Scalability limitations of optical access and metro networks due to the polarization-dependent gain of semiconductor optical amplifiers

I. Roudas[*a] and N. Antoniades[b]
[a] Department of Electrical and Computer Engineering, University of Patras, Rio 26500, Greece.
[b] Department of Engineering Science and Physics, City University of New York, College of Staten Island, Staten Island, NY 10314, USA.

ABSTRACT

This article presents, for the first time, the derivation of approximate analytical formulae for the probability density function and the cumulative density function of the optical signal-to-noise ratio variation in optical local and metropolitan area networks due to the weakly polarization-dependent gain of cascaded semiconductor optical amplifiers. The cumulative density function is used to calculate the outage probability and derive specifications for the maximum allowable value of polarization-dependent gain per semiconductor optical amplifier in order to achieve a given network size.

Keywords: Optical communications, optical signal-to-noise ratio (OSNR), polarization-dependent gain (PDG), semiconductor optical amplifiers (SOA), optical metropolitan area networks (MAN).

1. INTRODUCTION

Semiconductor optical amplifiers (SOAs) are increasingly considered for multi-channel amplification in optical local area networks (LANs) and metropolitan area networks (MANs) due to their low cost, small size, wide bandwidth, and capability of operation in several wavelength bands [1]. However, SOAs can cause severe distortions of wavelength division multiplexed (WDM) optical signals, i.e., due to their chirp, self-gain modulation, cross-gain modulation, self-phase modulation, cross-phase modulation and four wave mixing, resulting in a decrease of the optical network performance [1]. Thanks to the small size of optical LANs and MANs, the aforementioned transmission impairments can be tolerated or mitigated to a certain extent with appropriate design and biasing of the SOAs. Then, the dominant residual signal degradation induced by the SOAs is related to their polarization-dependent gain (PDG).

PDG leads to a variation of the optical signal-to-noise ratio (OSNR) at the output of a chain of semiconductor optical amplifiers. This, in turn, can cause significant fluctuations of the bit error rate and outages in the performance of optical communications systems and networks [2]. This effect becomes more pronounced as the number of cascaded SOAs increases and sets an upper limit to the scalability of the optical communications systems and networks.

The impact of PDG and polarization-dependent loss (PDL), in the presence or absence of polarization mode dispersion (PMD), was extensively studied, both theoretically and experimentally, in long-haul and ultra-long-haul optical communications systems, e.g., [3]-[26]. In contrast, there is no analogous study for the case of optical LANs and MANs because polarization effects are usually considered negligible. However, if these networks contain even a small number of SOAs concatenated in series, the impact of PDG can become significant, since commercially available SOAs for these applications exhibit PDG typically of 0.5-1.5 dB [27].

A recent comprehensive study [21] derived analytical expressions for the outage probability due to polarization-dependent loss (PDL) in long-haul terrestrial optical communications systems, where the number of cascaded PDL elements is large and the PDL can be considered to be continuously distributed along the system.

This paper extends the model of [21] to the case of optical LANs and MANs with a small number of SOAs exhibiting weakly PDG. The formalism of [21] is altered to consider lumped PDG elements along the system and is simplified by using the approximate PDL vector concatenation rule of [19], [20] for the case of weakly PDG. It is shown that the OSNR variation after N SOAs, in the case of weakly PDG, is approximately equal to the sum of N independent,

[*] roudas@ece.upatras.gr; phone (30) 2610-996-484; fax (30) 2610-997-342.

Optical Transmission Systems and Equipment for Networking V, edited by Benjamin B. Dingel, Ken-ichi Sato, Werner Weiershausen, Achyut K. Dutta, Proc. of SPIE Vol. 6388, 63880J, (2006) · 0277-786X/06/$15 · doi: 10.1117/12.684893

uniformly distributed random variables. Therefore, it is straightforward to derive analytical formulae for the probability density function (pdf) and the cumulative density function (cdf) of the ONSR variation. The latter is used to calculate the outage probability and derive specifications for the maximum allowable PDG per SOA as a function of the number of SOAs which can be cascaded.

The remainder of the paper is organized as follows: in Section 2, the optical communications system is described and the equivalent system model is outlined and formulated. Section 3 presents analytical and simulation results that demonstrate the validity of the model and a plot of the maximum allowable PDG per SOA, as a function of the number of SOAs which can be cascaded.

2. THEORETICAL MODEL

2.1. System description

The system topology under study consists of a chain of N stages (Fig. 1 (a)). Each stage is composed of an SOA with average gain $g_{0,i}$, followed by passive optical components and short spans of optical fibers with total insertion loss l_i $\left(0 \le l_i \le 1\right)$, such that $g_{0,i} l_i = 1$, $i = 1,\ldots,N$. In addition, SOAs exhibit weakly PDG. This is due to the difference in the confinement factors of the transverse electric (TE) and transverse magnetic (TM) modes, since the active region is not rotationally symmetric [28].

In the subsequent analysis, a number of simplifying assumptions, which are frequently satisfactory in practice, are adopted: (i) SOAs are operating at the linear (i.e., unsaturated) regime. (ii) All transmission impairments other than component insertion loss/gain and SOA PDG are neglected due to the small size of the network. (iii) Due to the birefringence of interconnecting fibers, the signal state of polarization (SOP) is fully randomized between consecutive SOAs. However, fiber PMD is not taken into account. (iv) There is no dynamic gain equalization or PDG equalization in order to minimize cost.

Conforming to the above assumptions, SOAs can be considered as equivalent partial polarizers with gain instead of insertion loss. Therefore, the models of [19]-[21], initially intended for the description of PDL, can be readily adapted to the problem under study.

The system topology can be reduced to the one shown in the simplified block diagram of Fig. 1 (b). The SOAs, the passive optical components and the optical fibers are eliminated since the optical attenuation induced by the passive optical components and the optical fibers is compensated fully by the average gain of the SOAs. SOAs are represented by independent equivalent noise sources with total power P_n at the input of each SOA followed by a PDG element.

For the derivation of OSNR statistics, two cases can be distinguished, according to whether the OSNR at the output of the SOA chain is calculated taking into account only the ASE noise parallel to the received signal SOP or both ASE noise components. The former definition of OSNR is better correlated to the error probability in optically preamplified direct-detection receivers when the ASE-ASE noise beating is negligible [18], [21]. However, in the case of CWDM optical LANs and MANs, employing optical multiplexers-demultiplexers with 13 nm bandwidth [29], the contribution of the ASE noise orthogonal to the received signal SOP in the error probability might be significant. Therefore, we focus here on the OSNR statistics when the total ASE noise is taken into account.

2.2. Definitions and notations

The PDG eigenaxes of the i-th SOA in Stokes space are denoted by the unit Stokes vectors $\pm\hat{p}_i$, $i = 1,\ldots,N$. The gains associated with these eigenaxes are $G_{\max,i}, G_{\min,i}$, respectively. Both the eigenaxes and the gains are assumed independent of frequency. The SOA PDG is defined in dB units as $\rho_{i,dB} = 10\log\rho_i$ where [2]

$$\rho_i = \frac{G_{\max,i}}{G_{\min,i}} \tag{1}$$

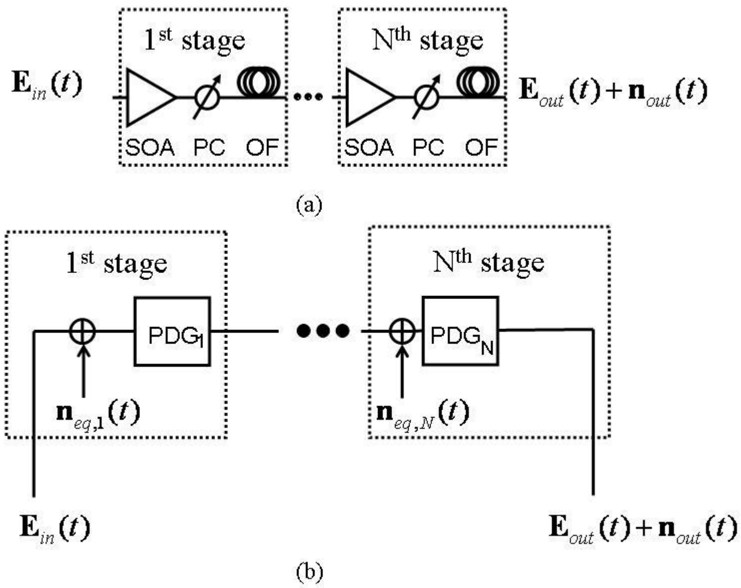

Fig. 1. (a) System topology; (b) Equivalent system model. (Symbols: SOA: semiconductor optical amplifier, PC: passive optical components, OF: optical fiber span, $\mathbf{E}_{in}(t)$: input signal electric field, $\mathbf{E}_{out}(t)$: output signal electric field, $\mathbf{n}_{eq,i}(t)$: equivalent input ASE noise electric field, $\mathbf{n}_{out}(t)$: output ASE noise electric field).

Following [21], it is possible to define the following auxiliary quantities:

- Average gain:

$$g_{0,i} = \frac{G_{\max,i} + G_{\min,i}}{2} \tag{2}$$

- PDG coefficient:

$$\Gamma_i = \frac{G_{\max,i} - G_{\min,i}}{G_{\max,i} + G_{\min,i}} \tag{3}$$

- PDG vector in Stokes space

$$\vec{\Gamma}_i = \Gamma_i \hat{p}_i \tag{4}$$

2.3. Mathematical formulation

In the following, we will use the first-order Taylor expansion of the transmittance of N concatenated stages in terms of Γ_i (cf. expressions (6), (10), (11) in [19])

$$T_{tot}\left(N\right)=1+\vec{\Gamma}_{tot}\left(N\right)\hat{s}+\mathcal{O}\left(\Gamma_i^{\,2}\right) \tag{5}$$

where $\hat{s}$ is the launched signal SOP at the input of the chain and we defined the total PDG vector of the SOA chain as

$$\vec{\Gamma}_{tot}\left(N\right)=\sum_{i=1}^{N}\vec{\Gamma}_i \tag{6}$$

The power of the optical signal at the output of the chain is (cf. expression (1) of [21] with slight changes in notation)

$$P_{s,out}\left(N\right)=T_{tot}\left(N\right)P_s\cong P_s\left[1+\vec{\Gamma}_{tot}\left(N\right)\hat{s}\right] \tag{7}$$

where P_s is the launched optical signal power.

The power of the total ASE noise at the output of the SOA chain is given by (cf. expression (12) of [21] with slight changes in notation)

$$P_{n,out,tot}\left(N\right)\cong NP_n \tag{8}$$

where P_n denotes the total equivalent ASE noise power at the input of the individual SOAs in both polarizations.

The OSNR after N stages taking into account both ASE noise components is given by

$$R\left(N\right)=\frac{P_{s,out}\left(N\right)}{P_{n,out,tot}\left(N\right)}\cong R_N\left[1+\vec{\Gamma}_{tot}\left(N\right)\hat{s}\right] \tag{9}$$

where R_N is the OSNR after N stages in the absence of PDG

$$R_N=\frac{P_s}{NP_n} \tag{10}$$

2.4. OSNR statistics

It is convenient to introduce the relative OSNR variation y defined as

$$y=\frac{R(N)-R_N}{R_N} \tag{11}$$

It can be shown that the length of the projection of a unit vector $\hat{p}_i$ with random direction in the Stokes space on a fixed unit vector $\hat{s}$ is uniformly distributed over the interval [0,1] [30], [31]. Then, the relative OSNR variation y can be expressed as a weighted sum of N independent, uniformly distributed random variables $\vec{\Gamma}_i\hat{s}$, each taking values in the interval $\left[-\Gamma_i,\Gamma_i\right]$.

We distinguish the following partial cases:

A. Chain of identical SOAs

In the case of identical SOAs with PDG coefficients $\Gamma_i = \Gamma$, the pdf of the relative OSNR variation at the output of the N-th stage can be expressed in a variety of closed forms, e.g., [30]

$$p_y(y) = \frac{1}{(2\Gamma)^N (N-1)!} \sum_{k=0}^{N} (-1)^k \binom{N}{k} \left[y + (N-2k)\Gamma \right]_+^{N-1} \tag{12}$$

where the "plus" function x_+^n vanishes for $x \le 0$ and equals x^n for $x \ge 0$ [30].

The cdf of the relative ONSR variation in the case of identical SOAs with PDG coefficients $\Gamma_i = \Gamma$ is [30]

$$F_y(y) = \frac{1}{(2\Gamma)^N N!} \sum_{k=0}^{N} (-1)^k \binom{N}{k} \left[y + (N-2k)\Gamma \right]_+^{N} \tag{13}$$

B. Chain of non-identical SOAs

In the case of non-identical SOAs with PDG coefficients Γ_i, the relative ONSR variation pdf cannot be calculated in closed form. However, it might be approximated in its central region by a Gaussian, in the case of a large number of concatenated SOAs, due to the central limit theorem [32]. It is straightforward to show that the relative ONSR variation has zero mean and variance given by

$$\sigma_y{}^2 = \frac{1}{3} \sum_{i=1}^{N} \Gamma_i^2 \tag{14}$$

The cdf of the relative ONSR variation in the case of non-identical SOAs with PDG coefficients Γ_i can be calculated analytically by

$$F_y(y) = F_u\left[\frac{1}{2}\left(y + \sum_{i=1}^{N} \Gamma_i \right) \right] \tag{15}$$

where $F_u(u)$ is the auxiliary cdf (cf. expression (26.57b) in [33])

$$F_u(u) = \frac{1}{N! \prod_{i=1}^{N} \Gamma_i} \sum_{\mathbf{v} \in C} \operatorname{sgn}(\mathbf{v}) \left(u - \mathbf{\Gamma v} \right)_+^N \tag{16}$$

In (16), $\mathbf{\Gamma} = (\Gamma_1, \ldots, \Gamma_N)$, C is the hypercube $\left\{ \mathbf{x} \in R^N ; 0 \le x_i \le 1 \text{ for } i = 1, \ldots, N \right\}$, the summation is over the 2^n vertices $\mathbf{v}$ of C, and we defined the hypervector sign function as

$$\operatorname{sgn}(\mathbf{v}) = (-1)^m \tag{17}$$

$$m = \sum_{i=1}^{N} v_i \tag{18}$$

Similar to [21], the outage probability can be approximately defined as the probability that the normalized OSNR $R(N)/R_N$ falls below a threshold χ (referred to as OSNR margin) and is given by

$$P_{\text{outage}} = F_y(\chi - 1) \qquad (19)$$

It must be stressed that the above definition of the outage probability is unconventional [2]. Formally, one needs to first calculate the error probability, e.g., generalizing the formalism of [22], [23]. However, this calculation is outside of the scope of the current paper and will be part of future work.

The validity of the theoretical expressions is confirmed by comparison with the simulation results of [26]. To accelerate the simulation, the random orientation of the PDG vector is chosen as follows: a constellation of n approximately equidistant points on the surface of the Poincaré sphere is a priori selected so the Poincaré sphere is sufficiently covered even with a small number of points n. The optimal configuration of n points on the surface of a sphere can be computed using various optimization algorithms (see e.g., [34] and the references therein). If n is the number of realizations of the PDG vector of each SOA and N is the number of spans, there are n^N possible combinations of orientations of the PDG vectors of SOAs of consecutive stages, i.e., the simulation is repeated n^N times.

3. RESULTS AND DISCUSSION

Fig. 2-3 show semi-logarithmic plots of the relative OSNR variation pdf, as given by (12), at the output of the SOA chain, for $N = 2-5$ cascaded stages (curves). The validity of the theoretical expression (12) is checked by comparison with Monte Carlo simulation for SOA PDG equal to $\rho_i = 0.5$ dB and $\rho_i = 1$ dB (points) (see [26] for details). It is observed that, for small values of $N\Gamma$, there is excellent agreement between the analytical and numerical results. For larger values of $N\Gamma$, the left tail of the numerical pdf decreases more rapidly than the right tail and the numerical results deviate from the theoretical prediction. This discrepancy is due to the fact that the actual relative OSNR variation lies in the interval $\left[\left\{(1-\Gamma)^N - 1\right\}/\Gamma, \left\{(1+\Gamma)^N - 1\right\}/\Gamma\right]$, whereas (12) takes values in the interval $\left[-N\Gamma, N\Gamma\right]$. This indicates that (13), (19) yield slightly pessimistic results for the calculation of outage probability.

Fig. 4 shows the maximum allowable PDG per SOA, calculated using (13), (19), as a function of the number of stages traversed in order to achieve an outage probability of 1/17,520, which corresponds to an outage time of 30 min per year [2]. The OSNR margin χ allocated for PDG is assumed 1 dB (solid curve) and 2 dB (dash-dotted curve). For example, if the OSNR margin allocated for PDG is 1 dB and the maximum allowable PDG per SOA is 0.5 dB, up to four SOAs can be cascaded in series. If the OSNR margin is increased to 2 dB, then the maximum allowable PDG per SOA, for a network containing four SOAs in series, is 0.89 dB.

4. SUMMARY

This article presents, for the first time, the derivation of approximate analytical formulae for the pdf and cdf of the relative OSNR variation in optical access and metro networks comprising N cascaded SOAs with small PDG coefficients. The cdf is used to calculate the outage probability and derive specifications for the maximum allowable PDG per SOA. For large values of $N\Gamma_i$, the aforementioned expression for the outage probability becomes pessimistic, so the specifications can be applied a fortiori in all cases.

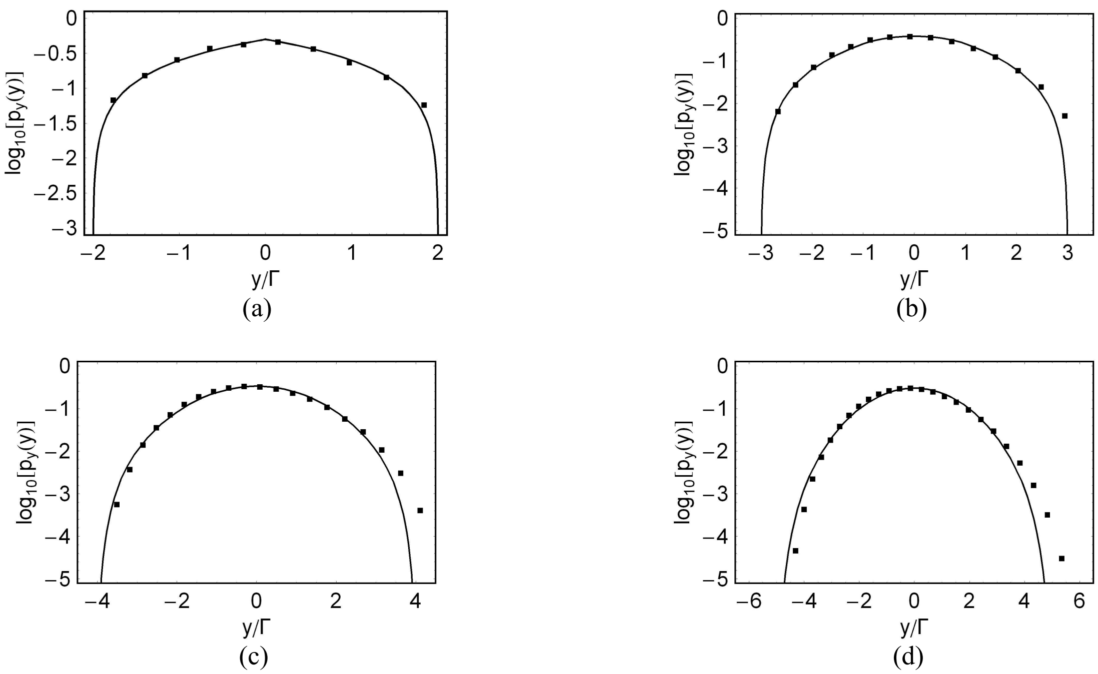

Fig. 2. Analytical pdf's of the relative OSNR variation for (a) $N=2$, (b) $N=3$, (c) $N=4$, (d) $N=5$ SOA stages plotted in logarithmic scale using (12) and verification by Monte Carlo simulation. (Symbols: curves: analysis, points: simulation). (Conditions: SOA PDG $\rho_i = 0.5$ dB, number of simulation runs: 122^N).

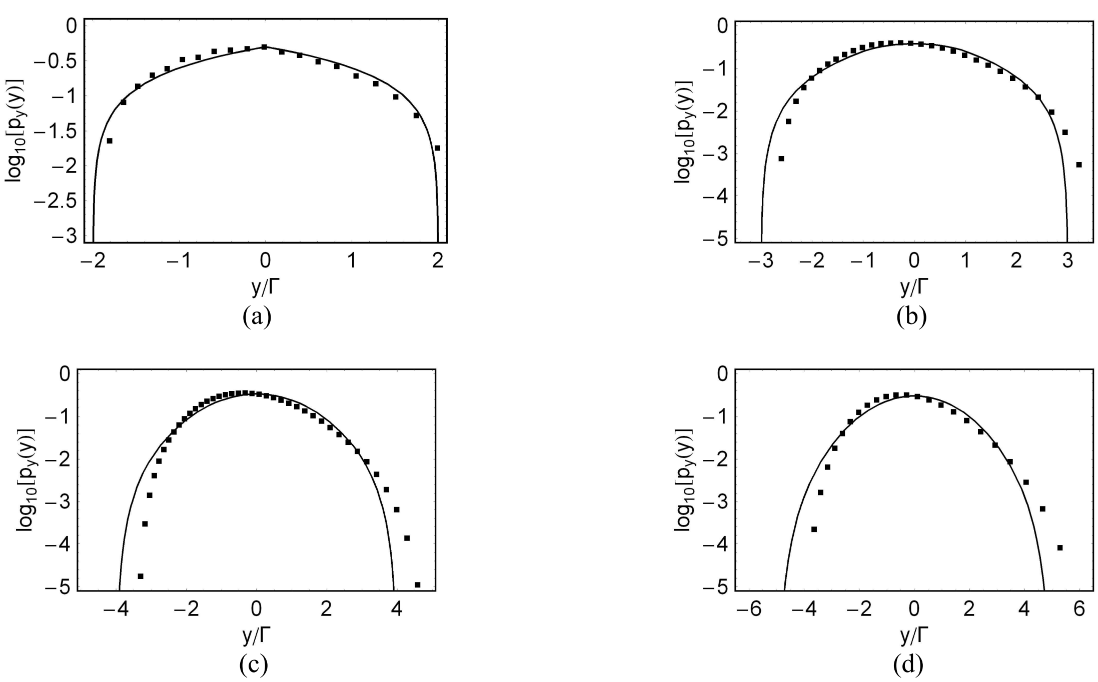

Fig. 3. Same as Fig. 2 for SOA PDG $\rho_i = 1$ dB.

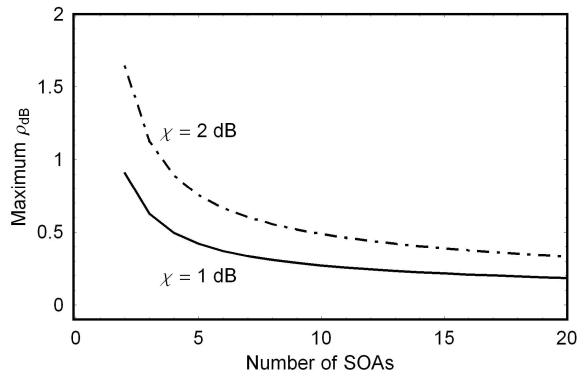

Fig. 4. Maximum allowable PDG per SOA as a function of the number of SOA stages for OSNR margin $\chi = 1$ dB (solid curve) and $\chi = 2$ dB (dash-dotted curve).

5. ACKNOWLEDGMENT

The authors wish to thank Drs. K. C. Reichmann, P. P. Iannone, N. J. Frigo, and A. M. Levine of AT&T Labs-Research, and Drs. B. R. Hemenway and M. Sauer of Corning Inc., for stimulating discussions.

REFERENCES

[1] D. R. Zimmerman and L. H. Spiekman, "Amplifiers for the masses: EDFA, EDWA, and SOA amplets for metro and access applications," *J. Lightwave Tech*., Vol. 22, No. 1, pp. 63–70, Jan. 2004.

[2] C. D. Poole and J. Nagel, "Polarization effects in lightwave systems," in *Optical Fiber Telecommunications IIIA*, I. P. Kaminow and T. L. Koch, Eds. San Diego, CA: Academic, 1997, ch. 6.

[3] M. G. Taylor, "Observation of new polarization dependence effect in long haul optically amplified system," *IEEE Phot. Tech. Lett.*, Vol. 5, No. 10, pp. 1244–1246, Oct. 1993.

[4] V. J. Mazurczyk and J. L. Zyskind, "Polarization dependent gain in erbium doped-fiber amplifiers," *IEEE Phot. Tech. Lett.*, Vol. 6, No. 5, pp. 616–618, May 1994.

[5] P. Wysocki and V. Mazurczyk, "Polarization dependent gain in erbium-doped fiber amplifiers: computer model and approximate formulas," *J. Lightwave Technol.*, Vol. 14, No. 4, pp. 572–584, Apr. 1996.

[6] F. Bruyère and O. Audouin, "Penalties in long-haul optical amplifier systems due to polarization dependent loss and gain," *IEEE Photon. Technol. Lett*., Vol. 6, pp. 654–656, May 1994.

[7] E. Lichtman, "Limitation imposed by polarization-dependent gain and loss on all-optical ultralong communication systems," *J. Lightwave Technol*., Vol. 13, pp. 906–913, May 1995.

[8] N. Gisin, "Statistics of polarization dependent losses," *Opt. Commun*., Vol. 114, No. 5–6, pp. 399–405, Feb. 1995.

[9] N. Gisin and B. Huttner, "Combined effects of polarization mode dispersion and polarization dependent losses in optical fibers," *Opt. Commun.*, Vol. 142, No. 1–3, pp. 119–125, Oct. 1997.

[10] B. Huttner, C. Geiser, and N. Gisin, "Polarization-induced distortions in optical fiber networks with polarization-mode dispersion and polarization-dependent losses," *IEEE J. Select. Topics Quantum Electron*., Vol. 6, pp. 317–329, Mar./Apr. 2000.

[11] P. Lu, L. Chen, and X. Bao, "Polarization mode dispersion and polarization-dependent loss for a pulse in single-mode fibers*," J. Lightwave Technol*., Vol. 19, pp. 856–860, June 2001.

[12] Y. Sun, I. T. Lima Jr., H. Jiao, J.Wen, H. Xu, H. Ereifej, G. M. Carter, and C. R. Menyuk, "Study of system performance in a 107-km dispersion managed recirculating loop due to polarization effects," *IEEE Photon. Technol. Lett*., Vol. 13, pp. 966–968, Sept. 2001.

[13] L.-S. Yan, Q. Yu, Y. Xie, and A. E. Willner, "Experimental demonstration of the system performance degradation due to the combined effect of polarization-dependent loss with polarization-mode dispersion," *IEEE Photon. Technol. Lett*., Vol. 14, pp. 224–226, Feb. 2002.

[14] B. Marks, Y. Sun, C. Menyuk, and G. Carter, "Polarization-state evolution in recirculating loops with polarization-dependent loss," *Opt. Lett*., Vol. 27, No. 21, pp. 1881–1883, 2002.

[15] M. Yu, C. Kan, M. Lewis, and A. Sizmann, "Statistics of signal-to-noise ratio and path-accumulated power due to concatenation of polarization dependent loss," *IEEE Photon. Technol. Lett*., Vol. 14, pp. 1418–1420, Oct. 2002.

[16] A. Mecozzi and M. Shtaif, "The statistics of polarization-dependent loss in optical communication systems," *IEEE Photon. Technol. Lett*., Vol. 14, pp. 313–315, Mar. 2002.

[17] A. Galtarossa and L. Palmieri, "The exact statistics of polarization-dependent loss in fiber-optic links," *IEEE Photon. Technol. Lett*., Vol. 15, pp. 57–59, Jan. 2003.

[18] C. Xie and L. F. Mollenauer, "Performance degradation induced by polarization-dependent loss in optical fiber transmission systems with and without polarization-mode dispersion," *J. Lightwave Technol*., Vol. 21, No. .9, pp. 1953–1957, Sept. 2003.

[19] M. Karlsson and M. Petersson, "Quaternion approach to PMD and PDL phenomena in optical fiber systems," *J. Lightwave Technol*., Vol. 22, No. 4, pp. 1137–1146, Apr. 2004.

[20] C. Vinegoni, M. Karlsson, M. Petersson, and H. Sunnerud, "The statistics of polarization-dependent loss in a recirculating loop," *J. Lightwave Technol*., Vol. 22, No. 4, pp. 968–976, April 2004.

[21] A. Mecozzi and M. Shtaif, "Signal-to-Noise-Ratio degradation caused by polarization-dependent loss and the effect of dynamic gain equalization," *J. Lightwave Technol*., Vol. 22, No. 8, pp. 1856–1871, Aug. 2004.

[22] I. T. Lima Jr., A. O. Lima, Y. Sun, H. Jiao, J. Zweck, C. R. Menyuk, and G. M. Carter, "A receiver model for optical fiber communication systems with arbitrarily polarized noise," *J. Lightwave Technol*., Vol. 23, No. 3, pp. 1478–1490, Mar. 2005.

[23] J. L. Rebola and A. V. T. Cartaxo, “Performance evaluation of optically preamplified receivers with partially polarized noise and arbitrary optical filtering: a rigorous approach,” *IEE Proc.-Optoelectron.*, Vol. 152, No. 5, pp. 251–262, Oct. 2005.
[24] N. Antoniades, K. C. Reichmann, P. P. Iannone and A. M. Levine, “Engineering methodology for the use of SOAs and CWDM transmission in the metro network environment,” *in Proc. Optical Fiber Commun. Conf. (OFC'2006)*, Anaheim, CA, USA, Mar. 2006, paper OTuG6.
[25] O. Liboiron-Ladouceur, K. Bergman, M. Borodisky, and M. Brodsky, “Effects of Cumulative PDG on the Scalability of SOA-based Optical Packet Switching Networks,” *in Proc. Optical Fiber Commun. Conf. (OFC'2006)*, Anaheim, CA, USA, Mar. 2006, paper JThB16.
[26] N. Antoniades, K. C. Reichmann, P. P. Iannone, N. J. Frigo, A. M. Levine, and I. Roudas, “Engineering methodology for the design of CWDM using SOAs in metro,” *IEEE Phot. Tech. Lett.*, submitted for publication.
[27] S. Park, R. Leavitt, R. Enck, V. Luciani, Y. Hu, P. J. S. Heim, D. Bowler, and M. Dagenais, “Semiconductor optical amplifier for CWDM operating over 1540–1620 nm,” *IEEE Phot. Tech. Lett.*, Vol. 17, No. 5, pp. 980–982, May 2005.
[28] M. J. O’Mahony, “Semiconductor laser optical amplifiers for use in future fiber systems,” *J. Lightwave Technol.,* Vol. 6, pp. 531–544, Apr. 1988.
[29] K. C. Reichmann, P. P. Iannone, X. Zhou, N. J. Frigo, and B. R. Hemenway, “240-km CWDM transmission using cascaded SOA Raman hybrid amplifiers with 70-nm bandwidth,” *IEEE Phot. Tech. Lett.*, Vol. 18, No. 2, pp. 328–330, Jan. 2006.
[30] W. Feller, *An introduction to probability theory and its applications,* 2d ed., Vol.2. New York: Wiley, 1971.
[31] C. D. Poole and D. L. Favin, “Polarization-mode dispersion measurements based on transmission spectra through a polarizer,” *J. Lightwave Technol*., Vol. 12, No. 6, pp. 917–929, Jun. 1994.
[32] A. Papoulis and S. U. Pillai, *Probability, Random Variables and Stochastic Processes*, 4th ed., McGraw-Hill, New York, 2002.
[33] N. Johnson, S. Kotz, and N. Balakrishnan, *Discrete univariate distributions*. 2d ed., Vol. 2, New York: Wiley, 1995.
[34] E. B. Saff and A. B. J. Kuijlaars, “Distributing many points on a sphere,” *Math. Intelligencer*, Vol. 19, No. 1, pp. 5–11, 1997.