Physics 53

Satellite Motion

—You know, it's at times likes this when I'm stuck in a Vogon airlock with a man from Betelgeuse, about to die from asphyxiation in deep space, that I wish I had listened to what my mother told me when I was young.
— Why, what did she say?
—I don't know. I didn't listen.

— Hitchhiker's Guide to the Galaxy

Radial and angular motion

We will be concerned here with the orbiting motion of a satellite, such as a planet around the sun, or the moon around the earth. We neglect the effects of all other bodies.

We will assume that the satellite's mass (m) is much smaller than that (M) of the body it moves around. Both bodies actually orbit around the CM of the system, but M moves only a small amount because the CM is so close to it. We ignore this motion as an approximation, and take the center of M to be the origin of our coordinate system.

For a system with two masses of comparable size, such as a double star, obviously this is a bad approximation and one must analyze the motion of both bodies around the CM.

Because gravity (a central and conservative force) is the only force doing work:

The total mechanical energy E of the system is conserved.

The gravitational force acts along the line between the two bodies, so there is no torque about the origin. This means that:

The total angular momentum L of the system about the origin is conserved.

This is still true if the CM is taken as origin even when we take account of the motion of both bodies.

We will find that the nature of the trajectory is determined by the values of these two conserved quantities.

The velocity of the satellite has in general two components, one parallel and one perpendicular to the position vector **r** of the satellite. We call these the radial and tangential velocities and write

$$\mathbf{v} = \mathbf{v}_r + \mathbf{v}_\perp .$$

The kinetic energy then breaks into two parts:

$$K = \tfrac{1}{2}mv_r^2 + \tfrac{1}{2}mv_\perp^2 .$$

The magnitude of the angular momentum of the satellite is given by

$$L = mrv_\perp$$

so we can solve for $v_\perp$ in terms of L and write the kinetic energy as:

$$K = \tfrac{1}{2}m{v_r}^2 + \frac{L^2}{2mr^2}.$$

Now we add the potential energy to get the total energy:

$$E = \tfrac{1}{2}m{v_r}^2 + \frac{L^2}{2mr^2} - G\frac{Mm}{r}.$$

This expression is essentially one-dimensional, since the only variables in it are the speed along the radial line between the two bodies and their distance apart along that line. The second term in the formula is actually the *tangential* part of the kinetic energy, but since L is constant we have been able to express it a function only of the distance r.

Because the energy is *formally* one-dimensional, we can use the methods given earlier (see Energy 2) to determine turning points in the radial motion. This will give us the distances of closest approach and furthest recession of the satellite.

The satellite is not actually at *rest* at these turning points. Its velocity component *parallel* to **r** is zero, but it will still have a velocity component *perpendicular* to **r**.

We define an *effective* potential energy by combining the terms that depend only on r:

$$U_{eff}(r) = \frac{L^2}{2mr^2} - G\frac{Mm}{r},$$

so that

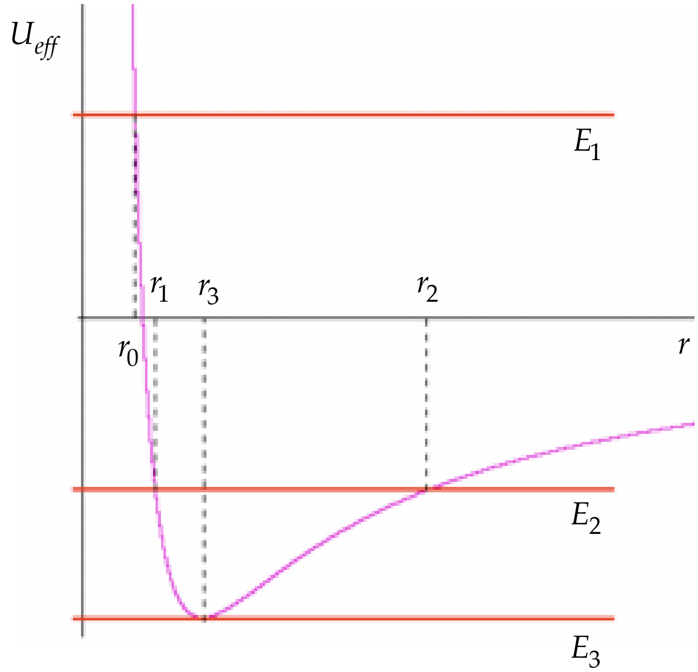
$$E = \tfrac{1}{2}m{v_r}^2 + U_{eff} .$$

At points where the total energy E is equal to U_{eff} the kinetic energy of *radial* motion along the line between the bodies is (instantaneously) zero. Those are the turning points in the radial motion.

We see that as $r \to \infty$, $U_{eff} \to 0$. This means that if the total energy E is positive (or zero) there is no turning point for large values of r, hence the motion is *unbound*. But if E is negative there are two turning points and we have bound orbits.

Types of orbits

Shown is a typical curve for U_{eff}, with three different possible values of the total energy E.



For positive energy E_1 there is only one turning point, at r_0. If the system has this energy, m approaches M to distance r_0 and then moves away, never to return. The motion is *unbound*; the trajectory can be shown to be a hyperbola passing around M. Comets that appear only once in our solar system are of this type.

For negative energy E_2 there are turning points at both r_1 and r_2. The distance between m and M varies back and forth between these values. The motion is *bound*, and the trajectory can be shown to be an ellipse with M at one focus. (Kepler's 1st law.)

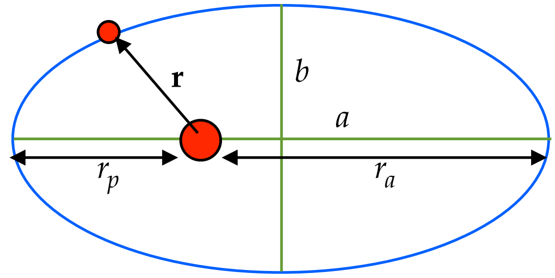
For negative energy E_3 the two turning points coincide and the ellipse becomes a circle of radius r_3.

There is a simple rule about the nature of the trajectories:

Positive (or zero) total energy gives unbound motion, while negative total energy gives bound motion.

This dichotomy between positive and negative *total* energy arises from our having chosen *potential* energy to be *zero* when the objects are infinitely far apart, so at finite distances it is negative.

Elliptical orbits



Shown is a typical elliptical orbit.

The turning points r_p and r_a are the distances of closest approach and furthest recession. These points are usually denoted by the Greek prefixes *peri* ("around") and *apo* ("from"). Thus a planet's point of closest approach to the sun is called its perihelion, and its point of furthest recession is its aphelion (*helios* is sun in Greek).

For satellites around the earth, the corresponding names are perigee and apogee.

We find the turning points by setting $U_{eff} = E$, remembering that E is negative.

$$\frac{L^2}{2mr^2} - G\frac{Mm}{r} = E, \text{ or } r^2 + \frac{GMm}{E}r - \frac{L^2}{2mE} = 0$$

The roots of this equation are r_p and r_a, so it must be the same as

$$(r - r_p)(r - r_a) = 0, \text{ or } r^2 - (r_p + r_a)r + r_p r_a = 0$$

Comparing the coefficients in these different two forms, we see that

$$r_p + r_a = -\frac{GMm}{E}, \; r_p r_a = -\frac{L^2}{2mE}$$

The turning points are related to the axes of the ellipse by

$$r_p + r_a = 2a, \; r_p r_a = b^2,$$

so we can write the energy and angular momentum of the orbit in terms of *a* and *b*:

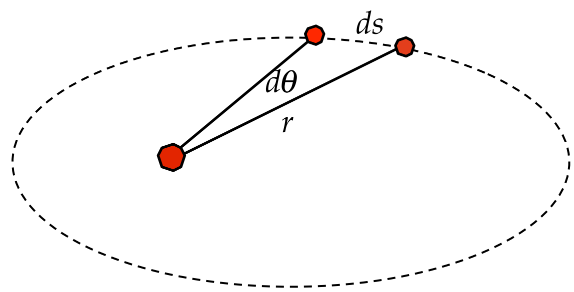
Elliptical orbit relations	Energy: $E = -\frac{GMm}{2a}$ Angular momentum: $L^2 = -2mEb^2$

The first of these is an important formula, showing that the energy depends *only* on the length of the orbit 2*a*. The second shows that *L* is proportional to *b*, so more eccentric orbits (smaller *b*) correspond to lower *L*. Maximum *L* occurs for a circular orbit, $b = a$.

Kepler's laws

In our analysis we did not prove that the orbits are ellipses (Kepler's 1st law), but it can be shown. We will now prove the other two laws.

Proof of the 1st law requires an extensive use of calculus, so we will not give it here.



Shown is an elliptical orbit and the position of the satellite at two times differing by dt. The satellite moves a distance ds along the orbit, while the line connecting the bodies turns through angle $d\theta$.

The area swept out by the line is that of the narrow triangle $dA = \frac{1}{2} r\, ds$. Since $ds = r\, d\theta$ we have $dA = \frac{1}{2} r^2\, d\theta$. The rate at which area is swept out is

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \omega$$

But $r\omega = v$ and $mr^2\omega = L$, so we find

$$\frac{dA}{dt} = \frac{L}{2m}$$

Since L is a constant, we have shown that the rate at which area is swept out is constant. This is Kepler's 2nd law, which we see is a simple consequence of conservation of angular momentum.

If we integrate this rate over the time for a complete revolution (the period T) we obtain the total area of the ellipse. Thus

$$\text{Area} = \frac{LT}{2m}$$

The area of an ellipse is πab. Substituting from the formulas given above relating E and L to a and b, we find after some algebra:

Kepler's 3rd law	$\frac{a^3}{T^2} = \frac{GM}{4\pi^2}$

This shows that the ratio a^3 / T^2 is the same for all objects orbiting a given mass M. This is Kepler's 3rd law.

By measuring a and T one can use this relation to determine the mass M. It is in this way that we determine relatively easily the mass of the sun and the masses of those planets (such as earth) having moons.

Actually, if we had taken into account the neglected motion of M around the CM of the system, what would appear in the above formula would be the total mass $M + m$ instead of M. It is this quantity that one determines experimentally by measuring T and a. To determine M and m separately with high accuracy one must make detailed observations of the motion of both bodies around the CM of the system.

If one knows m, then determination of M is easier. We now know the masses of Mercury and Venus (which have no moons) with high accuracy because we have been able to place satellites of known mass in orbit around them, and measure the orbit parameters.