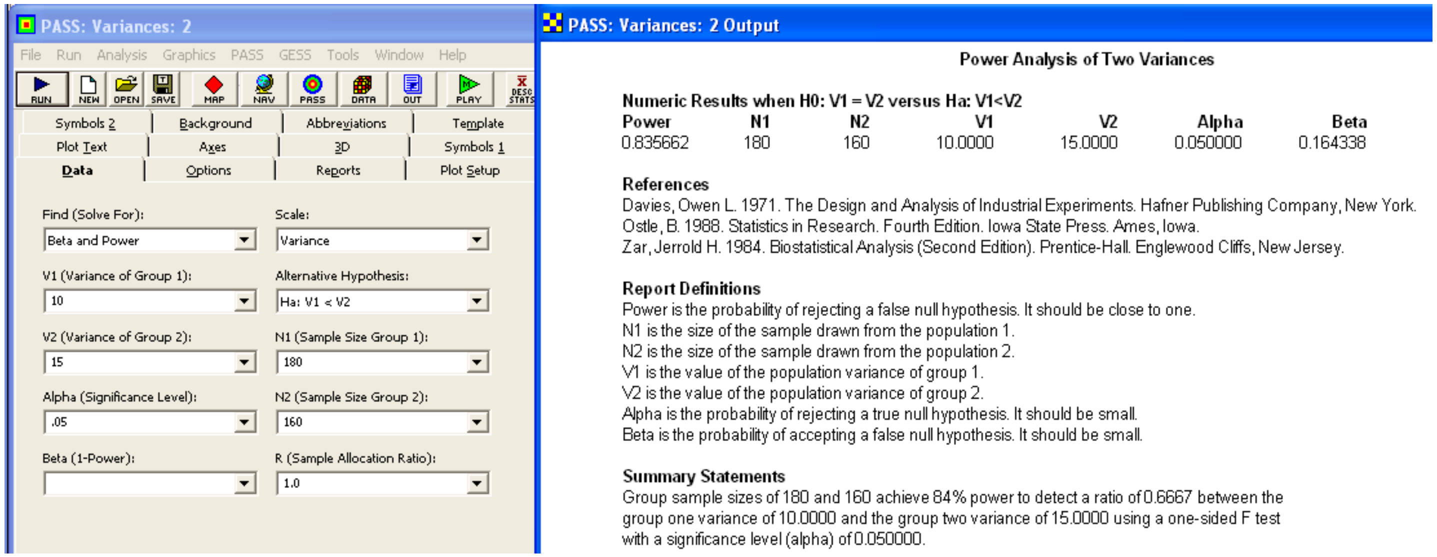
By the $F$ test method the power is

$$\begin{aligned} \pi &= P\left(\frac{10}{15} F_{0.95,2(6)(15),2(8)(10)} < F < \infty\right) \\ &= P\left(0.86 < F < \infty\right) \\ &= 0.837. \end{aligned}$$

MINITAB V16 supports the two-sample Poisson method but only for equal sample sizes.
By the F test method using **Piface**> **Two variances (F Test)** with $n_1 = 2 \times 6 \times 15 = 180$ and $n_2 = 2 \times 8 \times 10 = 160$:




By the F test method using **PASS**> **Variance**> **Variance: 2 Groups**:




**Example 5.8** What minimum total counts are required for the two-sample counts test to detect a factor of two difference between the count rates with 90% power? Assume that the two sample sizes will be equal.
**Solution:** The hypotheses to be tested are $H_0 : \lambda_1/\lambda_2 = 1$ versus $H_A : \lambda_1/\lambda_2 > 1$. From Equation 5.45, which is expressed in terms of the ratio of the two means, the number of

count events $x_1$ required to reject $H_0$ when $\lambda_1/\lambda_2 = 2$ is

$$
\begin{aligned}
x_1 &= \left(1+\frac{\lambda_1}{\lambda_2}\right)\left(\frac{z_\alpha + z_\beta}{\ln\left(\lambda_1/\lambda_2\right)}\right)^2 \\
&= (1+2)\left(\frac{1.645+1.282}{\ln\left(2\right)}\right)^2 \\
&= 54.
\end{aligned}
$$

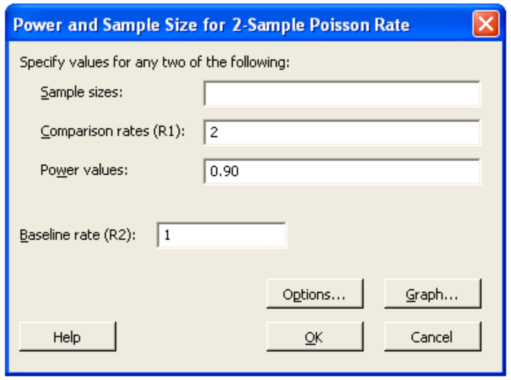
Because $\lambda_2 = \lambda_1/2$, the corresponding number of $x_2$ counts is $x_2 = 54/2 = 27$.

From **MINITAB (V16)**> **Stat**> **Power and Sample Size**> **2-Sample Poisson Rate**:

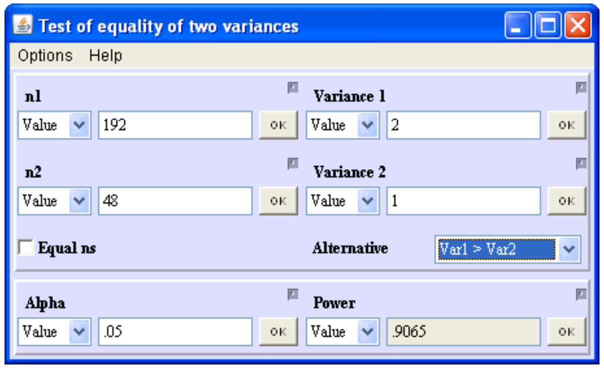
```
MTB > Power;
SUBC>   TwoRate;
SUBC>   RCompare 2;
SUBC>   Power 0.90;
SUBC>   RBaseline 1;
SUBC>   Alternative 1;
SUBC>   Alpha 0.05;
SUBC>   Length 1.0;
SUBC>   GPCurve.
```

**Power and Sample Size**

```
Test for 2-Sample Poisson Rate

Testing comparison rate = baseline rate (versus >)
Calculating power for baseline rate = 1
Alpha = 0.05
"Lengths" of observation for sample 1, sample 2 = 1, 1


Comparison  Sample  Target
      Rate    Size   Power  Actual Power
         2      26     0.9      0.903039

The sample size is for each group.
```




By trial and error using the F test method in **Piface**> **Two variances (F Test)**, 90% power is obtained with $x_1 = 192/\left(2 \times 2\right) = 48$ and $x_2 = 48/\left(2 \times 2\right) = 24$:



## 5.3 Tests for Many Poisson Counts

**Example 5.9** In a test for differences between mean counts from five different processes, determine the power to reject $H_0 : \lambda_i = \lambda_j$ for all $i, j$ pairs when $\lambda_1 = \lambda_2 = \lambda_3 = 16$, $\lambda_4 = 9$, $\lambda_5 = 25$ and $n = 3$ units from each process are inspected. The number of counts will be reported for each unit. Assume that the test will be performed using one-way ANOVA applied to the square root transformed counts.
**Solution:** After the square root transform, the transformed treatment means are $\lambda'_1 = \lambda'_2 = \lambda'_3 = 4$, $\lambda'_4 = 3$, and $\lambda'_5 = 5$. The grand transformed mean is $\overline{\lambda'} = 4$, so the treatment biases relative to the grand mean are $0, 0, 0, -1$, and $1$, respectively. The ANOVA $F$ test noncentrality parameter is then

$$\phi = \frac{E(SS_{Treatment})}{E(MS_\epsilon)} = \frac{3\left(0^2 + 0^2 + 0^2 + (-1)^2 + (1)^2\right)}{\left(\frac{1}{2}\right)^2} = 24$$

where $E\left(MS_\epsilon\right) = (\sigma')^2 = \left(\frac{1}{2}\right)^2$ is the error variance of the transformed counts. The ANOVA will have $df_{Treatment} = 4$ and $df_\epsilon = 15 - 1 - 4 = 10$, so the $F$ test critical value will be $F_{0.95,4,24} = 3.48$. The power to reject $H_0$ is then given by Equation 8.1:

$$\begin{aligned} F_{1-\alpha} &= F_{1-\pi,\phi} \\ 3.48 &= F_{1-\pi,24} \\ 3.48 &= F_{0.11,24}, \end{aligned}$$

so the power is $\pi = 0.89$ to reject $H_0$ for the specified set of means.

**Example 5.10** In a test for differences among the means of five Poisson populations, determine the probability of rejecting $H_0 : \lambda_i = \lambda_0$ for all $i$ when $\lambda_i = \{16, 16, 16, 12, 20\}$. The number of units inspected is $n_i = 4$ for all $i$.
**Solution:** Given the Poisson means specified under $H_A$, the value of $\lambda_0$ under $H_0$ is given by

$$\lambda_0 = \frac{1}{5}\left(16 + 16 + 16 + 12 + 20\right) = 16.$$

With $n_i = n = 4$, the noncentrality parameter is given by

$$\begin{aligned} \phi &= n \sum_{i=1}^{k} \frac{\left(\lambda_{A,i} - \lambda_0\right)^2}{\lambda_0} \\ &= 4\left(\frac{(0)^2}{16} + \frac{(0)^2}{16} + \frac{(0)^2}{16} + \frac{(-4)^2}{16} + \frac{(4)^2}{16}\right) = 8. \end{aligned}$$

The power is determined from Equation 5.61:

$$\chi^2_{0.95} = 9.49 = \chi^2_{0.395,8}$$

where the central and noncentral $\chi^2$ distributions both have $\nu = 5 - 1 = 4$ degrees of freedom, so the power is $\pi = 0.605$.

## 5.4 Correcting for Background Counts

**Example 5.11** In a two-sample test for counts, what common sample size $n = n_1 = n_2$ is required to distinguish $\lambda_1 = \lambda_2 = 6$ from $\lambda_1 = 6$, $\lambda_2 = 15$ with 90% power in the presence of a background count rate of $\lambda_0 = 10$?
**Solution:** From Equation 5.55, modified to account for the background count rate, the necessary sample size to reject $H_0 : \lambda_1 = \lambda_2$ in favor of $H_A : \lambda_1 < \lambda_2$ with 90% power and $\alpha = 0.05$ is given by

$$\begin{aligned} n &= \frac{1}{2}\left(\frac{z_\alpha + z_\beta}{(\sqrt{\lambda_2 + \lambda_0} - \sqrt{\lambda_1 + \lambda_0})}\right)^2 \qquad (5.1) \\ &= \frac{1}{2}\left(\frac{1.645 + 1.282}{\sqrt{25} - \sqrt{16}}\right)^2 = 5. \end{aligned}$$

# Chapter 6

# Regression

## 6.1 Linear Regression

**Example 6.1** Designed experiments frequently involve two or three equally weighted levels of $x$. Compare the sample sizes required for these two important special cases if they must both deliver a $\beta_1$ confidence interval half-width $\delta$ and the observations are taken over the same $x$ range from $x_{min}$ to $x_{max}$. For the three-level case, assume that the middle level will be midway between $x_{min}$ and $x_{max}$.

**Solution:** The subscripts 2 and 3 will be used to indicate parameters from the two-level and three-level cases, respectively. For the two-level case, from Equation 6.12 with $k_2 = 2$ and $\Delta x_2 = x_{max} - x_{min}$, the sample size per $x$ level will be

$$\begin{aligned} n_2 &\geq 2\left(\frac{t_{\alpha/2}\widehat{\sigma}_\epsilon}{\delta\Delta x_1}\right)^2 \\ &\geq 2\left(\frac{t_{\alpha/2}\widehat{\sigma}_\epsilon}{\delta\left(x_{max} - x_{min}\right)}\right)^2. \end{aligned} \tag{6.1}$$

For the three-level case with $k_3 = 3$ and $\Delta x_3 = \frac{1}{2}\left(x_{max} - x_{min}\right)$ the sample size per $x$ level will be

$$\begin{aligned} n_3 &\geq \frac{1}{2}\left(\frac{t_{\alpha/2}\widehat{\sigma}_\epsilon}{\delta\Delta x_2}\right)^2 \\ &\geq 2\left(\frac{t_{\alpha/2}\widehat{\sigma}_\epsilon}{\delta\left(x_{max} - x_{min}\right)}\right)^2. \end{aligned} \tag{6.2}$$

Because $n_2 = n_3$, $N_2 = 2n_2$, and $N_3 = 3n_2$, the two experiments appear to have the same ability to resolve $\beta_1$ even though the three-level experiment requires 50% more observations! This means that the middle observations in the three-level experiment are effectively wasted for the purpose of estimating $\beta_1$. This statement is not entirely true because the middle observations in the three-level experiment do add error degrees of freedom, which potentially decrease $n_3$ compared to $n_2$ for the same $\delta$. In general, the purpose of using three levels of $x$ in an experiment is not to improve the precision of the $\beta_1$ estimate; rather, three levels are used to allow a linear lack of fit test, which is not possible using just two levels of $x$.

**Example 6.2** Compare the sample sizes required to estimate the slope parameter with equal precision for two experiments if $x$ is uniformly distributed over the interval from $x_{min}$ to $x_{max}$ in the first experiment and if $x$ has two levels, $x_{min}$ and $x_{max}$, in the second experiment.
**Solution:** From Equations 6.9 and 6.13 the ratio of the total number of observations required by the two experiments is

$$\frac{N_{\text{uniform } x}}{N_{\text{two levels of } x}} \simeq \frac{12\left(\frac{t_{\alpha/2}\widehat{\sigma}_{\epsilon}}{\delta(x_{max}-x_{min})}\right)^2}{4\left(\frac{t_{\alpha/2}\widehat{\sigma}_{\epsilon}}{\delta\Delta x}\right)^2}$$

where the $t_{\alpha/2}$ values may differ a bit because of the difference in error degrees of freedom. Both experiments cover the same $x$ range, so $\Delta x = x_{max} - x_{min}$ and the sample size ratio reduces to:

$$\frac{N_{\text{uniform } x}}{N_{\text{two levels of } x}} \simeq 3.$$

That is, three times as many observations are required in an experiment that uses uniformly distributed $x$ values than if the $x$ values are concentrated at the ends of the $x$ range. Because we saw in Example 6.1 that the experiment with three evenly spaced, equally weighted levels of $x$ requires 1.5 times as many observations as the two-level experiment, other methods of taking evenly spaced, equally weighted observations of $x$ must give experiment sample size ratios between 1.5 and 3. Obviously, the two-level equally weighted method is the most efficient method for specifying $x$ values for an experiment.

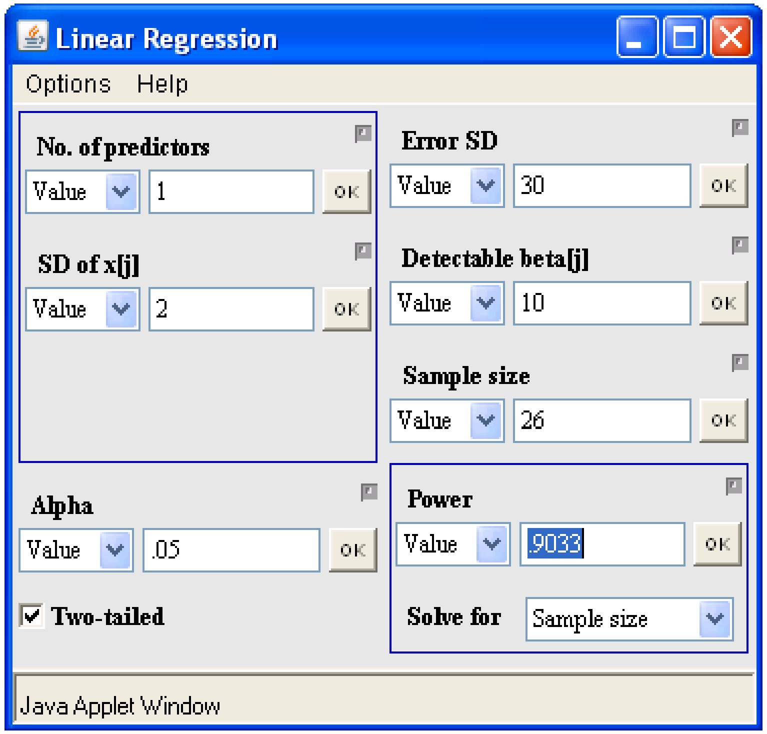
**Example 6.3** For an experiment to be analyzed by linear regression with a single predictor, how many observations are required to reject $H_0 : \beta_1 = 0$ in favor of $H_A : \beta_1 \neq 0$ with 90% power for $\beta_1 = 10$ when a) the distribution of $x$ values will be normal with $\mu_x \simeq 15$ and $\sigma_x \simeq 2$; b) an equal number of observations will be taken at $x = 10$ and $x = 20$; c) uniformly distributed values of $x$ will be used over the interval $10 \leq x \leq 20$; and d) an equal number of observations will be taken at $x = 10$, 15, and 20. Experience with the process tells us the standard error of the model is expected to be $\sigma_{\epsilon} = 30$.
**Solution:**

a) With $t \simeq z$ in Equation 6.19, the first iteration to find $N$ gives

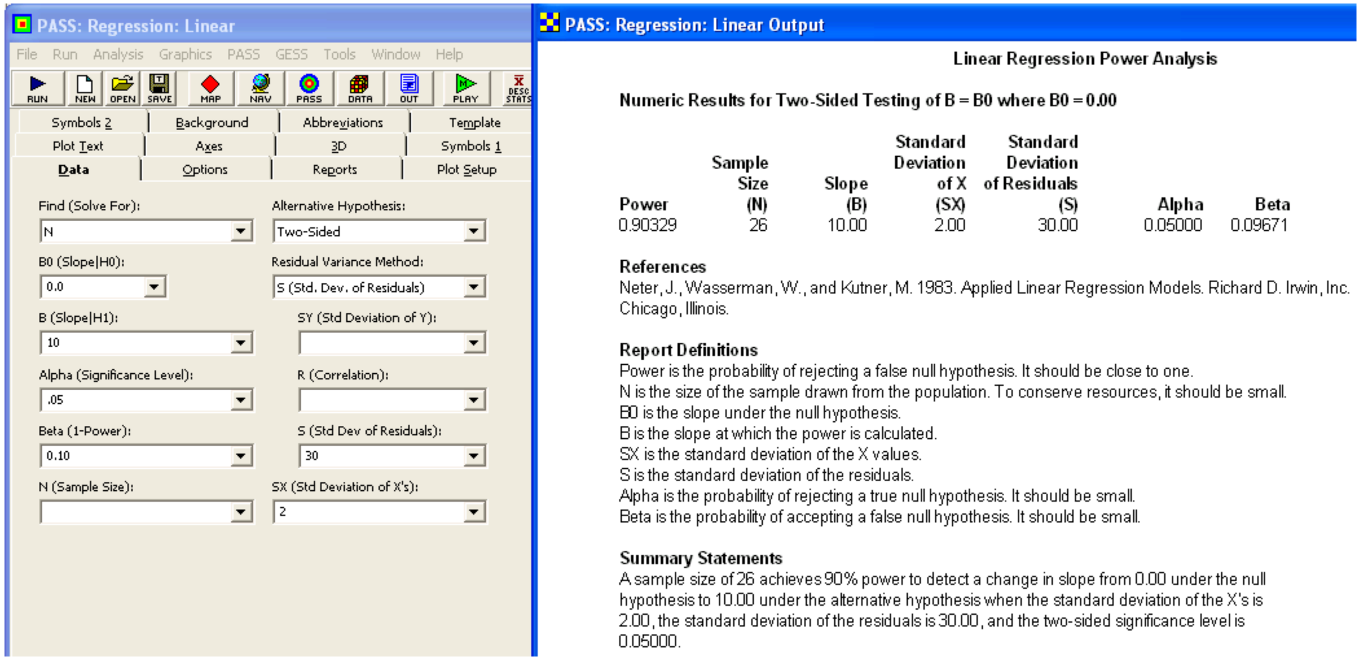
$$N = (z_{0.025} + z_{0.10})^2 \left(\frac{30}{10 \times 2}\right)^2 = 24.$$

Further iterations indicate that the required sample size is $N = 26$.

From **Piface**> **Linear regression**:




From **PASS**> **Regression**> **Linear Regression**:




b) The standard deviation of the $x$ values will be

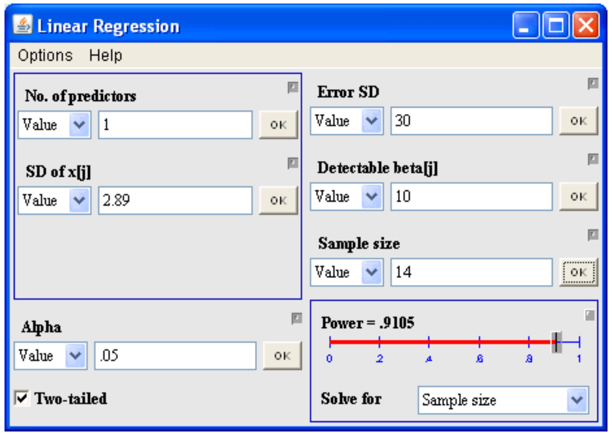
$$\sigma_x = \sqrt{\frac{SS_x}{N}} = \sqrt{\frac{1}{N}\frac{N}{2}\left((-5)^2 + (5)^2\right)} = 5.$$

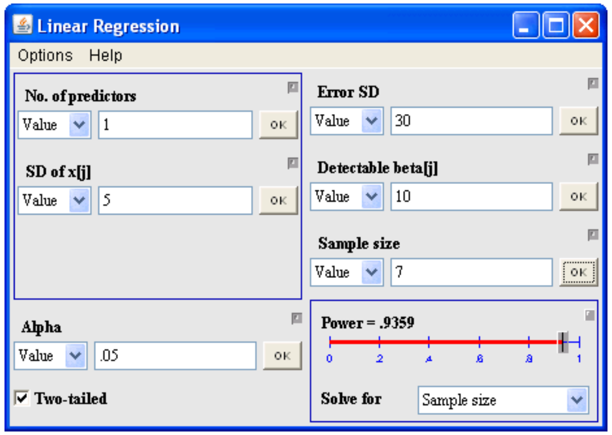
The first iteration to find $N$, with $t \simeq z$, gives

$$N = (z_{0.025} + z_{0.10})^2 \left(\frac{30}{10 \times 5}\right)^2 = 4.$$

Further iterations indicate that $N = 7$ observations are required.

From **Piface**> **Linear regression**:




c) For uniformly distributed $x$, the standard deviation of the $x$ values is

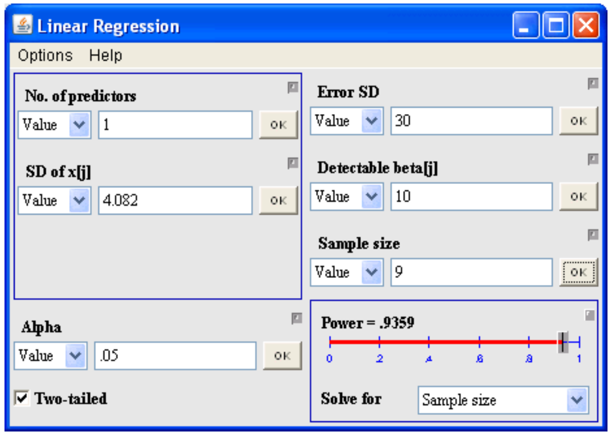
$$\sigma_x = \frac{x_{\max} - x_{\min}}{\sqrt{12}} = \frac{10}{\sqrt{12}} = 2.89.$$

With $t \simeq z$, the first iteration to find $N$ gives

$$N = (z_{0.025} + z_{0.10})^2 \left( \frac{30}{10 \times 2.89} \right)^2 = 12.$$

Further iterations indicate that $N = 14$ observations are required.

From **Piface**> **Linear regression**:




d) The standard deviation of the $x$ values will be

$$\sigma_x = \sqrt{\frac{SS_x}{N}} = \sqrt{\frac{1}{N}\frac{N}{3}\left((-5)^2 + (0)^2 + (5)^2\right)} = 4.0825.$$

The first iteration to find $N$, with $t \simeq z$, gives

$$N = (z_{0.025} + z_{0.10})^2 \left(\frac{30}{10 \times 4.0825}\right)^2 = 6.$$

Further iterations indicate that $N = 9$ observations are required.

From **Piface**> **Linear regression**:

**Example 6.4** What is the power to reject $H_0$ for the situation described in Example 6.3a if the sample size is $N = 20$?
**Solution:** From Equation 6.18 with $SS_x = N\sigma_x^2$ and $df_\epsilon = 20 - 2 = 18$,

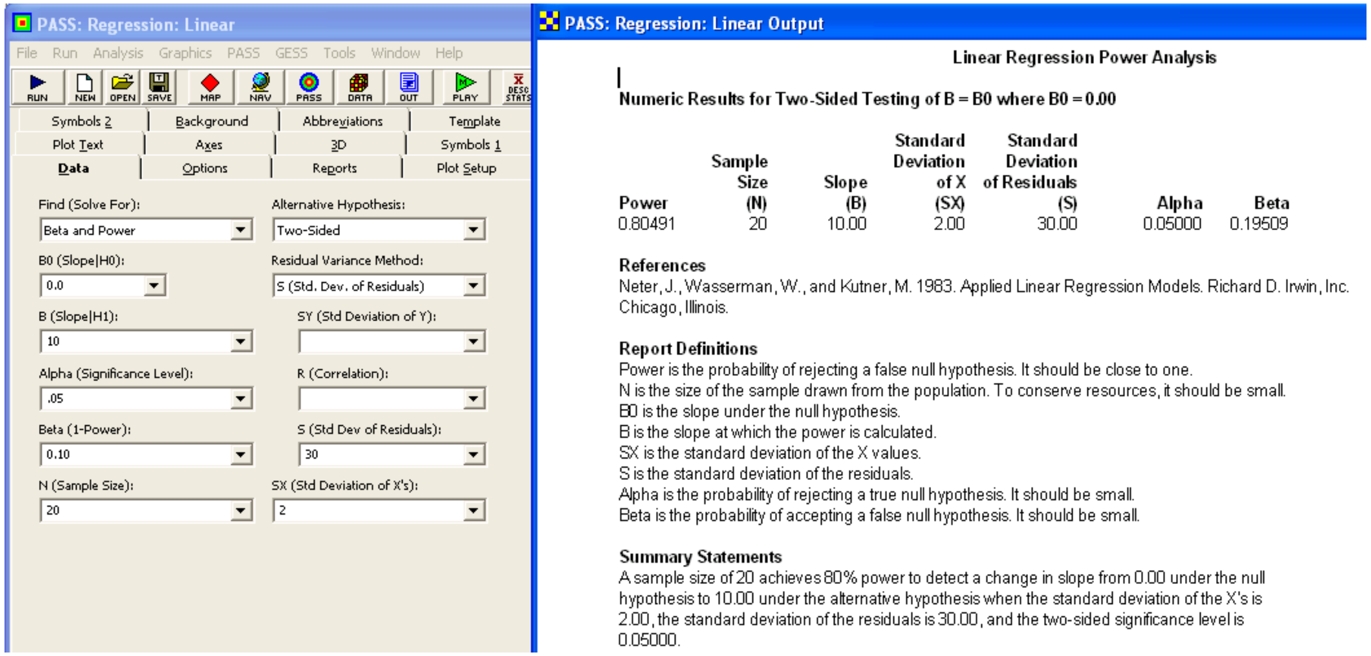
$$\begin{aligned} t_\beta &= \frac{|\beta_1|\sqrt{N}\sigma_x}{\sigma_\epsilon} - t_{0.025,18} \\ &= \frac{10\sqrt{202}}{30} - 2.10 \\ &= 0.881. \end{aligned}$$

The power, as given by Equation 6.17, is

$$\begin{aligned} \pi &= P(-\infty < t < 0.881) \\ &= 0.805. \end{aligned}$$

From **Piface**> **Linear regression**:


From **PASS**> **Regression**> **Linear Regression**:



## 6.2 Logistic Regression

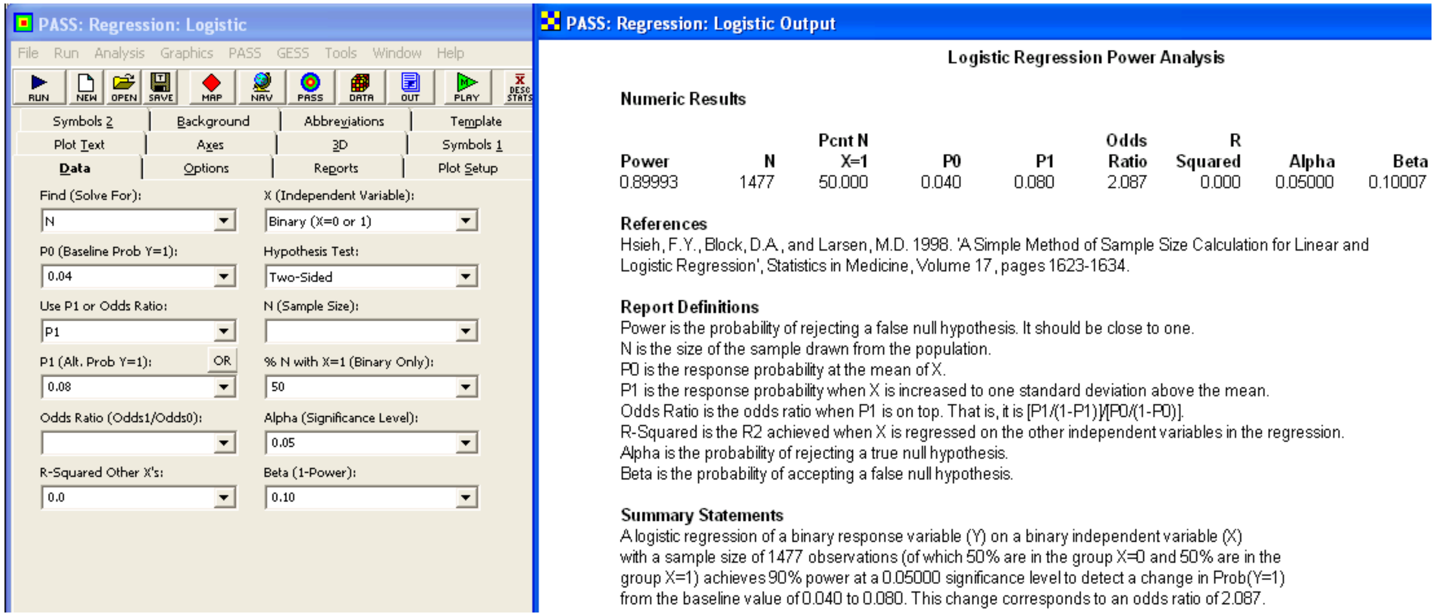
**Example 6.5** What sample size is required for an experiment to be analyzed by logistic regression if $H_0 : \beta_1 = 0$ should be rejected in favor of $H_A : \beta_1 \neq 0$ with 90% power when $x$ is dichotomous with associated proportions $p_1 = 0.04$ and $p_2 = 0.08$?
**Solution:** The odds ratio for the given proportions is

$$OR = \frac{p_1/(1-p_1)}{p_2/(1-p_2)} = \frac{0.04/0.96}{0.08/0.92} = 0.479.$$

The required sample size is given by Equation 4.97:

$$n = \left(\frac{z_{0.025} + z_{0.10}}{\ln(0.479)}\right)^2 \left(\frac{1}{0.04(0.96)} + \frac{1}{0.08(0.92)}\right) = 770.$$

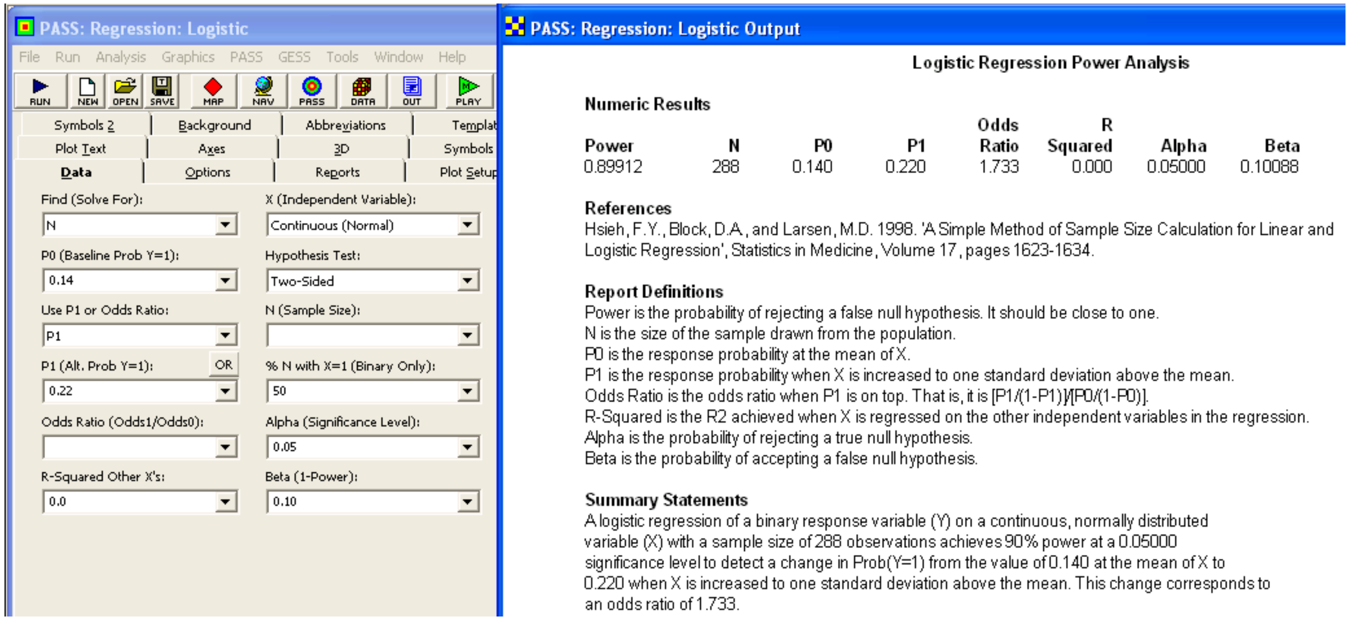
From **PASS**> **Regression**> **Logistic Regression** the total sample size is:




**Example 6.6** What sample size is required for an experiment to be analyzed by logistic regression if $H_0 : \beta_1 = 0$ should be rejected in favor of $H_A : \beta_1 \neq 0$ with 90% power when $x$ is normally distributed with expected success proportions $p(x = \mu) = 0.14$ and $p(x = \mu + \sigma) = 0.22$.
**Solution:** From Equation 6.22 the required sample size is

$$n = \frac{(1.96 + 1.282)^2}{0.14(0.86)\left(\ln\left(\frac{0.14/0.86}{0.22/0.78}\right)\right)^2} = 289.$$

From **PASS**> **Regression**> **Logistic Regression**:


PASS: Regression: Logistic
File Run Analysis Graphics PASS GESS Tools Window Help
RUN NEW OPEN SAVE MAP NAV PASS DATA OUT PLAY
Symbols 2 Background Abbreviations Templat
Plot Text Axes 3D Symbols
Data Options Reports Plot Setup
Find (Solve For):
N
X (Independent Variable):
Continuous (Normal)
P0 (Baseline Prob Y=1):
0.14
Hypothesis Test:
Two-Sided
Use P1 or Odds Ratio:
P1
N (Sample Size):
P1 (Alt. Prob Y=1):
OR
0.22
% N with X=1 (Binary Only):
50
Odds Ratio (Odds1/Odds0):
Alpha (Significance Level):
0.05
R-Squared Other X's:
0.0
Beta (1-Power):
0.10
PASS: Regression: Logistic Output
Logistic Regression Power Analysis
Numeric Results
Power N P0 P1 Odds Ratio R Squared Alpha Beta
0.89912 288 0.140 0.220 1.733 0.000 0.05000 0.10088
References
Hsieh, F.Y., Block, D.A., and Larsen, M.D. 1998. 'A Simple Method of Sample Size Calculation for Linear and Logistic Regression', Statistics in Medicine, Volume 17, pages 1623-1634.
Report Definitions
Power is the probability of rejecting a false null hypothesis. It should be close to one.
N is the size of the sample drawn from the population.
P0 is the response probability at the mean of X.
P1 is the response probability when X is increased to one standard deviation above the mean.
Odds Ratio is the odds ratio when P1 is on top. That is, it is [P1/(1-P1)]/[P0/(1-P0)].
R-Squared is the R2 achieved when X is regressed on the other independent variables in the regression.
Alpha is the probability of rejecting a true null hypothesis.
Beta is the probability of accepting a false null hypothesis.
Summary Statements
A logistic regression of a binary response variable (Y) on a continuous, normally distributed variable (X) with a sample size of 288 observations achieves 90% power at a 0.05000 significance level to detect a change in Prob(Y=1) from the value of 0.140 at the mean of X to 0.220 when X is increased to one standard deviation above the mean. This change corresponds to an odds ratio of 1.733.

# Chapter 7

# Correlation and Agreement

## 7.1 Pearson's Correlation

**Example 7.1** Determine the number of paired observations required to obtain the following confidence interval for the population correlation:

$$P(0.9 < \rho < 0.99) = 0.95.$$

**Solution:** The Fisher's $Z$-transformed confidence interval is

$$\begin{aligned} P(Z_{0.9} < Z_\rho < Z_{0.99}) &= 0.95 \\ P(1.472 < Z_\rho < 2.647) &= 0.95. \end{aligned}$$

Then with $\alpha = 0.05$ in Equation 7.10, the required sample size is

$$\begin{aligned} n &= 4\left(\frac{1.96}{2.647 - 1.472}\right)^2 + 3 \\ &= 15. \end{aligned}$$

**Example 7.2** An experiment is planned to test $H_0 : \rho = 0.9$ versus $H_A : \rho < 0.9$ on the basis of $n = 28$ paired observations. Determine the power of the test to reject $H_0$ when $\rho = 0.7$.
**Solution:** Under $H_0$ following Fisher's transform we have

$$(\mu_Z)_0 = \frac{1}{2} ln\left(\frac{1+0.9}{1-0.9}\right) = 1.472$$

and by Equation 7.4

$$\sigma_Z = \frac{1}{\sqrt{28-3}} = 0.2.$$

For the one-sided left-tailed test, the critical value of $Z$ that distinguishes the accept/reject regions is given by

$$\begin{aligned} Z_{A/R} &= (\mu_Z)_0 - z_\alpha \sigma_z \\ &= 1.472 - z_{0.05}\,(0.2) \\ &= 1.472 - 1.645\,(0.2) \\ &= 1.143. \end{aligned}$$

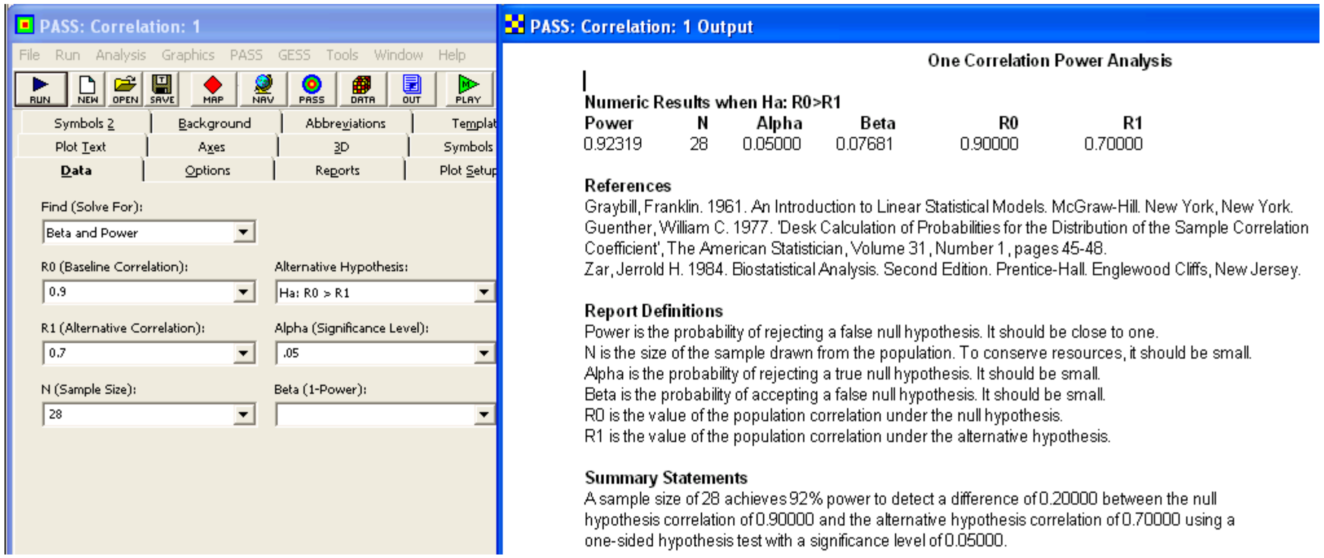
The corresponding $Z$ value under $H_A$ when $\rho = 0.7$ is

$$(\mu_Z)_A = \frac{1}{2} ln \left( \frac{1+0.7}{1-0.7} \right) = 0.867.$$

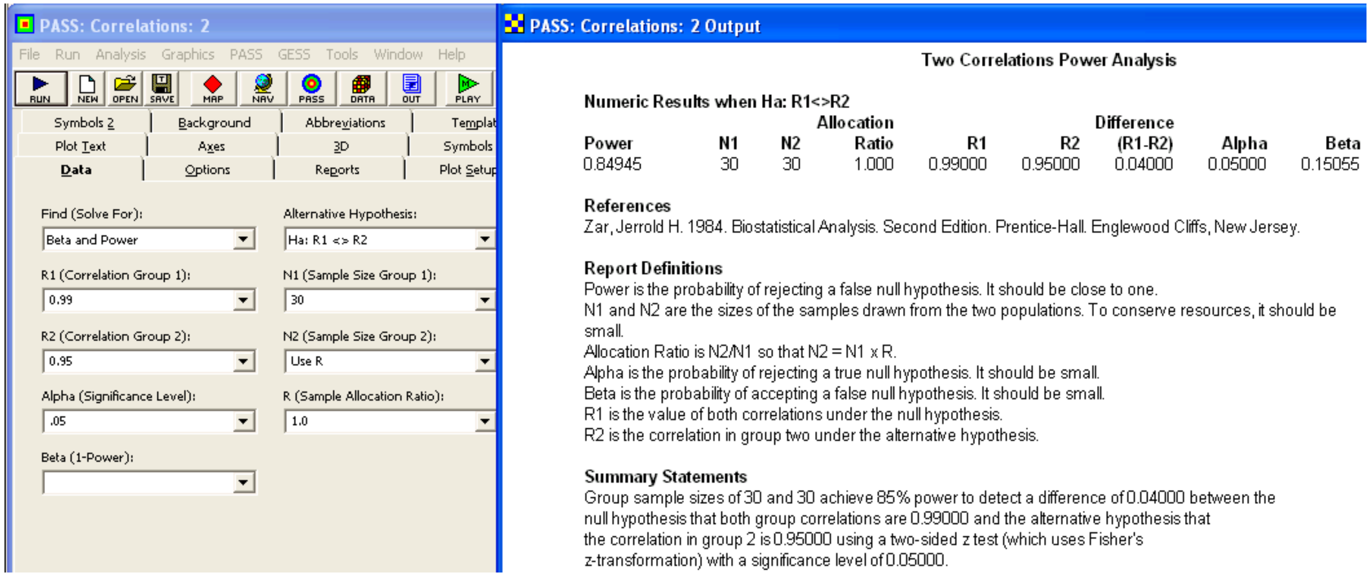
Then the power to reject $H_0$ when $\rho = 0.7$ is

$$\begin{aligned} \pi &= \Phi\left(-\infty < Z < Z_{A/R}; (\mu_Z)_A, \sigma_Z\right) \\ &= \Phi\left(-\infty < Z < 1.143; 0.867, 0.2\right) \\ &= \Phi\left(-\infty < z < 1.38\right) \\ &= 0.916. \end{aligned}$$

From **PASS**> **Correlations**> **Correlations: One**:




**Example 7.3** Find the power to reject $H_0 : \rho_1 = \rho_2$ in favor of $H_A : \rho_1 \neq \rho_2$ when $\rho_1 = 0.99$, $\rho_2 = 0.95$, and $n_1 = n_2 = 30$.




**Solution:** The Fisher-transformed difference between the two correlations under $H_A$ is

$$\begin{aligned}\Delta Z &= Z_1 - Z_2 \\ &= \frac{1}{2} ln\left(\frac{1+0.99}{1-0.99}\right) - \frac{1}{2} ln\left(\frac{1+0.95}{1-0.95}\right) \\ &= 0.815.\end{aligned}$$

From Equations 7.11 and 7.14 with $\alpha = 0.05$ the power is

$$\begin{aligned}\pi &= \Phi\left(-\infty < z < \left(\frac{0.815}{\sqrt{\frac{2}{30-3}}} - 1.96\right)\right) \\ &= \Phi\left(-\infty < z < 1.03\right) \\ &= 0.85.\end{aligned}$$

From **PASS**> **Correlations**> **Correlations: Two**:

**Example 7.4** What sample size should be drawn from two populations to perform the two-sample test for correlation ($H_0 : \rho_1 = \rho_2$ versus $H_A : \rho_1 \neq \rho_2$) with 90% power to reject $H_0$ when $\rho_1 = 0.9$ and $\rho_2 = 0.8$?

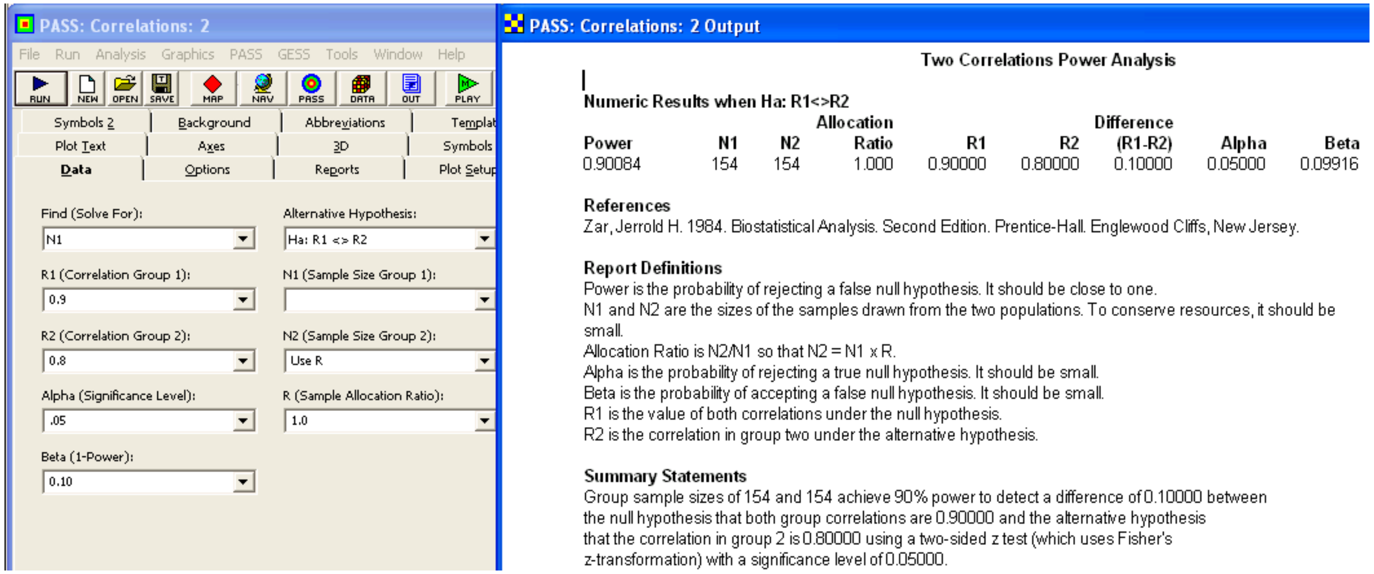
**Solution:** The Fisher-transformed difference between the two correlations is

$$\begin{aligned}\Delta Z &= Z_1 - Z_2 \\ &= \frac{1}{2}ln\left(\frac{1+0.9}{1-0.9}\right) - \frac{1}{2}ln\left(\frac{1+0.8}{1-0.8}\right) \\ &= 0.374.\end{aligned}$$

From Equation 7.15 with $\alpha = 0.05$ and $\beta = 0.10$ the required common sample size is

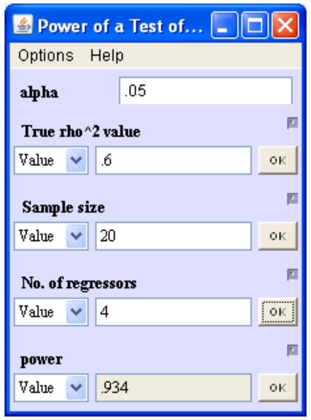
$$\begin{aligned}n &= 2\left(\frac{1.96+1.28}{0.374}\right)^2 + 3 \\ &= 154.\end{aligned}$$

From **PASS**> **Correlations**> **Correlations: Two**:




**Example 7.5** Determine the power to reject $H_0 : \rho^2 = 0$ when in fact $\rho^2 = 0.6$ based on a sample of $n = 20$ observations taken with four random covariates.
**Solution:** The regression model for $y$ as a function of the four predictors will have $df_{model} = k = 4$ model degrees of freedom and $df_\epsilon = n - k - 1 = 20 - 4 - 1 = 15$ error degrees of freedom. The $F$ distribution noncentrality parameter from Equation 7.19 with $\rho^2 = 0.6$ is

$$\phi = 20\frac{0.6}{1-0.6} = 30.$$




From Equation 7.18 we have

$$\begin{aligned} F_{0.95} &= F_{1-\pi,30} \\ 3.056 &= F_{0.024,30}, \end{aligned}$$

so the power is $\pi = 1 - 0.024 = 0.976$.

From **Piface**> **R-square (multiple correlation)**:(I can't explain the discrepancy between my solution and the solution from Piface. There is a comment in Piface's **Help**> **This Dialog** that there is a discrepancy between it and the references.)

## 7.2 Intraclass Correlation

**Example 7.6** An experiment will be performed to determine the single-rater intraclass correlation in a one-way design with $r = 2$ observations per subject. How many subjects must be sampled if the desired confidence interval for $ICC$ is $P(0.7 < ICC < 0.9) = 0.95$?
**Solution:** By Equation 7.31, the desired confidence interval for $ICC$ transforms into the following confidence interval for $Z_{ICC}$:

$$P(0.867 < Z_{ICC} < 1.472) = 0.95.$$

Then, from Equation 7.36 with $r = 2$ observations per subject and $\alpha = 0.05$, the number of subjects required is

$$\begin{aligned} n &= 4\left(\frac{1.96}{1.472 - 0.867}\right)^2 + \frac{3}{2} \\ &= 44. \end{aligned}$$

**Example 7.7** Confirm the answer to Example 7.6 using the method of Donner and Koval.

**Solution:** Assuming that $\widehat{ICC} = 0.8$, the sample size according to Donner and Koval is given by Equation 7.38:

$$\begin{aligned} n &= \frac{8}{2(2-1)}\left(\frac{1.96(1-0.8)(1+(2-1)0.8)}{0.9-0.7}\right)^2 \\ &= 50, \end{aligned}$$

which is in reasonable agreement with the sample size determined by the Fisher's transformation method.

**Example 7.8** How many subjects are required in an experiment to reject $H_0 : ICC = 0.6$ with 80% power when $ICC = 0.8$ and two raters will rate each subject? Confirm the sample size by calculating the exact power.
**Solution:** From Equation 7.31 the $Z_{ICC}$ values corresponding to $ICC = 0.6$ and $ICC = 0.8$ are $Z_0 = 0.693$ and $Z_1 = 1.099$, respectively. From Equation 7.42 with $r = 2$ and $\alpha = 0.05$, the approximate sample size is

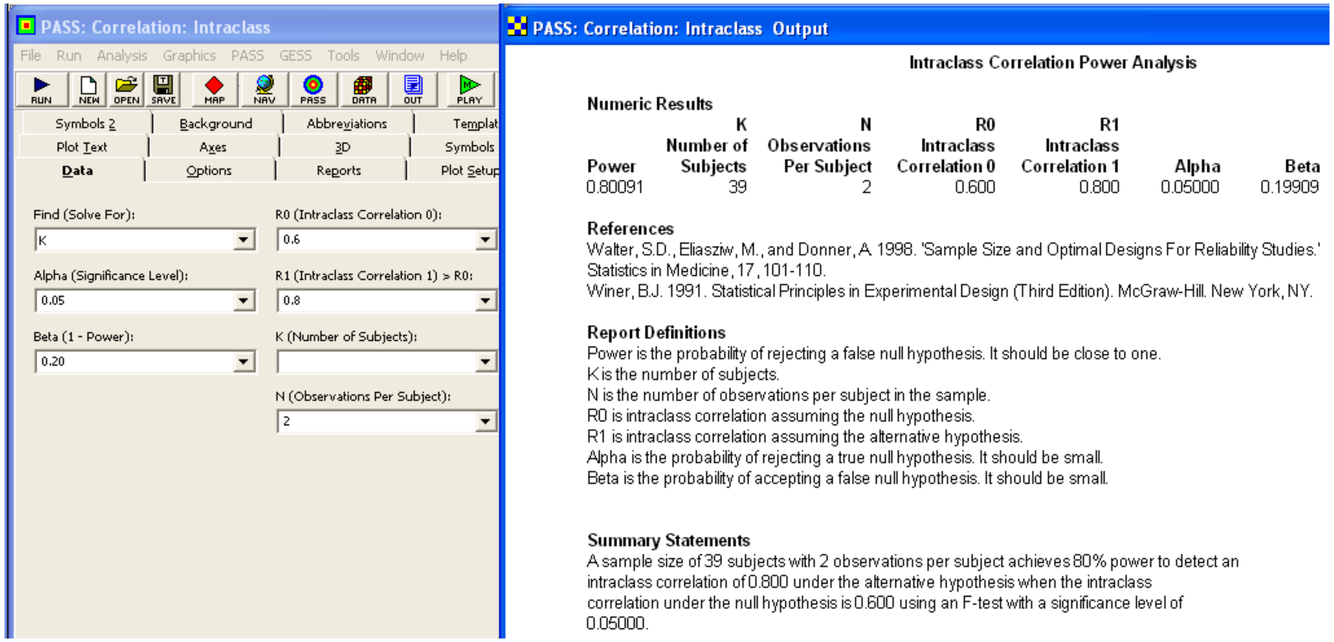
$$\begin{aligned} n &= \left(\frac{z_{0.05} + z_{0.20}}{Z_1 - Z_0}\right)^2 + \frac{3}{2} \\ &= \left(\frac{1.645 + 0.84}{1.099 - 0.693}\right)^2 + \frac{3}{2} \\ &= 39. \end{aligned}$$

The exact power is given by Equation 7.39 where the $F$ distribution has $df_1 = 39 - 1 = 38$ numerator degrees of freedom and $df_2 = 39(2-1) = 39$ denominator degrees of freedom. The power is given by

$$\begin{aligned} \pi &= P\left(\frac{1+2\left(\frac{0.6}{1-0.6}\right)}{1+2\left(\frac{0.8}{1-0.8}\right)} F_{0.95} < F < \infty\right) \\ &= P(0.760 < F < \infty) \\ &= 0.80, \end{aligned}$$

which is in excellent agreement with the target power.

From **PASS**> **Correlation**> **Intraclass Correlation**:




## 7.3 Cohen's Kappa

**Example 7.9** How many units should two operators evaluate in an attribute inspection agreement experiment to be analyzed using Cohen's $\kappa$ if the true value of the unknown $\kappa$ must be determined to within $\pm 0.10$ with 95% confidence? A preliminary experiment indicated that $\kappa \simeq 0.85$ and $p_e \simeq 0.5$.
**Solution:** With $\alpha = 0.05$ and $\delta = 0.10$ in Equation 7.55, the required sample size is

$$n = \frac{0.85\,(1-0.85)}{1-0.5}\left(\frac{1.96}{0.10}\right)^2 = 98.$$

**Example 7.10** Calculate the power to reject $H_0 : \kappa = 0.4$ in favor of $H_A : \kappa > 0.4$ when $\kappa = 0.7$ if a sample of size $n = 70$ is allocated to $k = 3$ categories in the ratio $0.4 : 0.5 : 0.1$.
**Solution:** The expected chance agreement by Equation 7.46 is $p_e = 0.4^2 + 0.5^2 + 0.1^2 = 0.42$. The two $\kappa$ values of interest have intermediate values not covered by the large- or small-$\kappa$ approximations, so it is necessary to estimate $\sigma_{\widehat{\kappa}}$ using Equation 7.47. Under $H_0$ with $\kappa = 0.4$ and $p_e = 0.42$ in Equation 7.44, we have

$$\begin{aligned} p_o &= 0.4\,(1-0.42) + 0.42 \\ &= 0.652, \end{aligned}$$

so

$$\begin{aligned} \sigma_{\widehat{\kappa}_0} &\simeq \frac{1}{1-0.42}\sqrt{\frac{0.652\,(1-0.652)}{70}} \\ &\simeq 0.0982. \end{aligned}$$

Under $H_A$ with $\kappa = 0.7$ we have

$$\begin{aligned} p_o &= 0.7\,(1-0.42)+0.42 \\ &= 0.826, \end{aligned}$$

so

$$\begin{aligned} \sigma_{\widehat{\kappa}_1} &\simeq \frac{1}{1-0.42}\sqrt{\frac{0.826\,(1-0.826)}{70}} \\ &\simeq 0.0781. \end{aligned}$$

Then with $\alpha = 0.05$, $z_\beta$ is given by Equation 7.58:

$$\begin{aligned} z_\beta &= \frac{(0.7-0.4)-1.645\,(0.0982)}{0.0781} \\ &= 1.77 \end{aligned}$$

and the power is given by Equation 7.57:

$$\begin{aligned} \pi &= \Phi\,(-\infty < z < 1.77) \\ &= 0.962. \end{aligned}$$

**Example 7.11** An experiment is to be performed to test for agreement between two methods of categorizing a dichotomous response. The hypotheses to be tested are $H_0 : \kappa = 0$ versus $H_A : \kappa > 0$ where $\kappa$ is Cohen's kappa. How many units must be inspected if the test should have 90% power to reject $H_0$ when $\kappa = 0.40$ and the total number of units to be inspected is evenly split between the two categories?
**Solution:** Because the units will be balanced between the two categories, the agreement expected by chance from Equation 7.51 is $p_e \simeq 0.5$. With $\alpha = 0.05$, $\pi = 0.90$, $\beta = 1 - \pi = 0.10$, and $\delta = 0.4 - 0 = 0.4$ in Equation 7.59, the required sample size is

$$n \simeq \frac{0.5}{1-0.5}\left(\frac{z_{0.05}+z_{0.10}}{\delta}\right)^2 = \left(\frac{1.645+1.282}{0.40}\right)^2 = 54.$$

**Example 7.12** An experiment is to be performed to test for agreement between two raters using a categorical four-state response. The hypotheses to be tested are $H_0 : \kappa = 0.8$ versus $H_A : \kappa > 0.8$. How many units must be inspected if the test should have 90% power to reject $H_0$ when $\kappa = 0.9$? The units to be inspected are evenly balanced across the four categories.
**Solution:** From Equation 7.51 with $k = 4$ categories, $p_e \simeq 0.25$. With $\alpha = 0.05$, $\beta = 1 - \pi = 0.10$, and $\delta = 0.9 - 0.8 = 0.1$ in Equation 7.60, the required sample size is

$$\begin{aligned} n &\simeq \frac{1}{1-0.25}\left(\frac{1.645\sqrt{0.8\times 0.2}+1.282\sqrt{0.9\times 0.1}}{0.9-0.8}\right)^2 \\ &\simeq 145. \end{aligned}$$

## 7.4 Receiver Operating Characteristic (ROC) Curves

**Example 7.13** What sample size is required to estimate the value of an ROC curve's $AUC$ to within ±0.05 with 95% confidence if the $AUC$ value is expected to be about 90%?
**Solution:** The desired confidence interval has the form

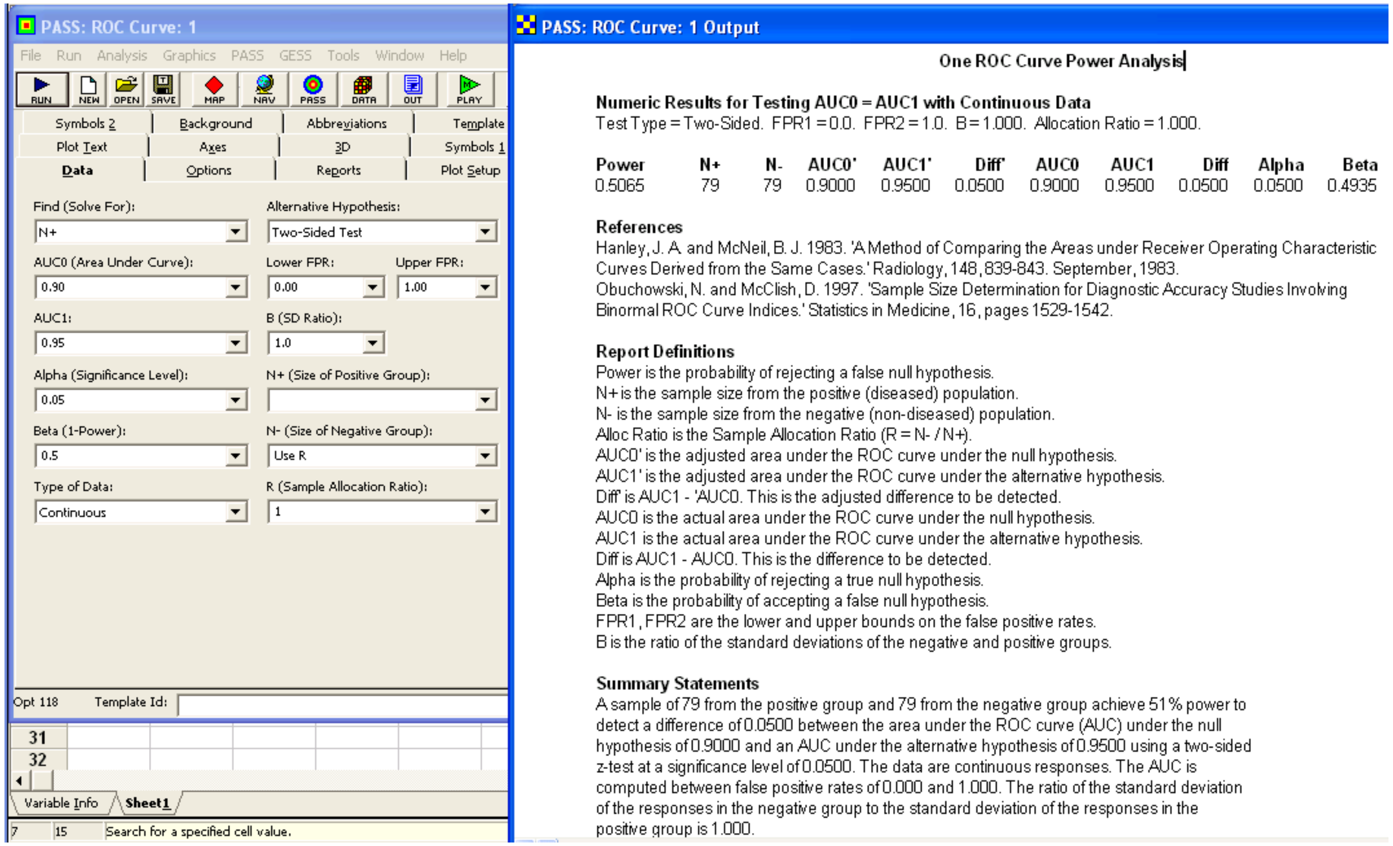
$$P\left(\widehat{AUC} - 0.05 < AUC < \widehat{AUC} + 0.05\right) = 0.95.$$

With $\alpha = 0.05$, $AUC = 0.90$, and $\delta = 0.05$ in Equation 7.66, the required sample size is

$$\begin{aligned} n &\simeq \frac{1 - AUC}{2}\left(\frac{z_{0.025}}{\delta}\right)^2 \\ &\simeq \frac{1 - 0.90}{2}\left(\frac{1.96}{0.05}\right)^2 \\ &\simeq 77. \end{aligned}$$

That is, about 77 positives and 77 negatives are required. The large-sample and large $AUC$ assumptions are reasonably satisfied, so this approximate sample size should be accurate.

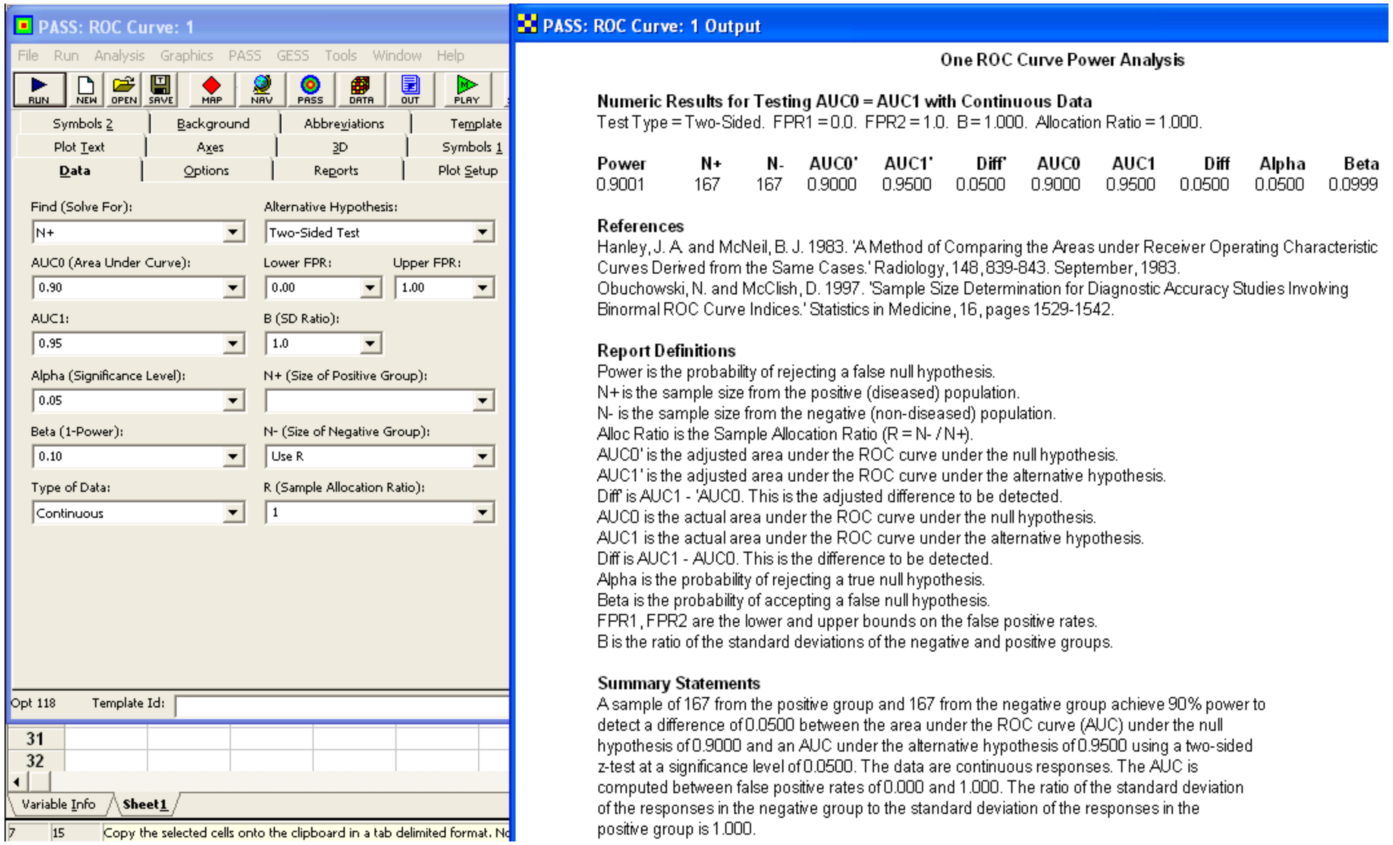
From **PASS**> **Diagnostic Tests**> **ROC Curve - 1 Test**:




**Example 7.14** What sample size is required to reject $H_0 : AUC = 0.9$ in favor of $H_A : AUC \neq 0.9$ with 90% power when $AUC = 0.95$?
**Solution:** With $\alpha = 0.05$ in Equation 7.67, the required sample size is approximately

$$
\begin{aligned}
n &= \left( \frac{\sqrt{\frac{1-0.90}{2}} z_{0.025} + \sqrt{\frac{1-0.95}{2}} z_{0.10}}{0.95 - 0.90} \right)^2 \\
&= \left( \frac{\sqrt{\frac{1-0.90}{2}} 1.96 + \sqrt{\frac{1-0.95}{2}} 1.282}{0.95 - 0.90} \right)^2 \\
&= 165.
\end{aligned}
$$

From **PASS**> **Diagnostic Tests**> **ROC Curve - 1 Test**:

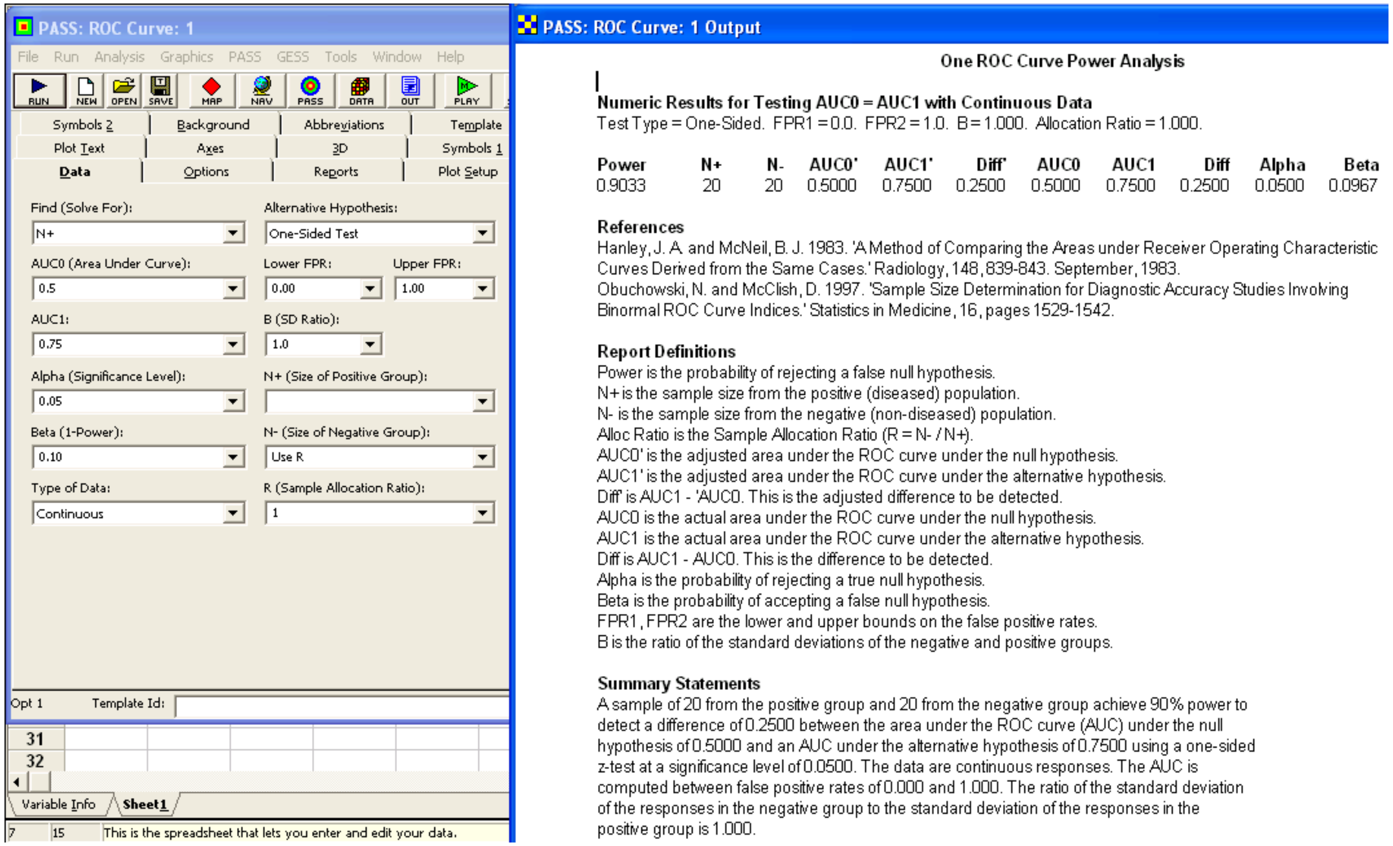



**Example 7.15** What sample size is required to reject $H_0 : AUC = 0.5$ versus $H_A : AUC > 0.5$ with 90% power when $AUC = 0.75$?
**Solution:** With $\beta = 0.10$ when $AUC = 0.75$ in Equation 7.67, the required sample size is approximately

$$
\begin{aligned}
n &= \left(\frac{\sqrt{\frac{1}{6}}z_{0.05} + \sqrt{\frac{1-0.75}{2}}z_{0.10}}{0.75 - 0.50}\right)^2 \\
&= \left(\frac{\sqrt{\frac{1}{6}}1.645 + \sqrt{\frac{1-0.75}{2}}1.282}{0.75 - 0.50}\right)^2 \\
&= 21.
\end{aligned}
$$

The large-sample assumption is only marginally satisfied, so this sample size may be somewhat inaccurate.

From **PASS**> **Diagnostic Tests**> **ROC Curve - 1 Test**:




## 7.5 Bland-Altman Plots

**Example 7.16** What minimum sample size is required to demonstrate the agreement between two methods to measure length by the Bland-Altman method if the limits of agreement are $LOA_{U/L} = \pm 3cm$ and the standard deviation of the differences had been estimated to be $\widehat{\sigma}_d = 0.65cm$ from historical data? Assume that the limits of agreement must cover 99% of the samples with 95% confidence and that there is no bias between the two methods, i.e., $\mu_d = 0$.
**Solution:** The two-sided normal distribution tolerance interval factor $k_2$ is given by

$$
\begin{aligned}
k_2 &= \frac{LOA}{\widehat{\sigma}_d} \\
&= \frac{3cm}{0.65cm} \\
&= 4.615.
\end{aligned}
$$

With $\alpha = 0.05$ and $Y = 0.99$ in Appendix E, Table E.7, the smallest sample size that gives $k_2 \leq 4.615$ is $n = 10$.

# Chapter 8

# Designed Experiments

## 8.1 One-Way Fixed Effects ANOVA

**Example 8.1** In a one-way classification design with four treatments and five observations per treatment, determine the power of the ANOVA to reject $H_0$ if the treatment means are $\mu_i = \{40, 55, 55, 50\}$. The four populations are expected to be normal and homoscedastic with $\sigma_\epsilon = 8$.
**Solution:** The grand mean is $\bar{\mu} = 50$ so the treatment biases relative to the grand mean are $\tau_i = \{-10, 5, 5, 0\}$. From Equation 8.2 the $F$ distribution noncentrality parameter is,

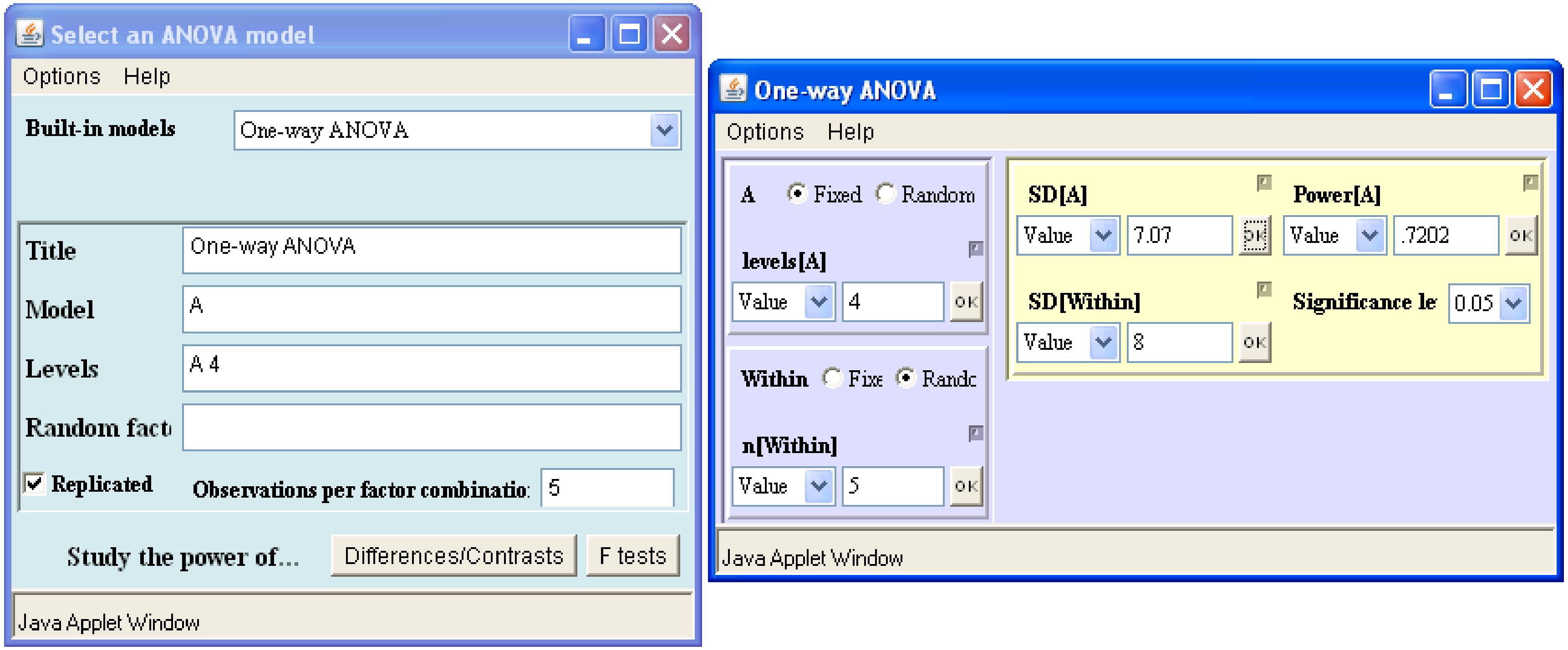
$$\phi = \frac{n \sum_{i=1}^{k} \tau_i^2}{\sigma_\epsilon^2} = \frac{5\left(-10^2 + (5)^2 + (5)^2 + (0)^2\right)}{8^2} = 11.72.$$

The $F$ statistic will have $df_{treatments} = 4 - 1 = 3$ and $df_\epsilon = 4\,(5-1) = 16$ degrees of freedom. The power is 72% as determined from Equation 8.1:

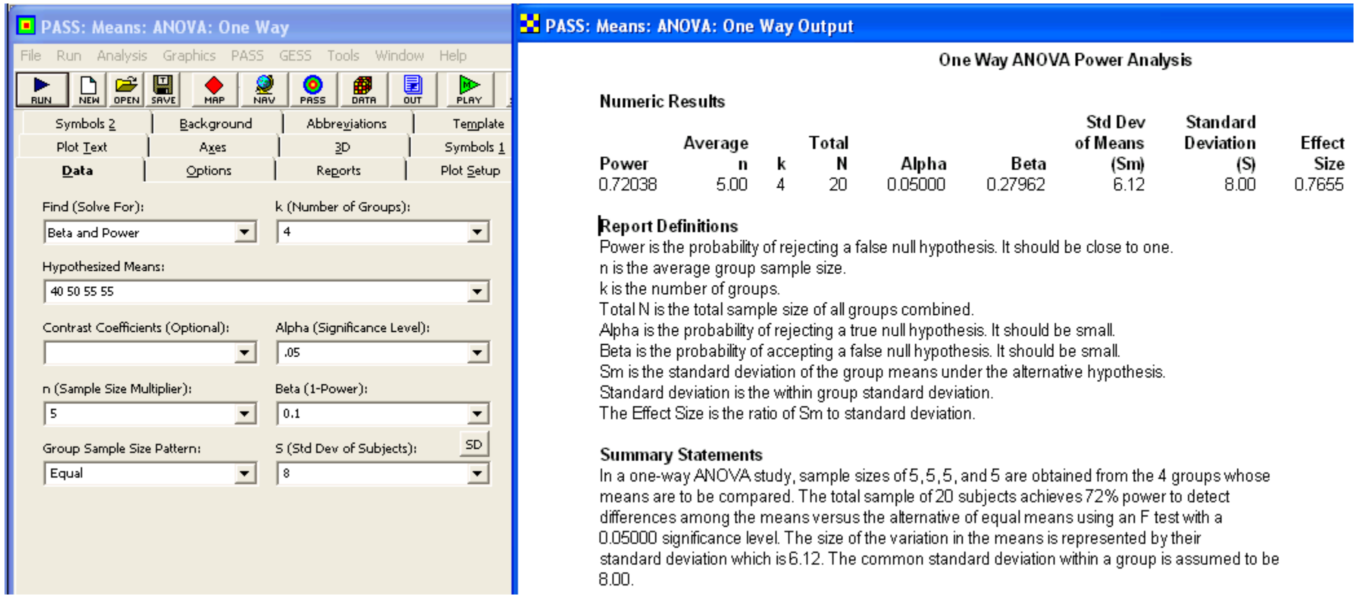
$$F_{0.95} = 3.239 = F_{0.280,11.72}.$$

From **Piface**> **Balanced ANOVA (any model)**> **One-way ANOVA** with:

$$s_A = \sqrt{\frac{\left((-10)^2 + (5)^2 + (5)^2 + (0)^2\right)}{4-1}} = 7.07$$




From **PASS**> **Means**> **Many Means**> **ANOVA: One-Way**:




**MINITAB**> **Stat**> **Power and Sample Size**> **One-Way ANOVA** cannot be used to solve this problem because it does not allow specification of the individual treatment means or the standard deviation of the treatment means. The steps required to calculate the power from the model and error degress of freedom and the noncentrality parameter using **MINITAB**> **Calc**> **Probability Distributions**> **F** are:

```
MTB  invcdf 0.95;
SUBC f 3 16.

F distribution with 3 DF in numerator and 16 DF in denominator

P(~X~=~x~)         x
       0.95   3.23887

MTB  cdf 3.23887;
SUBC f 3 16 11.72.

F distribution with 3 DF in numerator and 16 DF in denominator and noncentrality parameter 11.72

      x  P(~X~=~x~)
3.23887     0.279568

Power = 1 - 0.280 = 0.720
```

**Example 8.2** Determine the power of the ANOVA to reject $H_0$ in a one-way classification design with four treatments and five observations per treatment if the treatment biases from the grand mean are $\tau_i = \{-12, 12, 0, 0\}$. The four populations are expected to be normal and homoscedastic with $\sigma_\epsilon = 8$.

**Solution:** From Equation 8.5 with $\delta = 24$, the $F$ distribution noncentrality parameter is

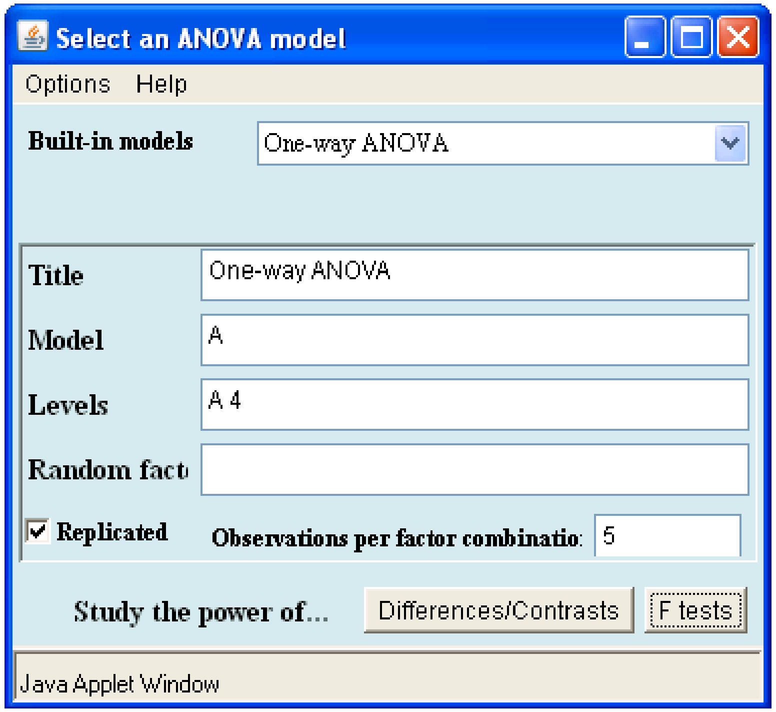
$$\phi = \frac{n}{2}\left(\frac{\delta}{\sigma_\epsilon}\right)^2 = \frac{5}{2}\left(\frac{24}{8}\right)^2 = 22.5.$$

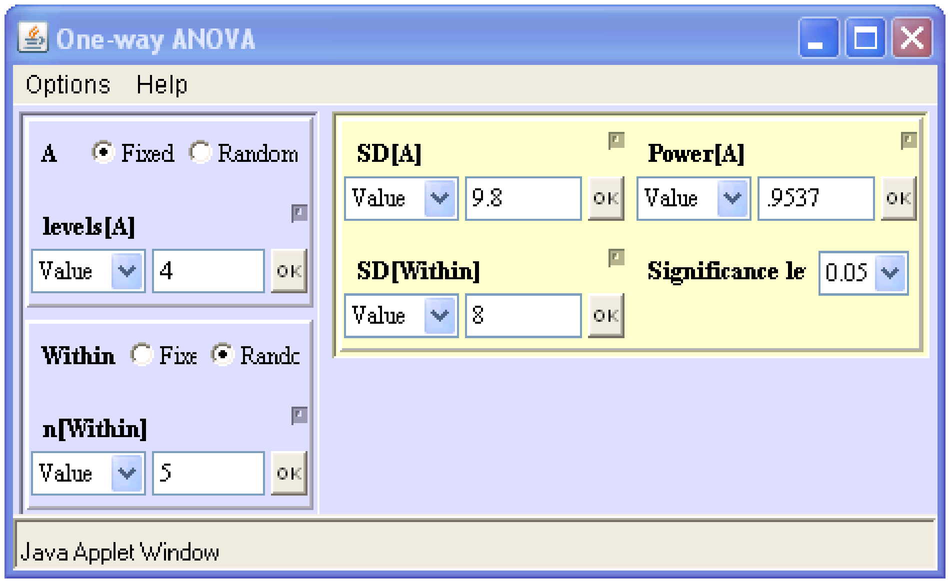
The $F$ statistic will have $df_{treatments} = 4 - 1 = 3$ and $df_\epsilon = 4\,(5-1) = 16$ degrees of freedom. The power is 95.4% as determined from Equation 8.1

$$F_{0.95} = 3.239 = F_{0.046,22.5}.$$

From **Piface> Balanced ANOVA (any model)> One-way ANOVA** with:

$$s_A = \sqrt{\frac{\left((-12)^2 + (12)^2 + (0)^2 + (0)^2\right)}{4-1}} = 9.80$$



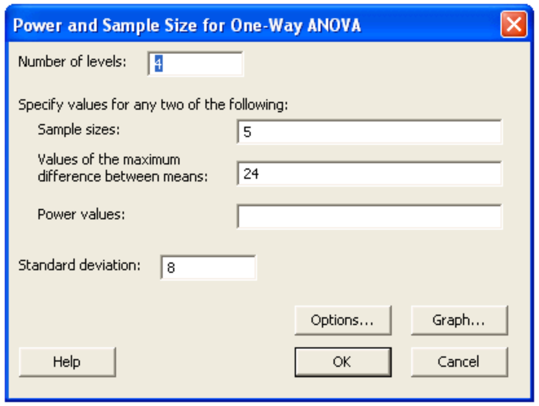




From **MINITAB> Stat> Power and Sample Size> One-Way ANOVA**:

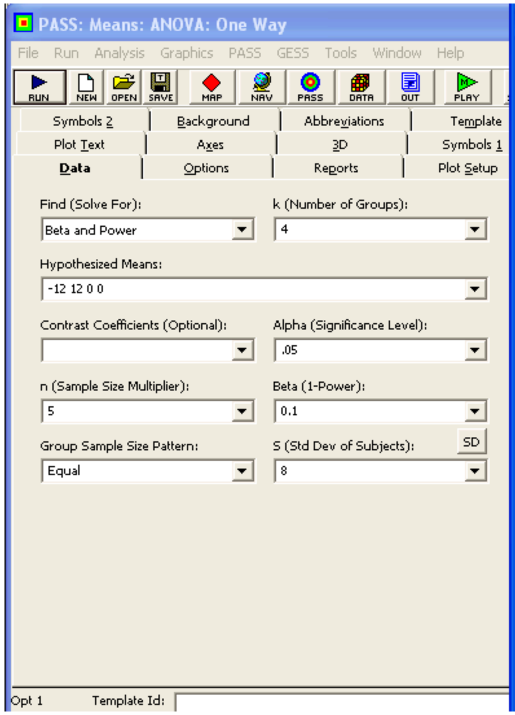
```
MTB > Power;
SUBC>   OneWay 4;
SUBC>     Sample 5;
SUBC>     MaxDifference 24;
SUBC>     Sigma 8;
SUBC>   GPCurve.
```

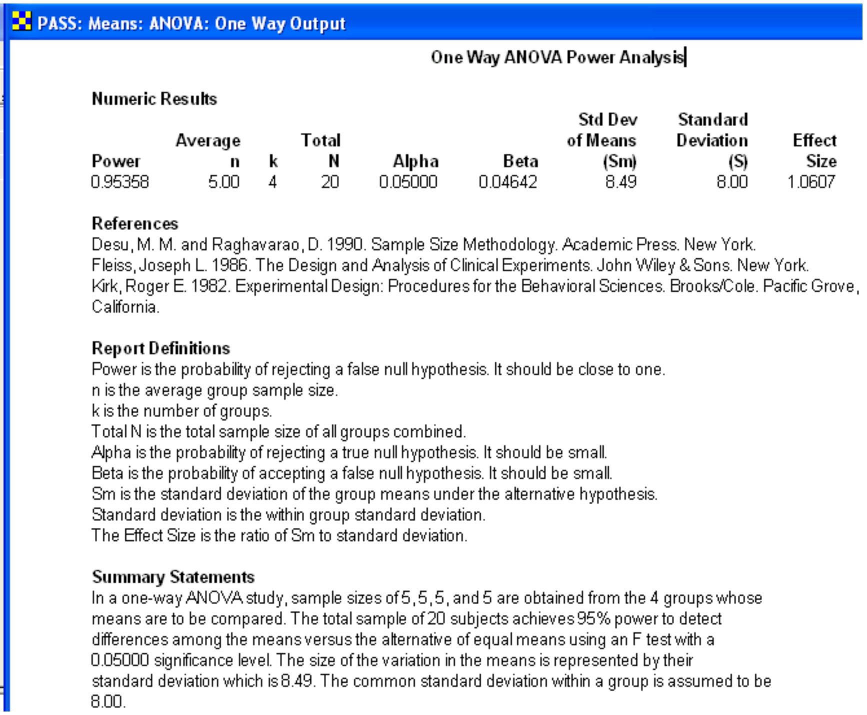
**Power and Sample Size**

```
One-way ANOVA

Alpha = 0.05  Assumed standard deviation = 8  Number of Levels = 4


   SS  Sample              Maximum
Means    Size    Power  Difference
  288       5  0.953578          24

The sample size is for each level.
```




From **PASS**> **Means**> **Many Means**> **ANOVA: One-Way**:



PASS: Means: ANOVA: One Way Output

One Way ANOVA Power Analysis

**Numeric Results**

| Power | Average n | k | Total N | Alpha | Beta | Std Dev of Means (Sm) | Standard Deviation (S) | Effect Size |
|---|---|---|---|---|---|---|---|---|
| 0.95358 | 5.00 | 4 | 20 | 0.05000 | 0.04642 | 8.49 | 8.00 | 1.0607 |

**References**

Desu, M. M. and Raghavarao, D. 1990. Sample Size Methodology. Academic Press. New York.
Fleiss, Joseph L. 1986. The Design and Analysis of Clinical Experiments. John Wiley & Sons. New York.
Kirk, Roger E. 1982. Experimental Design: Procedures for the Behavioral Sciences. Brooks/Cole. Pacific Grove, California.

**Report Definitions**

Power is the probability of rejecting a false null hypothesis. It should be close to one.
n is the average group sample size.
k is the number of groups.
Total N is the total sample size of all groups combined.
Alpha is the probability of rejecting a true null hypothesis. It should be small.
Beta is the probability of accepting a false null hypothesis. It should be small.
Sm is the standard deviation of the group means under the alternative hypothesis.
Standard deviation is the within group standard deviation.
The Effect Size is the ratio of Sm to standard deviation.

**Summary Statements**

In a one-way ANOVA study, sample sizes of 5, 5, 5, and 5 are obtained from the 4 groups whose means are to be compared. The total sample of 20 subjects achieves 95% power to detect differences among the means versus the alternative of equal means using an F test with a 0.05000 significance level. The size of the variation in the means is represented by their standard deviation which is 8.49. The common standard deviation within a group is assumed to be 8.00.

**Example 8.3** In a one-way classification design with four treatments and five observations per treatment, determine the power of the ANOVA to reject $H_0$ if the treatment biases from the grand mean are $\tau_i = \{18, -6, -6, -6\}$. The four populations are expected to be normal and homoscedastic with $\sigma_\epsilon = 8$.

**Solution:** From Equation 8.6 with $\delta = 24$, the $F$ distribution noncentrality parameter is

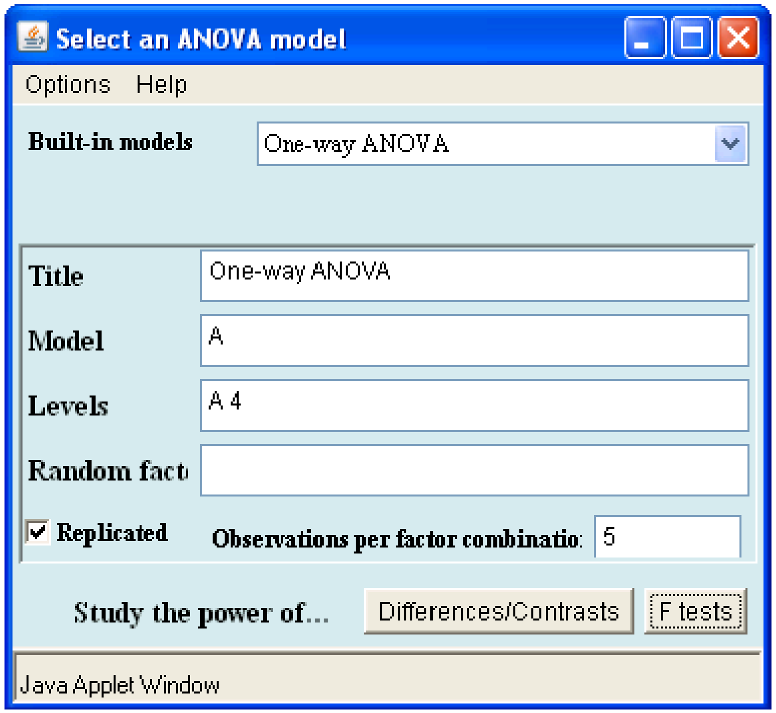
$$\phi = \frac{n(k-1)}{k}\left(\frac{\delta}{\sigma_\epsilon}\right)^2 = \frac{5 \times 3}{4}\left(\frac{24}{8}\right)^2 = 33.75.$$

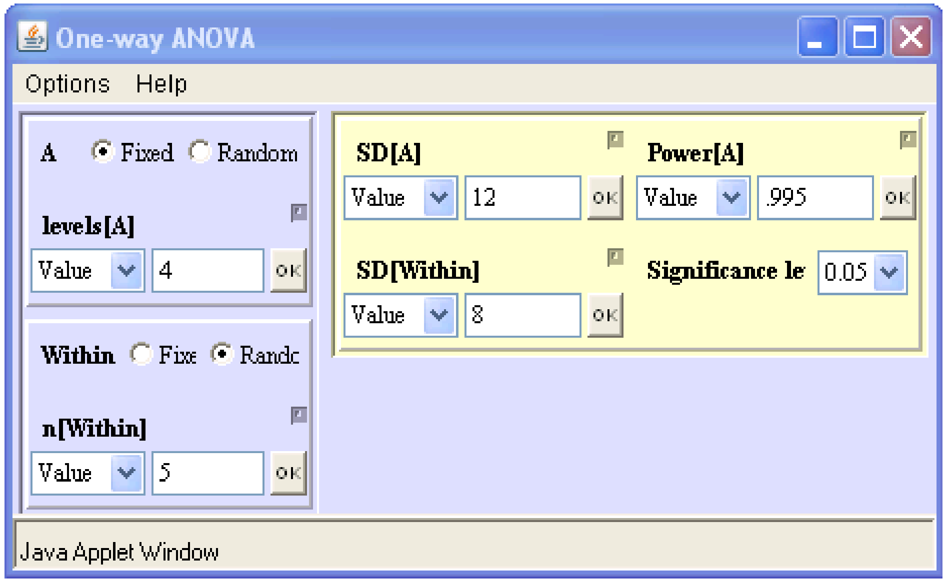
The $F$ statistic will have $df_{treatments} = 4 - 1 = 3$ and $df_\epsilon = 4(5-1) = 16$ degrees of freedom. The power is 99.5% as determined from Equation 8.1:

$$F_{0.95} = 3.239 = F_{0.005,33.75}.$$

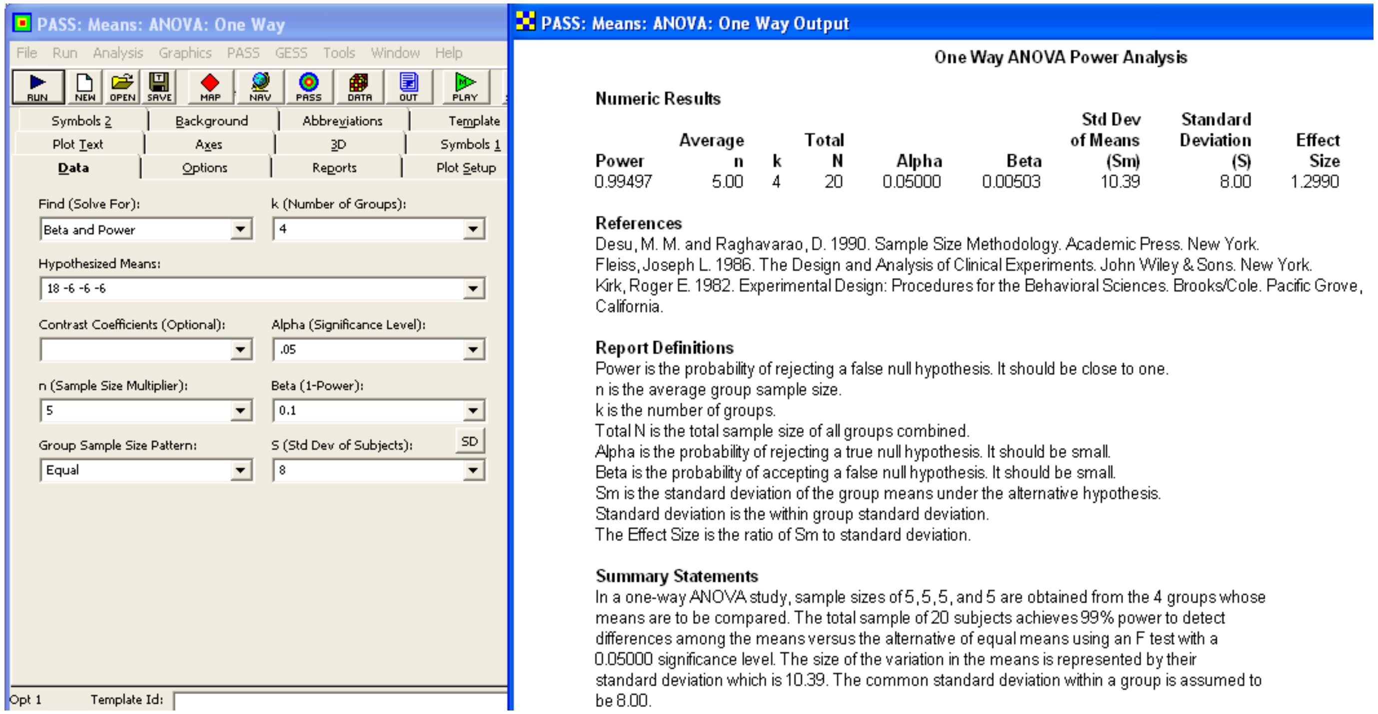
From **Piface**> **Balanced ANOVA (any model)**> **One-way ANOVA** with:

$$s_A = \sqrt{\frac{\left((18)^2 + (-6)^2 + (-6)^2 + (-6)^2\right)}{4-1}} = 12.0$$




From **PASS**> **Means**> **Many Means**> **ANOVA: One-Way**:




**Example 8.4** Determine the power to reject $H_0$ by one-way ANOVA when the treatment means are $\mu_i = \{50, 30, 40, 40, 40\}$ and the sample sizes are $n_i = \{12, 12, 20, 20, 15\}$. The five populations are expected to be normal and homoscedastic with $\sigma_\epsilon = 13$.
**Solution:** The grand mean of the experimental data is expected to be

$$\frac{\sum n_i \mu_i}{\sum n_i} = \frac{(50 \times 12) + \cdots + (40 \times 15)}{12 + \cdots + 15} = 40.$$

The treatment biases relative to the grand mean are $\tau_i = \{10, -10, 0, 0, 0\}$ so the noncentrality parameter is

$$\phi = \frac{12\,(10)^2 + 12\,(-10)^2 + 20\,(0)^2 + 20\,(0)^2 + 15\,(0)^2}{13^2} = 14.2.$$

The ANOVA will have $df_{treatments} = 5 - 1 = 4$ and $df_\epsilon = \sum n_i - 1 - 4 = 74$ degrees of freedom. The power is 84.7% as determined from Equation 8.1:

$$F_{0.95} = 2.495 = F_{0.153,14.2}.$$

## 8.2 Randomized Block Design

**Example 8.5** Recalculate the power for Example 8.1 if the experiment is built as a randomized block design and the standard deviation of the population of block biases is $\sigma_{blocks} = 4$.
**Solution:** The $F$ distribution noncentrality parameter ($\phi = 11.72$) and the treatment degrees of freedom ($df_{treatments} = 3$) will be unchanged from the original solution, but if the experiment is built in five blocks with one replicate in each block, the new error degrees of freedom for the RBD will be

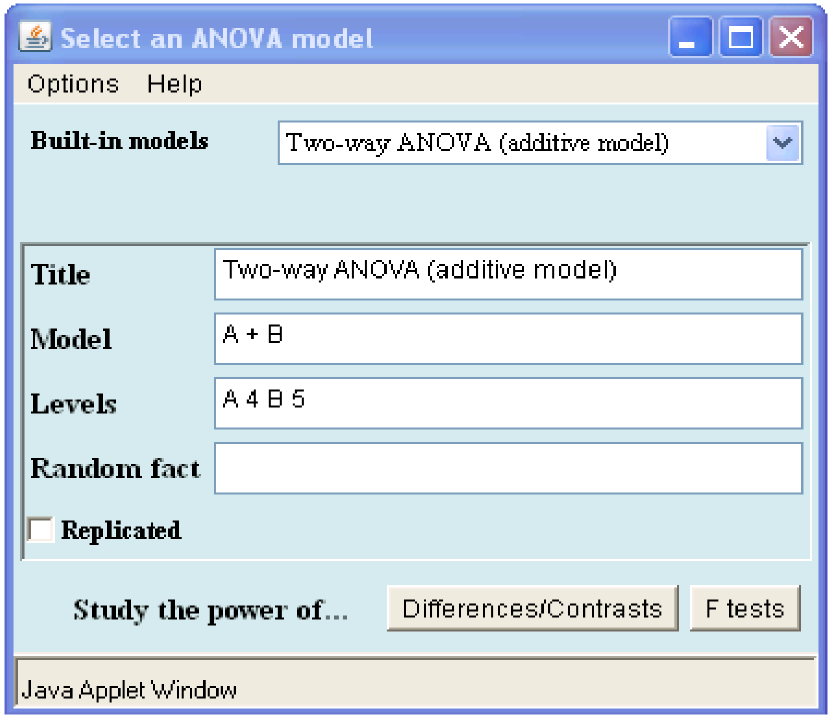
$$\begin{aligned} df_{\epsilon} &= df_{total} - df_{treatments} - df_{blocks} \\ &= 19 - 3 - 4 \\ &= 12. \end{aligned}$$

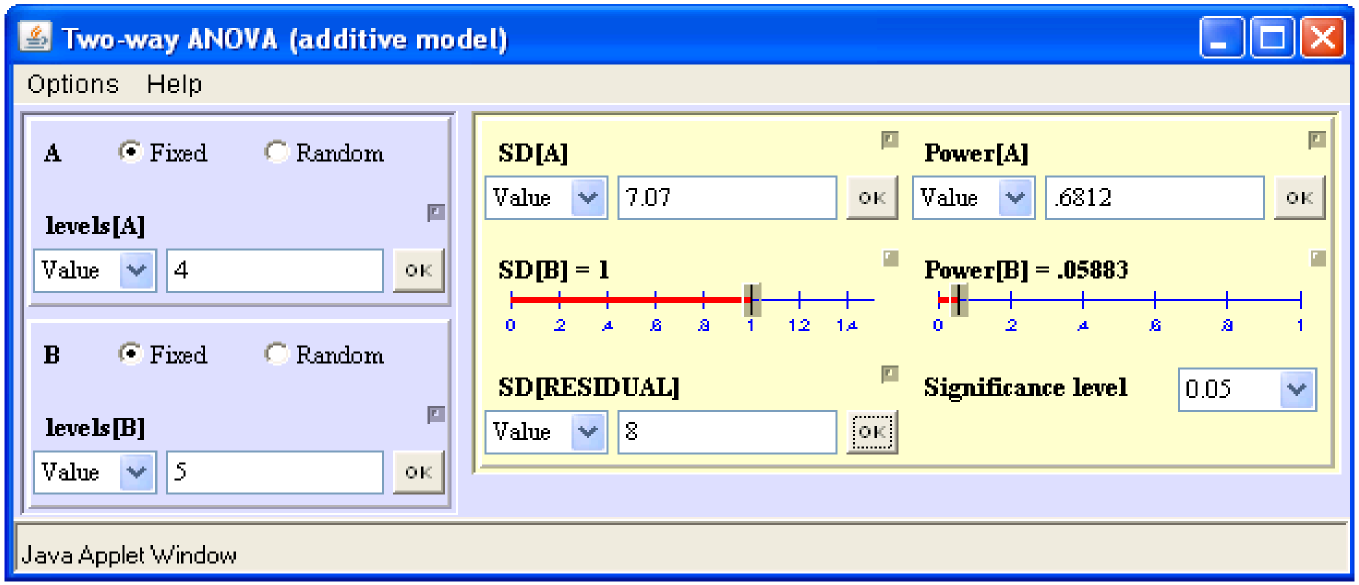
The power of the RBD is 68% as determined from

$$F_{0.95} = 3.490 = F_{0.32,11.72}.$$

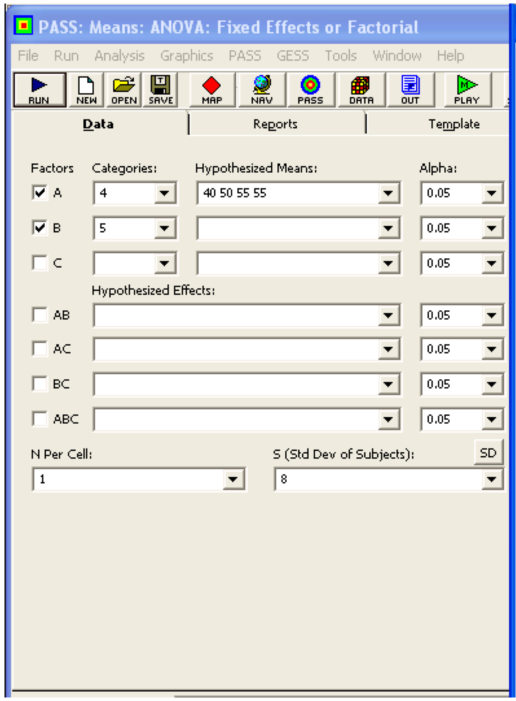
This is slightly lower than the original power (72%) because the RBD has fewer error degrees of freedom than the CRD. The RBD's power is not affected by the block variation because it separates that variation from the error variation that is used to determine the power.

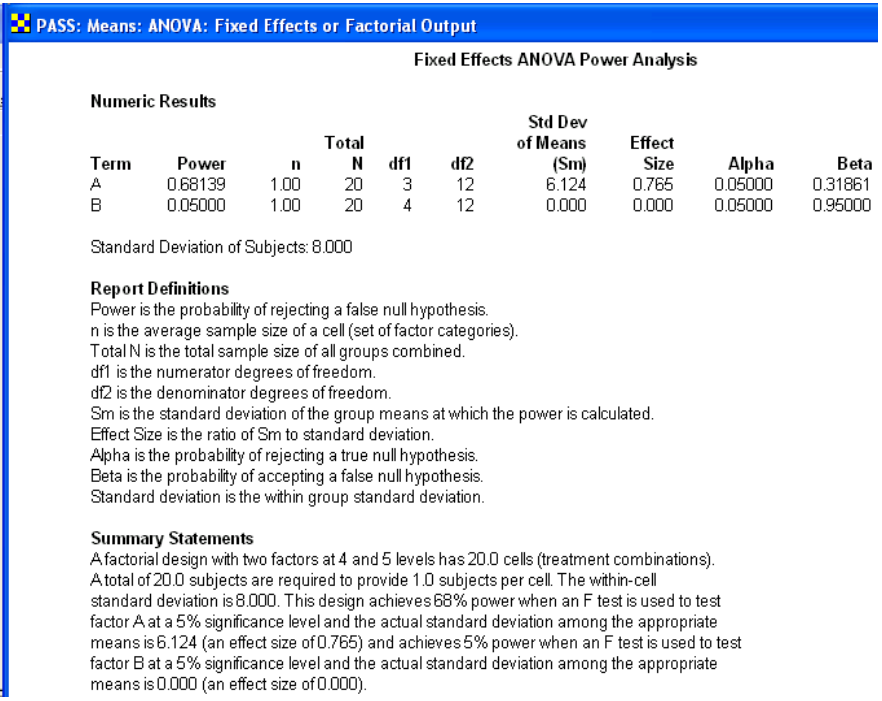
From **Piface**> **Balanced ANOVA (any model)**> **Two-way ANOVA (additive model)** with:



From **PASS**> **Means**> **Many Means**> **ANOVA: Fixed Effect**:



PASS: Means: ANOVA: Fixed Effects or Factorial Output

Fixed Effects ANOVA Power Analysis

Numeric Results

| Term | Power | n | Total N | df1 | df2 | Std Dev of Means (Sm) | Effect Size | Alpha | Beta |
|---|---|---|---|---|---|---|---|---|---|
| A | 0.68139 | 1.00 | 20 | 3 | 12 | 6.124 | 0.765 | 0.05000 | 0.31861 |
| B | 0.05000 | 1.00 | 20 | 4 | 12 | 0.000 | 0.000 | 0.05000 | 0.95000 |

Standard Deviation of Subjects: 8.000

Report Definitions
Power is the probability of rejecting a false null hypothesis.
n is the average sample size of a cell (set of factor categories).
Total N is the total sample size of all groups combined.
df1 is the numerator degrees of freedom.
df2 is the denominator degrees of freedom.
Sm is the standard deviation of the group means at which the power is calculated.
Effect Size is the ratio of Sm to standard deviation.
Alpha is the probability of rejecting a true null hypothesis.
Beta is the probability of accepting a false null hypothesis.
Standard deviation is the within group standard deviation.

Summary Statements
A factorial design with two factors at 4 and 5 levels has 20.0 cells (treatment combinations). A total of 20.0 subjects are required to provide 1.0 subjects per cell. The within-cell standard deviation is 8.000. This design achieves 68% power when an F test is used to test factor A at a 5% significance level and the actual standard deviation among the appropriate means is 6.124 (an effect size of 0.765) and achieves 5% power when an F test is used to test factor B at a 5% significance level and the actual standard deviation among the appropriate means is 0.000 (an effect size of 0.000).

**Example 8.6** A 40-run experiment was performed using an RBD with $k = 5$ treatments and $r = 8$ blocks. The ANOVA table from the experiment is shown in Table 8.2. Calculate the blocking efficiency and the increase in the number of runs required to obtain the same estimation precision for treatment means using a CRD.
**Solution:** The blocking efficiency as determined from Equation 8.13 is

$$\begin{aligned} E &= \frac{(7 \times 14) + (8 \times 4 \times 4)}{(5 \times 8 - 1)\,4} \\ &= 1.45. \end{aligned}$$

That is, the CRD will require about 45% more runs than the RBD because it ignores the variation associated with block effects. Because the number of runs in the RBD was $kr = 40$, the number of runs required for the CRD to obtain the same estimation precision for the treatment means would be $Ekr = 1.45 \times 40 = 58$. Apparently the blocking was beneficial and should be used in future studies.

## 8.3 Balanced Full Factorial Design with Fixed Effects

**Example 8.7** A $2 \times 3 \times 5$ full factorial experiment with four replicates is planned. The experiment will be blocked on replicates and the ANOVA model will include main effects and two-factor interactions. Determine the power to detect a difference $\delta = 300$ units between two levels of each study variable if the standard error of the model is expected to

be $\sigma_\epsilon = 500$.
**Solution:** If the three study variables are given the names $A$, $B$, and $C$ and have $a = 2$, $b = 3$, and $c = 5$ levels, respectively, then the degrees of freedom associated with the terms in the model will be $df_{blocks} = 3$, $df_A = 1$, $df_B = 2$, $df_C = 4$, $df_{AB} = 2$, $df_{AC} = 4$, $df_{BC} = 8$, and

$$\begin{aligned} df_\epsilon &= df_{total} - df_{model} \\ &= (2 \times 3 \times 5 \times 4 - 1) - (3 + 1 + 2 + 4 + 2 + 4 + 8) \\ &= 119 - 24 \\ &= 95. \end{aligned}$$

From Equation 8.2, the $F$ distribution noncentrality parameter for variable $A$ with treatment biases $\alpha_1 = -150$ and $\alpha_2 = 150$ is

$$\begin{aligned} \phi_A &= \frac{bcn \sum_{i=1}^{2} \alpha_i^2}{\sigma_\epsilon^2} \\ &= \frac{3 \times 5 \times 4 \times \left((-150)^2 + 150^2\right)}{500^2} \\ &= 10.8. \end{aligned}$$

The distribution of $F_A$ will have $df_A = 1$ numerator and $df_\epsilon = 95$ denominator degrees of freedom, so the power associated with $A$ is given by Equation 8.1:

$$F_{0.95} = 3.942 = F_{1-\pi_A,10.8},$$

which is satisfied by $\pi_A = 0.908$ or 90.8%.

Similarly, the $F$ distribution noncentrality parameter for $B$ with biases $\beta_1 = -150$, $\beta_2 = 150$, and $\beta_3 = 0$ is

$$\begin{aligned} \phi_B &= \frac{acn \sum_{i=1}^{3} \beta_i^2}{\sigma_\epsilon^2} \\ &= \frac{2 \times 5 \times 4 \times \left((-150)^2 + 150^2 + 0^2\right)}{500^2} \\ &= 7.2. \end{aligned}$$

The distribution of $F_B$ will have $df_B = 2$ numerator and $df_\epsilon = 95$ denominator degrees of freedom, so the power associated with $B$ is given by

$$F_{0.95} = 3.093 = F_{1-\pi_B,7.2},$$

which is satisfied by $\pi_B = 0.654$.

Finally, the $F$ distribution noncentrality parameter for $C$ with biases $\gamma_1 = -150$, $\gamma_2 = 150$, and $\gamma_3 = \gamma_4 = \gamma_5 = 0$ is

$$\begin{aligned}\phi_C &= \frac{abn \sum_{i=1}^{5} \gamma_i^2}{\sigma_\epsilon^2} \\ &= \frac{2 \times 3 \times 4 \times \left((-150)^2 + 150^2 + 0^2 + 0^2 + 0^2\right)}{500^2} \\ &= 4.32.\end{aligned}$$

The distribution of $F_C$ will have $df_C = 4$ numerator and $df_\epsilon = 95$ denominator degrees of freedom, so the power associated with $C$ is given by

$$F_{0.95} = 2.469 = F_{1-\pi_C, 4.32},$$

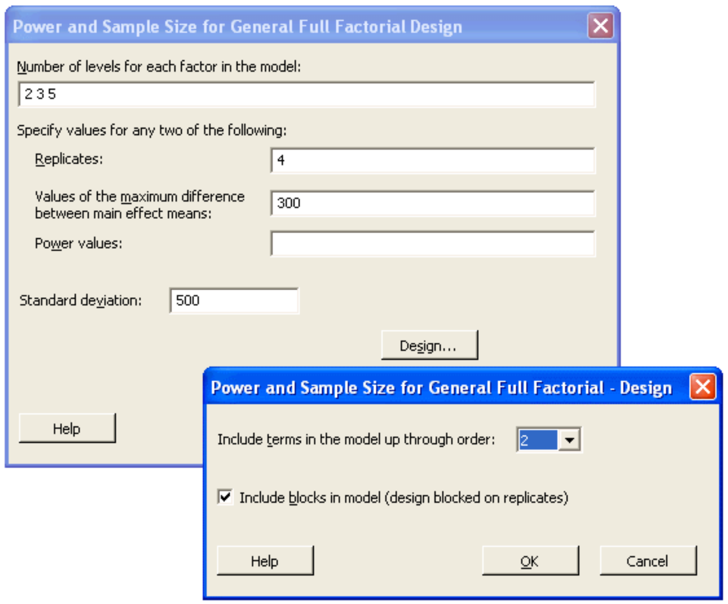
which is satisfied by $\pi_C = 0.328$. These three power calculations confirm by example that the power to detect a variable effect decreases as the number of variable levels increases.

From **MINITAB**> **Stat**> **Power and Sample Size**> **General Full Factorial Design** (MINITAB only reports the power for the variable with the most levels, which in this case is C):

```
MTB > Power;
SUBC>  FDesign;
SUBC>   NLevels 2 3 5;
SUBC>   Reps 4;
SUBC>   MaxDifference 300;
SUBC>   Sigma 500;
SUBC>   TOrder 2;
SUBC>   FitB;
SUBC>   Alpha 0.05;
SUBC>   GPCurve.
```

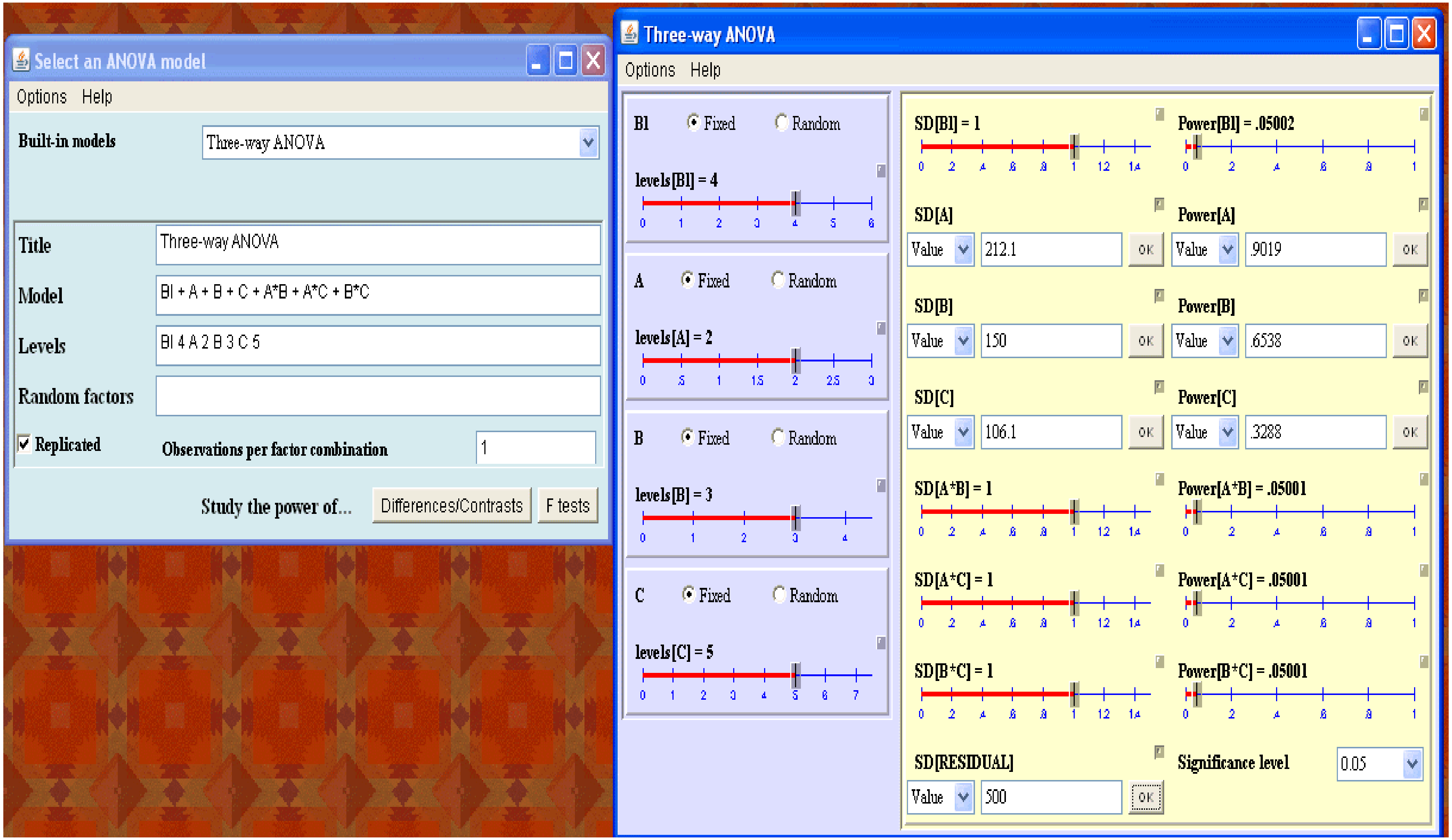
**Power and Sample Size**

```
General Full Factorial Design

Alpha = 0.05  Assumed standard deviation = 500

Factors: 3  Number of levels: 2, 3, 5

Include terms in the model up through order:  2
Include blocks in model.


   Maximum        Total
Difference  Reps   Runs     Power
       300     4    120  0.328605
```

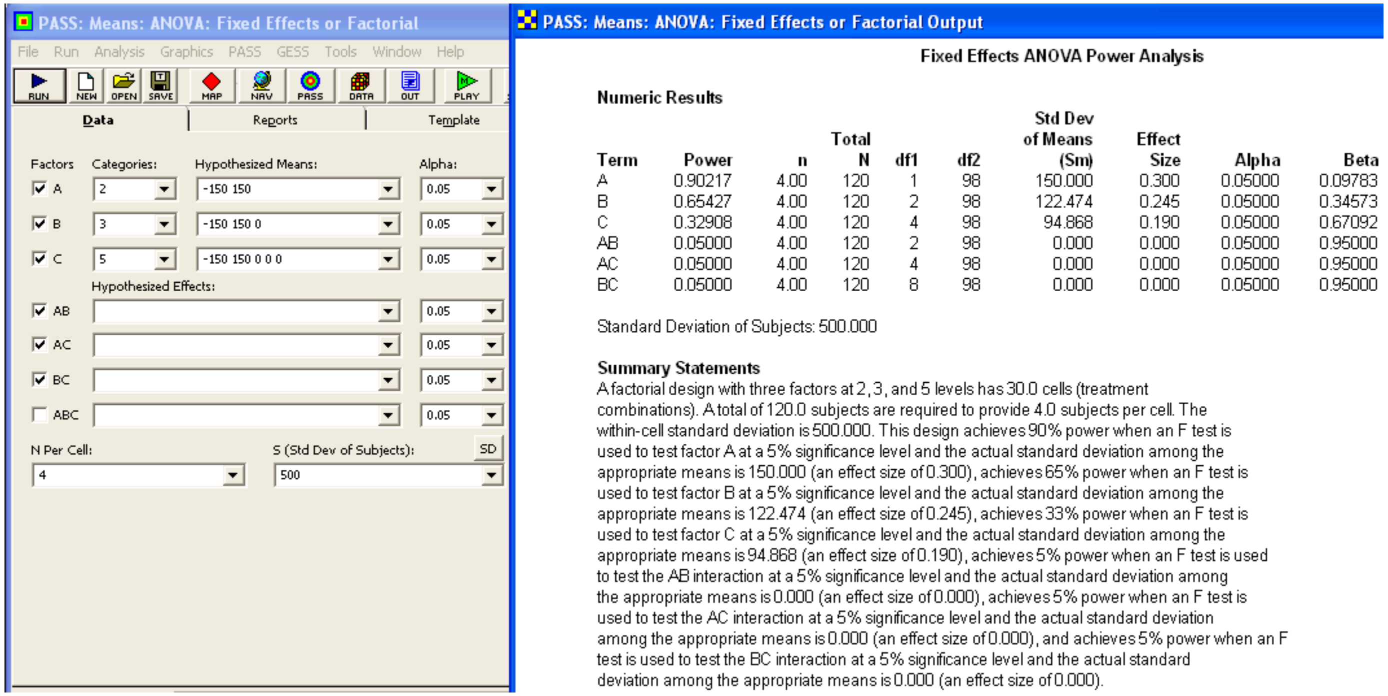


From **Piface**> **Balanced ANOVA (any model)**> **Three-way ANOVA** with

$$s_A = \sqrt{\left(2\,(150)^2\right)/\,(2-1)} = 212.1$$
$$s_B = \sqrt{\left(2\,(150)^2 + (0)^2\right)/\,(3-1)} = 150.0$$
$$s_C = \sqrt{\left(2\,(150)^2 + 3\,(0)^2\right)/\,(5-1)} = 106.1$$




From **PASS**> **Means**> **Many Means**> **ANOVA: Fixed Effect**:




Fixed Effects ANOVA Power Analysis

**Numeric Results**

| Term | Power | n | Total N | df1 | df2 | Std Dev of Means (Sm) | Effect Size | Alpha | Beta |
|---|---|---|---|---|---|---|---|---|---|
| A | 0.90217 | 4.00 | 120 | 1 | 98 | 150.000 | 0.300 | 0.05000 | 0.09783 |
| B | 0.65427 | 4.00 | 120 | 2 | 98 | 122.474 | 0.245 | 0.05000 | 0.34573 |
| C | 0.32908 | 4.00 | 120 | 4 | 98 | 94.868 | 0.190 | 0.05000 | 0.67092 |
| AB | 0.05000 | 4.00 | 120 | 2 | 98 | 0.000 | 0.000 | 0.05000 | 0.95000 |
| AC | 0.05000 | 4.00 | 120 | 4 | 98 | 0.000 | 0.000 | 0.05000 | 0.95000 |
| BC | 0.05000 | 4.00 | 120 | 8 | 98 | 0.000 | 0.000 | 0.05000 | 0.95000 |

Standard Deviation of Subjects: 500.000

**Summary Statements**

A factorial design with three factors at 2, 3, and 5 levels has 30.0 cells (treatment combinations). A total of 120.0 subjects are required to provide 4.0 subjects per cell. The within-cell standard deviation is 500.000. This design achieves 90% power when an F test is used to test factor A at a 5% significance level and the actual standard deviation among the appropriate means is 150.000 (an effect size of 0.300), achieves 65% power when an F test is used to test factor B at a 5% significance level and the actual standard deviation among the appropriate means is 122.474 (an effect size of 0.245), achieves 33% power when an F test is used to test factor C at a 5% significance level and the actual standard deviation among the appropriate means is 94.868 (an effect size of 0.190), achieves 5% power when an F test is used to test the AB interaction at a 5% significance level and the actual standard deviation among the appropriate means is 0.000 (an effect size of 0.000), achieves 5% power when an F test is used to test the AC interaction at a 5% significance level and the actual standard deviation among the appropriate means is 0.000 (an effect size of 0.000), and achieves 5% power when an F test is used to test the BC interaction at a 5% significance level and the actual standard deviation among the appropriate means is 0.000 (an effect size of 0.000).

MINITAB doesn't have a built in capability to do sample size and power calculations for multi-way ANOVA, however, the custom macro *power.mac* (posted at *www.mmbstatistical.com/Sampl* can be used to calculate the power for one design variable at a time in a balanced multi-way ANOVA. For the first variable:

```
MTB  %power c1
Executing from file: C:\Program Files\Minitab 15\English\Macros\power.MAC

Do you want to specify your design from the terminal or from a column?

If from terminal a column will be created in the column specified.
Otherwise the column specified will be the input.

(terminal=1, column=2)
DATA 1

How many runs are in one replicate?
DATA 30

How many replicates?
DATA 4

How many levels does the variable have?
DATA 2
```

```
How many model degrees of freedom are there?
DATA 24

What is the standard deviation?
DATA 500

What is the smallest difference that you
want to detect between two levels?
DATA 300


N          120.000
Runs       30.0000
reps       4.00000
Levels     2.00000
dfmodel    24.0000
dferror    95.0000
Fcrit      3.94122
lambda     10.8000
sigma      500.000
delta      300.000
Power      0.901990
```

## 8.4 Random and Mixed Models

Piface can calculate sample size and power for random and mixed models but MINITAB and PASS can not.

**Example 8.8** A balanced full factorial experiment is to be performed using $a = 3$ levels of a fixed variable $A$, $b = 5$ randomly selected levels of a random variable $B$, and $n = 4$ replicates. Determine the power to reject $H_0 : \alpha_i = 0$ *for all* $i$ when the $A$-level biases are $\alpha_i = \{-20, 20, 0\}$ with $\sigma_B = 25$, $\sigma_{AB} = 0$, and $\sigma_\epsilon = 40$. Assume that the $AB$ interaction term will be included in the ANOVA even though its expected variance component is 0.
**Solution:** The ANOVA table with the equations for the expected mean squares is shown in Table 8.3. From the ANOVA table, the error mean square used for testing the $A$ effect (that is, the denominator of $F_A$) is

$$\begin{aligned} MS_{\epsilon(A)} &= MS_{AB} \\ &= \widehat{\sigma}_\epsilon^2 + n\widehat{\sigma}_{AB}^2. \end{aligned}$$

The noncentrality parameter for the test of the fixed effect $A$ is given by Equation 8.15:

$$\begin{aligned} \phi_A &= \frac{N}{a} \frac{\sum_{i=1}^{a} \alpha_i^2}{MS_{\epsilon(A)}} \\ &= \frac{3 \times 5 \times 4}{3} \frac{(-20)^2 + (20)^2 + (0)^2}{(40)^2 + 4\,(0)^2} \\ &= 10. \end{aligned}$$

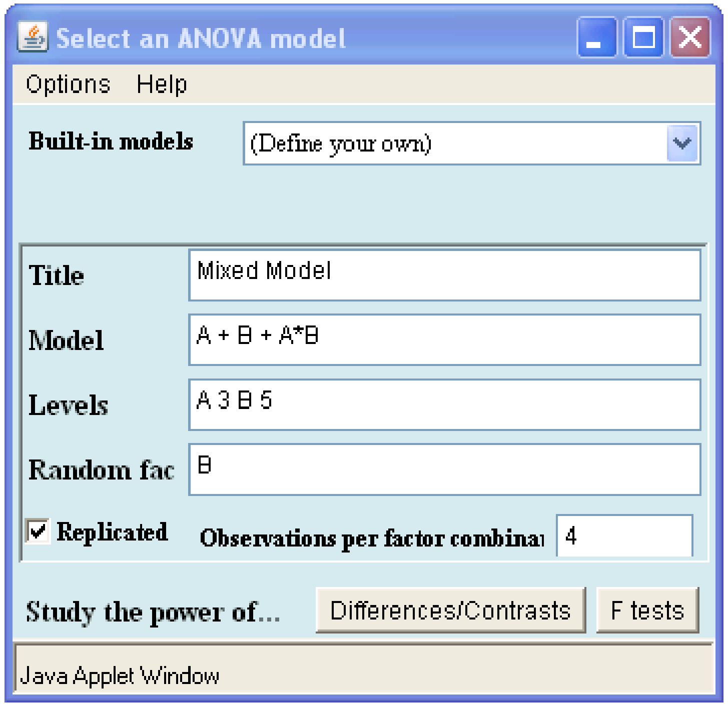
With $df_A = 2$, $df_{AB} = 8$, and $\alpha = 0.05$ in Equation 8.1

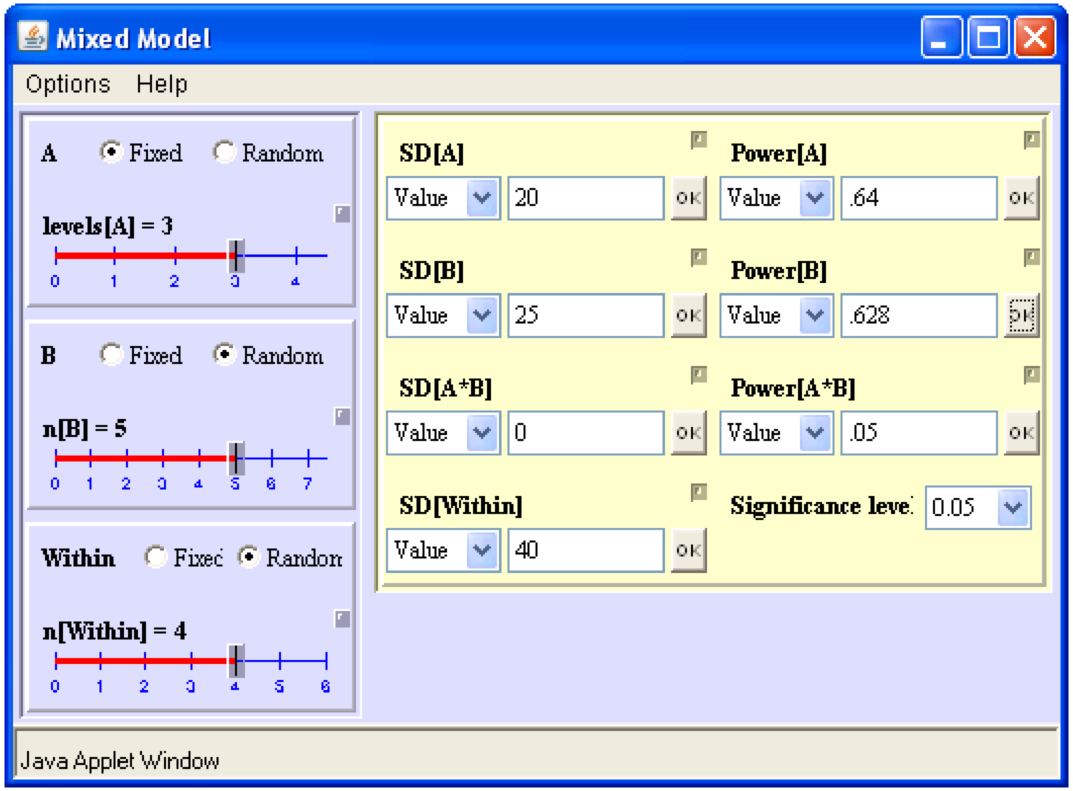
$$F_{0.95} = 4.459 = F_{1-\pi,10.0},$$

which is satisfied by $\pi = 0.640$.

From **Piface**> **Balanced ANOVA (any model)** with:

$$s_A = \sqrt{\left((-20)^2 + (20)^2 + (0)^2\right) / (3-1)} = 20$$




**Example 8.9** Determine the power to reject $H_0 : \sigma_B^2 = 0$ when $\sigma_B = 25$, $\sigma_{AB} = 0$, and $\sigma_\epsilon = 40$ for Example 8.8. Retain the $AB$ interaction term in the model even though its variance component is 0.

**Solution:** The ANOVA table with the equations for the expected mean squares is shown in Table 8.3. From Equation 8.19 under the specified conditions, the expected $F_B$ value is approximately

$$\begin{aligned} E(F_B) &\simeq \frac{E(MS_B)}{E(MS_{AB})} \\ &\simeq \frac{\sigma_\epsilon^2 + n\sigma_{AB}^2 + an\sigma_B^2}{\sigma_\epsilon^2 + n\sigma_{AB}^2} \\ &\simeq \frac{(40)^2 + 4(0)^2 + 3\times 4\times (25)^2}{(40)^2 + 4(0)^2} \\ &\simeq 5.69. \end{aligned}$$

With $df_B = 4$, $df_{AB} = 8$, and $\alpha = 0.05$, the critical $F$ value for the test for the $B$ effect is $F_{0.95,4,8} = 3.838$, so from Equation 8.20 the power is approximately

$$\begin{aligned} \pi &\simeq P\left(\frac{3.838}{5.69} < F < \infty\right) \\ &\simeq P(0.675 < F < \infty) \\ &\simeq 0.618. \end{aligned}$$

See Piface solution to Example 8.8.

## 8.5 Nested Designs

**Example 8.10** $A$ is a fixed variable with three levels and $B$ is a random variable with four levels nested within each level of $A$. The nested design is crossed with a five-level fixed variable $C$ and one replicate of the experiment will be built. The model to be fitted is: $A + B(A) + C + AC + BC$. Find the power to detect a difference $\delta = 40$ between two levels of $A$ assuming that the standard deviations for the random effects are $\sigma_B = 12$, $\sigma_{BC} = 4$, and $\sigma_\epsilon = 10$.
**Solution:** The ANOVA table with the equations for the expected mean squares is shown in Table 8.4 where the $\alpha_i$ are the $A$-level biases, the $\tau_i$ are the $C$-level biases, and the $\gamma_i$ are the $AC$ interaction biases. The error mean square used for testing the $A$ effect is $MS_{\epsilon(A)} = MS_{B(A)}$. The noncentrality parameter for the test of the fixed effect $A$ is given by Equation 8.15:

$$\begin{aligned} \phi_A &= \frac{a\times b\times c\times n}{a} \frac{\sum_{i=1}^{a}\alpha_i^2}{\sigma_\epsilon^2 + n\sigma_{BC}^2 + cn\sigma_{B(A)}^2} \\ &= \frac{3\times 4\times 5\times 1}{3} \frac{(-20)^2 + (20)^2 + (0)^2}{(10)^2 + (4)^2 + 5(12)^2} \\ &= 19.14. \end{aligned}$$

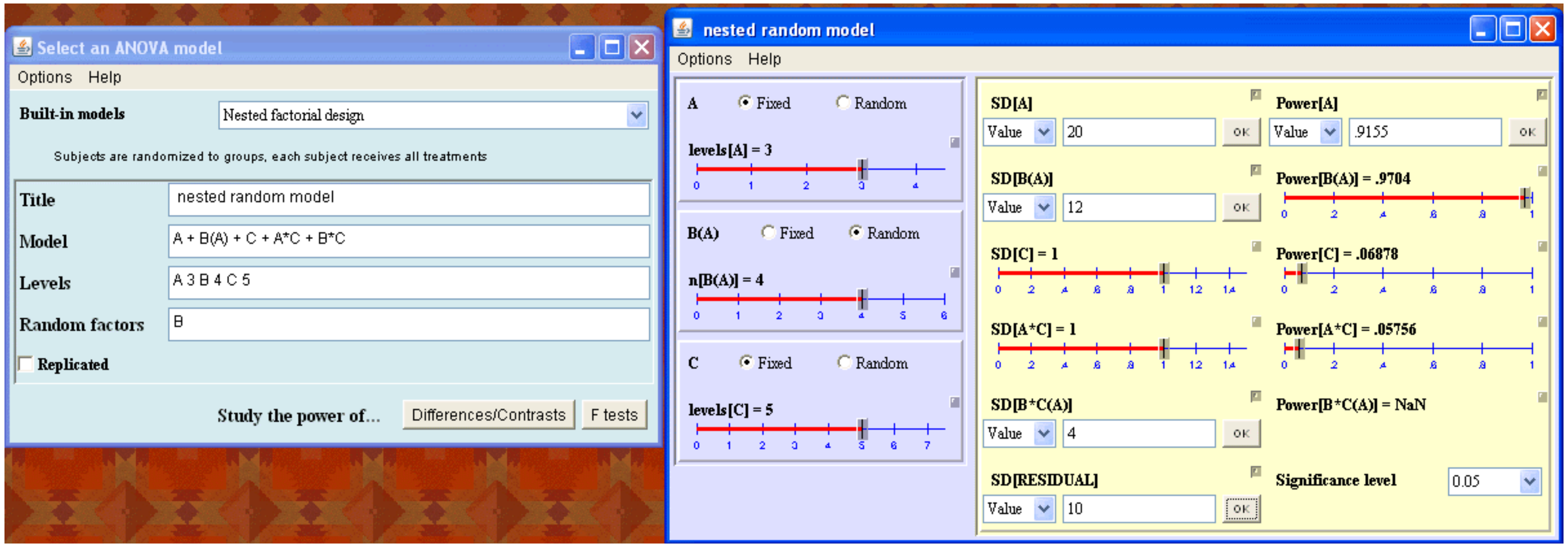
With $df_A = 2$, $df_{B(A)} = 3(4-1) = 9$, and $\alpha = 0.05$ in Equation 8.1:

$$F_{0.95} = 4.256 = F_{1-\pi,19.14}$$

which is satisfied by $\pi = 0.915$.

From **Piface> Balanced ANOVA (any model)** with:

$$s_A = \sqrt{\frac{2\,(20)^2 + (0)^2}{3-1}} = 20.0$$




## 8.6 Two-Level Factorial Designs

**Example 8.11** Use the method of Equation 8.24 to determine the number of replicates required to detect an effect of size $\delta = 6$ with 90% power in a $2^4$ experiment when $\sigma_\epsilon = 10$. Assume that the ANOVA model will include main effects and two-factor interactions.
**Solution:** With $t \simeq z$ in the first iteration of Equation 8.24, the number of replicates required to deliver 90% power to detect the difference $\delta = 6$ between two levels of a design variable is

$$\begin{aligned} n &\geq \frac{1}{2^{4-2}}(1.96 + 1.282)^2 \left(\frac{10}{6}\right)^2 \\ &\geq 8. \end{aligned}$$

Another iteration (not shown) confirms that $n = 8$ is the correct number of replicates.
See Example 8.12.

**Example 8.12** Use the method of Equation 8.21 to confirm the solution to Example 8.11.
**Solution:** By Equation 8.22 the $F$ distribution noncentrality parameter is

$$\phi = 8 \times 2^{4-2}\left(\frac{6}{10}\right)^2 = 11.5.$$

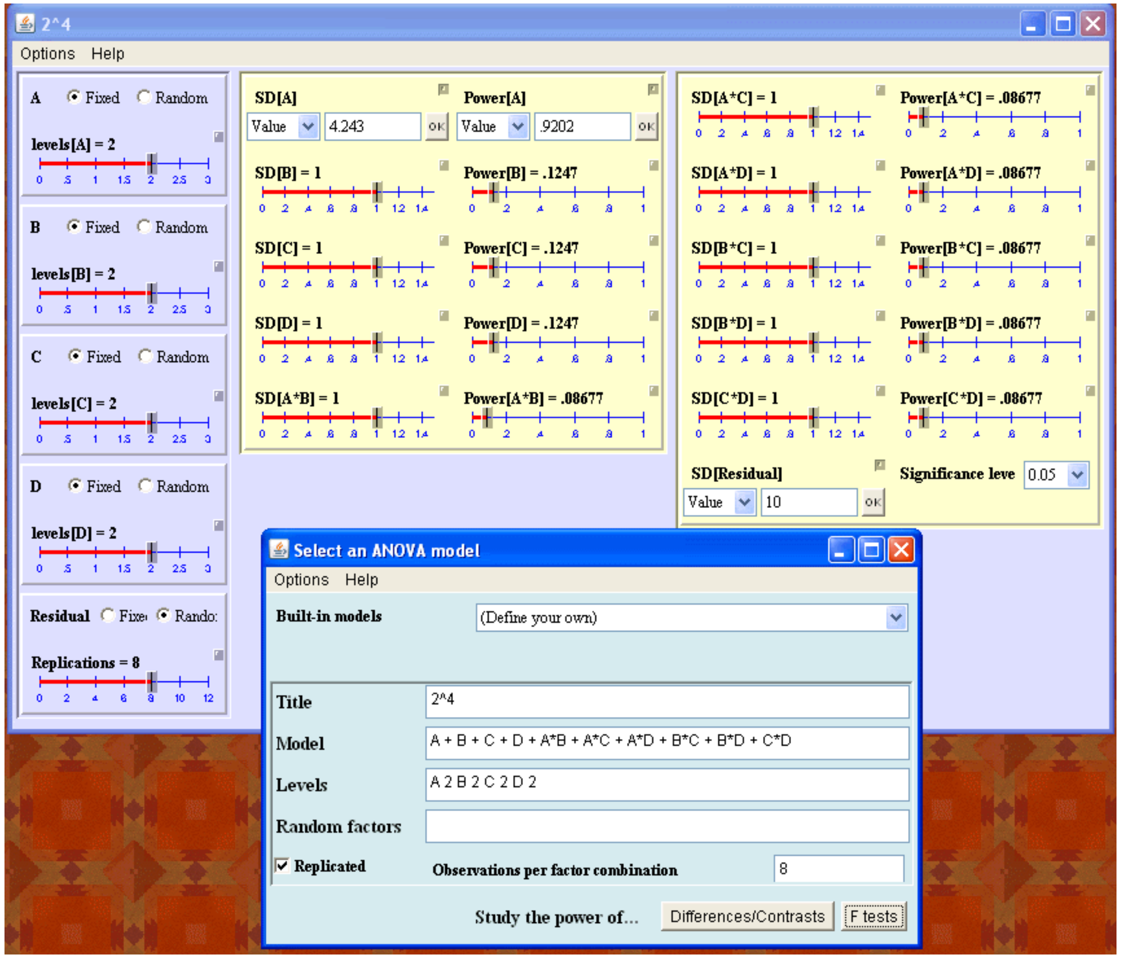
The central and noncentral $F$ distributions will have $df_i = 1$ numerator and $df_\epsilon = df_{total} - df_{model} = \left(8 \times 2^4 - 1\right) - (4+6) = 117$ denominator degrees of freedom. The power, determined from the condition

$$\begin{aligned} F_{0.95} &= F_{1-\pi,11.5} \\ 3.922 &= F_{0.080,11.5} \end{aligned}$$

is $\pi = 0.920$ or 92.0%. This value is slightly larger than the 90% goal because the calculated value of $n$ was fractional and was rounded up to the nearest integer. With $n = 7$ the power is slightly less than 90%.

From **Piface**> **Balanced ANOVA (any model)** with:

$$s_A = \sqrt{\frac{2\,(3)^2}{2-1}} = 4.243$$




From **MINITAB**> **Stat**> **Power and Sample Size**> **2-Level Factorial Design** (with 5 terms removed from the model: four three-factor interactions and the one four-factor interaction):

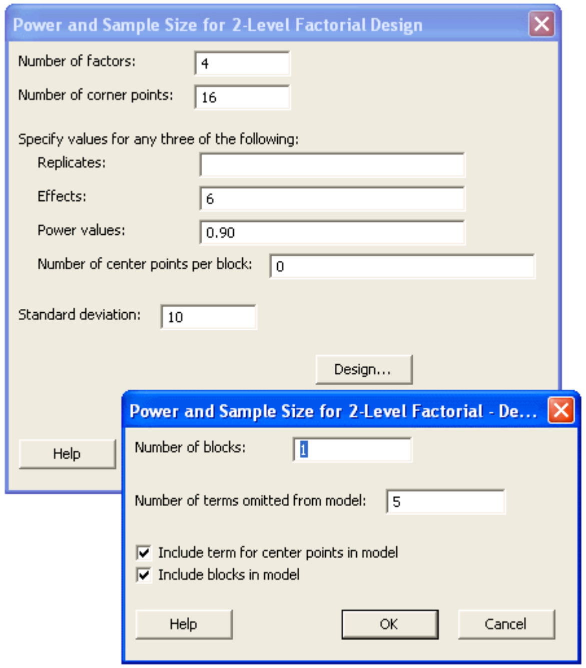
```
MTB > Power;
SUBC>   FFDesign 4 16;
SUBC>     Effect 6;
SUBC>     Power 0.90;
SUBC>     CPBlock 0;
SUBC>     Sigma 10;
SUBC>     Omit 5;
SUBC>     FitC;
SUBC>     FitB;
SUBC>   GPCurve.
```

**Power and Sample Size**

```
2-Level Factorial Design

Alpha = 0.05  Assumed standard deviation = 10

Factors:    4  Base Design: 4, 16
Blocks:  none

Number of terms omitted from model: 5


Center                  Total  Target
Points  Effect  Reps    Runs   Power   Actual Power
     0       6     8     128     0.9       0.920162
```

**Power Curve for 2-Level Factorial Design**

```
MTB >
```

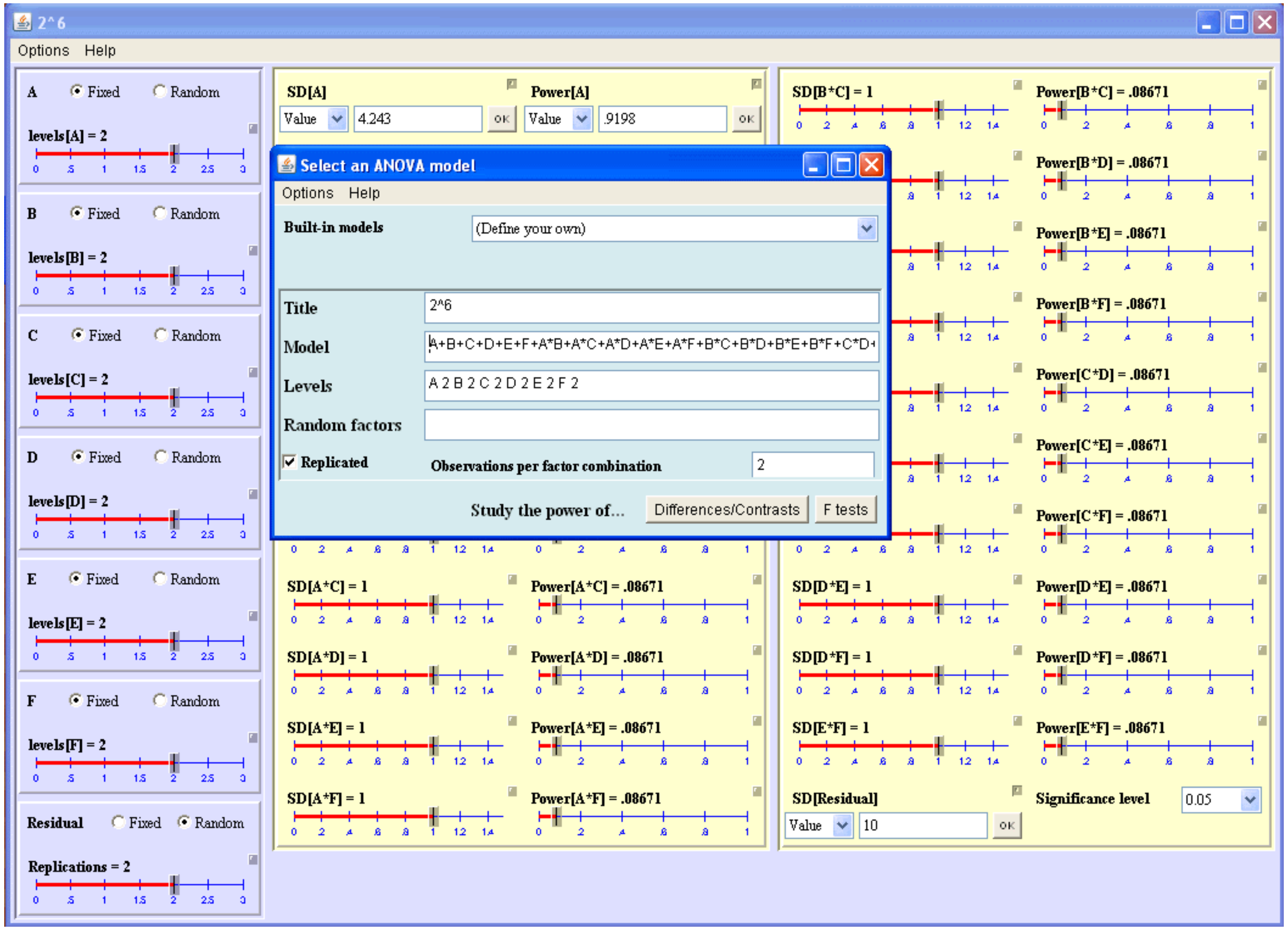



**Example 8.13** Suppose that two more two-level variables were added to the $2^4$ experiment with $n = 8$ replicates from Example 8.11 without any increase in the total number of runs. Calculate the power for the resulting $2^6$ experiment.
**Solution:** The $2^6$ experiment must have $n = 2$ replicates to maintain the same number of runs as the original experiment. Because $8 \times 2^4 = 2 \times 2^6$, the $F$ distribution noncentrality parameter will be unchanged. The new error degrees of freedom for the $F$ distributions will be $df_\epsilon = \left(2 \times 2^6 - 1\right) - (6 + 15) = 106$. The power, determined from

$$
\begin{aligned}
F_{0.95} &= F_{1-\pi,11.5} \\
3.931 &= F_{0.081,11.5}
\end{aligned}
$$

is $\pi = 0.919$ or 91.9%. This example confirms that adding variables to a $2^k$ design without increasing the total number of observations has little effect on the power provided that the error degrees of freedom remains large.

From **Piface**> **Balanced ANOVA (any model)**:

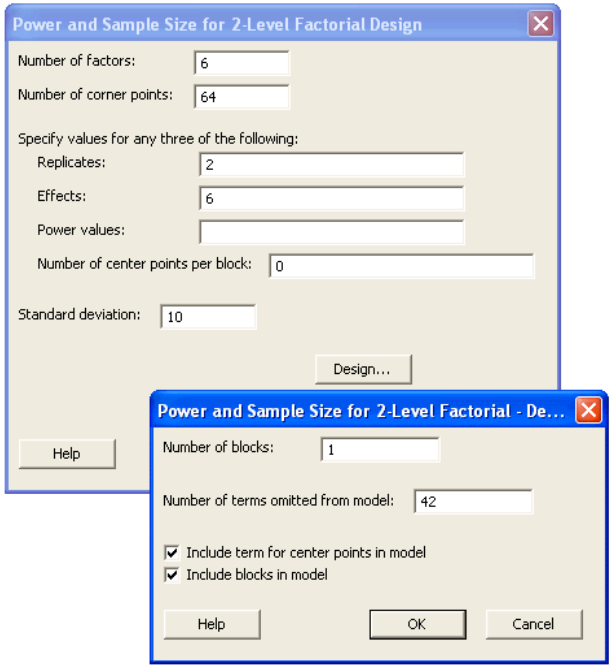



From **MINITAB**> **Stat**> **Power and Sample Size**> **2-Level Factorial Design** (with 42 terms removed from the model: 20 three-factor interactions, 15 four-factor interactions, 6 five-factor interactions, and the one six-factor interaction):

```
MTB > Power;
SUBC>   FFDesign 6 64;
SUBC>     Reps 2;
SUBC>     Effect 6;
SUBC>     CPBlock 0;
SUBC>     Sigma 10;
SUBC>     Omit 42;
SUBC>     FitC;
SUBC>     FitB;
SUBC>   GPCurve.
```

**Power and Sample Size**

```
2-Level Factorial Design

Alpha = 0.05  Assumed standard deviation = 10

Factors:    6   Base Design: 6, 64
Blocks:  none

Number of terms omitted from model: 42


Center                Total
Points  Effect  Reps   Runs     Power
     0       6     2    128  0.919727
```




**Example 8.14** Derive a simplified expression for the total number of observations required for a $2^k$ experiment to detect a difference $\delta$ between two levels of a design variable assuming $\alpha = 0.05$ and $\beta = 0.10$. Under what conditions should this expression be valid?
**Solution:** From Equation 8.25 the total number of replicates required for a $2^k$ design to have 90% power to detect a difference $\delta$ between two levels of a design variable is approximately

$$\begin{aligned} n2^k &\geq 4\left(z_{0.025} + z_{0.10}\right)^2 \left(\frac{\sigma_\epsilon}{\delta}\right)^2 \\ &\geq 42\left(\frac{\sigma_\epsilon}{\delta}\right)^2. \end{aligned} \tag{8.1}$$

This condition will be strictly valid when $df_\epsilon$ is large so that the $t \simeq z$ approximation is well satisfied.

**Example 8.15** How many replicates of a $2^3$ design are required to determine the regression coefficient for a main effect with precision $\delta = 300$ with 95% confidence when the standard error of the model is expected to be $\sigma_\epsilon = 600$?
**Solution:** If the error degrees of freedom are sufficiently large that $t_{0.025} \simeq z_{0.025}$ then

$$\begin{aligned} n &\geq \frac{1}{2^3}\left(\frac{1.96 \times 600}{300}\right)^2 \\ &\geq 2. \end{aligned}$$

With only $2 \times 2^3 = 16$ total runs, the $t_{0.025} \simeq z_{0.025}$ assumption is not satisfied. Another iteration shows that the transcendental sample size condition is satisfied for $n = 3$ replicates of the $2^3$ design.

**Example 8.16** What is the power for the $2_{IV}^{4-1}$ design with two replicates to detect a difference of $\delta = 10$ between two levels of a design variable if $\sigma_\epsilon = 5$?
**Solution:** With two replicates the total number of experimental runs will be $2\left(2^{4-1}\right) = 16$. Because the experiment design is resolution IV, the model can include main effects and only three of the six possible two-factor interactions, so $df_{model} = 4 + 3 = 7$. Then, the error degrees of freedom will be $df_\epsilon = (16 - 1) - 7 = 8$. The $F$ distribution noncentrality parameter associated with a difference of $\delta = 10$ between two levels of a design variable is given by a slightly modified form of Equation 8.22:

$$\begin{aligned} \phi &= n2^{(k-p)-2}\left(\frac{\delta}{\sigma_\epsilon}\right)^2 \\ &= 2 \times 2^{(4-1)-2}\left(\frac{10}{5}\right)^2 \\ &= 16.0 \end{aligned} \tag{8.2}$$

where $p = 1$ accounts for the half-fractionation of the full factorial design. Then, by Equation 8.21

$$\begin{aligned} F_{0.95} &= F_{1-\pi,16} \\ 5.318 &= F_{0.063,16}. \end{aligned}$$

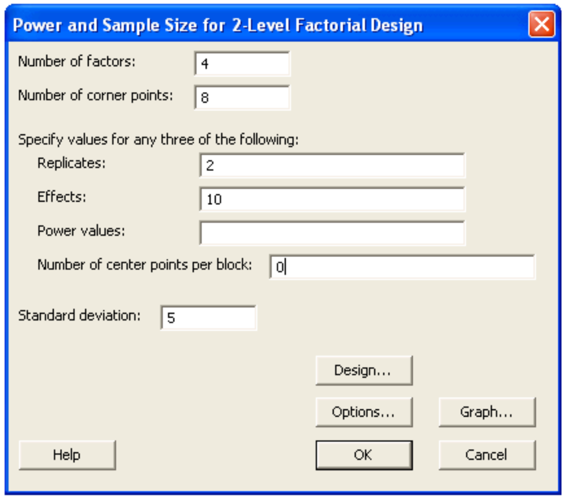
The power is $\pi = 1 - 0.063 = 0.937$.
From **MINITAB**> **Stat**> **Power and Sample Size**> **2-Level Factorial Design**:

**Power Curve for 2-Level Factorial Design**

```
MTB > Power;
SUBC>   FFDesign 4 8;
SUBC>     Reps 2;
SUBC>     Effect 10;
SUBC>     CPBlock 0;
SUBC>     Sigma 5;
SUBC>     FitC;
SUBC>     FitB;
SUBC>   GPCurve.
```

**Power and Sample Size**

```
2-Level Factorial Design

Alpha = 0.05  Assumed standard deviation = 5

Factors:    4   Base Design: 4, 8
Blocks:   none


Center                    Total
Points  Effect  Reps      Runs     Power
     0      10     2        16  0.936743
```



**Example 8.17** How many replicates of a $2_V^{5-1}$ design are required to have 90% power to detect a difference $\delta = 0.4$ between two levels of a design variable? Assume that ten of the fifteen possible terms will drop out of the model and that the standard error will be $\sigma_\epsilon = 0.18$.
**Solution:** If a model with main effects and two factor interactions is fitted to one replicate of the $2_V^{5-1}$ design, there will not be any degrees of freedom left to estimate the error, so either the experiment must be replicated or some terms must be dropped from the model. Under the assumption that the number of replicates is large, so that we can take $t \simeq z$ in the first iteration of Equation 8.24, we have

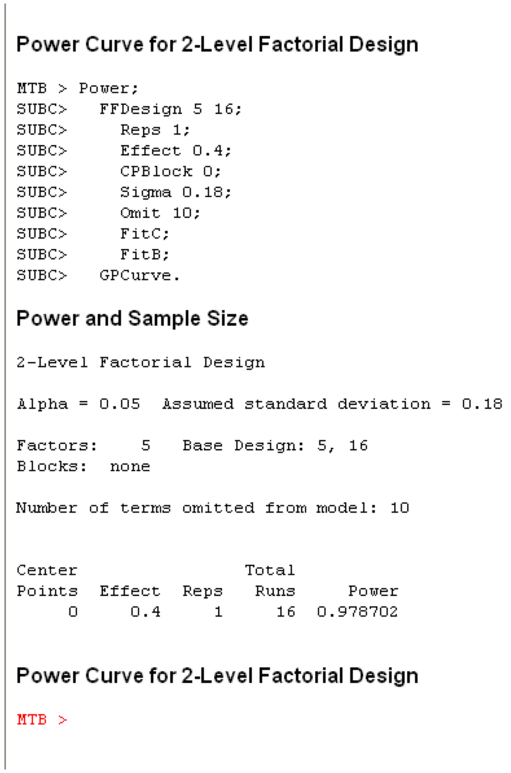
$$\begin{aligned} n &\geq \frac{1}{2^{(5-1)-2}} \left(z_{0.025} + z_{0.10}\right)^2 \left(\frac{0.18}{0.4}\right)^2 \\ &\geq 0.532. \end{aligned}$$

Obviously, the $t \simeq z$ approximation is not satisfied, so at least one more iteration is required. If only one replicate of the half-fractional factorial design is built and ten of the fifteen possible terms are dropped from the model, the error degrees of freedom will be $df_\epsilon = 15 - 10 = 5$. Then, for the second iteration of Equation 8.24, we have

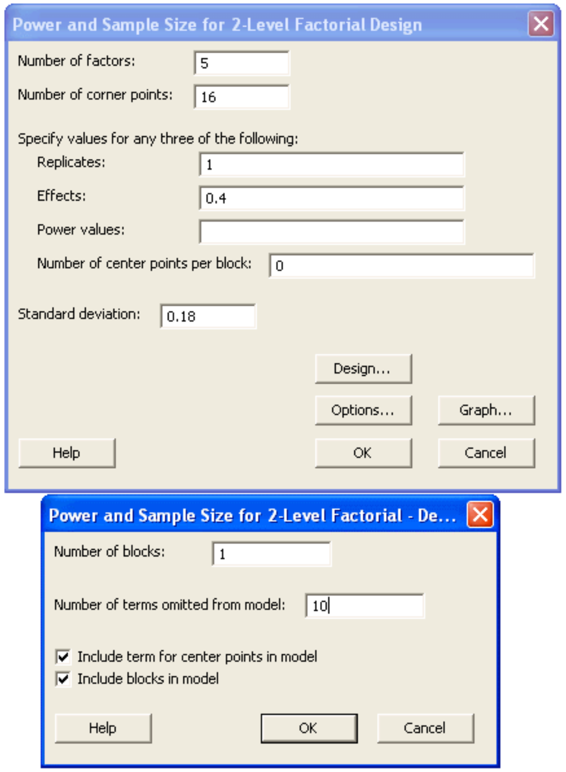
$$\begin{aligned} n &\geq \frac{1}{2^{(5-1)-2}} \left(t_{0.025} + t_{0.10}\right)^2 \left(\frac{0.18}{0.4}\right)^2 \\ &\geq \frac{1}{2^{(5-1)-2}} \left(2.228 + 1.372\right)^2 \left(\frac{0.18}{0.4}\right)^2 \\ &\geq 0.656, \end{aligned}$$

which rounds up to $n = 1$. Calculation of the power (not shown) confirms $\pi = 0.98$ for one replicate.

From **MINITAB**> **Stat**> **Power and Sample Size**> **2-Level Factorial Design**:



```
Power Curve for 2-Level Factorial Design

MTB > Power;
SUBC>   FFDesign 5 16;
SUBC>     Reps 1;
SUBC>     Effect 0.4;
SUBC>     CPBlock 0;
SUBC>     Sigma 0.18;
SUBC>     Omit 10;
SUBC>     FitC;
SUBC>     FitB;
SUBC>   GPCurve.

Power and Sample Size

2-Level Factorial Design

Alpha = 0.05  Assumed standard deviation = 0.18

Factors:    5   Base Design: 5, 16
Blocks:  none

Number of terms omitted from model: 10


Center                Total
Points  Effect  Reps   Runs     Power
     0     0.4     1     16  0.978702

Power Curve for 2-Level Factorial Design

MTB >
```




**Example 8.18** How many replicates of a 9-variable 12-run Plackett-Burman design are required to detect a difference $\delta = 7000$ between two levels of a variable with 90% power if the standard error is expected to be $\sigma_\epsilon = 4000$?
**Solution:** Plackett-Burman designs are resolution III, so their models may contain only main effects. If the experiment is run with only one replicate, then $df_\epsilon = (12 - 1) - 9 = 2$ and the large-sample approximation is obviously not satisfied. If enough replicates are run so that the large-sample approximation is satisfied, then with $\alpha = 0.05$, $\beta = 0.10$ and $t \simeq z$, the approximate number of replicates required is

$$\begin{aligned} n &\geq \frac{4}{12}(1.96 + 1.282)^2 \left(\frac{4000}{7000}\right)^2 \\ &\geq 2. \end{aligned}$$

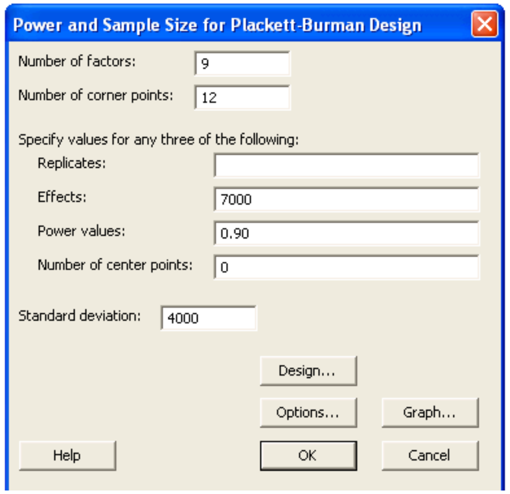
Another iteration confirms that two replicates are sufficient to achieve 90% power.

From **MINITAB**> **Stat**> **Power and Sample Size**> **Plackett-Burman Design**:

```
MTB > Power;
SUBC>   PBDesign 9 12;
SUBC>     Effect 7000;
SUBC>     Power 0.90;
SUBC>     CPBlock 0;
SUBC>     Sigma 4000;
SUBC>     FitC;
SUBC>   GPCurve.
```

**Power and Sample Size**

```
Plackett-Burman Design

Alpha = 0.05  Assumed standard deviation = 4000

Factors:            9   Design: 12
Center pts (total): 0


Center                  Total  Target
Points  Effect  Reps    Runs   Power  Actual Power
     0    7000     2      24     0.9      0.975221
```




## 8.7 Two-Level Factorial Designs with Centers

**Example 8.19** Calculate the power to detect a difference $\delta = 1400$ between two levels of a study variable in a $2^3$ design with three replicates built in blocks with two center points per block. Include terms for main effects, two-factor interactions, lack of fit, and blocks in the model. The standard error is expected to be $\sigma_\epsilon = 1000$.
**Solution:** The experiment will have $3\left(2^3+2\right) = 30$ total observations, so the total degrees of freedom will be $df_{total} = 29$. The degrees of freedom for the model will be

$$\begin{aligned} df_{model} &= df_{blocks} + df_{main\,effects} + df_{interactions} + df_{LOF} \\ &= 2+3+3+1 \\ &= 9, \end{aligned}$$

so the error degrees of freedom will be

$$df_\epsilon = df_{total} - df_{model} = 29 - 9 = 20.$$

The power $\pi$ to reject $H_0 : \delta = 0$ for the main effect of any one of the study variables is given by Equation 8.21 with one numerator and twenty denominator degrees of freedom where the $F$ distribution noncentrality parameter, as given in Equation 8.22, is

$$\begin{aligned} \phi &= 3 \times 2^{3-2}\left(\frac{1400}{1000}\right)^2 \\ &= 11.76. \end{aligned}$$

With $\alpha = 0.05$ and

$$\begin{aligned} F_{1-\alpha} &= F_{1-\pi,\phi} \\ F_{0.95} &= 4.351 = F_{1-\pi,8.17}, \end{aligned}$$

we find the power to be $\pi = 0.903$.

From **MINITAB**> **Stat**> **Power and Sample Size**> **2-Level Factorial Design**:

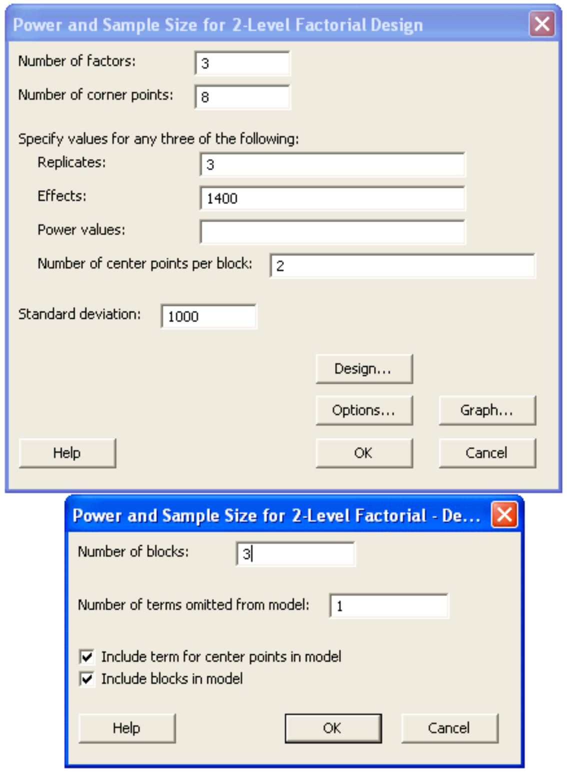
```
MTB > Power;
SUBC>   FFDesign 3 8;
SUBC>     Reps 3;
SUBC>     Effect 1400;
SUBC>     CPBlock 2;
SUBC>     Sigma 1000;
SUBC>     Blocks 3;
SUBC>     Omit 1;
SUBC>     FitC;
SUBC>     FitB;
SUBC>   GPCurve.
```

**Power and Sample Size**

```
2-Level Factorial Design

Alpha = 0.05  Assumed standard deviation = 1000

Factors: 3   Base Design: 3, 8
Blocks:  3

Number of terms omitted from model: 1
Including a term for center points in model.
Including blocks in model.


Center
Points
   Per                  Total
 Block  Effect  Reps     Runs      Power
     2    1400     3       30   0.903298
```

**Power Curve for 2-Level Factorial Design**

```
MTB >
```




**Example 8.20** Determine the ratio of the precisions of the estimates for the lack of fit and main effects in a $2^4$ plus centers design when two center points are used per replicate.
**Solution:** From Equation 8.39 with $k = 4$ and $n_0 = 2$ the ratio of the lack of fit and main effect precision estimates will be

$$\begin{aligned} \frac{\delta_{**}}{\delta} &= \sqrt{1 + \frac{2^4}{2}} \\ &= 3. \end{aligned}$$

That is, the confidence interval for the lack of fit estimate will be three times wider than the confidence interval for the main effects.

## 8.8 Response Surface Designs

**Example 8.21** How many replicates of a three-variable Box-Behnken design are required to estimate the regression coefficients associated with main effects, two-factor interactions, and quadratic terms to within $\delta = \pm 2$ with 95% confidence if the standard error is expected to be $\sigma_\epsilon = 5$?

**Solution:** A first estimate for the number of replicates required to estimate the regression coefficients associated with main effects is given by Equation 8.43 with $t_{0.025} \simeq 2$ and, from Table 8.5 for the $BB\,(3)$ design, $SS_{Main\ Effects} = 8$ is

$$\begin{aligned} n &\geq \frac{1}{8}\left(\frac{2 \times 5}{2}\right)^2 \\ &\geq 4. \end{aligned}$$

With $n = 4$ replicates, the error degrees of freedom will be

$$\begin{aligned} df_\epsilon &= df_{total} - df_{model} \\ &= (4 \times 15 - 1) - 9 \\ &= 50, \end{aligned}$$

so the approximation for $t_{0.025}$ is justified. Another iteration with $n = 3$ replicates indicates that the precision of the regression coefficient estimates would be slightly greater than $\delta = 2$, so $n = 4$ replicates are required.

From Table 8.5 for two-factor interactions, $SS_{Interaction} = 4$, so the number of replicates required to estimate the regression coefficients associated with two-factor interactions with confidence interval half-width $\delta = 2$ with 95% confidence is

$$\begin{aligned} n &\geq \frac{1}{4}\left(\frac{t_{0.025} \times 5}{2}\right)^2 \\ &\geq 6. \end{aligned}$$

From Table 8.5 for quadratic terms, $SS_{Quadratic} = 3.694$, so the number of replicates required to estimate the regression coefficients associated with quadratic terms is

$$\begin{aligned} n &\geq \frac{1}{3.694}\left(\frac{t_{0.025} \times 5}{2}\right)^2 \\ &\geq 7. \end{aligned}$$

# Chapter 9

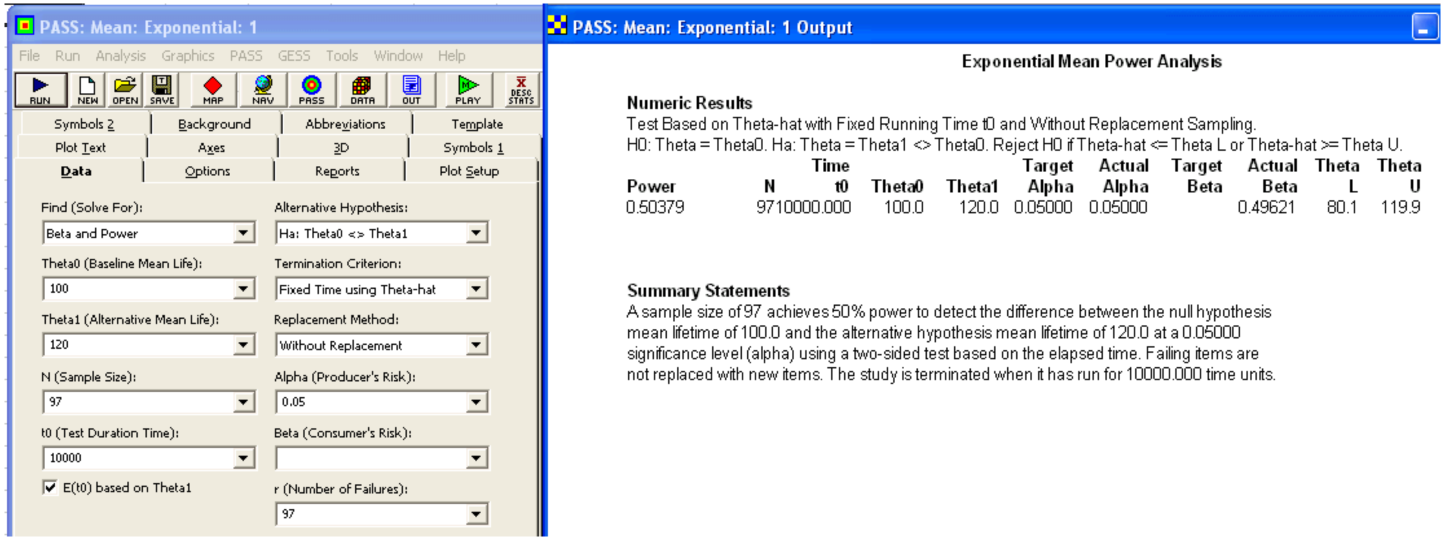
# Reliability and Survival

## 9.1 Reliability Parameter Estimation

**Example 9.1** How many units must be tested to failure to determine, with 20% precision and 95% confidence, the exponential mean life $\mu$?
**Solution:** From Equation 9.6 with $\alpha = 0.05$ and $\delta = 0.2$, the required number of failures is

$$r = \left(\frac{1.96}{0.20}\right)^2 = 97.$$

From **PASS**> **Means**> **One**> **Inequality (Exponential)**:

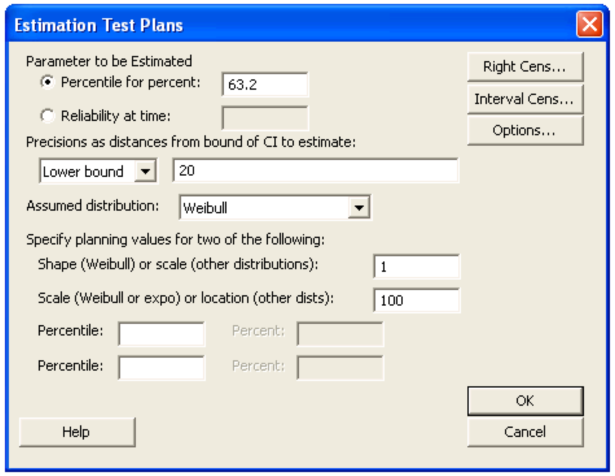


From **MINITAB**> **Stat**> **Reliability/Survival**> **Test Plans**> **Estimation**:

```
MTB > Etestplan;
SUBC>   EPtile 63.2;
SUBC>   Dlower 20;
SUBC>   Weibull;
SUBC>   SetS 1;
SUBC>   ScLocation 100;
SUBC>   TwoSided.
```

**Estimation Test Plans**

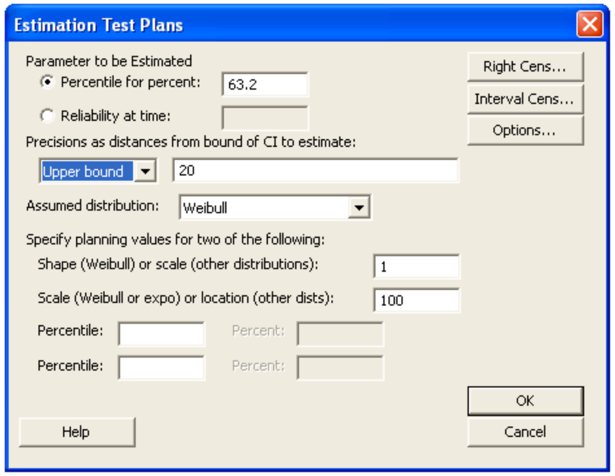
```
Uncensored data

Estimated parameter: 63.2th percentile
Calculated planning estimate = 99.9672
Target Confidence Level = 95%
Precision in terms of the lower bound of a
two-sided confidence interval.

Planning distribution: Exponential
Scale = 100


                          Actual
           Sample  Confidence
Precision    Size       Level
       20      78     95.1330
```




```
MTB > Etestplan;
SUBC>   EPtile 63.2;
SUBC>   Dupper 20;
SUBC>   Weibull;
SUBC>   SetS 1;
SUBC>   ScLocation 100;
SUBC>   TwoSided.
```

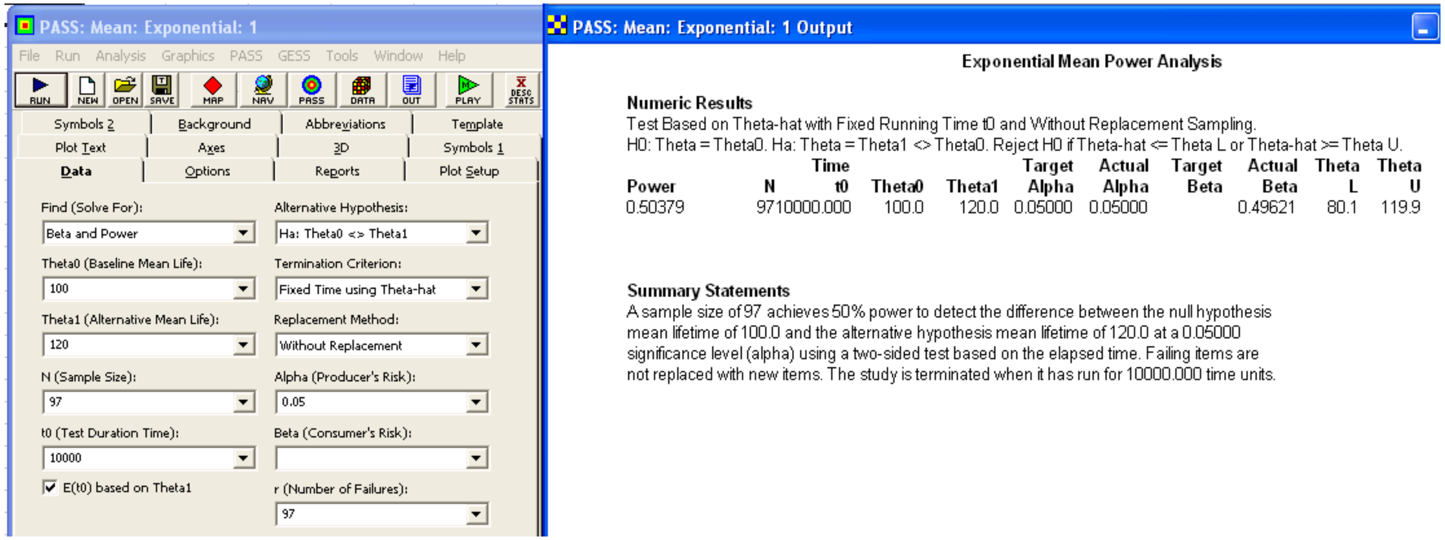
**Estimation Test Plans**

```
Uncensored data

Estimated parameter: 63.2th percentile
Calculated planning estimate = 99.9672
Target Confidence Level = 95%
Precision in terms of the upper bound of a
two-sided confidence interval.

Planning distribution: Exponential
Scale = 100


                          Actual
           Sample  Confidence
Precision    Size       Level
       20     116     95.0499
```




**Example 9.2** How many units must be tested to failure to determine, with 20% precision and 95% confidence, any failure percentile under the assumption that the reliability distribution is exponential?
**Solution:** The conditions required to estimate the failure percentiles are the same as those in Example 9.1, so the same number of failures required is $r = 97$.

From **PASS**> **Means**> **One**> **Inequality (Exponential)**:




**Example 9.3** How many units must be tested to failure in an experiment to determine, with 95% confidence, the exponential reliability to within 10% of its true value if the expected reliability is 80%?
**Solution:** From Equation 9.12 with $\alpha = 0.05$, $\delta = 0.10$, and $\widehat{R} = 0.80$, the required number of failures is

$$r = \left(\frac{1.96 \ln(0.80)}{0.10}\right)^2 = 20.$$

**Example 9.4** How many units must be tested to failure to estimate, with 20% precision and 95% confidence, the Weibull scale factor if the shape factor is known to be $\beta = 2$?
**Solution:** The goal of the experiment is to obtain a confidence interval for the Weibull scale factor of the form given by Equation 9.17 with $\delta = 0.20$ and $\alpha = 0.05$. From Equation 9.19 the required number of failures is

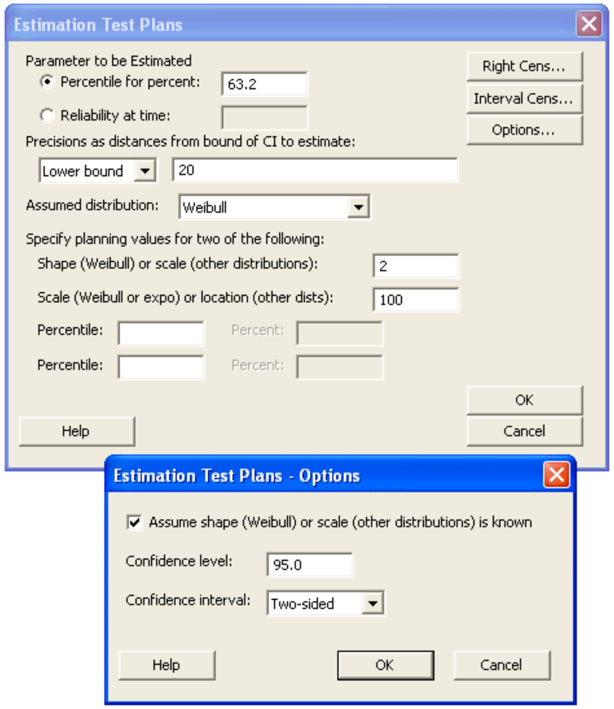
$$r = \left(\frac{1.96}{2 \times 0.20}\right)^2 = 25.$$

The Weibull scale parameter is the 63.2% percentile. From **MINITAB**> **Stat**> **Reliability/Survival**> **Test Plans**> **Estimation**:

```
MTB > Etestplan;
SUBC>   EPtile 63.2;
SUBC>   Dlower 20;
SUBC>   Weibull;
SUBC>   SetS 2;
SUBC>   ScLocation 100;
SUBC>   TwoSided.
```

**Estimation Test Plans**

```
Uncensored data

Estimated parameter: 63.2th percentile
Calculated planning estimate = 99.9836
Target Confidence Level = 95%

Precision in terms of the lower bound of a
two-sided confidence interval.

Planning distribution: Weibull
Scale = 100, Shape (true value) = 2


                       Actual
            Sample  Confidence
Precision     Size       Level
       20       20     95.4090
```




The MINITAB solution for the upper bound is $n = 29$, so the average of the two sample sizes is consistent with the approximate solution.

**Example 9.5** How many units must be tested to failure to estimate, with 95% confidence, the Weibull shape parameter to within 20% of its true value?
**Solution:** The goal of the experiment is to produce a 95% confidence interval for $\beta$ of the form given by Equation 9.20 with $\delta = 0.20$. From Equation 9.24 with $\alpha = 0.05$, the required number of failures is

$$r = 6\left(\frac{1.96}{\pi \times 0.20}\right)^2 = 59.$$

**Example 9.6** How many units must be tested to failure to estimate the Weibull reliability with 5% precision and 95% confidence when the expected reliability is 90%? Assume that the Weibull shape factor is known.
**Solution:** The desired confidence interval will have the form of Equation 9.31. From Equation 9.35 with $\delta = 0.05$, $\alpha = 0.05$, and $\widehat{R} = 0.90$, the required number of failures is

$$r = \left(\frac{1.645}{0.05}\left(\frac{1-0.9}{0.9}\right)\right)^2 = 19.$$

**Example 9.7** An experiment is planned to estimate, with 95% confidence, the time at which 10% of units will fail to within 1000 hours. The life distribution is expected to be normal with $\widehat{\sigma}_t = 2000$ and all units will be tested to failure.

**Solution:** With $z_{\alpha/2} = z_{0.025} = 1.96$ and $z_f = z_{0.10} = 1.282$ in Equation 9.40, the sample size is

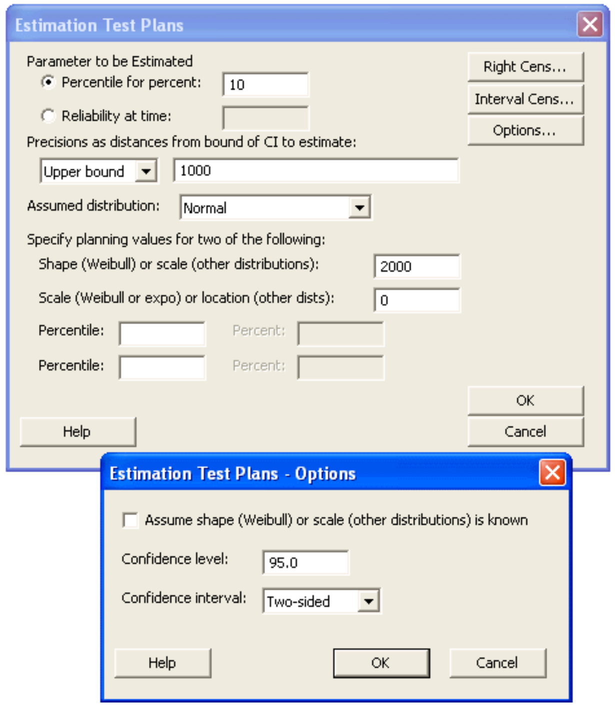
$$\begin{aligned} n &= \left(\frac{1.96 \times 2000}{1000}\right)^2 \left(1 + \frac{(1.282)^2}{2}\right) \\ &= 28. \end{aligned}$$

From **MINITAB**> **Stat**> **Reliability/Survival**> **Test Plans**> **Estimation**:

```
MTB > Etestplan;
SUBC>   EPtile 10;
SUBC>   Dupper 1000;
SUBC>   Normal;
SUBC>   ShScale 2000;
SUBC>   ScLocation 0;
SUBC>   TwoSided.
```

**Estimation Test Plans**

```
Uncensored data

Estimated parameter: 10th percentile
Calculated planning estimate = -2563.10
Target Confidence Level = 95%
Precision in terms of the upper bound of a
two-sided confidence interval.

Planning distribution: Normal
Location = 0, Scale = 2000


                        Actual
           Sample   Confidence
Precision    Size        Level
     1000      28      95.0065
```



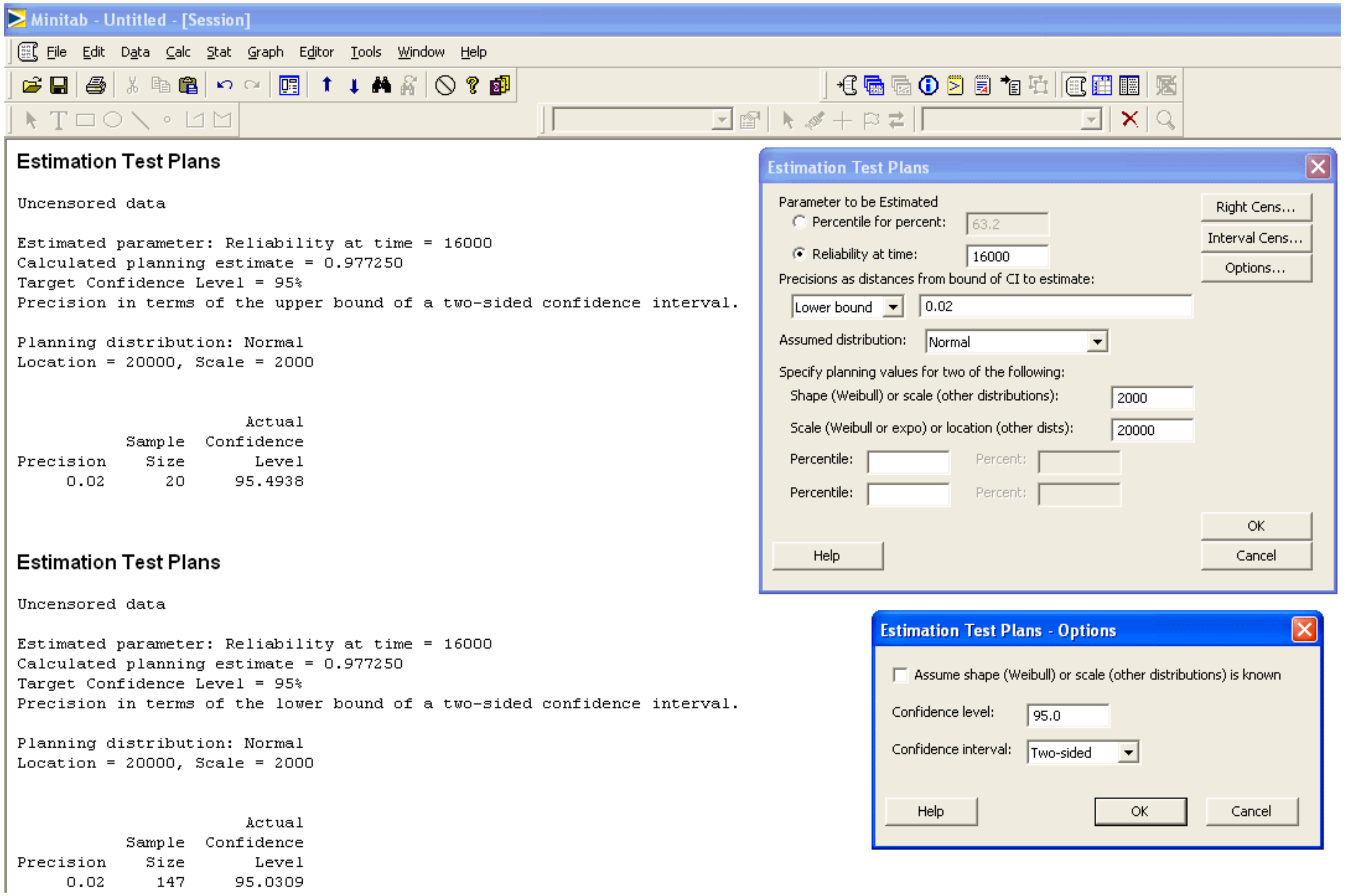

**Example 9.8** What sample size is required to estimate, with 95% confidence, the 24000 hour failure probability of a product to within 2% if the life distribution is expected to be normal with $\mu \simeq 20000$ and $\sigma \simeq 2000$?
**Solution:** With $x = 24000$ and $\widehat{z} = (24000 - 20000)/2000 = 2$, the required confidence interval for the 24 hour failure probability has the form

$$P\left(\widehat{\Phi}(2) - 0.02 < \Phi(x = 24000; \mu, \sigma) < \widehat{\Phi}(2) + 0.02\right) = 0.95.$$

From Equation 9.44 the required sample size to obtain this interval is

$$\begin{aligned} n &= \left(\frac{z_{0.025}\varphi(2)}{0.02}\right)^2 \left(1+\frac{1}{2}2^2\right) \\ &= \left(\frac{1.96\times 0.0540}{0.02}\right)^2 (3) \\ &= 85. \end{aligned}$$

From **MINITAB**> **Stat**> **Reliability/Survival**> **Test Plans**> **Estimation** the calculated sample sizes are $n = 20$ for the upper bound and $n = 147$ for the lower bound. Their average, $n = 84$, is in good agreement with the approximate solution.



## 9.2 Reliability Demonstration Tests

**Example 9.9** How many units must be tested for 200 hours without any failures to show, with 95% confidence, that the $MTTF$ of a system exceeds 400 hours. The life distribution is exponential and the test is time terminated.
**Solution:** We must determine the value of $n$ with $r = 0$ failures in $t' = 200$ hours of testing such that

$$P(400 < \mu < \infty) = 0.95.$$

From the $f'$ equation for the exponential distribution from Table 9.1 with $\mu_0 = 400$, the $t' = 200$ hour failure probability is

$$\begin{aligned} f' &= 1 - e^{-t'/\mu_0} \\ &= 1 - e^{-200/400} \\ &= 0.3935. \end{aligned}$$

With $r = 0$ and $\alpha = 0.05$ the smallest value of $n$ that satisfies Equation 9.46 is $n = 6$ because

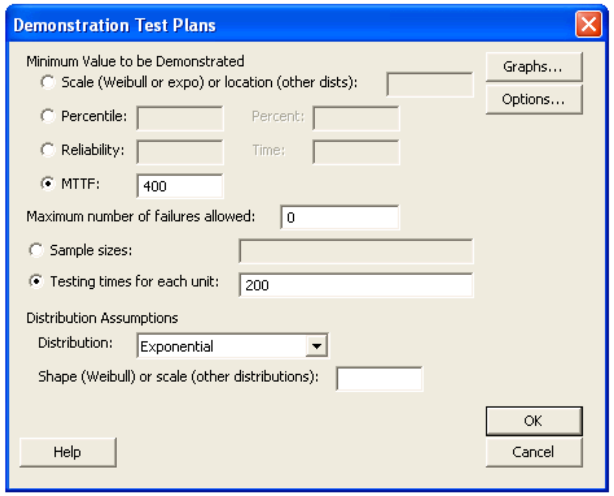
$$(b(0; 6, 0.3935) = 0.04977) < (\alpha = 0.05).$$

From **MINITAB**> **Stat**> **Reliability/Survival**> **Test Plans**> **Demonstration:**

```
MTB > DtestPlan 0;
SUBC>   MTTF 400 ;
SUBC>   TTime 200 ;
SUBC>   Exponential;
SUBC>   GPOPGraph.
```

**Demonstration Test Plans**

```
Reliability Test Plan
Distribution: Exponential
MTTF Goal = 400, Target Confidence Level = 95%


                                 Actual
Failure  Testing  Sample  Confidence
   Test     Time    Size       Level
      0      200       6     95.0213
```




**Example 9.10** Determine how long ten units must be life tested with no more than one failure during the test period to demonstrate, with 90% confidence, that the mean life is greater than 1000 hours. Assume that the life distribution is exponential.
**Solution:** The goal of the experiment is to demonstrate that

$$P(1000 < \mu < \infty) = 0.90.$$

With $n = 10$ and $r = 1$, Equation 9.46 gives

$$\sum_{x=0}^{1} b\left(x; n = 10, f^{'}\right) = 0.10 \tag{9.48}$$

which is satisfied by $f' = 0.337$. From the $t'$ equation for the exponential distribution from Table 9.1, the required duration of the test in hours is

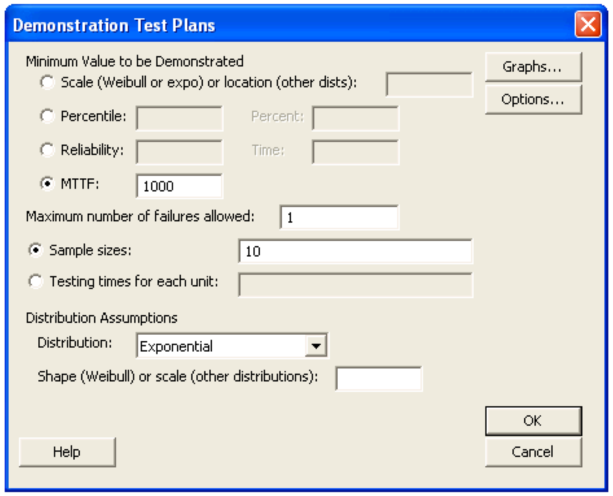
$$\begin{aligned} t' &= -\mu_0 ln\left(1 - f'\right) \\ &= -1000 ln\left(1 - 0.337\right) \\ &= 411. \end{aligned}$$

From **MINITAB**> **Stat**> **Reliability/Survival**> **Test Plans**> **Demonstration:**

```
MTB > DtestPlan 1;
SUBC>   MTTF 1000 ;
SUBC>   Sample 10;
SUBC>   Exponential;
SUBC>   GPOPGraph;
SUBC>   Confidence 90.
```

**Demonstration Test Plans**

```
Reliability Test Plan
Distribution: Exponential
MTTF Goal = 1000,  Actual Confidence Level = 90%


Failure  Sample  Testing
   Test    Size     Time
      1      10  410.751
```




**Example 9.11** Determine how long ten units must be life tested with no more than one failure during the test period to demonstrate, with 90% confidence, that the Weibull scale factor is at least 1000 hours. Assume that the Weibull shape factor is known to be $\beta = 2$.
**Solution:** The design parameters of the RDT are the same as in Example 9.10, so Equation 9.48 still applies and the end-of-test failure probability is $f' = 0.337$. From the $t'$ equation for the Weibull distribution from Table 9.1, the required duration of the test in hours is

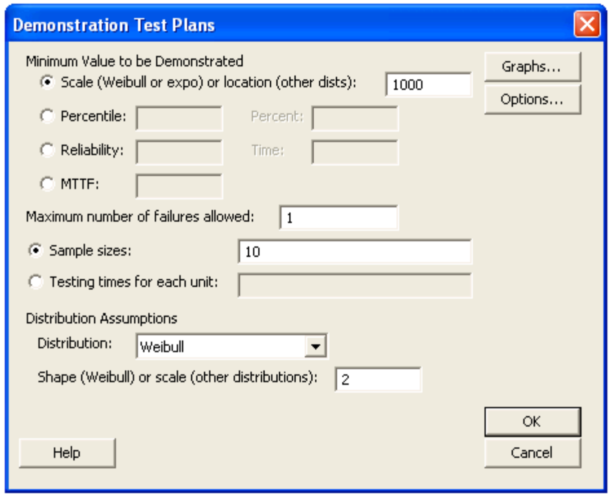
$$\begin{aligned} t' &= \eta_0 \left(-ln\left(1 - f'\right)\right)^{\frac{1}{\beta}} \\ &= 1000 \left(-ln\left(1 - 0.337\right)\right)^{\frac{1}{2}} \\ &= 641. \end{aligned}$$

From **MINITAB**> **Stat**> **Reliability/Survival**> **Test Plans**> **Demonstration:**

```
MTB > DtestPlan 1;
SUBC>   ScLocation 1000;
SUBC>   Sample 10;
SUBC>   Weibull;
SUBC>   ShScale 2;
SUBC>   GPOPGraph;
SUBC>   Confidence 90.
```

**Demonstration Test Plans**

```
Substantiation Test Plan
Distribution: Weibull, Shape = 2
Scale Goal = 1000,  Actual Confidence Level = 90%


Failure  Sample  Testing
   Test    Size     Time
      1      10  640.898
```




**Example 9.12** Determine how long ten units must be life tested with no more than one failure during the test period to demonstrate, with 90% confidence, that the mean life is at least 1000 hours. Assume that the life distribution is normal with $\sigma = 100$.
**Solution:** The design parameters of the RDT are the same as in Example 9.10, so Equation 9.48 applies and the failure probability is $f' = 0.337$. From the equation for $t'$ from Table 9.1, the required duration of the test in hours is

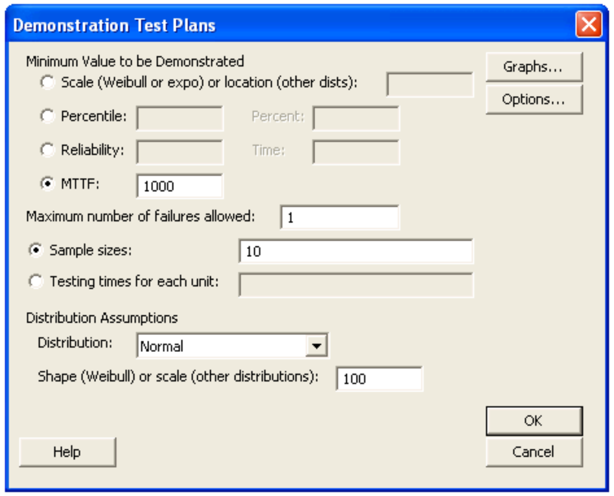
$$\begin{aligned} t' &= \mu + z_{f'}\sigma \\ &= 10000 + z_{0.337}\,(100) \\ &= 10000 + (-0.42 \times 100) \\ &= 958. \end{aligned}$$

From **MINITAB**> **Stat**> **Reliability/Survival**> **Test Plans**> **Demonstration:**

```
MTB > DtestPlan 1;
SUBC>   MTTF 1000 ;
SUBC>   Sample 10;
SUBC>   Normal;
SUBC>   ShScale 100;
SUBC>   GPOPGraph;
SUBC>   Confidence 90.
```

**Demonstration Test Plans**

```
Reliability Test Plan
Distribution: Normal, Scale = 100
MTTF Goal = 1000,  Actual Confidence Level = 90%


Failure  Sample  Testing
   Test    Size     Time
      1      10  957.892
```




**Example 9.13** How many units must be tested for 400 hours without any failures to demonstrate 90% reliability at 600 hours , with 95% confidence? Assume that the reliability distribution is exponential.
**Solution:** In terms of the 600 hour failure probability, the goal of the experiment is to demonstrate

$$P\left(0 < f\left(600\right) < 0.10\right) = 0.95$$

based on a sample of size $n$ tested to $t' = 400$ hours with $r = 0$ failures. From Table 9.2 the equation for $f'$ for the exponential distribution gives

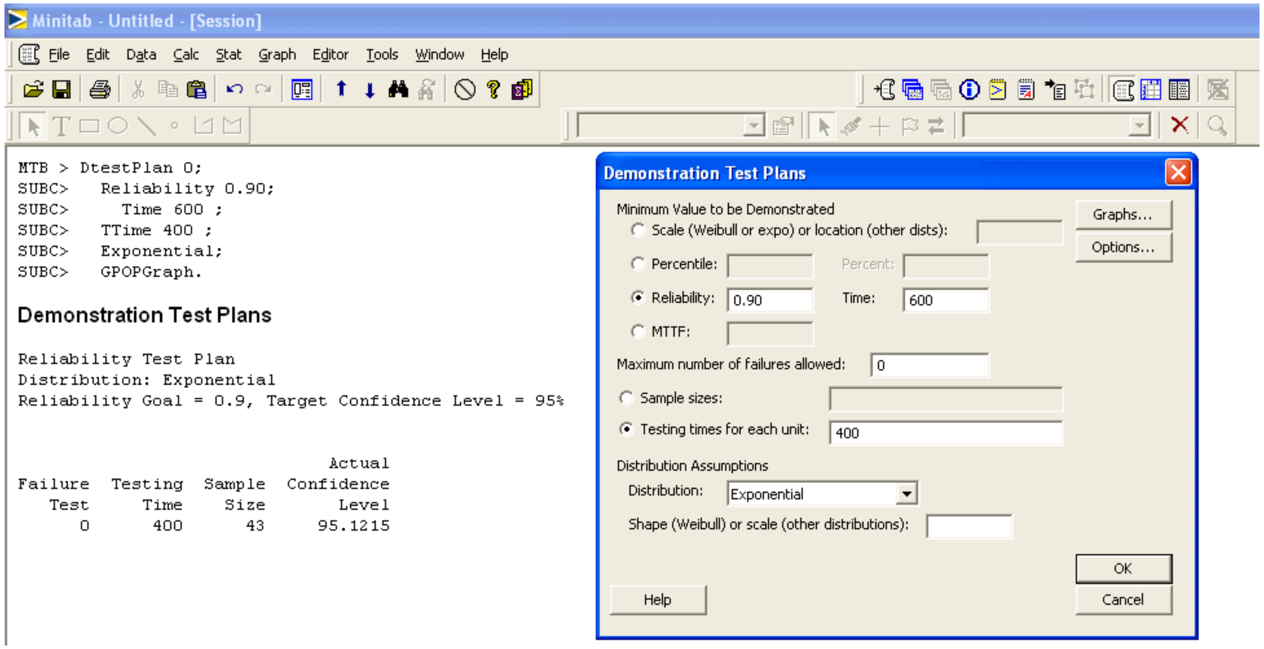
$$\begin{aligned} f' &= 1 - \left(1 - f_0\right)^{t'/t_0} \\ &= 1 - \left(1 - 0.10\right)^{400/600} \\ &= 0.0678. \end{aligned}$$

With $r = 0$, $f' = 0.0678$, and $\alpha = 0.05$, Equation 9.46 gives

$$b\left(0; n, 0.0678\right) \leq 0.05,$$

which is satisfied by $n = 43$.

From **MINITAB**> **Stat**> **Reliability/Survival**> **Test Plans**> **Demonstration:**




**Example 9.14** How long must ten units be life tested with no more than one failure during the test period to demonstrate, with 80% confidence, that the 3000-hour reliability is at least 90%. Assume that the life distribution is Weibull with $\beta = 1.8$.

**Solution:** The goal of the experiment is to demonstrate that

$$P\left(0.90 < R\left(3000\right) < 1\right) = 0.80$$

or in terms of the failure probability

$$P\left(0 < f\left(3000\right) < 0.10\right) = 0.80.$$

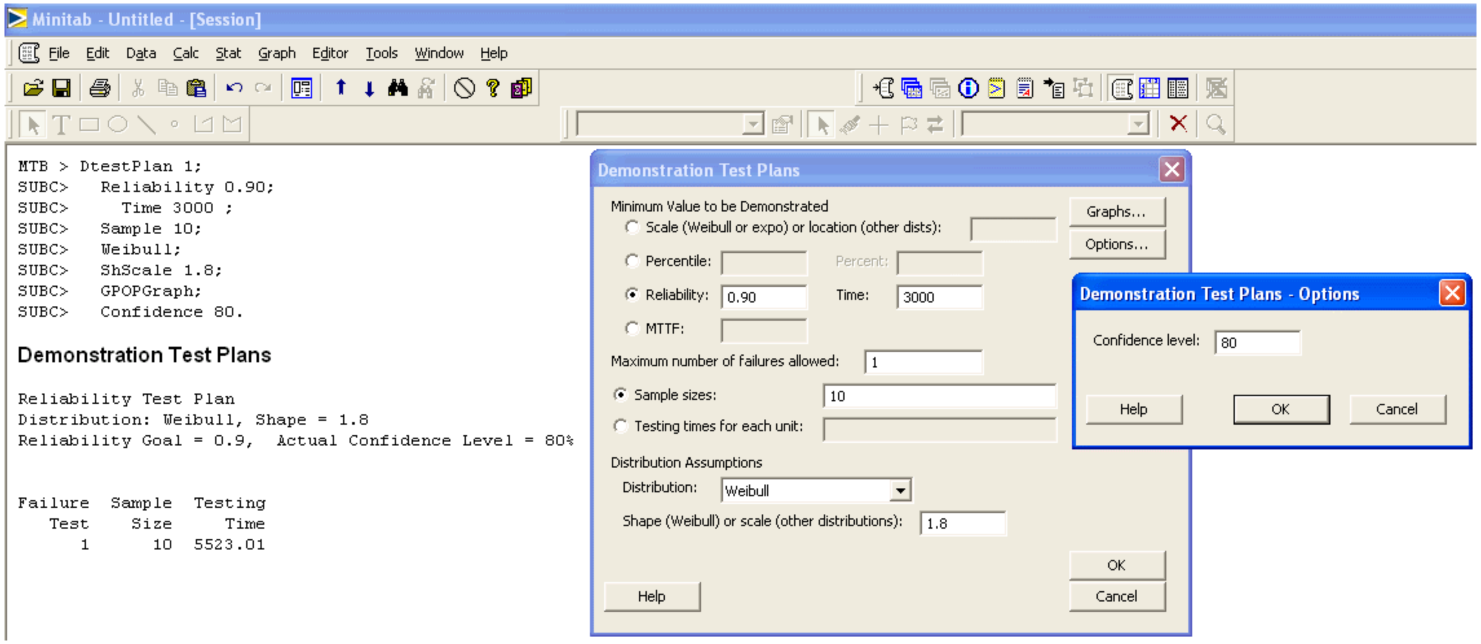
With $n = 10$, $r = 1$, and $\alpha = 0.20$, Equation 9.46 becomes

$$\sum_{x=0}^{1} b\left(x; 10, f'\right) \leq 0.20,$$

which is satisfied by $f' = 0.271$. From the equation for $t'$ for the Weibull distribution from Table 9.2 with $f_0 = 0.10$ and $t_0 = 3000$, the test time is

$$\begin{aligned} t' &= t_0 \left(\frac{ln\left(1-f'\right)}{ln\left(1-f_0\right)}\right)^{1/\beta} \\ &= 3000 \left(\frac{ln\left(1-0.271\right)}{ln\left(1-0.10\right)}\right)^{1/1.8} \\ &= 5523. \end{aligned}$$

From **MINITAB> Stat> Reliability/Survival> Test Plans> Demonstration:**




**Example 9.15** How many units must be tested for 140 hours with no more than one failure to demonstrate that the 100 hour reliability is at least 95% with 90% confidence? Assume that the reliability distribution is normal with $\sigma = 20$.
**Solution:** The goal of the experiment is to demonstrate

$$P(0 < f(100) < 0.05) = 0.90$$

based on a sample of size $n$ tested to $t' = 140$ hours with no more than $r = 1$ failures. From Table 9.2, the equation for $z_{f'}$ for the normal distribution gives

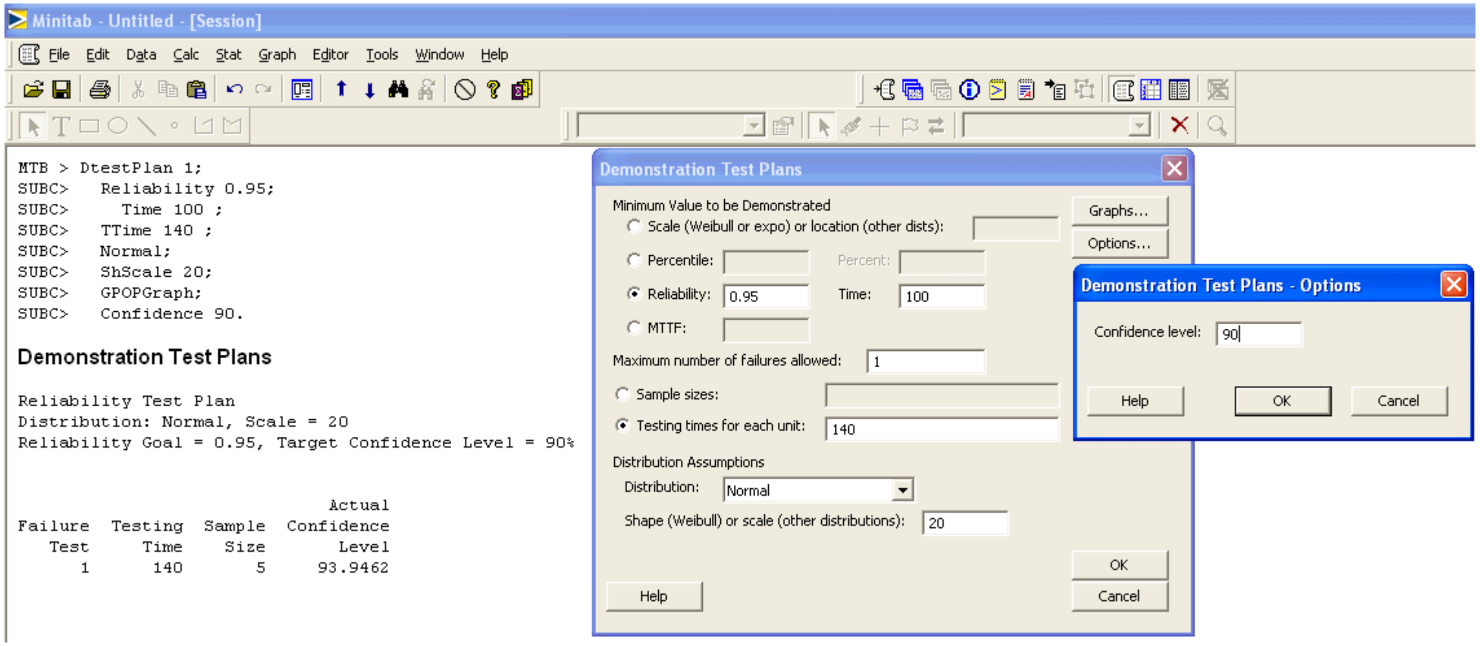
$$\begin{aligned} z_{f'} &= z_{f_0} + \left(\frac{t' - t_0}{\sigma}\right) \\ &= z_{0.05} + \left(\frac{140 - 100}{20}\right) \\ &= -1.645 + 2.0 \\ &= 0.355, \end{aligned}$$

which is satisfied by $f' = \Phi(-\infty < z < 0.355) = 0.639$. With $r = 1$, $f' = 0.639$, and $\alpha = 0.10$, Equation 9.46 becomes

$$\sum_{x=0}^{1} b(x; n, 0.639) \leq 0.10,$$

which is satisfied by $n = 5$.

From **MINITAB**> **Stat**> **Reliability/Survival**> **Test Plans**> **Demonstration:**




**Example 9.16** How many units must be tested without any failures to $t_0$ hours to demonstrate 90% reliability at $t_0$ hours with 95% confidence? Assume that the distribution is Weibull.
**Solution:** The goal of the experiment is to demonstrate

$$P(0 < f(t_0) < 0.10) = 0.95.$$

With $f_0 = 0.10$ and $t' = t_0$ in the Weibull equation for $f'$ from Table 9.2

$$\begin{aligned} f' &= 1 - (1 - 0.10)^{\left(\frac{t_0}{t_0}\right)^{\beta}} \\ &= 0.10. \end{aligned}$$

With $f' = 0.10$, $r = 0$, and $\alpha = 0.05$, Equation 9.46 is

$$b(0; n, 0.10) \le 0.05,$$

which is satisfied by $n = 29$. This is just a case of the rule of three: $n = 3/f_0$.

**Example 9.17** How long must 50 units be tested without any failures to demonstrate that the time at which the first 1% of the population fails exceeds 400 cycles? Assume that the life distribution is exponential and use the 95% confidence level.
**Solution:** The goal of the experiment is to demonstrate that

$$P(400 < t_{0.01} < \infty) = 0.95.$$

From Equation 9.46 with $r = 0$, $n = 50$, and $\alpha = 0.05$

$$b\,(0; 50, f') \leq 0.05,$$

which is satisfied by $f' = 0.058$. From the exponential form of $t'$ from Table 9.2 with $t_0 = 400$, the required duration of the test in cycles is

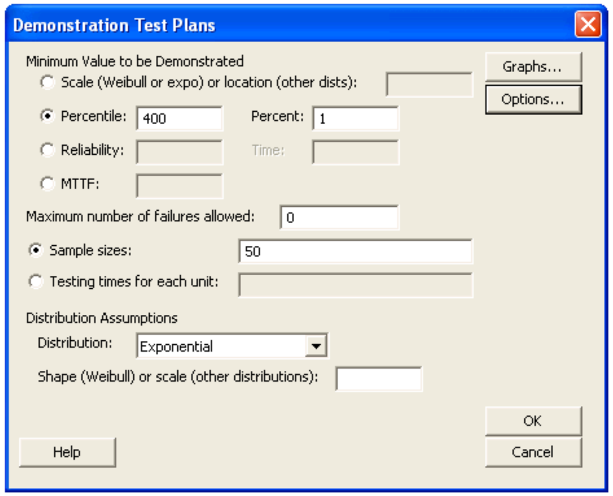
$$\begin{aligned} t' &= t_0 \frac{ln\,(1-f')}{ln\,(1-f_0)} \\ &= 400 \frac{ln\,(1-0.058)}{ln\,(1-0.01)} \\ &= 2380. \end{aligned}$$

From **MINITAB**> **Stat**> **Reliability/Survival**> **Test Plans**> **Demonstration:**

```
MTB > DtestPlan 0;
SUBC>   PTile 1;
SUBC>     Time 400;
SUBC>   Sample 50;
SUBC>   Exponential;
SUBC>   GPOPGraph.
```

**Demonstration Test Plans**

```
Reliability Test Plan
Distribution: Exponential
Percentile Goal = 400,  Actual Confidence Level = 95%


Failure  Sample  Testing
   Test    Size     Time
      0      50  2384.58
```




**Example 9.18** How many units must be tested to 30,000 cycles without any failures to demonstrate, with 95% confidence, that the 20,000 cycle reliability is at least 90%? The life distribution is known to be Weibull with $\beta = 3.1$.
**Solution:** The goal of the experiment is to demonstrate that

$$P\,(20000 < t_{0.10} < \infty) = 0.95.$$

The equation for $f'$ for the Weibull distribution from Table 9.2 with $f_0 = 0.10$, $t_0 = 20000$, $t' = 30000$, and $\beta = 3.1$ gives

$$\begin{aligned} f' &= 1-(1-f_0)^{(t'/t_0)^{\beta}} \\ &= 1-(1-0.10)^{(30000/20000)^{3.1}} \\ &= 0.3095. \end{aligned}$$

Then, with $r = 0$, $f' = 0.3095$, and $\alpha = 0.05$, Equation 9.46 gives

$$b(0; n, 0.3095) \leq 0.05,$$

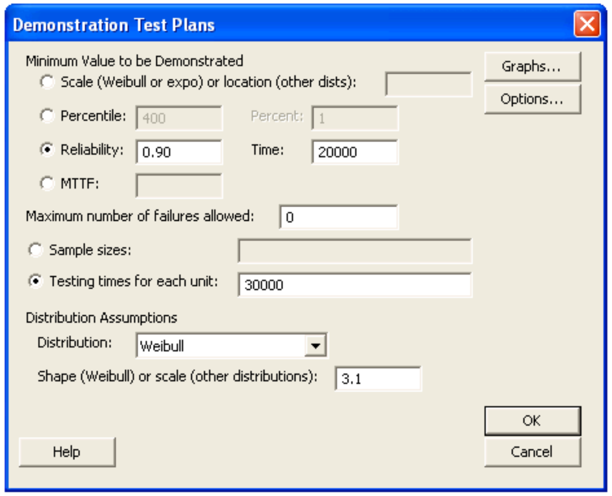
which is satisfied by $n = 9$.

From **MINITAB**> **Stat**> **Reliability/Survival**> **Test Plans**> **Demonstration:**

```
MTB > DtestPlan 0;
SUBC>    Reliability 0.90;
SUBC>      Time 20000 ;
SUBC>    TTime 30000 ;
SUBC>    Weibull;
SUBC>    ShScale 3.1;
SUBC>    GPOPGraph.
```

**Demonstration Test Plans**

```
Reliability Test Plan
Distribution: Weibull, Shape = 3.1
Reliability Goal = 0.9, Target Confidence Level = 95%


                                     Actual
Failure  Testing  Sample  Confidence
   Test     Time    Size       Level
      0    30000       9     96.4305
```




## 9.3 Two-Sample Reliability Tests

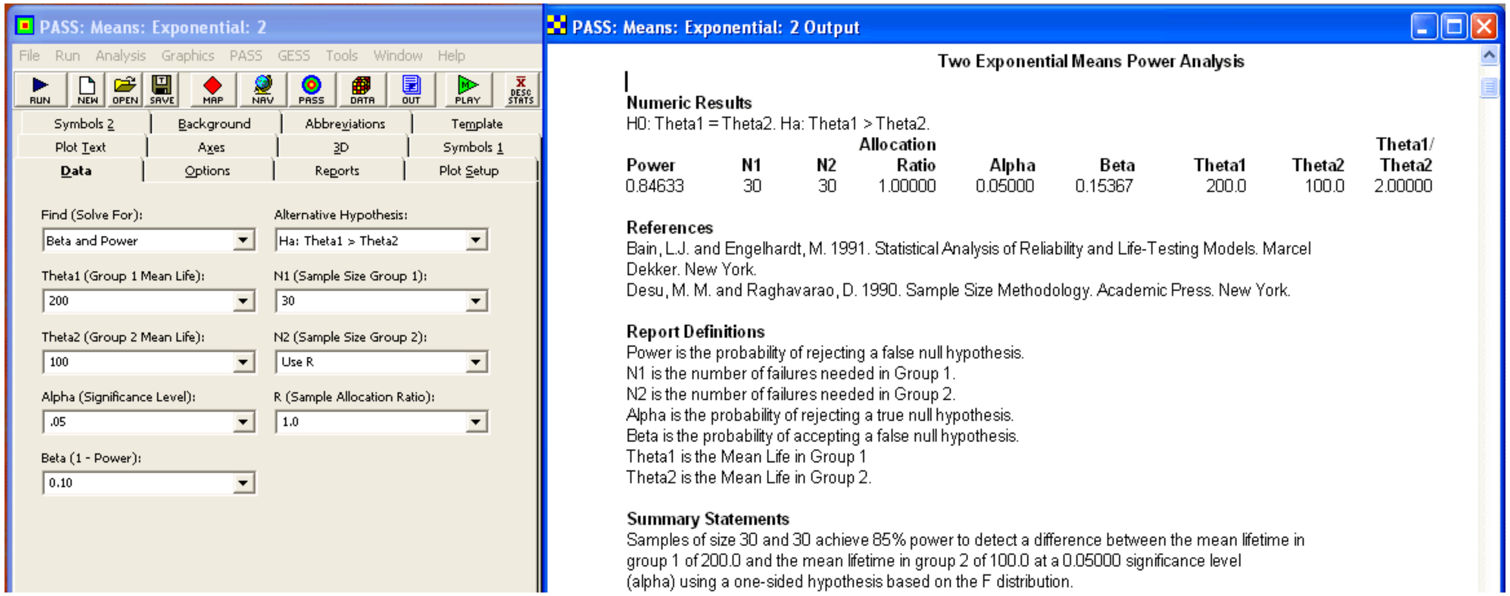
**Example 9.19** A reliability experiment is to be performed to compare the mean life of two different product designs. Determine the power to reject $H_0 : \mu_1 = \mu_2$ in favor of $H_A : \mu_1 > \mu_2$ when $\mu_1 = 200$ hours and $\mu_2 = 100$ hours using two different strategies: a) $n_1 = n_2 = 30$ units, all tested to failure and b) $n_1 = 40$, $n_2 = 20$, and the test will be suspended when 90% of the units from one of the two designs have failed. Assume that both life distributions are exponential

**Solution:**

a) With $n_1 = n_2 = 30$ units tested to failure, the $F$ test critical value will be $F_{0.95,60,60} = 1.534$ and by Equation 9.55 the power will be

$$\begin{aligned}\pi &= P\left(\left(\frac{100}{200} \times 1.534\right) < F < \infty\right) \\ &= P(0.767 < F < \infty) \\ &= 0.846\end{aligned}$$

From **PASS**> **Means**> **Two**> **Independent**> **Inequality (Exponential)**:




b) Under $H_A$ in the second strategy, the second treatment group has fewer units with lower mean life so they should be exhausted first. The time at which 90% or 18 of these units will have failed is expected to be about $t = -100 \ln(0.1) = 230$ hours. At the same time about $40\left(1 - e^{-230/200}\right) = 27$ of the units from the first treatment group are expected to fail. If the test is suspended then, with $x_1 = 27$ and $x_2 = 18$, then the power will be

$$
\begin{aligned}
\pi &= P\left(\left(\frac{100}{200} \times F_{0.95,54,36}\right) < F < \infty\right) \\
&= P(0.842 < F < \infty) \\
&= 0.721.
\end{aligned}
$$

Under the second strategy, the test will end much earlier (i.e., when the 18th unit with 100 hour mean life fails versus when the 30th unit with 200 hour mean life fails); however, at the penalty of reduced experimental power.

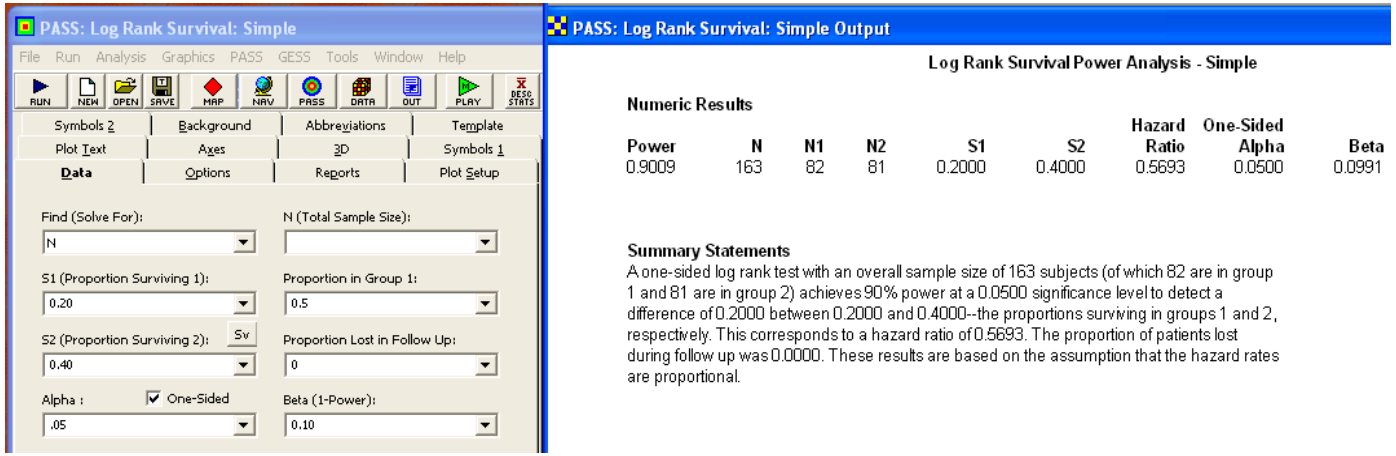
**Example 9.20** Determine how many units must be included in a study to compare the survival rates of two treatments using the log-rank test if the control treatment is expected to have about 20% survivors at the end of the study and the study should have 90% power to reject $H_0$ if the experimental treatment has 40% survivors at the end of the study. Assume that the hazard rates are proportional and that the sample sizes will be equal.
**Solution:** From the expected end-of-study conditions under $H_A$ the log-hazard ratio is estimated to be

$$r_A \simeq \frac{\ln(0.40)}{\ln(0.20)} = 0.5693,$$

so the required sample size is

$$n_1 = n_2 = \left(\frac{z_{0.05} + z_{0.10}}{\ln(0.5693)}\right)^2 \left(\frac{1}{1-0.2} + \frac{1}{1-0.4}\right) = 79.$$

From **PASS**> **Survival and Reliability**> **Log-Rank Survival (Simple)**:




**Example 9.21** Compare the power of the log-rank test to the power of the two-sample test for exponential mean life for Example 9.19b.
**Solution:** Because the hazard rate of an exponential distribution is constant, the proportional hazards assumption is satisfied. At 230 hours with $s_1(t') = 2/20 = 0.10$ and $s_2(t') = 13/40 = 0.325$ the hazard ratio under $H_A$ will be

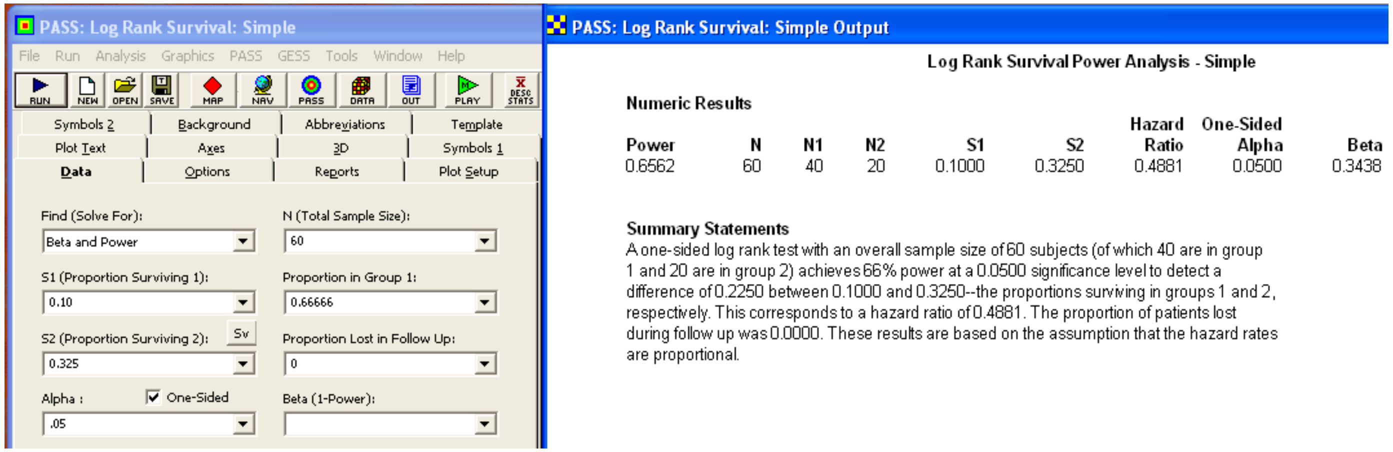
$$r_A = \frac{\ln(0.325)}{\ln(0.10)} = 0.488.$$

With $n_2/n_1 = 2$, $d_1(t') = 18$, and $d_2(t') = 27$, the $z_\beta$ value from Equation 9.64 is

$$z_\beta = \frac{1 - 0.488}{1 + 2(0.488)}\sqrt{2(18+27)} - 1.96 = 0.50.$$

Then the power for the log-rank test is $\pi = \Phi(-\infty < z < 0.50) \simeq 0.69$, which is slightly less than the power for the two-sample exponential test for mean life, which was $\pi = 0.72$. The two-tailed test was used here to match the power obtained in Example 9.19.

From **PASS**> **Survival and Reliability**> **Log-Rank Survival (Simple)**:




**Example 9.22** Compare the sample size calculated by Lachin's method to that of Schoenfeld's method in Example 9.20.
**Solution:** From the information given in the problem statement and Equation 9.66 the sample size by Lachin's method must be

$$n_1 = n_2 = \frac{(1.645 + 1.282)^2}{2 - 0.2 - 0.4} \left( \frac{1 + 0.5693}{1 - 0.5693} \right)^2 = 82,$$

which is in good agreement with Schoenfeld's method, $n = 79$.

See the NCSS/PASS solution shown in Example 9.20. The manual calculation by Schoenfeld's method is in excellent agreement with PASS which uses the same method.

## 9.4 Interference

**Example 9.23** A random sample of component strengths gave $n_S = 100$, $\widehat{\mu}_S = 600$, and $\widehat{\sigma}_S = 60$ and a random sample of loads gave $n_L = 36$, $\widehat{\mu}_L = 450$, and $\widehat{\sigma}_L = 40$. Both distributions are known to be normal. Determine the 90% upper confidence limit for the interference failure rate.
**Solution:** The point estimate for $\widehat{z}_f$ is given by Equation 9.73:

$$\widehat{z}_f = \frac{450 - 600}{\sqrt{40^2 + 60^2}} = -2.08$$

and the corresponding point estimate for the interference failure rate is

$$\widehat{f} = \Phi\left(-\infty < z < -2.08\right) = 0.0188.$$

The approximate standard deviation of the $\widehat{z}_f$ distribution is given by Equation 9.74:

$$\widehat{\sigma}_{\widehat{z}_f} = \sqrt{\frac{1}{40^2 + 60^2} \left( \frac{40^2}{36} + \frac{60^2}{100} + \frac{1}{2} \left( \frac{450 - 600}{40^2 + 60^2} \right)^2 \left( \frac{40^4}{36} + \frac{60^4}{100} \right) \right)} = 0.178.$$

Then, from Equation 9.76 with $z_{0.10} = 1.282$

$$\begin{aligned} \widehat{z}_{f_U} &= -2.08 + 1.282 \times 0.178 \\ &= -1.85, \end{aligned}$$

so from Equation 9.75 the 90% upper confidence limit for the interference failure probability is

$$\begin{aligned} \widehat{f}_U &= \Phi\left(-\infty < z < -1.85\right) \\ &= 0.032. \end{aligned}$$

That is, on the basis of the sample data, we can claim that the one-sided upper 90% confidence interval for the interference failure rate is

$$P\left(0 < f < 0.032\right) = 0.90.$$

**Example 9.24** What sample size is required to demonstrate that the interference failure probability is less than 0.1% with 90% confidence if the strength distribution is known to be normal with $\mu_S = 20$ and $\sigma_S = 2$ and the load distribution is expected to be normal with $\widehat{\mu}_L = 13$ and $\widehat{\sigma}_L = 1$?
**Solution:** The point estimate for the interference failure probability determined from the $S$ parameters and the $L$ parameter estimates is

$$\begin{aligned} \widehat{f} &= \Phi\left(-\infty < z < \widehat{z}_f\right) \\ &= \Phi\left(-\infty < z < \frac{13-20}{\sqrt{1^2+2^2}}\right) \\ &= \Phi\left(-\infty < z < -3.13\right) \\ &= 0.000874. \end{aligned}$$

The sample size required to study the load distribution is given by swapping the relevant $S$ and $L$ subscripts in Equation 9.78:

$$\begin{aligned} n_L &= \left(\frac{z_\alpha}{\widehat{z}_{f_U} - \widehat{z}_f}\right)^2 \left(\frac{\widehat{\sigma}_L^2}{\widehat{\sigma}_L^2 + \sigma_S^2}\right)\left(1 + \frac{\widehat{\sigma}_L^2}{2}\left(\frac{\widehat{\mu}_L - \mu_S}{\widehat{\sigma}_L^2 + \sigma_S^2}\right)^2\right) \qquad (9.1) \\ &= \left(\frac{z_{0.10}}{z_{0.001} - z_{0.000874}}\right)^2 \left(\frac{1^2}{1^2+2^2}\right)\left(1 + \frac{1^2}{2}\left(\frac{13-20}{1^2+2^2}\right)^2\right) \\ &= \left(\frac{1.282}{-3.09 - (-3.13)}\right)^2 \left(\frac{1^2}{1^2+2^2}\right)\left(1 + \frac{1^2}{2}\left(\frac{13-20}{1^2+2^2}\right)^2\right) \\ &= 407. \end{aligned}$$

**Example 9.25** What sample size is required to determine the 95% two-sided confidence interval for the exponential-exponential interference failure rate if the confidence limits must be within 50% of the predicted mean failure rate?
**Solution:** The goal of the experiment is to obtain a confidence interval for the exponential-exponential interference failure rate $f$ of the form

$$\Phi\left(0.50\widehat{f} < f < 1.50\widehat{f}\right) = 0.95.$$

With $z_{0.025} = 1.96$ and $\delta = 0.50$ in Equation 9.90, the required equal sample sizes are

$$n_L = n_S = 2\left(\frac{1.96}{0.50}\right)^2 = 31.$$

**Example 9.26** How many measurements of mating components in a device must be taken to demonstrate, with 95% confidence, that their true interference failure rate does not exceed the observed failure rate by 20% if the two distributions are known to be Weibull with $\beta_S = 2.5$ and $\beta_L = 1.5$, respectively?
**Solution:** The goal of the experiment is to acquire sufficient information to demonstrate the following one-sided upper confidence interval for the interference failure rate $f$:

$$P\left(0 < f < \widehat{f}\,(1 + 0.2)\right) = 0.95.$$

With $\delta = 0.2$ and $\alpha = 0.05$ in Equation 9.100, we obtain the sample size

$$\begin{aligned} n &= \left(\frac{1.645}{0.2 \times \Gamma\left(1 + \frac{2.5}{1.5}\right)}\right)^2 \left(1 + \frac{2.5^2}{1.5^2}\right) \\ &= 113. \end{aligned}$$

# Chapter 10

# Statistical Quality Control

## 10.1 Statistical Process Control

**Example 10.1** Evaluate the following control chart run rule: A process is judged to be out of control if at least two of three consecutive observations falls beyond the same $2\sigma$ limit on the chart.
**Solution:** The rule is easy to identify on the chart, so it satisfies the first condition for a valid run rule. If the process is in control and the distribution of the statistic (call it $w$) is approximately normal, then the probability that any point on the chart falls above $\mu_w + 2\sigma_w$ is $p = \Phi(2 < z < \infty) = 0.023$. The probability that at least $x = 2$ of $n = 3$ consecutive points fall above that limit is given by the binomial probability

$$\sum_{x=2}^{3} b(x; n = 3, p = 0.023) = 0.0016.$$

Because this pattern could also appear on the bottom half of the chart, the type I error rate for this rule is $\alpha = 2(0.0016) = 0.0032$, which is acceptably low, so the rule meets the second requirement for a valid control chart rule. If the process mean shifted to $\mu_w + 2\sigma_w$, then the probability that an observation would fall beyond $\mu_w + 2\sigma_w$ is $p = 0.5$ and the corresponding power of the rule is

$$\pi = \sum_{x=2}^{3} b(x; n = 3, p = 0.5) = 0.5.$$

This meets the third requirement of a valid control chart rule. Because all three conditions are satisfied, that is: 1) the rule is easy to recognize, 2) it has a low type I error rate, and 3) it has good power to detect shifts in the process, then it is a valid control chart run rule.

**Example 10.2** One of the weaknesses of defects charts when the sampling unit is small is that it is not possible to declare a process to be out of control on the lower side of the chart with a single observation. Evaluate the following special run rule for defects charts: If a defects chart's sampling unit size is sufficient to deliver $\lambda \geq 3$, then the process is out of control if two consecutive sampling units have 0 defects.
**Solution:** The chart obviously meets the first and third conditions for valid control chart run rules, but it is not clear if the second condition (low type I error rate) is satisfied. If the mean defect rate is $\lambda = 3$, then the probability of a sampling unit having 0 defects when the process is in control $(H_0 : \lambda = 3)$ is $Poisson(x = 0; \lambda = 3) = 0.05$, a rather common occurrence. Under the same conditions, the probability of observing two consecutive zeros is $b(x = 2; n = 2, p = 0.05) = 0.0025$, but this is just the type I error rate for the rule. Because $\alpha = 0.0025$ is acceptably low, the rule meets all three conditions for a valid control chart run rule.

**Example 10.3** When Walter Shewhart invented control charts, he expected that an operator would be using about four run rules to interpret at most three control charts. If each of Shewhart's run rules had $\alpha_i \simeq 0.004$, what is the expected overall type I error rate?
**Solution:** From Equation 10.4

$$\alpha_{FAMILY} \simeq 3 \times 4 \times 0.004 = 0.048.$$

That is, with three simultaneous charts and four run rules, Shewhart expected about 5% of the sampling intervals to result in type I errors.

**Example 10.4** What is the minimum sample size required to have a positive lower control limit on a defectives chart if the process fraction defective is expected to be $p = 0.01$?
**Solution:** From Equation 10.6 the required sample size is

$$\begin{aligned} n &> \left(\frac{9(1-0.01)}{0.01}\right) \\ &> 891. \end{aligned}$$

**Example 10.5** What is the smallest sampling unit size for a defects chart that will deliver no more than about 5% zero-defect observations when the process delivers 0.6 defects per unit?
**Solution:** From Equation 10.13 the mean number of defects per sampling unit is the value of $\lambda$ that satisfies the condition

$$Poisson\,(x = 0; \lambda) = 0.05, \tag{10.1}$$

which is $\lambda = 3$. Consequently, the sampling unit size must be $3/0.6 = 5$ units.

**Example 10.6** Calculate the power to reject $H_0 : \mu = 30$ when $\mu = 32$ if an $\bar{x}$ chart is kept using $n = 4$ and $\sigma_x = 2$. Also determine the corresponding $ARL$.
**Solution:** The $\bar{x}$ chart control limits will fall at

$$UCL/LCL = 30 \pm \frac{3 \times 2}{\sqrt{4}} = 33/27.$$

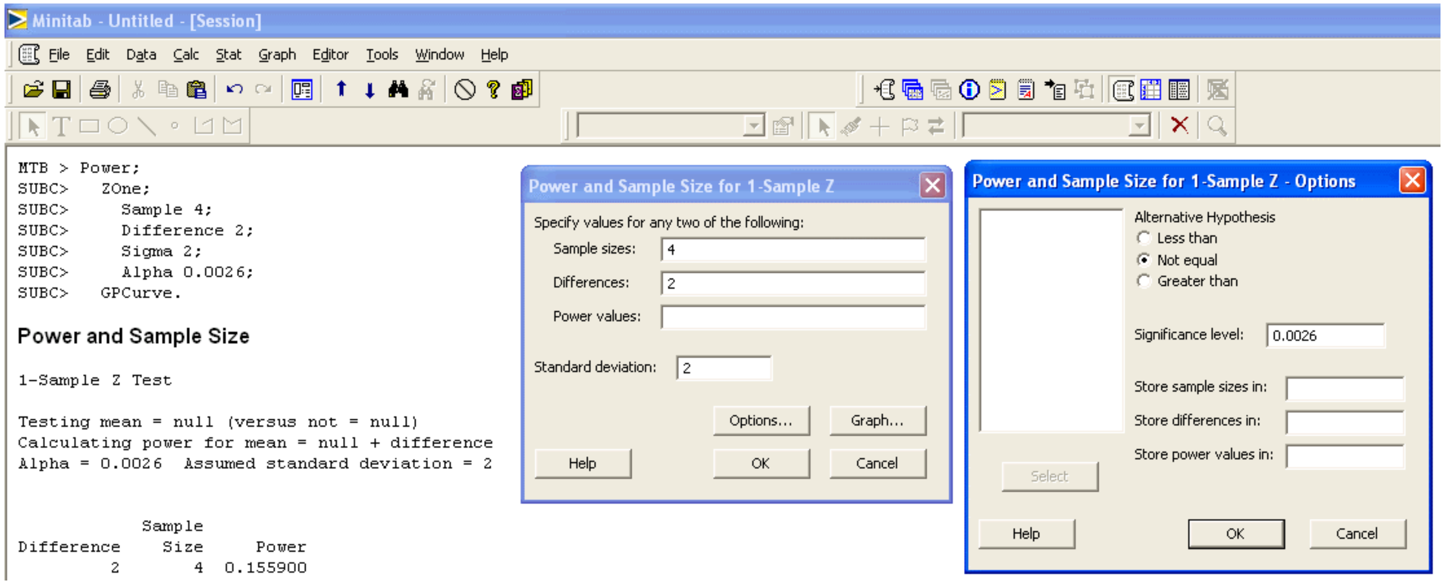
Assuming that the only out-of-control rule used is one point beyond three sigma limits, the power is given by Equation 10.18

$$\begin{aligned} \pi &= 1 - \Phi\left(27 < \bar{x} < 33; \mu_x = 32, \sigma_{\bar{x}} = \frac{2}{\sqrt{4}} = 1\right) \\ &= 1 - \Phi\,(-5 < z < 1) \\ &= 0.16. \end{aligned}$$

Under the same conditions, the average number of subgroups that will have to be drawn after a shift from $\mu = 30$ to $\mu = 32$ to detect the shift is

$$ARL = \frac{1}{0.16} = 6.3.$$

From **MINITAB**> **Stat**> **Power and Sample Size**> **1-Sample Z**:

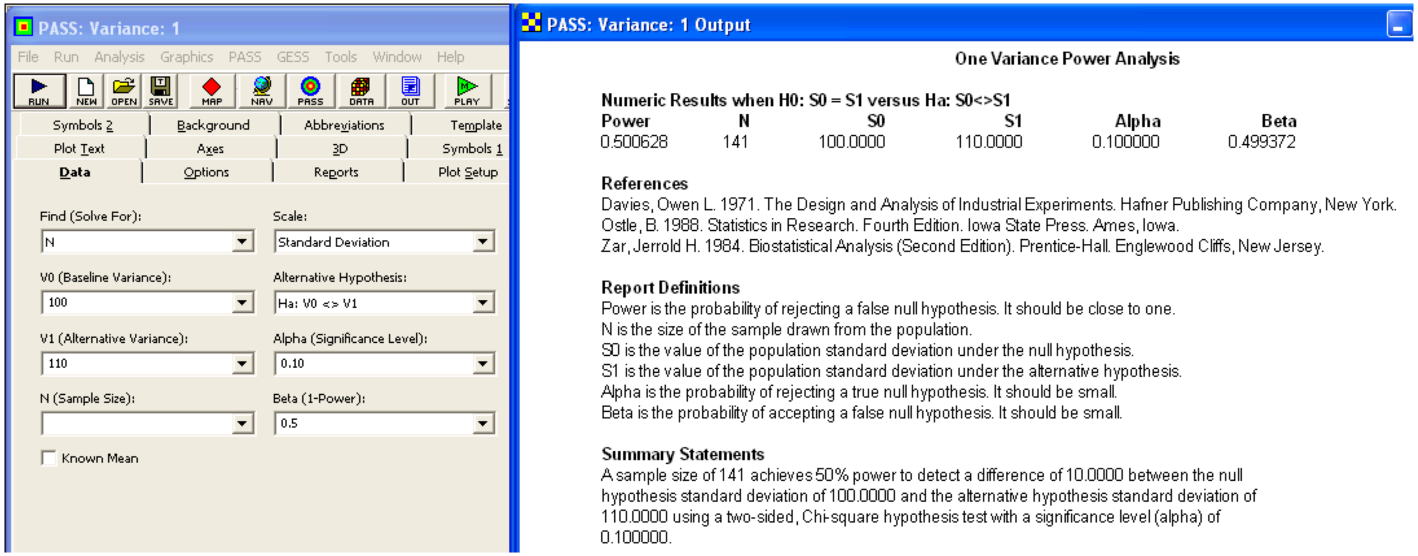



## 10.2 Process Capability

**Example 10.7** What sample size is required to determine $c_p$ to within 10% of its true value with 90% confidence?
**Solution:** With $\delta = 0.10$ and $\alpha = 0.10$ in Equation 10.26 the required sample size is

$$n = \frac{1}{2}\left(\frac{1.645}{0.10}\right)^2 = 136.$$

The $c_p$ value is inversely proportional to the standard deviation, so a standard deviation calculator can be used to determine the sample size required for a confidence interval for $c_p$. From **PASS**> **Variance**> **Variance: 1 Group**:




**Example 10.8** What sample size is required to estimate $c_{pk}$ to within 5% of its true value with 90% confidence if $c_{pk} = 1.0$ is expected?
**Solution:** From Equation 10.30 with $\delta = 0.05$ and $\alpha = 0.05$ the required sample size is

$$n \simeq \left(\frac{1.645}{0.05}\right)^2 \left(\frac{1}{9\,(1.0)^2} + \frac{1}{2}\right) = 662.$$

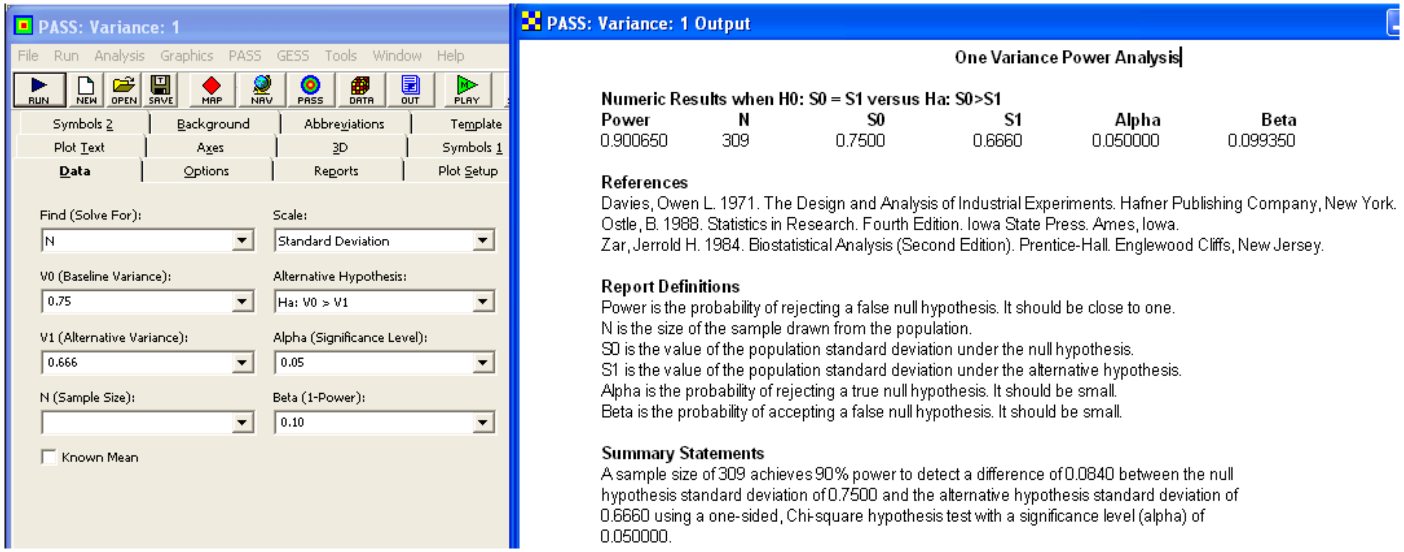
**Example 10.9** What sample size is required to estimate $c_{pk}$ to within 5% of its true value with 90% confidence if $c_{pk}$ is expected to be very large?
**Solution:** From Equation 10.31 with $\delta = 0.05$ and $\alpha = 0.05$ the required sample size is

$$n \simeq \frac{1}{2}\left(\frac{1.645}{0.05}\right)^2 = 541.$$

**Example 10.10** Determine the sample size required to reject $H_0 : c_p = 1.33$ in favor of $H_A : c_p > 1.33$ with 90% power when $c_p = 1.5$.
**Solution:** With $(c_p)_0 = 1.33$, $(c_p)_1 = 1.5$, $\alpha = 0.05$, and $\beta = 0.10$ in Equation 10.33, the required sample size is

$$n \simeq \frac{1}{2}\left(\frac{1.645 + 1.282}{\ln\left(\frac{1.5}{1.33}\right)}\right)^2 = 297.$$

The hypothesis test for $c_p$ can be performed using a sample size calculator for the standard deviation. By setting the standard deviations to the reciprocals of $c_p$ in **PASS**> **Variance**> **Variance: 1 Group**:




**Example 10.11** Determine the sample size required to reject $H_0 : c_{pk} = 1.33$ in favor of $H_A : c_{pk} > 1.33$ with 90% power when $c_{pk} = 1.5$.
**Solution:** With $(c_{pk})_0 = 1.33$, $(c_{pk})_1 = 1.5$, $\alpha = 0.05$, and $\beta = 0.10$ in Equation 10.35, the sample size required to reject $H_0$ is

$$\begin{aligned} n &= \left( \frac{1.645\,(1.33)\sqrt{\frac{1}{9\times 1.33^2}+\frac{1}{2}}+1.282\,(1.5)\sqrt{\frac{1}{9\times 1.5^2}+\frac{1}{2}}}{1.5-1.33} \right)^2 \\ &= 326. \end{aligned}$$

As expected, this value is comparable to the $n = 297$ sample size required for the test of $c_p$ determined in Example 10.10 for similar conditions.

**Example 10.12** Determine the sample size for Example 10.11 using the large sample approximation and compare the result to the original sample size.
**Solution:** From the information given in the original problem statement and Equation 10.36 the approximate sample size is

$$\begin{aligned} n &\simeq \frac{1}{2}\left( \frac{1.645\,(1.33)+1.282\,(1.5)}{1.5-1.33} \right)^2 \\ &\simeq 292. \end{aligned}$$

This value is about 10% lower than the more accurate value calculated in Example 10.11.

## 10.3 Tolerance Intervals

**Example 10.13** What sample size is required to be 95% confident that at least 99% of a population of continuous measurement values falls within the extreme values of the sample?

**Solution:** With $\alpha = 0.05$ and $p_U = 0.01$ the required sample size is approximately

$$\begin{aligned} n &\simeq \frac{\chi^2_{0.95,4}}{2 \times 0.01} \\ &\simeq 475. \end{aligned}$$

Further iterations indicate that the smallest value of $n$ for which $\alpha \leq 0.05$ is $n = 473$, which leads to the following nonparametric tolerance interval for $x$:

$$P\left(0.99 < P\left(x_{min} \leq x \leq x_{max}\right) < 1\right) = 0.9502.$$

**Example 10.14** What sample size $n$ is required to be 95% confident that at least 99% of a population of continuous measurement values falls below the maximum value of the sample?
**Solution:** With $\alpha = 0.05$ and $p_U = 0.01$ the required sample size is

$$\begin{aligned} n &\simeq \frac{\chi^2_{0.95,2}}{2 \times 0.01} \\ &\simeq 300. \end{aligned}$$

**Example 10.15** Determine the sample size required to obtain a 95% confidence two-sided 99% coverage normal distribution tolerance interval with tolerance limits $UTL/LTL = \bar{x} \pm 3.5s$.
**Solution:** The desired tolerance interval has the form

$$P\left(0.99 \leq \Phi\left(\bar{x} - 3.5s \leq x \leq \bar{x} + 3.5s\right) \leq 1\right) = 0.95.$$

From Appendix E.7, as sample of size $n = 25$ gives $k_2 = 3.46$. A spreadsheet (not shown) was set up to calculate $k_2$ as a function of $n$ using Equation 10.47 with $p = 0.01$ and $\alpha = 0.05$. The spreadsheet indicated that the sample size $n = 24$ delivers $k_2 = 3.485$ and that $n = 23$ delivers $k_2 = 3.514$, so $n = 24$ should be used to be conservative. These approximate $k_2$ values differ from the exact values given in Appendix E.7 in the thousandths place.

**Example 10.16** Determine the sample size required to obtain a 95% confidence 99% coverage normal distribution tolerance interval with one-sided upper tolerance limit $UTL = \bar{x} + 3s$.
**Solution:** The required interval has the form

$$P\left(0.99 < \Phi\left(-\infty < x \leq UTL\right) < 1\right) = 0.95 \tag{10.2}$$

where $UTL = \bar{x} + k_1 s$ with $k_1 = 3$. From Table E.7 of Appendix E with $\alpha = 0.05$ and $Y = 0.99$, the required sample size is $n = 35$.

## 10.4 Acceptance Sampling

**Example 10.17** Design the single sampling plan for attributes that will accept 95% of lots when the process fraction defective is 1% and accept only 10% of lots when the process fraction defective is 5%.
**Solution:** From the problem statement, the lots are coming from a continuous process, so the sampling plan will be Type B with points on the OC curve at $(AQL, 1 - \alpha) =$

$(0.01, 0.95)$ and $(RQL, \beta) = (0.05, 0.10)$. From Table 10.1 with $RQL/AQL = 0.05/0.01 = 5.0$, the acceptance number must be $c = 3$. Then, from the RQL condition, the required sample size is approximately

$$\begin{aligned} n &\simeq \frac{\chi^2_{0.90,8}}{2\,(0.05)} \\ &\simeq \frac{13.36}{2 \times 0.05} \\ &\simeq 134. \end{aligned}$$

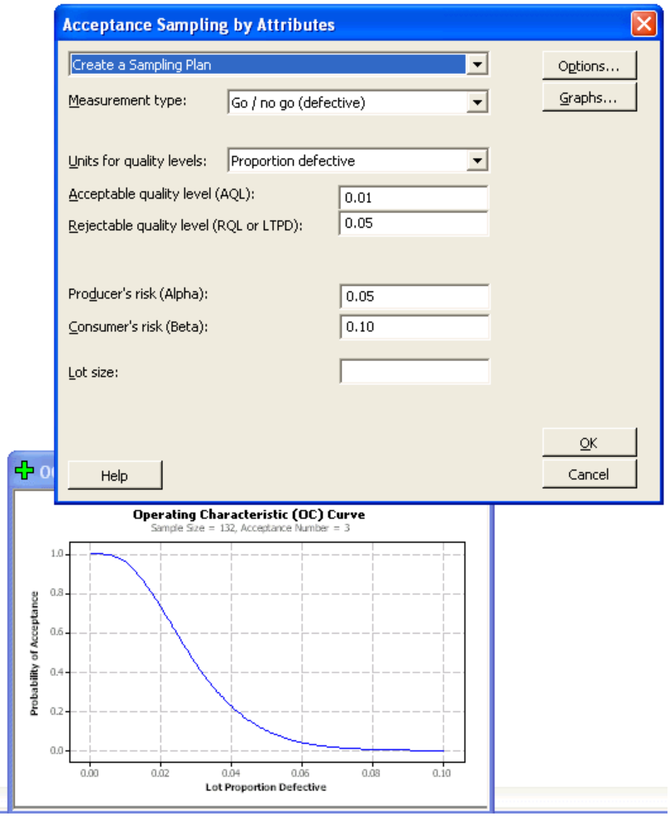
The exact sampling plan that meets the specifications in the problem statement is $n = 132$ and $c = 3$.

From **MINITAB**> **Stat**> **Quality Tools**> **Acceptance Sampling by Attributes**:

```
MTB > AASAMPLING 1;
SUBC>   AQL 0.01;
SUBC>   RQL 0.05;
SUBC>   CREATE;
SUBC>     ALPHA 0.05;
SUBC>     BETA 0.10;
SUBC>   PROPORTION;
SUBC>   GOC.
```

**Acceptance Sampling by Attributes**

```
Measurement type:  Go/no go
Lot quality in proportion defective
Use binomial distribution to calculate probability of acceptance


Acceptable Quality Level (AQL)          0.01
Producer's Risk (Alpha)                 0.05

Rejectable Quality Level (RQL or LTPD)  0.05
Consumer's Risk (Beta)                  0.1


Generated Plan(s)

Sample Size         132
Acceptance Number     3

Accept lot if defective items in 132 sampled <= 3;  Otherwise reject.


Proportion  Probability  Probability
 Defective    Accepting    Rejecting
      0.01        0.956        0.044
      0.05        0.099        0.901
```




**Example 10.18** Find the $c = 0$ plans that meet a) the $AQL$ requirement and b) the $RQL$ requirement from Example 10.17. Plot the three OC curves on the same graph.
**Solution:**

a) The sample size for the $c = 0$ plan that meets the $AQL$ requirement $(p, P_A) = (0.01, 0.95)$ is approximately

$$
\begin{aligned}
n &\simeq \frac{\chi^2_{\alpha,2}}{2 \times AQL} \\
&\simeq \frac{0.1026}{2 \times 0.01} \\
&\simeq 6.
\end{aligned}
$$

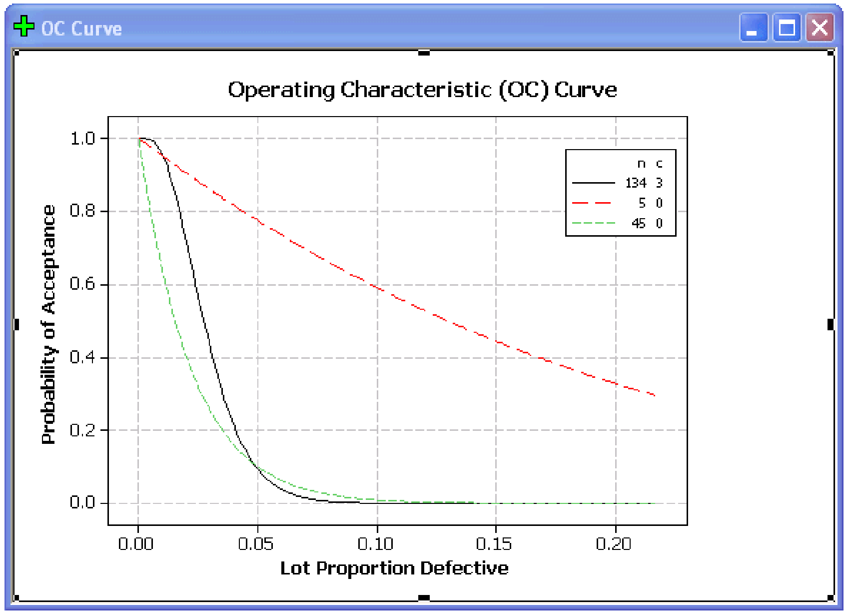
A few binomial calculations indicate that the exact sample size is $n = 5$ because $(b\,(0; 5, 0.01) = 0.951) > (1 - \alpha = 0.95)$.

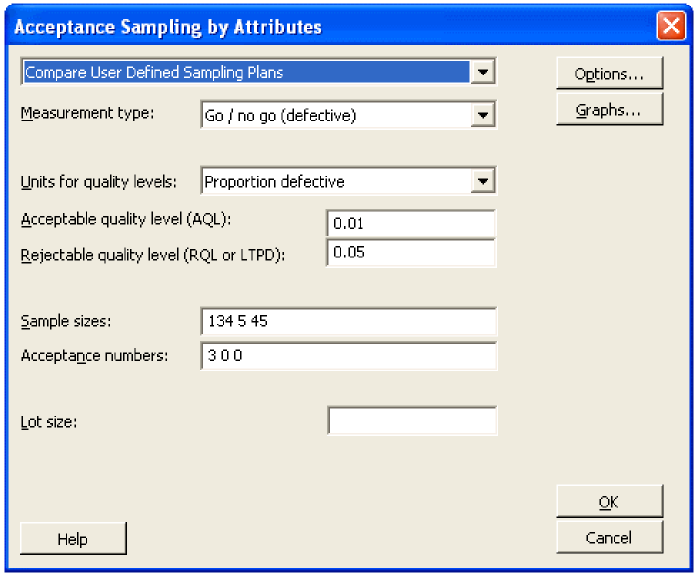
b) The sample size for the $c = 0$ plan that meets the $RQL$ requirement $(p, P_A) = (0.05, 0.10)$ is approximately

$$
\begin{aligned}
n &\simeq \frac{\chi^2_{1-\beta,2}}{2 \times RQL} \\
&\simeq \frac{4.61}{2 \times 0.05} \\
&\simeq 46.
\end{aligned}
$$

The exact sample size is $n = 45$ because $(b\,(0; 45, 0.05) = 0.099) < (\beta = 0.10)$.

From **MINITAB**> **Stat**> **Quality Tools**> **Acceptance Sampling by Attributes**:



**Example 10.19** Determine the sampling plan for lots of size $N = 50$ that will accept 95% of the lots with $D \leq 1$ defectives and reject 90% of the lots with $D \geq 5$ defectives.
**Solution:** The sampling plan must meet the simultaneous conditions given by Equation 10.57 with $D_1 = 1$ and $\alpha \leq 0.05$:

$$\sum_{x=0}^{c} h\left(x; D_1 = 1, N = 50, n\right) \geq 0.95 \tag{10.3}$$

and Equation 10.58 with $D_2 = 5$ and $\beta \leq 0.10$:

$$\sum_{x=0}^{c} h\left(x; D_2 = 5, N = 50, n\right) < 0.10. \tag{10.4}$$

The acceptance number $c$ is not specified, so different values of $c$ must be considered. The approximate sample size for the $c = 0$ sampling plan to meet the condition in Equation 10.63 is given by Equation 10.61:

$$n \simeq 50\left(1 - 0.10^{1/5}\right) = 19;$$

however, the condition in Equation 10.62 is not satisfied because

$$\left(h\left(x = 0; D_1 = 1, N = 50, n = 19\right) = 0.525\right) \ngeq 0.95.$$

Iterations with a hypergeometric calculator show that with $c = 1$ Equation 10.63 is satisfied when $n = 29$ because

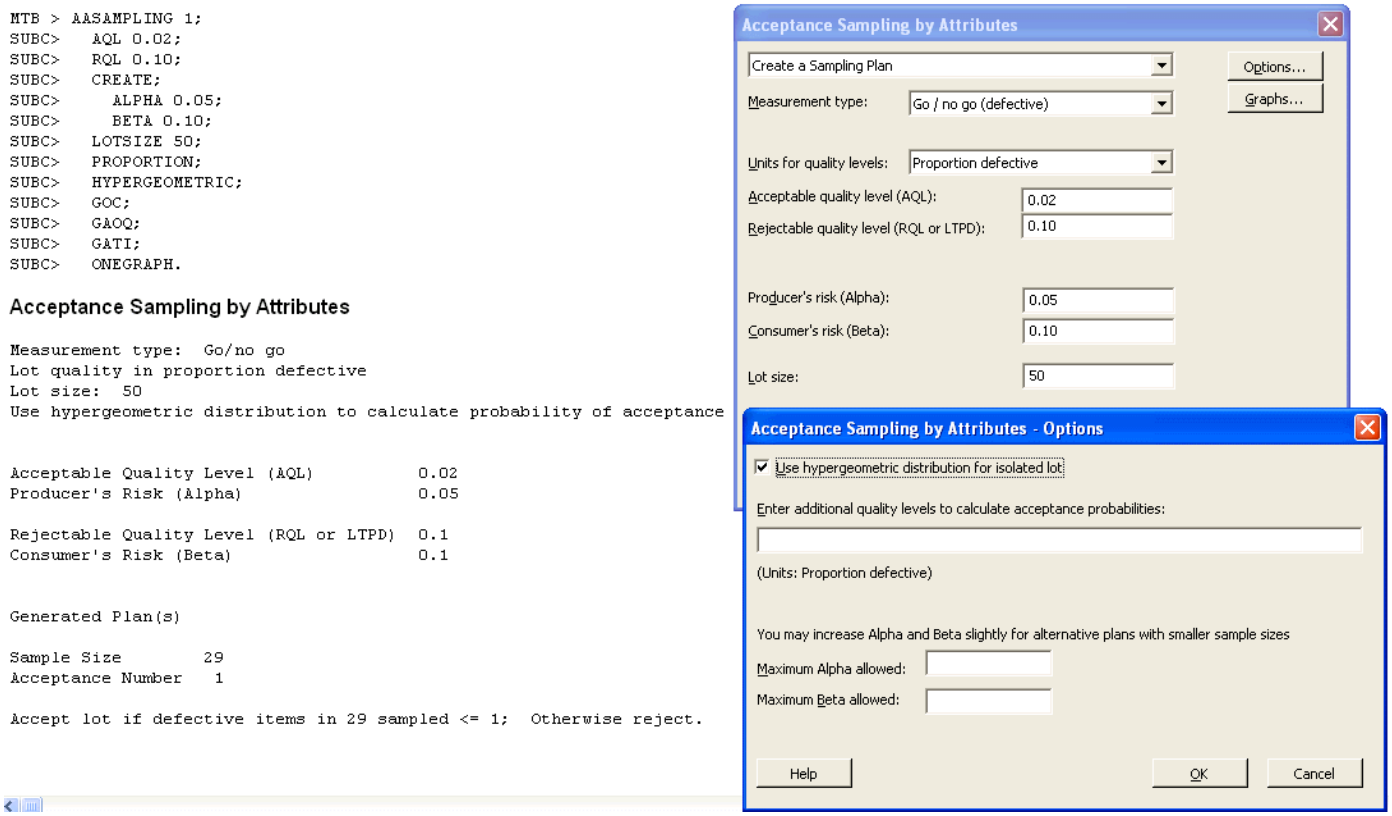
$$\begin{aligned}
\left(\sum_{x=0}^{1} h\left(x; D_1 = 5, N = 50, n = 28\right) = 0.109\right) &\nleq 0.10 \\
\left(\sum_{x=0}^{1} h\left(x; D_1 = 5, N = 50, n = 29\right) = 0.092\right) &\leq 0.10
\end{aligned}$$

and Equation 10.62 is satisfied because

$$\left(\sum_{x=0}^{1} h\left(x; D_1 = 1, N = 50, n = 29\right) = 1\right) \geq 0.95.$$

The sampling plan that meets the requirements is $n = 29$ with $c = 1$.

From **MINITAB**> **Stat**> **Quality Tools**> **Acceptance Sampling by Attributes**:




**Example 10.20** A 100% inspection process for large lots is to be replaced with a $c = 0$ sampling plan. What fraction of each lot must be inspected if lots that contain five or more defectives must be rejected 90% of the time?
**Solution:** From Equation 10.61 with $D = 5$ and $P_A = 0.10$, the fraction of each lot that must be inspected is

$$\begin{aligned} \frac{n}{N} &\simeq 1 - 0.10^{1/5} \\ &\simeq 0.37. \end{aligned}$$

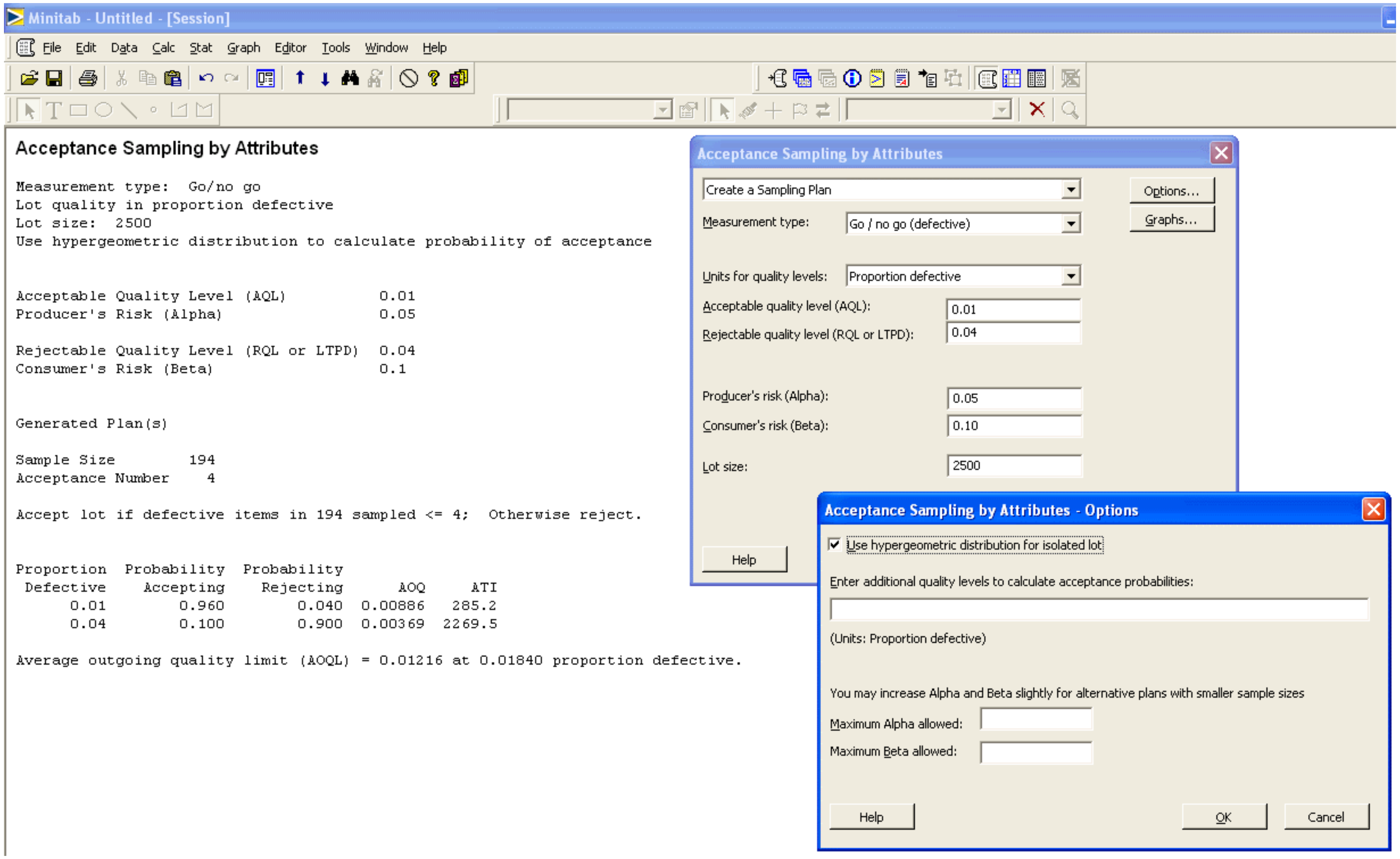
**Example 10.21** The calculated sample size in Example 4.2 was quite large compared to the lot size, which violates the small-sample approximation assumption. Repeat that example using the small-lot-size method.
**Solution:** The solution in the example indicated that 30% of the lot needed to be inspected. From Equation 10.61, which takes the relatively large sample size into account, the fraction of the lot that has to be inspected is more accurately

$$\begin{aligned} \frac{n}{N} &\simeq 1 - 0.05^{1/10} \\ &\simeq 0.259. \end{aligned}$$

**Example 10.22** Create the OC curves for normal, tightened, and reduced inspection under ANSI/ASQ Z1.4 using general inspection level II, single sampling, $N = 1000$, and $AQL = 1\%$.
**Solution:** The sampling plans determined for code letter J from the standard were normal $(n = 200, c = 5)$, tightened $(n = 200, c = 3)$, and reduced $(n = 80, c = 2)$. The operating characteristic curves were calculated using Equation 10.54 and are shown in Figure 10.6. For reference, the figure also shows the OC curve for the sampling plan determined for the same conditions using the Squeglia zero acceptance number sampling standard, which is often used instead of ANSI/ASQ Z1.4.

**Example 10.23** Determine the optimal rectifying inspection sampling plan for $LTPD = 0.04$ with $\beta = 0.10$ when the lot size is $N = 2500$ and the historical process fraction defective is $p = 0.01$.
**Solution:** A spreadsheet was used to solve Equations 10.66 and 10.67 as a function of acceptance number $c$ as shown in Table 10.2. For the specified conditions, the sampling plan that minimizes $ATI$ when $p = 0.01$ is $n = 232$ with $c = 5$. By comparison, the Dodge-Romig LTPD tables indicate a sampling plan with $n = 230$ and $c = 5$, which is in excellent agreement with the calculated plan.

From **MINITAB**> **Stat**> **Quality Tools**> **Acceptance Sampling by Attributes**:




**Example 10.24** Find the rectifying inspection plan with $LTPD = 0.04$ and $\beta = 0.1$ for a lot size of $N = 50$.
**Solution:** For the given conditions, the spreadsheet method gives $n = 58$, which exceeds the lot size. From Equation 10.70 the sample size required for the $c = 0$ plan is

$$\begin{aligned} n &\simeq 50\left(1 - 0.1^{\frac{1}{50\times 0.04}}\right) \\ &\simeq 35. \end{aligned}$$

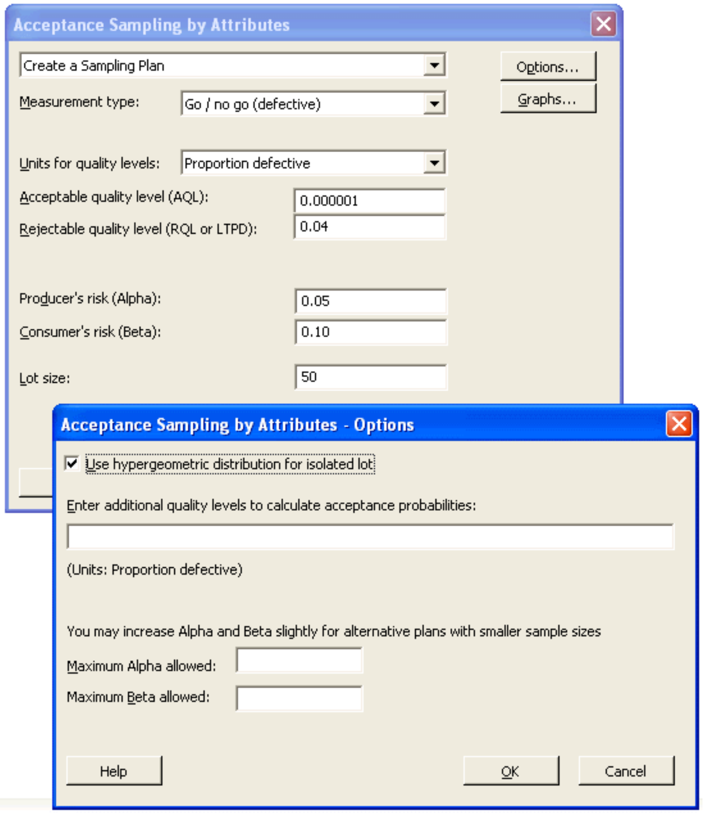
By comparison, the corresponding Dodge-Romig plan calls for $n = 34$ and is independent of the historical fraction defective.

From **MINITAB**> **Stat**> **Quality Tools**> **Acceptance Sampling by Attributes**:

```
MTB > AASAMPLING 1;
SUBC>   AQL 0.000001;
SUBC>   RQL 0.04;
SUBC>   CREATE;
SUBC>     ALPHA 0.05;
SUBC>     BETA 0.10;
SUBC>   LOTSIZE 50;
SUBC>   PROPORTION;
SUBC>   HYPERGEOMETRIC;
SUBC>   GOC;
SUBC>   GAOQ;
SUBC>   GATI;
SUBC>   ONEGRAPH.
```

**Acceptance Sampling by Attributes**

```
Measurement type:  Go/no go
Lot quality in proportion defective
Lot size:  50
Use hypergeometric distribution to calculate probability of acceptance


Acceptable Quality Level (AQL)          0.000001
Producer's Risk (Alpha)                 0.05

Rejectable Quality Level (RQL or LTPD)  0.04
Consumer's Risk (Beta)                  0.1


Generated Plan(s)

Sample Size        34
Acceptance Number   0

Accept lot if defective items in 34 sampled <= 0;  Otherwise reject.
```




**Example 10.25** What sample size is required for a $c = 1$ rectifying inspection single-sampling plan to obtain 1% $AOQL$ if the lot size is $N = 300$? For what value of incoming fraction defective will $AOQ$ be a maximum?
**Solution:** From Equation 10.76 with $A_1 = 0.839$ the required sample size is

$$n = \frac{1}{\frac{0.01}{0.839} + \frac{1}{300}} = 66.$$

$AOQ$ will be at its maximum value, $AOQL$, when the incoming fraction defective is

$$p_c = \frac{\chi_1^2}{2n} = \frac{3.24}{2 \times 66} = 0.0245.$$

**Example 10.26** Determine the sampling plan that minimizes the $ATI$ for lots of size $N = 1000$ with $AOQL = 0.02$ when the historical defective rate is $p = 0.01$.
**Solution:** A spreadsheet was used to solve for $n$ and $ATI$ as a function of $c$ using Equations 10.76 and 10.67. The results from the spreadsheet, shown in Table 10.4, indicate that the sampling plan that minimizes $ATI$ is given by $n = 65$ and $c = 2$. By comparison, this is exactly the same plan indicated in the Dodge-Romig tables for these conditions.

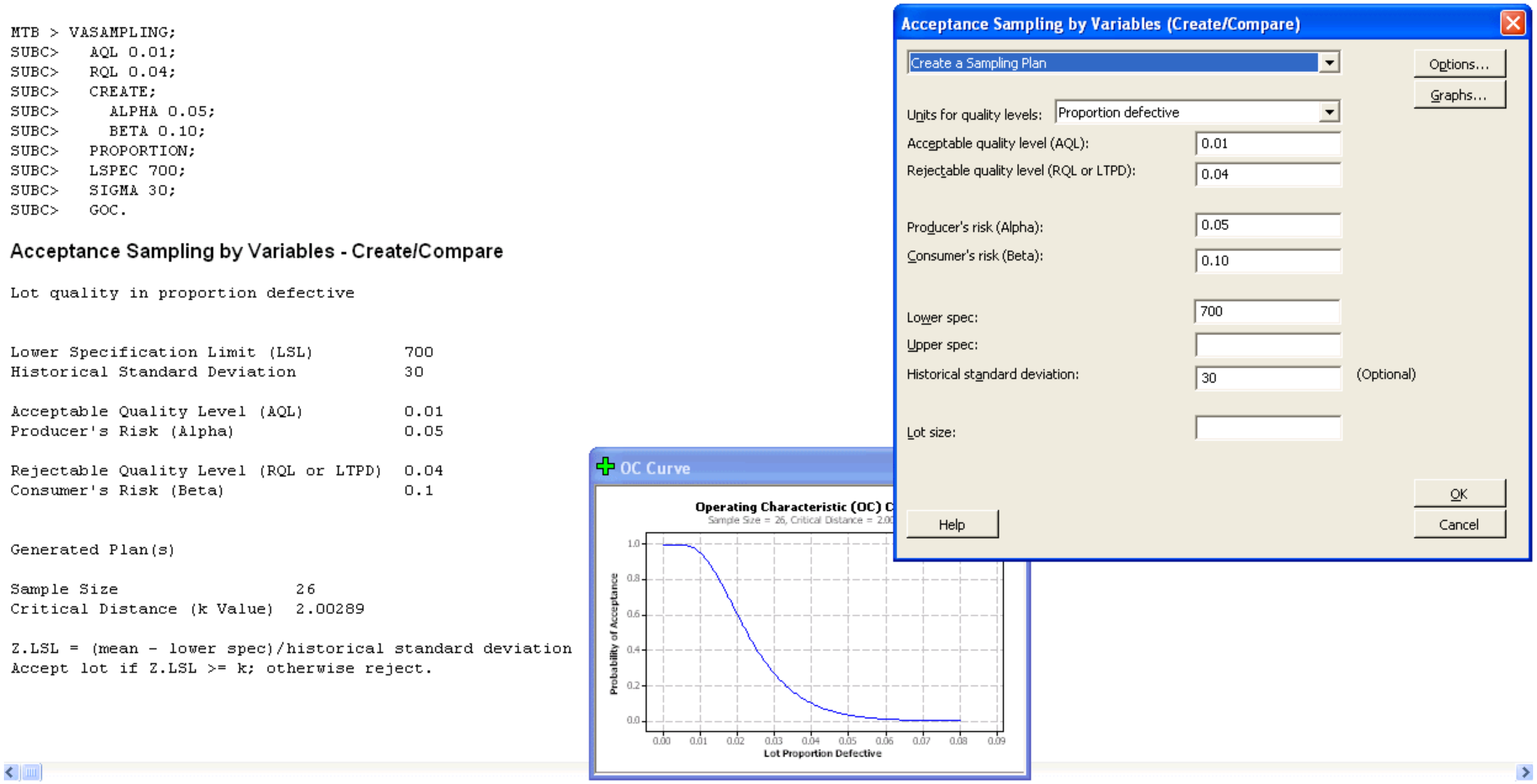
**Example 10.27** Find the single sampling plan for variables that will accept 95% of the lots with 1% defectives and reject 90% of the lots with 4% defectives when $\sigma = 30$ and the specification is one-sided with $USL = 700$.
**Solution:** The two specified points on the OC curve are $(p_0, 1-\alpha) = (0.01, 0.95)$ and $(p_1, \beta) = (0.04, 0.10)$. From Equation 10.79 the required sample size is

$$\begin{aligned} n &= \left(\frac{z_{0.05} + z_{0.10}}{z_{0.01} - z_{0.04}}\right)^2 \\ &= \left(\frac{1.645 + 1.282}{2.33 - 1.75}\right)^2 \\ &= 26. \end{aligned}$$

The critical value of $\bar{x}_{A/R}$ is

$$\begin{aligned} \bar{x}_{A/R} &= \mu_0 + z_\alpha \sigma_{\bar{x}} \\ &= (USL - z_{p_0}\sigma_x) + z_\alpha \frac{\sigma_x}{\sqrt{n}} \\ &= (700 - 2.33 \times 30) + 1.645 \frac{30}{\sqrt{26}} \\ &= 640. \end{aligned}$$

From **MINITAB**> **Stat**> **Quality Tools**> **Acceptance Sampling by Variables**> **Create/Compare:**




**Example 10.28** Determine the sample size ratio for attributes and variables inspection plans that will accept 95% of the lots with 0.1% defectives and reject 95% of the lots with 0.4% defectives.

**Solution:** The two points on the OC curve are $(p_0 = 0.001, 1-\alpha = 0.95)$ and $(p_1 = 0.004, \beta = 0.05)$. Because $\alpha = \beta = 0.05$ and both $p_0$ and $p_1$ are relatively small, from Equation 10.81 the ratio of the attributes- to variables-based sample sizes is approximately

$$\begin{aligned}\frac{n_{attributes}}{n_{variables}} &\simeq \frac{1}{4}\left(\frac{z_{0.001} - z_{0.004}}{\sqrt{0.004} - \sqrt{0.001}}\right)^2 \\ &\simeq \frac{1}{4}\left(\frac{3.090 - 2.652}{\sqrt{0.004} - \sqrt{0.001}}\right)^2 \\ &\simeq 48.\end{aligned}$$

So, the attributes plan sample size will have to be about 48 times larger than the variables plan sample size to obtain the same performance for acceptable and rejectable quality levels!

**Example 10.29** Find the single sampling plan for variables that will accept 95% of the lots with 1% defectives and reject 90% of the lots with 4% defectives. The specification is one-sided and $\sigma$ is unknown.
**Solution:** The two specified points on the OC curve are $(p_0, 1-\alpha) = (0.01, 0.95)$ and $(p_1, \beta) = (0.04, 0.10)$. From Equation 10.86, the condition that determines the sample size is

$$t_{0.95,n-1,-z_{0.01}\sqrt{n}} = t_{0.10,n-1,-z_{0.04}\sqrt{n}},$$

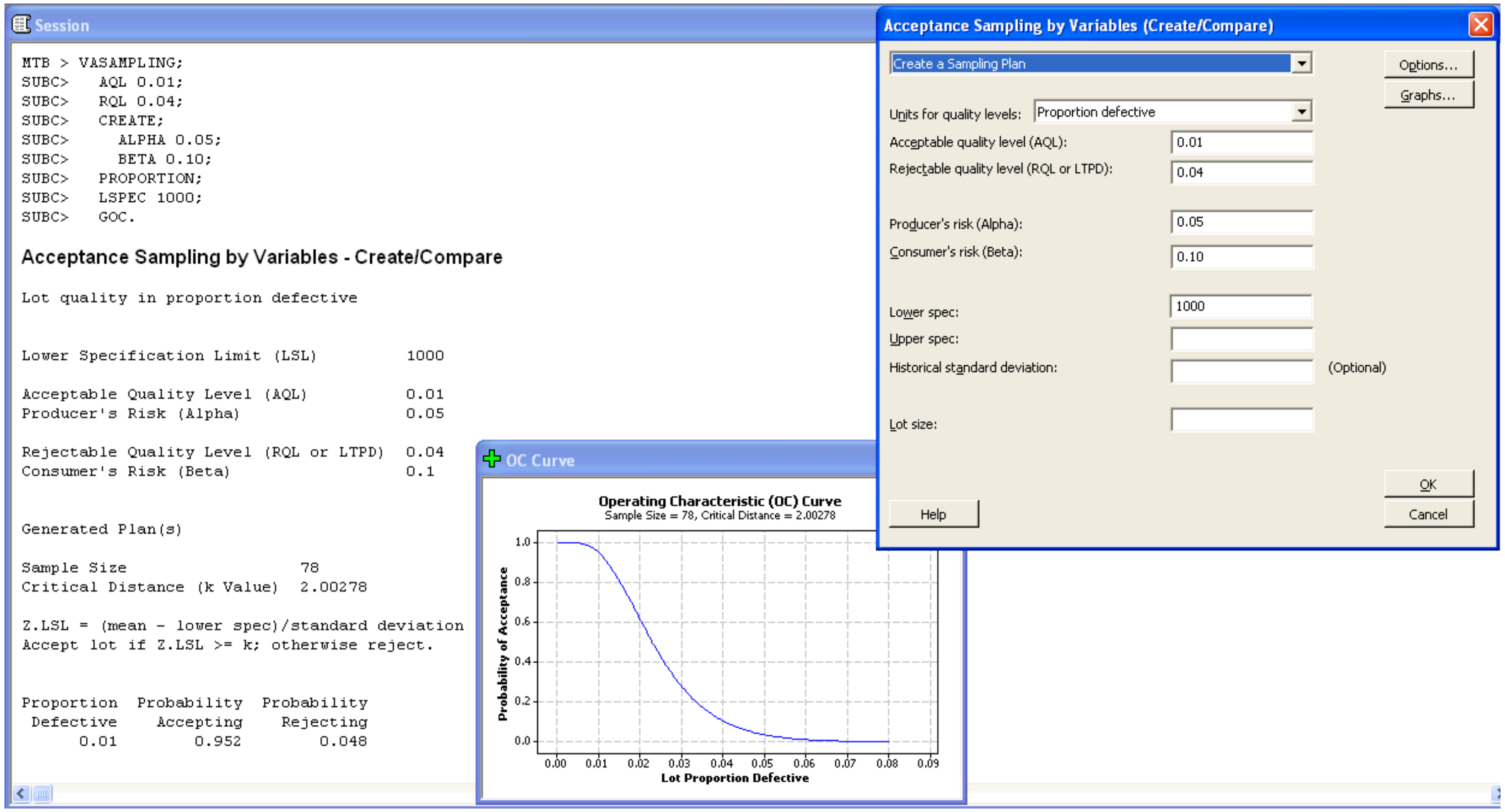
which is satisfied by $n = 78$ because

$$t_{0.95,77,-20.58} = t_{0.10,77,-15.46} = -17.75.$$

The accept/reject value of $k$ for the test is

$$k = \frac{17.75}{\sqrt{78}} = 2.01.$$

From **MINITAB**> **Stat**> **Quality Tools**> **Acceptance Sampling by Variables**> **Create/Compare:**




**Example 10.30** Plot the OC curves for the normal, tightened, and reduced sampling plans under ANSI/ASQ Z1.9 using a one-sided specification with Form 1, code letter F, and $AQL = 1\%$.

**Solution:** The sampling plans determined from the standard were normal $(n = 10, k = 1.72)$, tightened $(n = 10, k = 1.84)$, and reduced $(n = 4, k = 1.34)$. The operating characteristic curves were calculated using Equations 10.84 and 10.85. For example, the OC curve for normal inspection is given by

$$
\begin{aligned}
t_{P_A,df,-z_p\sqrt{n}} &= -k\sqrt{n} \\
t_{P_A,9,-z_p\sqrt{10}} &= -1.72\sqrt{10} \\
t_{P_A,9,-z_p\sqrt{10}} &= -5.439.
\end{aligned}
$$

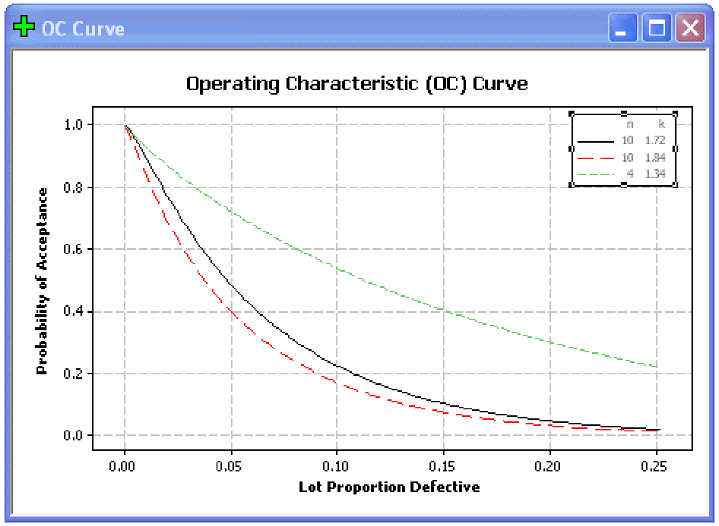
The OC curves are shown in Figure 10.9 and are in excellent agreement with the OC curves in the standard.

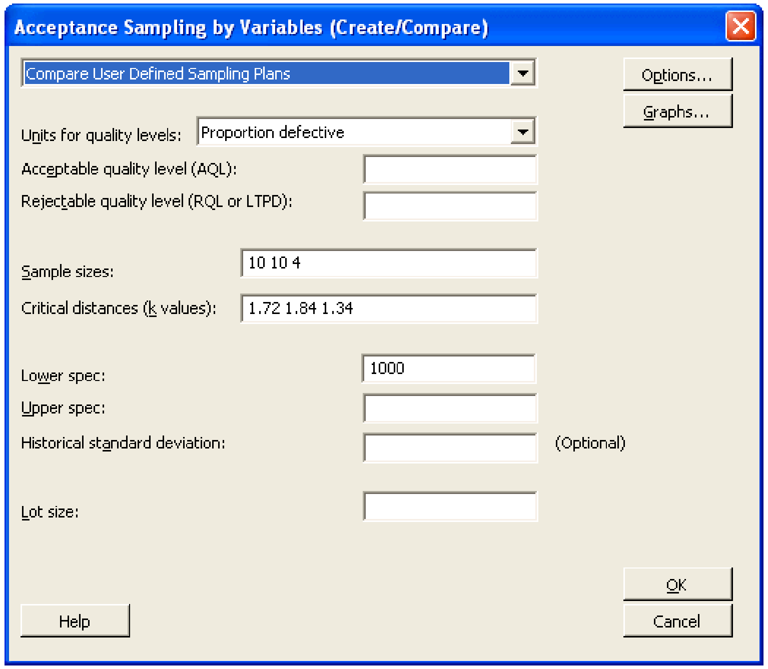
From **MINITAB**> **Stat**> **Quality Tools**> **Acceptance Sampling by Variables**> **Create/Compare:**

```
MTB > VASAMPLING;
SUBC>   COMPARE;
SUBC>     SIZE 10 10 4;
SUBC>     CDISTANCE 1.72 1.84 1.34;
SUBC>   PROPORTION;
SUBC>   LSPEC 1000;
SUBC>   GOC.
```

Acceptance Sampling by Variables - Create/Compare

```
Lot quality in proportion defective
```




## 10.5 Gage R&R Studies

# Chapter 11

# Resampling Methods

## 11.1 Software Requirements

## 11.2 Monte Carlo

**Example 11.1** How many samples should be drawn from a Poisson population to estimate the mean with 20% precision and 95% confidence? The mean is expected to be $\lambda = 5$.
**Solution:** The sample size must be sufficient to deliver the following confidence interval for $\lambda$:

$$\begin{aligned} P\left(0.8\widehat{\lambda} < \lambda < 1.2\widehat{\lambda}\right) &= 0.95 \\ P\left(4 < \lambda < 6\right) &= 0.95 \end{aligned}$$

where $\widehat{\lambda} = x/n$ and $x$ is the number of counts observed in $n$ sampling units. A MINITAB macro was used to draw 10000 random samples, all of size $n$, from a Poisson distribution with $\lambda = 5$, and calculate $\widehat{\lambda}$ for each sample. The $2.5^{th}$ and $97.5^{th}$ $\widehat{\lambda}$ percentiles were used to estimate the 95% confidence limits. Figure 11.1 shows the Monte Carlo confidence limits as a function of sample size. The sample size that delivers the desired confidence interval width is $n = 19$, which is in good agreement with the sample size given by Equation 5.11:

$$\begin{aligned} n &= \frac{1}{\lambda}\left(\frac{z_{\alpha/2}}{\delta}\right)^2 \\ &= \frac{1}{5}\left(\frac{1.96}{0.2}\right)^2 \\ &= 20. \end{aligned}$$

**Example 11.2** Use the Monte Carlo method to confirm the answer to Example 9.25, that samples of size $n = 31$ are required to estimate, with 95% confidence, the exponential–exponential interference failure rate to within ±50%.
**Solution:** A MINITAB macro was written that draws random samples of size $n = 31$ for exponential load and strength distributions, where the load distribution has mean $\mu_L$,

where $\mu_L$ comes from a uniform distribution with $1 \leq \mu_L \leq 10$, and the strength distribution has mean $\mu_S = k\mu_L$, where $k$ comes from a uniform distribution with $3 \leq k \leq 10$. The macro loops 1000 times, collecting the ratio of the empirical interference failure rate $\widehat{f}$ to the known failure rate $f$ from each pass. Figure 11.2 shows the histogram of $\widehat{f}/f$. About 97% of the observed $\widehat{f}$ values fall within ±50% of their known $f$ values, which is consistent with the intended 95% confidence level.

**Example 11.3** An experiment is proposed to study the defective rate ($p$) of a process using a sample of size $n = 300$ units. The process is considered to be acceptable when $p \leq 0.01$, but $p \geq 0.03$ is unacceptable. Determine the acceptance number and power for the sampling plan.
**Solution:** The hypotheses to be tested are $H_0 : p \leq 0.01$ versus $H_A : p > 0.01$. The decision to reject $H_0$ or not is based on the observed number of defectives $x$ in a sample of size $n = 300$. The critical value of $x$ that distinguishes the accept and reject regions should be chosen such that $\alpha \leq 0.05$ for $p = 0.01$. To determine this critical value, MINITAB was used to select 10000 random samples from a binomial population with $n = 300$ and $p = 0.01$. The frequencies and cumulative percentages by $x$ are shown in the $H_0$ columns of Table 11.1. From the $CumPct$ column, the critical value of $x$ must be set to $x_{A/R} = 6.5$ to provide good protection against type I errors ($\alpha = 1 - 0.9692 = 0.0308$). The frequency statistics under $H_A$ in the Figure were created by selecting 10000 random samples from a binomial population with $n = 300$ and $p = 0.03$. From the $CumPct$ column, the type II error probability relative to $x_{A/R} = 6.5$ is $\beta = 0.206$, so the power to reject $H_0$ when $p = 0.03$ is $\pi = 1 - \beta = 0.794$. This is in excellent agreement with the power calculated for the one-sample proportion test (see Section 4.1.2), which is $\pi = 0.797$.

**Example 11.4** Tukey's quick test is a nonparametric two-sample test for location that is easy to perform using dotplots. The samples must be comparable in size and they must be slipped, that is, one sample must have the largest observation and the other sample must have the smallest observation. The test statistic, $T$, is the number of slipped or nonoverlapping points. The Tukey test rejects $H_0 : \mu_1 = \mu_2$ when $T \geq 7$. For example, in Figure 11.3 $T = 7 + 10 = 17$, so there is sufficient evidence to reject $H_0$.

Use the Monte Carlo method to determine the power of Tukey's quick test as a function of effect size and sample size assuming that the two populations are normal and homoscedastic.
**Solution:** A MINITAB macro was written to draw 1000 random samples of equal sample size from two independent homoscedastic normal populations and count the number of times that $H_0$ was rejected by the Tukey quick test. Figure 11.4 shows that the power of the test is low for all sample sizes until the difference between the means is greater than 1.5 to 2.5 standard deviations. When the sample size is $n \geq 20$, Tukey's quick test has power $\pi \geq 0.90$ for differences between the means of 1.5 standard deviations or greater.

## 11.3 Bootstrap

**Example 11.5** The following yield strength values (in thousands of $psi$) for a material were obtained in a pilot study: {56, 23, 25, 68, 35, 31, 13, 15, 48, 37, 57, 69, 50, 76, 50, 19, 88, 33, 10, 21}. What sample size is required to estimate, with 95% confidence, the mean yield strength to within ±5000 psi?
**Solution:** The bootstrap percentile confidence interval width was studied as a function of sample size using 1000 resamples for sample sizes from $n = 30$ to $100$. Figure 11.5 shows the confidence interval width, given by

$$2\delta = \widehat{\theta}^*_{0.975} - \widehat{\theta}^*_{0.025},$$

versus sample size. The sample size required to obtain the desired precision of the estimate is $n = 75$.

**Example 11.6** The following data were obtained from a pilot study: {56, 48, 44, 62, 50, 47, 49, 57, 48, 55, 96, 47, 46, 47, 49, 72, 46, 61}. Determine the sample size required to obtain 90% power to reject $H_0 : \mu = 50$ when $\mu = 52$. The population is not normal, so assume that the experimental data will be analyzed using the bootstrap method.

**Solution:** If the population distribution is normal, the one-sample Student's $t$ test would be an appropriate method of analysis. Because the normality assumption is not satisfied, the preferred method of analysis using the bootstrap method uses the analogous bootstrap-$t$ distribution given by

$$t^* = \frac{\bar{x}^* - \mu_0}{s^*/\sqrt{n}} \tag{11.4}$$

where $\bar{x}^*$ and $s^*$ are the mean and standard deviation of bootstrap samples. Figure 11.6 shows the bootstrap distributions of $t^*$ with samples of size $n = 18$ under $H_0 : \mu = 50$ and $H_A : \mu = 56$, where the sample data were shifted to the appropriate population means before bootstrapping using transformations of the form

$$x_i' = x_i - \bar{x} + \mu.$$

From the bootstrap-$t$ distribution under $H_0$, the acceptance interval for $H_0$ is $-4.29 \leq t^* \leq 1.62$. The acceptance interval is skewed because the original sample is skewed. From the bootstrap distribution under $H_A$, the power is $\pi = 0.802$, which does not meet the 90% power requirement.

Figure 11.7 shows the bootstrap-$t$ test power versus sample size for samples from size $n = 16$ to 30. (Figure 11.6 was constructed from 1000 bootstrap samples. Each point in Figure 11.7 was constructed from 10000 bootstrap samples to reduce the noise in the power versus sample size plot.) The sample size required to obtain 90% power to reject $H_0$ when $\mu = 56$ is $n = 23$.