Scalars, Vectors and Tensors

A **scalar** is a physical quantity that it represented by a dimensional number at a particular point in space and time. Examples are hydrostatic pressure and temperature.

A **vector** is a bookkeeping tool to keep track of two pieces of information (typically *magnitude* and *direction*) for a physical quantity. Examples are position, force and velocity. The vector has three components.

$$\text{velocity vector} \qquad \vec{v} = v_1\vec{x}_1 + v_2\vec{x}_2 + v_3\vec{x}_3 = \sum_i v_i\vec{x}_i = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

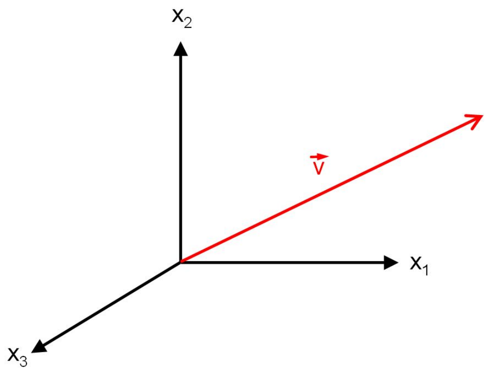


Figure 1: The Velocity Vector

The magnitude of the velocity is a scalar $v \equiv |\vec{v}|$

What happens when we need to keep track of *three pieces* of information for a given physical quantity?

We need a **tensor**. Examples are stress and strain. The tensor has nine components.

$$\text{stress tensor} \qquad \sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \tag{1-25}$$

For stress, we keep track of a *magnitude*, *direction* and which *plane the component acts on.*

The Stress Tensor

$$\text{stress tensor} \qquad \sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \tag{1-25}$$

The first subscript keeps track of the plane the component acts on (described by its unit normal vector), while the second subscript keeps track of the direction. Each component represents a magnitude for that particular plane and direction.

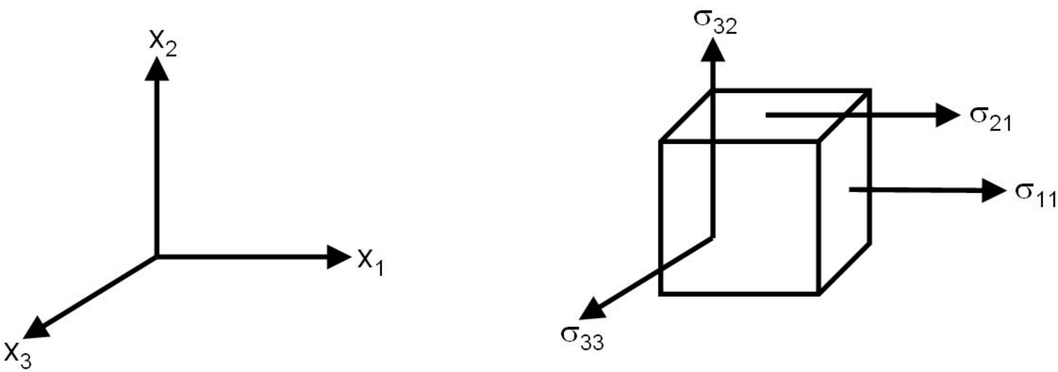


Figure 2: Four of the nine components of the stress tensor acting on a small cubic fluid element.

$$\text{The stress tensor is always } \textbf{symmetric} \qquad \sigma_{ij} = \sigma_{ji} \tag{1-26}$$

Thus there are only six independent components of the stress tensor.

Tensor calculus will not be required in this course.

The Stress Tensor

EXAMPLE: SIMPLE SHEAR

$$\sigma_{12} = \sigma_{21} \equiv \sigma = F/A \tag{1-27}$$

$$\sigma_{13} = \sigma_{31} = \sigma_{32} = \sigma_{32} = 0 \tag{1-28}$$

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma & 0 \\ \sigma & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix} \tag{1-29}$$

EXAMPLE: SIMPLE EXTENSION

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix} \tag{1-30}$$

EXAMPLE: HYDROSTATIC PRESSURE

$$\sigma_{ij} = \begin{bmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{bmatrix} \tag{1-30}$$

The minus sign is because pressure compresses the fluid element.

Polymers (and most liquids) are nearly incompressible. For an incompressible fluid, the hydrostatic pressure does not affect any properties. For this reason only differences in normal stresses are important.

In simple shear, the first and second **normal stress differences** are:

$$N_1 \equiv \sigma_{11} - \sigma_{22} \tag{1-32}$$

$$N_2 \equiv \sigma_{22} - \sigma_{33} \tag{1-33}$$

The Extra Stress Tensor is defined as

$$\tau_{ij} \equiv \begin{cases} \sigma_{ij} + P & \text{for } i = j \\ \sigma_{ij} & \text{for } i \neq j \end{cases} \tag{1-35}$$

Strain and Strain Rate Tensors

Strain is a dimensionless measure of local deformation. Since we want to relate it to the stress tensor, we had best define the strain tensor to be symmetric.

$$\gamma_{ij}(t_1, t_2) \equiv \frac{\partial u_i(t_2)}{\partial x_j(t_1)} + \frac{\partial u_j(t_2)}{\partial x_i(t_1)} \tag{1-37}$$

$\vec{u} = u_1\vec{x}_1 + u_2\vec{x}_2 + u_3\vec{x}_3$ is the displacement vector of a fluid element at time t_2 relative to its position at time t_1.

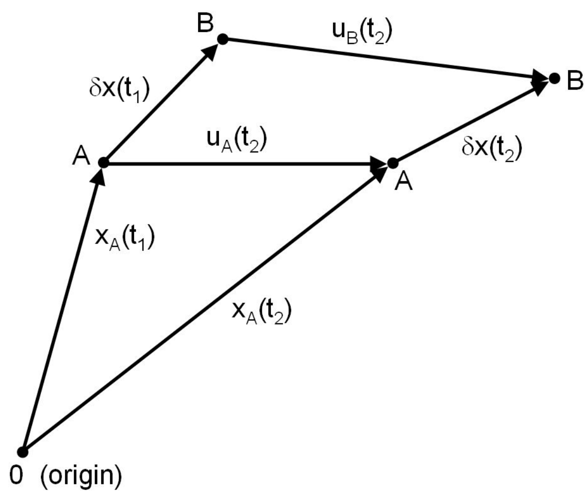


Figure 3: Displacement Vectors for two Fluid Elements A and B.

The **strain rate tensor** (or rate of deformation tensor) is the time derivative of the strain tensor.

$$\dot{\gamma}_{ij} \equiv d\gamma_{ij}/dt \tag{1-38}$$

The components of the local velocity vector are $v_i = du_i/dt$ (1-39). Since the coordinates x_i and time t are independent variables, we can switch the order of differentiations.

$$\dot{\gamma}_{ij} \equiv \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \tag{1-40}$$

Strain and Strain Rate Tensors
EXAMPLE: SIMPLE SHEAR

$$\gamma_{ij} = \begin{bmatrix} 0 & \gamma & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Where the scalar $\gamma = \partial u_1/\partial x_2 + \partial u_2/\partial x_1$ (1-41)

$$\dot{\gamma}_{ij} = \begin{bmatrix} 0 & \dot{\gamma} & 0 \\ \dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1-51}$$

Where the scalar $\dot{\gamma} \equiv d\gamma/dt$.

EXAMPLE: SIMPLE EXTENSION

$$\gamma_{ij} = \begin{bmatrix} 2\varepsilon & 0 & 0 \\ 0 & -\varepsilon & 0 \\ 0 & 0 & -\varepsilon \end{bmatrix} \tag{1-47}$$

Where the scalar $\varepsilon \equiv \partial u_1/\partial x_1$.

$$\dot{\gamma}_{ij} = \begin{bmatrix} 2\dot{\varepsilon} & 0 & 0 \\ 0 & -\dot{\varepsilon} & 0 \\ 0 & 0 & -\dot{\varepsilon} \end{bmatrix} \tag{1-48}$$

Where the scalar $\dot{\varepsilon} \equiv d\varepsilon/dt$.

The Newtonian Fluid

Newton's Law is written in terms of the extra stress tensor

$$\tau_{ij} = \eta \dot{\gamma}_{ij} \tag{1-49}$$

Equation (1-49) is valid for all components of the extra stress tensor in **any flow** of a Newtonian fluid. All low molar mass liquids are Newtonian (such as water, benzene, etc.)

EXAMPLE: SIMPLE SHEAR

$$\dot{\gamma}_{ij} = \begin{bmatrix} 0 & \dot{\gamma} & 0 \\ \dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1-48}$$

$$\tau_{ij} = \begin{bmatrix} 0 & \eta\dot{\gamma} & 0 \\ \eta\dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1-48}$$

The normal components are all zero

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = 0 \tag{1-53}$$

and the normal stress differences are thus both zero for the Newtonian fluid

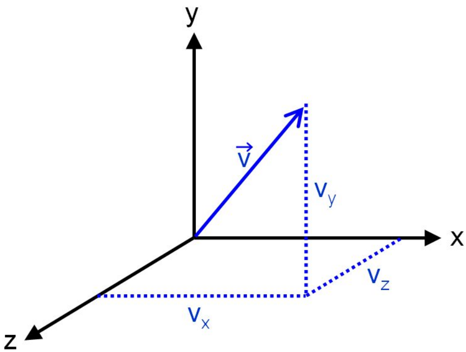
$$N_1 = N_2 = 0 \tag{1-54}$$

EXAMPLE: SIMPLE EXTENSION

$$\dot{\gamma}_{ij} = \begin{bmatrix} 2\dot{\varepsilon} & 0 & 0 \\ 0 & -\dot{\varepsilon} & 0 \\ 0 & 0 & -\dot{\varepsilon} \end{bmatrix} \tag{1-48}$$

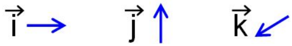
$$\tau_{ij} = \eta \begin{bmatrix} 2\dot{\varepsilon} & 0 & 0 \\ 0 & -\dot{\varepsilon} & 0 \\ 0 & 0 & -\dot{\varepsilon} \end{bmatrix} \tag{1-55}$$

Vector Calculus (p. 1)



velocity $\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$

$\vec{i}$, $\vec{j}$, $\vec{k}$ are unit vectors in the x, y, z directions



acceleration $\vec{a} = \frac{d\vec{v}}{dt} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$

$\vec{a} = \frac{d\vec{v}}{dt}$ involves *vectors* so it represents *three* equations:

$$a_x = \frac{dv_x}{dt} \quad a_y = \frac{dv_y}{dt} \quad a_z = \frac{dv_z}{dt}$$

$$\therefore \quad \vec{a} = \frac{dv_x}{dt}\vec{i} + \frac{dv_y}{dt}\vec{j} + \frac{dv_z}{dt}\vec{k}$$

The vector is simply notation (bookkeeping).

Divergence n**.** $\quad \vec{\nabla}\cdot\vec{v} \equiv \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

If $\frac{\partial v_x}{\partial x} > 0$ and $\frac{\partial v_y}{\partial y} > 0$ and $\frac{\partial v_z}{\partial z} > 0$
and $v_x > 0$ and $v_y > 0$ and $v_z > 0$
then you have an explosion! $\quad \therefore$ the name divergence.

Vector Calculus (p. 2)

Gradient *n.* $\vec{\nabla} P \equiv \frac{\partial P}{\partial x}\vec{i} + \frac{\partial P}{\partial y}\vec{j} + \frac{\partial P}{\partial z}\vec{k}$

3-D analog of a derivative

Gradient operator makes a vector from the scalar P.

(whereas divergence made a scalar from the vector $\vec{v}$).

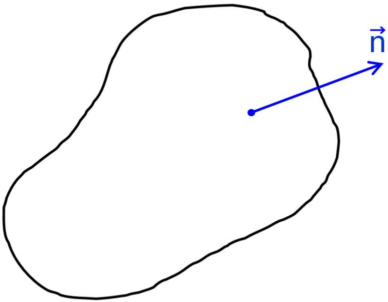
Laplacian *n.* $\nabla^2 \vec{v} \equiv \frac{\partial^2 \vec{v}_x}{\partial x^2} + \frac{\partial^2 \vec{v}_y}{\partial y^2} + \frac{\partial^2 \vec{v}_z}{\partial z^2}$

$$\begin{aligned}\nabla^2 \vec{v} \;=\; & \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right)\vec{i} \\ & + \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right)\vec{j} \\ & + \left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right)\vec{k}\end{aligned}$$

Laplacian operator makes a vector from a the vector $\vec{v}$.

Also written as $\vec{\nabla} \cdot \vec{\nabla} \vec{v}$, divergence of the gradient of $\vec{v}$.

Conservation of Mass in a Fluid (p. 2)
THE EQUATION OF CONTINUITY



Consider a Control Volume

$\rho =$ density (scalar)

$\vec{v} =$ velocity vector

$\vec{n} =$ unit normal vector on surface

$\rho(\vec{v} \cdot \vec{n})dA =$ net flux out of the control volume through small area dA.

$\int_S \rho(\vec{v} \cdot \vec{n})dA =$ net flow rate (mass per unit time) out of control volume.

$\int_V \rho dV =$ total mass inside the control volume.

$\frac{d}{dt}\int_V \rho dV =$ rate of accumulation of mass inside the control volume.

Mass Balance: $\frac{d}{dt}\int_V \rho dV = -\int_S \rho(\vec{v} \cdot \vec{n})dA$

Conservation of Mass in a Fluid (p. 2)

$$\frac{d}{dt}\int_V \rho dV = -\int_S \rho(\vec{v}\cdot\vec{n})dA$$

Integral mass balance can be written as a differential equation using the Divergence Theorem.

Divergence Theorem:

$$\int_S \rho(\vec{v}\cdot\vec{n})dA = \int_V \vec{\nabla}\cdot(\rho\vec{v})dV$$

$$\int_V \left(\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v})\right) dV = 0$$

Control volume was chosen arbitrarily

$$\therefore \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0$$

Derived for a control volume but applicable to any point in space.

Cornerstone of continuum mechanics!!!

Applies for both gases and liquids.

Conservation of Mass in a Liquid

A *liquid is incompressible*

Density ρ is always the same

$\frac{\partial \rho}{\partial t} = 0 \qquad \rho = \text{constant}$

Integral Mass Balance

$$\frac{d}{dt}\int_V \rho dV = -\int_S \rho(\vec{v}\cdot\vec{n})dA$$

becomes

$$\int_S (\vec{v}\cdot\vec{n})dA = 0$$

Continuity Equation $\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0$

becomes $\vec{\nabla}\cdot\vec{v} = 0$ for incompressible liquids

In Cartesian coordinates:

$$\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0 \tag{1-57}$$

Incompressible Continuity Equation for Liquids

Cartesian Coordinates: x, y, z

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

Cylindrical Coordinates: r, θ, z

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$$

Spherical Coordinates: r, θ, φ

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(v_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial \varphi} = 0$$

All are simply $\vec{\nabla} \cdot \vec{v} = 0$