Scattering Cross-sections in Quantum Electrodynamics

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This Thesis entitled Scattering Cross-sections in Quantum Electrodynamics by Ashish Chandra is approved for the degree of Master of Science from IIT Hyderabad

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Abstract

Our understanding of the fundamental interactions -electroweak and strong forces is described by a theory which is now known as the Standard Model of particle physics. In this thesis, I study the electroweak sector of the Standard Model. Using the Feynman rules of quantum electrodynamics, I derive cross-sections for $e^-e^+ \longrightarrow \mu^-\mu^+$ and for Compton Scattering i.e, $\gamma + e^- \longrightarrow \gamma + e^-$. First I obtain the cross-section for the scattering with unpolarized leptons and then to get a more detailed understanding of the process I re-study the scattering with polarized leptons. I give the details of the calculations and present some of the results involving gamma matrices and spinor sums in the appendix.

Contents

Chapter 1

Introduction

We know that particle numbers are not conserved when we relativistically collide them. We can not describe the physics behind this processes using single particle quantum mechanics. But quantum field theory combines two of the major themes of modern physics special relativity and quantum mechanics to describe this type of interactions. Quantum electrodynamics is an extension of quantum mechanics for analysis of the system with many particles. It gives the mathematical and conceptual framework for elementary particle physics. This theory supplies essential tools to nuclear physics, atomic physics, and astrophysics. Also, quantum field theory has led to new bridges between physics and mathematics. Since this thesis is about scattering cross-section in quantum electrodynamics, so it is instructive to start with an introduction of quantum electrodynamics (QED). It is a field theory of interaction between light and matter. It also gives full information on the interaction between charged particles (leptons). This theory allows us to predict how subatomic charged particles are created or destroyed. QED is also termed a gauge-invariant theory because its predictions are not affected by variations in space or time. The practical value of electromagnetic interactions gives the same result as QED theory predicts.

1.1 Background

Quantum Field Theory (QFT) is the theoretical framework for describing the phenomenon in particle physics. QFT treats particles as excited states of the underlying physical field. In QFT we describe the interactions using Feynman diagrams. QFT is not only mathematically rich but also well verified with high accuracy in experiments.

1.2 Objectives

The main objectives for my master's project are

1. Scattering cross-section for $e^-e^+ \longrightarrow \mu^-\mu^+$ interaction.

2. Scattering cross-section for Compton scattering.

1.3 Approach

We used Feynman rules of Quantum Electrodynamics (QED) to calculate scattering amplitude for $e^-e^+ \longrightarrow \mu^-\mu^+$ and Compton scattering. We squared and summed over all spins to get the expression for the differential cross-section. We used trace technology and center of the mass frame to simplify our differential cross-section. To get total cross-section, we integrated our differential cross-section over $d\Omega$ (solid angle).

Chapter 2

Lowest order interactions in Quantum Electrodynamics

2.1 Introduction

Quantum Electrodynamics (QED) is the theory of interaction between leptons (described by Dirac field $\psi(x)$) and photons (described by the electromagnetic field $A^\mu(x)$). Here these fields are operators in Heisenberg picture. The Lagrangian density in QED is defined as

$$\mathcal{L}_{QED} = \mathcal{L}_{Dirac} + \mathcal{L}_{Maxwell} + \mathcal{L}_{Int}$$

$$\mathcal{L}_{QED} = \overline{\psi}(i\partial\!\!\!/ - m)\psi - \frac{1}{4}(F_{\mu\nu})^2 - e\overline{\psi}\gamma^\mu\psi A_\mu$$

Where the first two terms on the last line describe free photons and electrons, and the third term treated as an interaction between Dirac and Electromagnetic field. This interacting Lagrangian gives the interacting Hamiltonian which interpreted as perturbation term. So we used time-dependent perturbation theory for interacting fields to calculate amplitude for the propagation of a particle. This amplitude includes interacting Hamiltonian, which helps us to use wick's theorem, to turn this type of amplitude into a sum of products of Feynman propagators. Wick's theorem helps us to set the Feynman rules for corresponding theory (scalar, electromagnetic theory), but for our purpose the Feynman diagram in QED is defined as

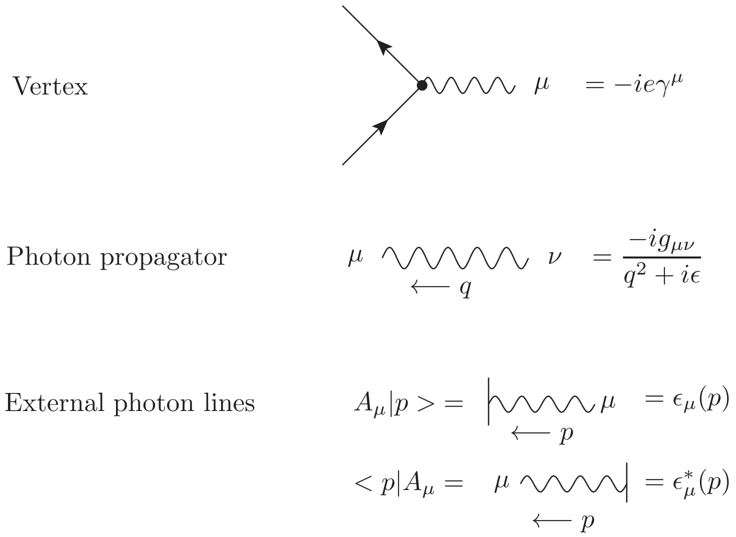

Figure 2.1: Feynman Rules in Quantum Electrodynamics

The symbol $\varepsilon_\mu(p)$ stands for the polarization vector of the initial or final state photon. Photons conventionally drawn as wavy lines. These Feynman rules simplify our problem of scattering amplitude and allow us to write directly scattering amplitude for a given interaction.

2.2 Unpolarized Scattering cross-section for $e^-e^+ \to \mu^-\mu^+$ interaction

Unpolarized scattering cross-section is defined as the sum of spins of scattered particles and average of incident particle spins. Here, we do not know what are the spins of incident and scattered particles have. The reaction $e^-e^+ \to \mu^-\mu^+$ is an elementary process but gives fundamental to the understanding of all reactions in e^-e^+ colliders. Using the Feynman rules we can at once draw the diagram for lowest order in α.

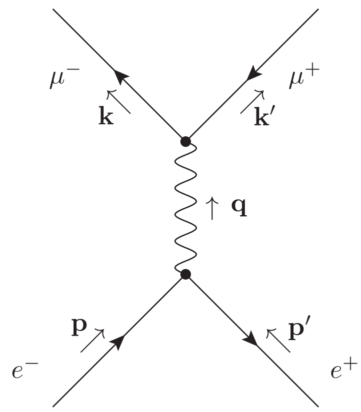

Figure 2.2: Feynman diagram for $e^-e^+ \to \mu^-\mu^+$ interaction

The interaction part of S-matrix gives the scattering amplitude as $-i\mathscr{M}$. So, using Feynman rules one can write the scattering amplitude for $e^-e^+ \to \mu^-\mu^+$ interaction.

$$-i\mathscr{M} = \overline{v}^{s'}(p')(ie\gamma^\mu)u^s(p)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\overline{u}^r(k)(ie\gamma^\nu)v^{r'}(k') \tag{2.1}$$

Differential scattering cross-section is directly proportional to the modulus square of scattering amplitude. So we need an Expression of $|\mathscr{M}|^2$ for calculating differential scattering cross-section.

$$|\mathscr{M}|^2 = \frac{e^4}{q^4}\left(\overline{v}^{s'}(p')(\gamma^\nu)u^s(p)\overline{u}^r(k)(\gamma_\nu)v^{r'}(k')\right)^\dagger\left(\overline{v}^{s'}(p')(\gamma^\mu)u^s(p)\overline{u}^r(k)(\gamma_\mu)v^{r'}(k')\right) \tag{2.2}$$

Since γ^μ and γ^0 are anti-hermitian and hermitian matrices hence, we can write

$$(\overline{v}\gamma^\mu u)^\dagger = u^\dagger(\gamma^\mu)^\dagger(\gamma^0)^\dagger v \tag{2.3}$$

$$u^\dagger(\gamma^\mu)^\dagger(\gamma^0)^\dagger v = u^\dagger(\gamma^\mu)^\dagger\gamma^0 v \tag{2.4}$$

$$u^\dagger(\gamma^\mu)^\dagger\gamma^0 v = u^\dagger\gamma^0\gamma^\mu v \tag{2.5}$$

$$u^\dagger\gamma^0\gamma^\mu v = \overline{u}\gamma^\mu v \tag{2.6}$$

$$|\mathscr{M}|^2 = \frac{e^4}{q^4}\left(\overline{u}^r(k)(\gamma_\nu)v^{r'}(k')\right)^\dagger\left(\overline{v}^{s'}(p')(\gamma^\nu)u^s(p)\right)^\dagger\left(\overline{v}^{s'}(p')(\gamma^\mu)u^s(p)\right)\left(\overline{u}^r(k)(\gamma_\mu)v^{r'}(k')\right) \tag{2.7}$$

Using equation 2.6 we can write $|\mathscr{M}|^2$ as

$$|\mathscr{M}|^2 = \frac{e^4}{q^4}\left(\overline{v}^{r'}(k')\gamma_\nu u^r(k)\overline{u}^s(p)\gamma^\nu v^{s'}(p')\overline{v}^{s'}(p')\gamma^\mu u^s(p)\overline{u}^r(k)\gamma_\mu v^{r'}(k')\right) \tag{2.8}$$

Since μ^- and μ^+ are the scattered particles but usually muon detectors are blind to detect polarization, So the measured differential cross-section is a sum over the muon spins r and r'.

$$\sum_r\sum_{r'}|\mathscr{M}|^2$$

e^- and e^+ are the incident unpolarized particles. So the differential cross-section is an average over the electron and positron spins s and s'.

$$\frac{1}{2}\sum_{s'}\frac{1}{2}\sum_s\sum_r\sum_{r'}|\mathscr{M}|^2$$

Sum over the polarization states of fermion

$$\sum_s u^s(p)\overline{u}^s(p) = \not{p} + m$$

Sum over the polarization states of antifermion

$$\sum_{s'} v^{s'}(p)\overline{v}^{s'}(p) = \not{p} - m$$

By summing over incoming particle spins, we will get (here we are writing equation in terms of components of a matrices)

$$\sum_s \sum_{s'} \overline{v_a}^{s'}(p')\gamma^\mu_{ab} u^s_b(p)\overline{u_c}^s \gamma^\nu_{cd} v^{s'}_d(p') \quad = \quad (\not{p}' - m_e)_{da}\gamma^\mu_{ab}(\not{p} + m_e)_{bc}\gamma^\nu_{cd} \tag{2.9}$$

Similarly for $\mu-$ and $\mu+$, we will get

$$\sum_r \sum_{r'} \overline{v}^{r'}(k')\gamma^\mu u^r(k)\overline{u}^r(k)\gamma^\nu v^{r'}(k') \quad = \quad (\not{k} + m_\mu)\gamma_\mu(\not{k}' - m_\mu)\gamma_\nu$$

Here m_μ is the mass of the muon particle.
Using the spin summing technique in equation (2.8) $\big($similar technique we have shown in equation (2.9)$\big)$, gives the expression for $\sum_{spin} |\mathscr{M}|^2$ as

$$\frac{1}{4}\sum_{spin} |\mathscr{M}|^2 = \frac{e^4}{4q^4} Tr\Big[(\not{p}' - m_e)\gamma^\mu(\not{p} + m_e)\gamma^\nu\Big] Tr\Big[(\not{k} + m_\mu)\gamma_\mu(\not{k}' - m_\mu)\gamma_\nu\Big] \tag{2.10}$$

Now, we will use trace technology to simplify our problem (here trace of whole matrices leads us to a scalar). First we will derive the solution of $Tr\Big[(\not{p}' - m_e)\gamma^\mu(\not{p} + m_e)\gamma^\nu\Big]$ and then similarly we can write solution for $Tr\Big[(\not{k} + m_\mu)\gamma_\mu(\not{k}' - m_\mu)\gamma_\nu\Big]$

$$\begin{aligned} Tr\left[(\gamma^\sigma p'_\sigma - m_e)\gamma^\mu(\gamma^\rho p_\rho + m_e)\gamma^\nu\right] &= Tr\left[(\gamma^\sigma p'_\sigma\gamma^\mu - m_e\gamma^\mu)(\gamma^\rho p_\rho\gamma^\nu + m_e\gamma^\nu)\right] \\ &= Tr\left[\gamma^\sigma p'_\sigma\gamma^\mu\gamma^\rho p_\rho\gamma^\nu + \gamma^\rho p'_\sigma\gamma^\mu m_e\gamma^\nu - m_e\gamma^\mu\gamma^\rho p_\rho\gamma^\nu - m_e^2\gamma^\mu\gamma^\nu\right] \\ &= Tr\left(\gamma^\sigma p'_\sigma\gamma^\mu\gamma^\rho p_\rho\gamma^\nu\right) + Tr\left(\gamma^\rho p'_\sigma\gamma^\mu m_e\gamma^\nu\right) - Tr\left(m_e\gamma^\mu\gamma^\rho p_\rho\gamma^\nu\right) \\ &\quad - Tr\left(m_e^2\gamma^\mu\gamma^\nu\right) \end{aligned}$$

Since trace of odd gamma matrices become zero. hence, we left with two terms

$$Tr\left[(\gamma^\sigma p'_\sigma - m_e)\gamma^\mu(\gamma^\rho p_\rho + m_e)\gamma^\nu\right] = Tr\left(\gamma^\sigma p'_\sigma\gamma^\mu\gamma^\rho p_\rho\gamma^\nu\right) - Tr\left(m_e^2\gamma^\mu\gamma^\nu\right)$$

The trace of $\left[\gamma^\sigma p'_\sigma\gamma^\mu\gamma^\rho p_\rho\gamma^\nu\right]$ matrices will not change if we interchange the matrices in trace, hence

$$Tr\left[(\gamma^\sigma p'_\sigma - m_e)\gamma^\mu(\gamma^\rho p_\rho + m_e)\gamma^\nu\right] = Tr\left(\gamma^\sigma\gamma^\mu\gamma^\rho\gamma^\nu p'_\sigma p_\rho\right) - Tr\left(m_e^2\gamma^\mu\gamma^\nu\right)$$

By using gamma matrices identity as

$$\begin{aligned} Tr[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] &= 4\left[g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right] \\ Tr\left(\gamma^\sigma\gamma^\mu\gamma^\rho\gamma^\nu p'_\sigma p_\rho\right) &= 4Tr\left[g^{\sigma\mu}g^{\rho\nu}p'_\sigma p_\rho - g^{\sigma\rho}g^{\mu\nu}p'_\sigma p_\rho + g^{\sigma\nu}g^{\mu\rho}p'_\sigma p_\rho\right] \\ Tr\left(\gamma^\sigma\gamma^\mu\gamma^\rho\gamma^\nu p'_\sigma p_\rho\right) &= 4Tr\left[p'^\mu p^\nu - g^{\mu\nu}p'^\rho p_\rho + p'^\nu p^\mu\right] \end{aligned}$$

By using gamma matrices identity

$$\begin{aligned} Tr\left[\gamma^\mu, \gamma^\nu\right] &= 2g^{\mu\nu} \\ Tr\left(\gamma^\mu\gamma^\nu\right) &= 4g^{\mu\nu} \end{aligned}$$

Final expression after simplifying equation 2.10

$$\begin{aligned} Tr\left[(\gamma^\sigma p'_\sigma - m_e)\gamma^\mu(\gamma^\rho p_\rho + m_e)\gamma^\nu\right] &= Tr\left(\gamma^\sigma p'_\sigma \gamma^\mu \gamma^\rho p_\rho \gamma^\nu\right) - Tr\left(m_e^2 \gamma^\mu \gamma^\nu\right) \\ &= 4Tr\left[p'^\mu p^\nu - g^{\mu\nu} p'^\rho p_\rho + p'^\nu p^\mu - 4m_e^2 g^{\mu\nu}\right] \end{aligned}$$

Similarly, we can get the expression for $Tr\left[(\gamma^\sigma k_\sigma + m_\mu)\gamma_\mu(\gamma^\rho k'_\rho - m_\mu)\gamma_\nu\right]$.

$$Tr\left[(\gamma^\sigma k_\sigma + m_\mu)\gamma_\mu(\gamma^\rho k'_\rho - m_\mu)\gamma_\nu\right] = 4Tr\left[k'_\mu k_\nu - g^{\mu\nu} k'^\rho k_\rho + k'_\nu k_\mu - 4m_\mu^2 g^{\mu\nu}\right]$$

Now, we can write $(1/4)\sum_{spin} |\mathscr{M}|^2$ in terms of above simplified formula

$$\frac{1}{4}\sum_{spin} |\mathscr{M}|^2 = \frac{4e^4}{q^4}\left[p'^\mu p^\nu - g^{\mu\nu} p'^\rho p_\rho + p'^\nu p^\mu - 4m_e^2 g^{\mu\nu}\right]\left[k'_\mu k_\nu - g^{\mu\nu} k'^\rho k_\rho + k'_\nu k_\mu - 4m_\mu^2 g^{\mu\nu}\right]$$

Here, we are taking $m_e = 0$ because incoming particles require high energy (Kinetic Energy) to produce larger mass particles. Then, we will get

$$\frac{1}{4}\sum_{spin} |\mathscr{M}|^2 = \frac{e^4}{q^4}\left[\left(p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu}[p' \cdot p]\right)\left(k'_\mu k_\nu + k'_\nu k_\mu - g_{\mu\nu}[k' \cdot k + m_\mu^2]\right)\right] \tag{2.11}$$

By solving above equation, We will get

$$\frac{1}{4}\sum_{spin} |\mathscr{M}|^2 = \frac{8e^4}{q^4}\left[(p' \cdot k)(p \cdot k') + (p' \cdot k')(p \cdot k) + (p \cdot p')m_\mu^2\right] \tag{2.12}$$

Since, we know that $\sum_{spin} |\mathscr{M}|^2$ is proportional to a differential cross-section which is a physical quantity. So, we must need to write above expression regarding energy and angle. The vectors p, p', k, k' and q concerning the basic kinematic variables-energies and angles. We need to take a particular frame of reference for which our equation become accessible. Now, we are making the simplest choice to evaluating cross-section in center-of-mass frame.

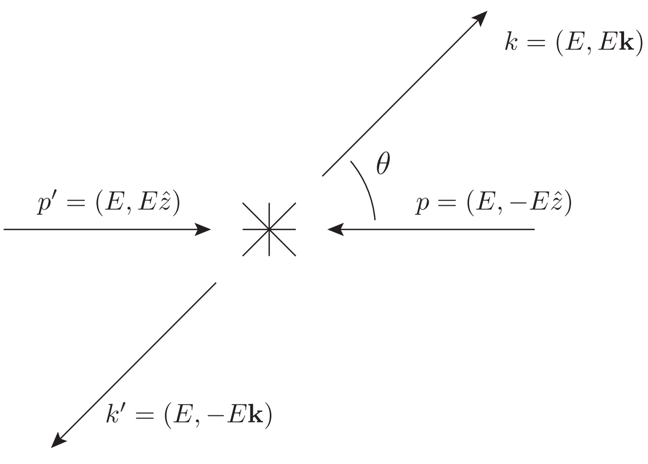

Figure 2.3: $e^-e^+ \to \mu^-\mu^+$ interaction in center-of-mass frame

To compute the modulus square matrix element, we need to write equation (2.12) in terms of energy and angle.

$$\begin{aligned} q^2 &= (p+p')^2 = 4E^2 \\ p\cdot p' &= 2E^2 \\ p\cdot k &= p'\cdot k' = E^2 - E|k|\cos\theta \\ p\cdot k' &= p'\cdot k = E^2 + E|k|\cos\theta \end{aligned}$$

Now, putting the values of $p\cdot k$, $p\cdot k'$ etc in the $(1/4)\sum_{spin}|\mathscr{M}|^2$ expression. It gives the following result

$$\frac{1}{4}\sum_{spin}|\mathscr{M}|^2 = \frac{e^4}{2E^2}\Big[(E-|k|\cos\theta)^2 + (E+|k|\cos\theta)^2 + 2m_\mu^2\Big] \tag{2.13}$$

The $\sum_{spin}|\mathscr{M}|^2$ expression is written in the center-of-mass energy form. Here $E = (E_{cm}/2)$

2.3 Total scattering cross-section

When a beam of particles strikes a target consisting of particles of a different type, some of the particles pass directly through the target while other deflected. Those deflected particles are said to interact when they collide with the target particles. The cross section is a measure of the effectiveness of the incident and target particles interaction. Larger the cross section, the more likely it is that the incident particles deflected.
In the center of mass frame, our differential cross-section is given by

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{|P||\mathscr{M}|^2}{2E_A 2E_B|v_A - v_B|(2\pi)^2 4E_{cm}} \tag{2.14}$$

Above formula comes when we take the interaction part of S-matrix. For our given interaction relative velocity and energy of the incoming particle in center of mass frame defined as

$$|v_A - v_B| = 2 \tag{2.15}$$

$$E_A = E_B = \frac{E_{cm}}{2} \tag{2.16}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{1}{2E_{cm}^2}\frac{|k|}{16\pi^2 E_{cm}}\frac{1}{4}\sum_{spins}|\mathscr{M}|^2 \tag{2.17}$$

Integrating differential scattering cross-section over solid angle ($d\Omega$) for total cross-section

$$\sigma_T = \frac{\alpha^2}{4E_{cm}^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\int_0^{2\pi}\int_0^{\pi}\left(1+\frac{m_\mu^2}{E^2}\right)\sin\theta d\theta d\phi$$

$$+\frac{\alpha^2}{4E_{cm}^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\left(1-\frac{m_\mu^2}{E^2}\right)\int_0^{2\pi}\int_0^{2\pi}\left(\frac{\cos(2\theta)+1}{2}\right)\sin\theta d\theta d\phi$$

By solving integration, our total scattering cross-section formula is

$$\sigma_T = \frac{4\pi}{3}\alpha^2\sqrt{1-\frac{m_\mu^2}{E^2}}\left[1+\frac{1}{2}\frac{m_\mu^2}{E^2}\right] \tag{2.18}$$

In the above expression when the energy of the incoming particle is less than the rest mass energy of the muon then the total cross-section will become zero. This information gives us that the energy of the incoming particle should always be greater than or equal to the rest mass energy of muons to produce them. The energy for which our total cross-section becomes finite called as threshold energy.

2.4 Result

In the high energy limit where$E >> m_\mu$, these formula reduce to

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} \xrightarrow[E>>m_\mu]{} \frac{\alpha^2}{4E_{cm}^2}(1+\cos^2\theta) \tag{2.19}$$

$$\sigma_T \xrightarrow[E>>m_\mu]{} \frac{4\pi\alpha^2}{3E_{cm}^2} \tag{2.20}$$

Here, we can see the angular dependence on the differential cross-section in the high energy limit. But the total scattering cross-section has energy dependence and strength of the electromagnetic interaction dependence. But as we can see from the above formula, differential cross-section has maximum value for $(\theta = 0, \pi)$. In the high-energy limit, E_{cm} is the only dimensionful quantity in this process, so dimensional analysis dictates that $\sigma_T \propto E_{cm}^{-2}$. Since, we knew intuitively from the beginning that $\sigma_T \propto \alpha^2$, we only had to work to get the factor of $(4\pi/3)$ in total scattering cross-section. Here, scattering cross-section is zero for $E_{cm} < 2m_\mu$. But, the threshold energy for which scattering cross-section is finite will be the rest mass energy of the incoming particle.

2.5 Polarized electron and positron annihilation

In our previous discussion, we have used unpolarized incident and scattered particles to find the average scattering cross-section. Now, we will take definite polarization of incident and scattered particles using helicity operator to calculate scattering cross-section. But, we can check average scattering cross-section by taking an average of all the possible scattering cross-section corresponding polarization of incident and scattered particles. Here, our Feynman diagram will be same for

$e^+e^- \to \mu^+\mu^-$ and lowest order in α. This calculation of polarized cross-section will help us to understand how the angular dependence appears in unpolarized cross-section. Here, we used helicity projection operator to project out the desired left and right-handed spinor for incoming and outgoing particles. Throughout this section, we work in high energy limit. Without putting helicity operator in scattering amplitude gives the same result as got for the unpolarized case.

2.6 Helicity

The helicity gives the projection of the directions of spin and the particle's momentum. If the 3-momentum **p** and spin both point in the same direction, the helicity has its maximum value (positive value), while if they point in opposite directions, the helicity has its maximum negative value. If **p** and spin are at right angles, the helicity is zero. Here, we are taking γ^5 matric as

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Now, we will take only one set of polarizations at a time. To do this, our projections operators onto right-and left-handed spinors, respectively

$$\frac{\mathbf{I}+\gamma^5}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\frac{\mathbf{I}-\gamma^5}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Now, we can make replacement in amplitude for right-handed spinor as

$$\overline{v}(p')\gamma^\mu u(p) \to \overline{v}(p')\gamma^\mu \left(\frac{\mathbf{I}+\gamma^5}{2}\right) u(p)$$

Scattering amplitude for $e^-_R e^+_L \to \mu^-_R \mu^+_L$ after replacement

$$i\mathscr{M} = \frac{ie^2}{q^2}\left(\overline{v}(p')\gamma^\mu \left(\frac{\mathbf{I}+\gamma^5}{2}\right) u(p)\right)\left(\overline{u}(k)\gamma_\nu \left(\frac{\mathbf{I}+\gamma^5}{2}\right) v(k')\right) \tag{2.21}$$

Here, we have taken right handed electron and the simplification of initial current density in Feynman diagram is given by

$$\overline{v}(p')\gamma^\mu \left(\frac{\mathbf{I}+\gamma^5}{2}\right) u(p) = v^\dagger(p')\gamma^{0\dagger}\gamma^\mu \left(\frac{\mathbf{I}+\gamma^5}{2}\right) u(p)$$

Since, we know that γ^μ, γ^0 and γ^5 are hermitian matrix

$$v^\dagger(p')\left[\left(\frac{\mathbf{I}+\gamma^5}{2}\right)\gamma^\mu\gamma^0\right]^\dagger u(p)$$

We know that right handed electron corresponds to a left-handed positron. Hence, the amplitude vanishes unless the electron and positron have their opposite helicity or equivalently unless their

spinor have the same helicity.
Now, the sum over the electron and positron spins in the modulus square amplitude.

$$\sum_{spin} \left|\overline{v}(p')\gamma^{\mu}\left(\frac{\mathbf{I}+\gamma^5}{2}\right)u(p)\right|^2 = \sum_{spin}\left[\overline{v}(p')\gamma^{\mu}\left(\frac{\mathbf{I}+\gamma^5}{2}\right)u(p)\overline{u}(p)\gamma^{\nu}\left(\frac{\mathbf{I}+\gamma^5}{2}\right)v(p')\right]$$

Since we know that the spins sums of ferimons and antifermions

$$\sum_{s} u^s(p)\overline{u}^s(p) = \not{p} + m$$

$$\sum_{s'} v^{s'}(p)\overline{v}^{s'}(p) = \not{p} - m$$

In high energy limit, we can take (m=0)

$$\sum_{s} u^s(p)\overline{u}^s(p) = \not{p}$$

$$\sum_{s'} v^{s'}(p)\overline{v}^{s'}(p) = \not{p}$$

By using above spins sum of ferimons and antifermions, we can write modulus square half amplitude as

$$\sum_{spin} |\overline{v}(p')\gamma^{\mu}\left(\frac{\mathbf{I}+\gamma^5}{2}\right)u(p)|^2 = \not{p}'\gamma^{\mu}\left(\frac{\mathbf{I}+\gamma^5}{2}\right)\not{p}\gamma^{\nu}\left(\frac{\mathbf{I}+\gamma^5}{2}\right)$$

Now, using trace technology

$$\begin{aligned} Tr\left[\not{p}'\gamma^{\mu}\left(\frac{\mathbf{I}+\gamma^5}{2}\right)\not{p}\gamma^{\nu}\left(\frac{\mathbf{I}+\gamma^5}{2}\right)\right] &= Tr\left[\frac{\gamma^{\sigma}p'_{\sigma}\gamma^{\mu}\gamma^{\rho}p_{\rho}\gamma^{\nu}}{4} + \frac{\gamma^{\sigma}p'_{\sigma}\gamma^{\mu}\gamma^{\rho}p_{\rho}\gamma^{\nu}}{4} + \frac{\gamma^{\sigma}p'_{\sigma}\gamma^{\mu}\gamma^{\rho}p_{\rho}\gamma^{\nu}\gamma^5}{4} \right.\\ &\qquad \left. + \frac{\gamma^{\sigma}p'_{\sigma}\gamma^{\mu}\gamma^5\gamma^{\rho}p_{\rho}\gamma^{\nu}\gamma^5}{4}\right] \\ &= \frac{1}{4}Tr\left[\gamma^{\sigma}\gamma^{\mu}\gamma^{\rho}\gamma^{\nu}p'_{\sigma}p_{\rho} + \gamma^{\sigma}\gamma^{\mu}\gamma^{\rho}\gamma^{\nu}p'_{\sigma}p_{\rho} + \gamma^{\sigma}\gamma^{\mu}\gamma^{\rho}\gamma^{\nu}\gamma^5 p'_{\sigma}p_{\rho}\right] \\ &= \frac{1}{4}Tr\left[2\gamma^{\sigma}\gamma^{\mu}\gamma^{\rho}\gamma^{\nu}p'_{\sigma}p_{\rho} + 2\gamma^{\sigma}\gamma^{\mu}\gamma^{\rho}\gamma^{\nu}p'_{\sigma}p_{\rho}\gamma^5\right] \qquad (2.22) \\ &= \frac{1}{2}Tr\Big(\gamma^{\sigma}\gamma^{\mu}\gamma^{\rho}\gamma^{\nu}p'_{\sigma}p_{\rho}\Big) + \frac{1}{2}Tr\Big(\gamma^{\sigma}\gamma^{\mu}\gamma^{\rho}\gamma^{\nu}p'_{\sigma}p_{\rho}\gamma^5\Big) \qquad (2.23) \end{aligned}$$

Now, using the gamma matrices identities

$$\begin{aligned} Tr[\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}] &= 4\left[g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right] \\ Tr[\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\gamma^5] &= -4i\epsilon^{\mu\nu\rho\sigma} \end{aligned}$$

Above identities makes the trace in Minkowski matrices, since p'_σ , p_ρ are vectors so we can take these out from trace.

$$\begin{aligned} Tr\left[\not{p}'\gamma^\mu\left(\frac{\mathbf{I}+\gamma^5}{2}\right)\not{p}\gamma^\nu\left(\frac{\mathbf{I}+\gamma^5}{2}\right)\right] &= \frac{1}{2}\left[Tr\left(\gamma^\sigma\gamma^\mu\gamma^\rho\gamma^\nu\right)+Tr\left(\gamma^\sigma\gamma^\mu\gamma^\rho\gamma^\nu\gamma^5\right)\right]p'_\sigma p_\rho \quad (2.24) \\ &= 2\Big[g^{\sigma\mu}g^{\rho\nu}-g^{\sigma\rho}g^{\mu\nu}+g^{\sigma\nu}g^{\mu\rho}-i\epsilon^{\mu\nu\rho\sigma}\Big]p'_\sigma p_\rho \quad (2.25) \end{aligned}$$

Final expression for modulus squared half amplitude

$$\sum_{spin}|\overline{v}(p')\gamma^\mu\left(\frac{\mathbf{I}+\gamma^5}{2}\right)u(p)|^2 = 2\left[p'^\mu p^\nu - g^{\mu\nu}(p'\cdot p)+p'^\nu p^\mu - i\epsilon^{\sigma\mu\rho\nu}p'_\sigma p_\rho\right] \quad (2.26)$$

Similarly, for muon modulus squared half amplitude given by

$$\sum_{spin}|\overline{u}(k)\gamma^\mu\left(\frac{\mathbf{I}+\gamma^5}{2}\right)v(k')|^2 = 2\left[k_\mu k'_\nu - g_{\mu\nu}(k\cdot k')+k^\nu k'_\mu - i\epsilon_{\sigma\mu\rho\nu}k^\sigma k'^\rho\right] \quad (2.27)$$

Further, we can write $|\mathscr{M}|^2$ for $e^-_R e^+_L \rightarrow \mu^-_R\mu^+_L$ interaction

$$\begin{aligned} \sum_{spin}|\mathscr{M}|^2 &= \frac{4e^4}{q^4}\left[p'^\mu p^\nu - g^{\mu\nu}(p'\cdot p)+p'^\nu p^\mu - i\epsilon^{\alpha\mu\beta\nu}p'_\alpha p_\beta\right] \\ &\quad\times\left[k_\mu k'_\nu - g_{\mu\nu}(k\cdot k')+k_\nu k'_\mu - i\epsilon_{\sigma\mu\rho\nu}k^\sigma k'^\rho\right] \quad (2.28) \\ &= \frac{4e^4}{q^4}\left[2(p\cdot k)(p'\cdot k')+2(p\cdot k')(p'\cdot k)-\epsilon^{\alpha\mu\beta\nu}\epsilon_{\sigma\mu\rho\nu}p'_\alpha p_\beta k^\rho k'^\sigma\right] \quad (2.29) \\ &= \frac{16e^4}{q^4}(p\cdot k')(p'\cdot k) \quad (2.30) \end{aligned}$$

Further, we can write differential cross-section for $e^-_R e^+_L \rightarrow \mu^-_R\mu^+_L$ interaction in center mass frame as

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{1}{2E^2_{cm}}\frac{|k|}{16\pi^2 E_{cm}}\frac{1}{4}\sum_{spins}|\mathscr{M}|^2 \quad (2.31)$$

In center of mass frame momentum product is given by

$$\begin{aligned} q^2 &= (p+p')^2 = 4E^2 \\ p\cdot p' &= 2E^2 \\ |k| &= \sqrt{E^2-m^2_\mu} \quad , \quad E=\frac{E_{cm}}{2} \\ p\cdot k &= p'\cdot k' = E^2 - E|k|\cos\theta \\ p\cdot k' &= p'\cdot k = E^2 + E|k|\cos\theta \end{aligned}$$

$$\begin{aligned} \left(\frac{d\sigma}{d\Omega}\right)_{cm} &= \frac{1}{2E^2_{cm}}\frac{|k|}{16\pi^2 E_{cm}}\frac{1}{4}\frac{16e^4}{q^4}(p\cdot k')(p'\cdot k) \\ \left(\frac{d\sigma}{d\Omega}\right)_{cm} &= \frac{1}{2E^2_{cm}}\frac{|k|}{\pi^2 E_{cm}}\frac{1}{4}\frac{e^4}{q^4}\left(E^2+E|k|\cos\theta\right)^2 \end{aligned}$$

In the high energy limit ($E >> m_\mu$) our final expression for differential cross-section is

$$\frac{d\sigma}{d\Omega}\left(e_R^- e_L^+ \to \mu_R^- \mu_L^+\right) = \frac{\alpha^2}{4E_{cm}^2}\left(1+\cos\theta\right)^2 \tag{2.32}$$

We can also get the other non vanishing helicity amplitude intuitively without repeating the whole process. For example this reaction $e_R^- e_L^+ \to \mu_L^- \mu_R^+$, the replacement in the modulus square amplitude of $\mu_L^- \mu_R^+$ will be γ^5 to $-\gamma^5$ on the left-hand side. Thus $\epsilon_{\rho\mu\sigma\nu}$ replaced by $-\epsilon_{\rho\mu\sigma\nu}$ on the right-hand side.

We can easily see that,

$$\frac{d\sigma}{d\Omega}\left(e_R^- e_L^+ \to \mu_L^- \mu_R^+\right) = \frac{\alpha^2}{4E_{cm}^2}\left(1-\cos\theta\right)^2 \tag{2.33}$$

Similarly, for other helicity amplitude

$$\frac{d\sigma}{d\Omega}\left(e_L^- e_R^+ \to \mu_R^- \mu_L^+\right) = \frac{\alpha^2}{4E_{cm}^2}\left(1-\cos\theta\right)^2 \tag{2.34}$$

$$\frac{d\sigma}{d\Omega}\left(e_L^- e_R^+ \to \mu_L^- \mu_R^+\right) = \frac{\alpha^2}{4E_{cm}^2}\left(1+\cos\theta\right)^2 \tag{2.35}$$

The other twelve helicity cross-sections are zero because we know that right handed electron corresponds to a left-handed positron and the amplitude vanishes unless the electron and positron have their opposite helicity. Hence, this gives four non zero helicity amplitude. Adding all sixteen contributions, and dividing by four will give us average unpolarized cross-section.

2.7 Crossing symmetry

Crossing symmetry defined as, the s-matrix element for any process involving a particle with momentum p in the initial state is equal to the s-matrix element for an identical process but particle replaced by anti-particle with momentum k ($k = -p$) in the final state. It is one of the most important elements of calculations which makes use of the analytical properties of the scattering amplitudes. It relates various amplitudes, for example, helicity amplitudes, in one channel to those in other channels, in which all incoming and outgoing particles have been interchanged.

Chapter 3

Compton scattering

In Compton scattering, a photon collides with an electron, loses some of its energy and deflected from its original direction of travel. In this scattering assuming that electron to be initially free. It is an inelastic scattering of a photon with an electrically charged particle. We will calculate the unpolarized cross-section for this section, to lowest order in α.

3.1 Feynman diagram

There are two diagrams that contribute to Compton scattering at tree-level

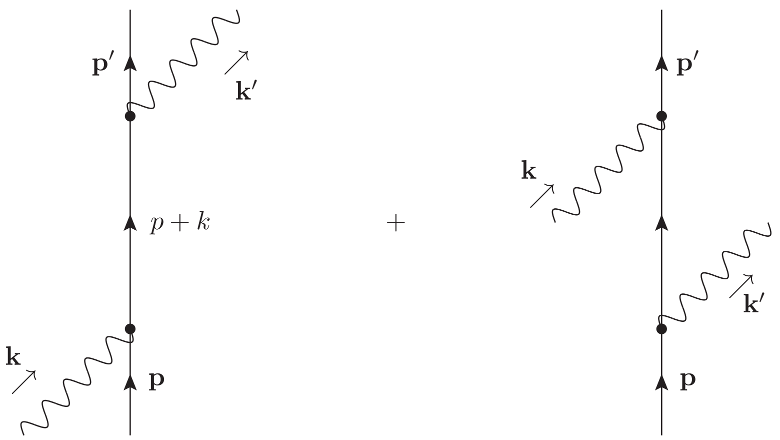

Figure 3.1: Feynman diagram for Compton scattering

In the Feynman diagram, p and k are the 4-momentum of the electron and photon before the collision, and p', k' their 4-momentum after the collision. Since the fermion portion of the diagrams is identical. Using $\varepsilon_\nu(k)$ and $\varepsilon^*_\mu(k^{'})$ to denote the polarization vector of the initial and final photon, we have the expression of Scattering amplitude for a given interaction.

$$
\begin{aligned}
-i\mathscr{M} &= \overline{u}(p')(-ie\gamma^{\mu})\varepsilon^{*}_{\mu}(k')\frac{i(\not{p}+\not{k}+m)}{(p+k)^2-m^2}(-ie\gamma^{\nu})\varepsilon_{\nu}(k)u(p) \\
&\quad +\overline{u}(p')(-ie\gamma^{\nu})\varepsilon_{\nu}(k)\frac{i(\not{p}-\not{k}'+m)}{(p-k)^2-m^2}(-ie\gamma^{\mu})\varepsilon_{\mu}*(k')u(p) \quad (3.1) \\
&= -ie^2\varepsilon_{\nu}(k)\varepsilon^{*}_{\mu}(k')\overline{u}(p')\left[\frac{\gamma^{\mu}(\not{p}+\not{k}+m)\gamma^{\nu}}{(p+k)^2-m^2}+\frac{\gamma^{\nu}(\not{p}-\not{k}'+m)\gamma^{\mu}}{(p-k')^2-m^2}\right]u(p) \quad (3.2)
\end{aligned}
$$

As we found $|\mathscr{M}|^2$ expression for $e^-e^+ \to \mu^-\mu^+$ interaction in chapter 2. Similarly, we will do for compton scattering but before this we need a simplified solution for $-i\mathscr{M}$. Since, $p^2 = m^2$ and $k^2 = 0$, the denominators of the propagators are

$$
\begin{aligned}
(p+k)^2 - m^2 &= 2p\cdot k \quad (3.3) \\
(p-k')^2 - m^2 &= -2p\cdot k' \quad (3.4)
\end{aligned}
$$

To simplify the numerators in $-i\mathscr{M}$ using Dirac gamma matrix algebra.

$$
(\not{p}+m)\gamma^{\nu}u(p) = (2p^{\nu} - \gamma^{\nu}\not{p} + \gamma^{\nu}m)u(p) = 2p^{\nu}u(p) \quad (3.5)
$$

Using above simplification of numerator in $-i\mathscr{M}$, we obtained

$$
-i\mathscr{M} = -ie^2\varepsilon_{\nu}(k)\varepsilon^{*}_{\mu}(k')\overline{u}(p')\left[\frac{\gamma^{\mu}\not{k}\gamma^{\nu}+2\gamma^{\mu}p^{\nu}}{2p\cdot k}+\frac{-\gamma^{\nu}\not{k}'\gamma^{\mu}+2\gamma^{\nu}p^{\mu}}{-2p\cdot k'}\right]u(p) \quad (3.6)
$$

Here, we are considering the scattering of an unpolarized photon by an unpolarized electron, without regard to their polarizations after the scattering. So, we will find the polarization sum to get the average cross-section.

Photon Polarization Sums

The amplitude expression retains freely specified spin and polarization states for the electrons and photons. Experimental Compton scattering involves unpolarized photons colliding with electrons, and so we must average over these states. We are considering an arbitrary QED process comprising an external photon with momentum k to get the expression for photon polarization sum. Since the scattering amplitude always contains $\varepsilon^{*}_{\mu}(k)$, so we can extract this factor and defined $\mathscr{M}(k)$ to be the rest of the scattering amplitude $\mathscr{M}$.

$$
\sum_{\epsilon}|\varepsilon^{*}_{\mu}(k)\mathscr{M}(k)|^2 = \sum_{\epsilon}\varepsilon^{*}_{\mu}\varepsilon_{\nu}\mathscr{M}^{\mu}(k)\mathscr{M}^{\nu *}(k) \quad (3.7)
$$

For simplicity, we took k vector in the 3-direction: $k^\mu = (k, 0, 0, k)$. Then the corresponding two transverse polarization vector are

$$\begin{aligned} \varepsilon_1^\mu &= (0, 1, 0, 0) && (3.8) \\ \varepsilon_2^\mu &= (0, 0, 1, 0) && (3.9) \end{aligned}$$

Now, we will sum our scattering amplitude $\mathscr{M}(k)$ over two transverse polarization vector. Then we have

$$\sum_\epsilon |\varepsilon_\mu^*(k)\mathscr{M}(k)|^2 = |\mathscr{M}^1(k)|^2 + |\mathscr{M}^2(k)|^2 \tag{3.10}$$

Classically, we know that the current density j^μ is conserved $\partial_\mu j^\mu = 0$. If the property still holds in the quantum theory, we can dot k_μ with $\mathscr{M}^\mu(k)$ to obtain

$$k_\mu \mathscr{M}^\mu(k) = 0 \tag{3.11}$$

The amplitude $\mathscr{M}$ vanishes when the polarization vector $\varepsilon_\mu(k)$ is replaced by k_μ. This relation is known as the Ward identity.
Now, we can see for $k^\mu = (k, 0, 0, k)$ the ward identity takes the form

$$\begin{aligned} k\mathscr{M}^0(k) - k\mathscr{M}^3(k) &= 0 \\ \mathscr{M}^0 &= \mathscr{M}^3 \\ \sum_\epsilon \varepsilon_\mu^* \varepsilon_\nu \mathscr{M}^\mu(k)\mathscr{M}^{\nu *}(k) &= |\mathscr{M}^1|^2 + |\mathscr{M}^2|^2 \\ \sum_\epsilon \varepsilon_\mu^* \varepsilon_\nu \mathscr{M}^\mu(k)\mathscr{M}^{\nu *}(k) &= -g_{\mu\nu}\mathscr{M}^\mu(k)\mathscr{M}^{\nu *}(k) \end{aligned}$$

So, photon polarization sum is given by

$$\sum_\epsilon \varepsilon_\mu^* \varepsilon_\nu \longrightarrow -g_{\mu\nu}$$

3.2 The Klein-Nishima Formula

We want to average the modulus square amplitude over the initial electron and photon polarizations, and sum over the final electron and photon polarizations.

$$\begin{aligned} \frac{1}{4}\sum_{spin} |\mathscr{M}|^2 &= \frac{e^4}{4} g_{\mu\rho} g_{\nu\sigma} \cdot Tr\Bigg[(\not{p}' + m) \cdot \left(\frac{\gamma^\mu \not{k} + 2\gamma^\mu p^\nu}{2p \cdot k} + \frac{\gamma^\nu \not{k}' \gamma^\mu - 2\gamma^\nu p^\mu}{2p \cdot k'} \right) \\ &\quad \times (\not{p} + m) \left(\frac{\gamma^\sigma \not{k} \gamma^\rho + 2\gamma^\rho p^\sigma}{2p \cdot k} + \frac{\gamma^\rho \not{k}' \gamma^\sigma - 2\gamma^\sigma p^\rho}{2p \cdot k'} \right) \Bigg] \end{aligned}$$

To simplify above expression we used the trace technology and Mandelstam variables:

$$\begin{aligned} s &= (p + k)^2 = 2p \cdot k + m^2 = 2p' \cdot k' + m^2; && (3.12) \\ t &= (p' - p)^2 = -2p \cdot p' + 2m^2 = -2k \cdot k'; && (3.13) \\ u &= (k' - p)^2 = -2k' \cdot p + m^2 = -2k \cdot p' + m^2 && (3.14) \end{aligned}$$

The momentum conservation at vertex implies $s + u + t = 2m^2$. Now, we can put Mandelstam variable to our scattering amplitude in lab frame.

$$\frac{1}{4}\sum_{spin}|\mathscr{M}|^2 = 2e^4\left[\frac{(s-m^2)t^3 + (3s^2 - 2m^2 s + 3m^4)t^2 + 4s(s-m^2)^2 t + 2(s-m^2)^4}{(s-m^2)^2(t+s-m^2)^2}\right]$$

Now, rewriting s, u and t in terms of $p \cdot k$, $p \cdot k'$ and $k \cdot k'$, we finally obtain

$$\frac{1}{4}\sum_{spin}|\mathscr{M}|^2 = 2e^4\left[\frac{p\cdot k'}{p\cdot k} + \frac{p\cdot k}{p\cdot k'} + 2m^2\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right) + m^4\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right)^2\right]$$

To get the expression for a differential cross-section, we must decide a frame of reference where our calculation becomes simpler. The easiest choice is lab frame because we can find the dynamics of particles after collision separately. But in the lab frame, the electron is initially at rest. But after the collision, The energy of the electron is typically ten orders of magnitude larger than that of the photon.

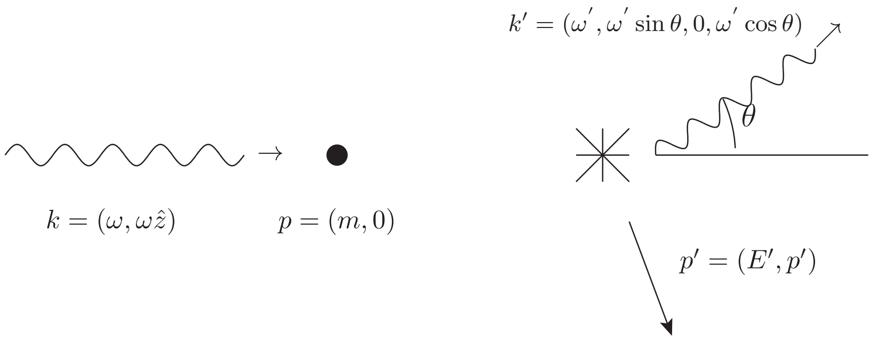

Figure 3.2: Compton scattering in Lab frame

Since, we know that Compton's formula for the shift in photon wavelength.

$$\omega' = \frac{\omega}{1 + \frac{\omega}{m}(1 - cos\theta)}$$

Now, the phase space integral in Lab frame is

$$\int d\Pi = \int \frac{d^3k'}{(2\pi)^3}\frac{1}{2\omega'}\frac{d^3p'}{(2\pi)^3}\frac{1}{2E'}(2\pi)^4\delta^{(4)}(k' + p' - k - p) \tag{3.15}$$

$$\int d\Pi = \frac{1}{8\pi}\int (d\cos\theta)\frac{(\omega')^2}{\omega m} \tag{3.16}$$

The differential cross-section is given by

$$d\sigma = \frac{\frac{1}{4}\sum_{spin}|\mathscr{M}|^2}{4E_A E_B|v_A - v_B|}\Pi_f\left(\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\right).(2\pi)^4\delta^{(4)}(p_A + p_B - \Sigma p_f) \tag{3.17}$$

Now, we will plug everything into our differential cross-section formula which we have written above. Since, in lab frame $|v_A - v_B| = 1$, then we find

$$\frac{d\sigma}{dcos\theta} = \frac{1}{2\omega}\frac{1}{2m}.\frac{1}{8\pi}\frac{(\omega')^2}{\omega m}.\left(\frac{1}{4}\sum_{spin}|\mathscr{M}|^2\right)$$

Now, our general cross-section formula is

$$\frac{d\sigma}{dcos\theta} = \frac{\pi\alpha^2}{m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - sin^2\theta\right]$$

Above expression correspond Klein-Nishina formula for differential cross-section
In the limit (this low energy limit) $\omega \longrightarrow 0$ and $\omega'/\omega \longrightarrow 1$, then the cross-section becomes

$$\frac{d\sigma}{dcos\theta} = \frac{\pi\alpha^2}{m^2}\left(1 + cos^2\theta\right)$$

This is the familiar Thomson cross section for scattering of classical electromagnetic radiation by a free electron where the energy of the photon is less than the rest mass energy of electron.

3.3 High Energy Behavior

In the centre of mass frame the 4-momenta of the particles may be written

$$\begin{aligned} k &= (\omega, 0, 0, \omega) \\ p &= (E, 0, 0, -\omega) \\ k' &= (\omega, \omega\sin\theta, 0, \omega\cos\theta) \\ p' &= (E, -\omega\sin\theta, 0, -\omega\cos\theta) \end{aligned}$$

To analyze the high energy behavior of the Compton scattering cross-section, it is easiest to work in the center-of-mass frame.

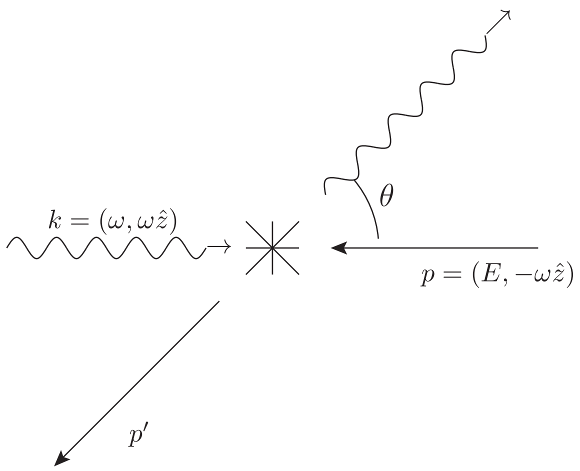

Figure 3.3: Compton scattering in center-of-mass frame

We can see that for $\theta = \pi$, the term $(p \cdot k/p \cdot k')$ becomes very large, while the other terms are all smaller. Thus for $E >> m$ and $\theta = \pi$, we have

$$\frac{1}{4}\sum_{spin}|\mathscr{M}|^2 = 2e^4 \cdot \frac{p \cdot k}{p \cdot k'} = 2e^4 \cdot \left(\frac{E+\omega}{E+\omega cos\theta}\right)$$

The differential cross-section in center of mass frame is given by

$$\frac{d\sigma}{dcos\theta} = \frac{1}{2} \cdot \frac{1}{2E} \cdot \frac{1}{2\omega}\left(\frac{\omega}{8\pi(E+\omega)}\right)\left[\frac{2e^4(E+\omega)}{(E+\omega\cos\theta)}\right]$$

$$\frac{d\sigma}{dcos\theta} = \frac{2\pi\alpha^2}{2m^2 + s(1+\cos\theta)}$$

Where $s = 2\omega(E+\omega) + m^2$
In the high energy limit $s >> m^2$, so we can drop the electron mass term if we supply an equivalent cuttoff near $\theta = \pi$. In this way, we can approximate the total Compton scattering cross-section. We find that the total cross-section behaves at high energy as

$$\sigma_{total} = \frac{2\pi\alpha^2}{s} log\left(\frac{s}{m^2}\right)$$

In the high energy limit $(s >> m^2)$ the differential cross-section has singularity at $(\theta = \pi)$ which means we have singularity for backward photon.

3.4 Result

We have two tree level Feynman diagram for Compton scattering which gives one amplitude for each diagram. We have shown that in the lab frame if the photon has more energy than the rest mass energy of the electron, Then the calculation for differential cross-section produces Klein-Nishina formula. But if photon has less energy than the rest mass energy of electron this condition produces Thomson cross section for scattering of classical electromagnetic radiation by a free electron. In further analysis, we have shown the singularity in differential cross-section for backward photon $(\theta = \pi)$. In the high energy limit, our total scattering cross-section is a logarithmic function of total energy.

Chapter 4

Appendix

4.1 Spin sum of fermions and antifermions

The general solution of the Dirac equation can be written as a linear combination of plane waves.

$$\Psi(x) = u(p)e^{-ip.x}$$

Above representation of Dirac equation correspond to positive frequency waves and it has two linearly independent solutions.

$$u^s(p) = \begin{pmatrix} \sqrt{p.\sigma}\xi^s \\ \sqrt{p.\overline{\sigma}}\xi^s \end{pmatrix}$$

Spinor conjugate of above spinor is given by

$$\overline{u}^s(p) = \begin{pmatrix} \xi^{s\dagger}\sqrt{p.\sigma} & \xi^{s\dagger}\sqrt{p.\overline{\sigma}} \end{pmatrix}$$

Solution of Dirac equation for negative frequency waves are

$$v^s(p) = \begin{pmatrix} \sqrt{p.\sigma}\eta^s \\ -\sqrt{p.\overline{\sigma}}\eta^s \end{pmatrix}$$

Now, We can easily evaluate the sum over the polarization states of a fermion.

$$\sum_{s=1,2} u^s(p)\overline{u}^s(p) = \begin{pmatrix} \sqrt{p.\sigma}\xi^s \\ \sqrt{p.\overline{\sigma}}\xi^s \end{pmatrix} \begin{pmatrix} \xi^{s\dagger}\sqrt{p.\sigma} & \xi^{s\dagger}\sqrt{p.\overline{\sigma}} \end{pmatrix}$$

Since we know

$$\sum_s \xi^s\xi^{s\dagger} = 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sum_{s=1,2} u^s(p)\overline{u}^s(p) = \begin{pmatrix} \sqrt{p.\sigma}\sqrt{p.\overline{\sigma}} & \sqrt{p.\sigma}\sqrt{p.\sigma} \\ \sqrt{p.\overline{\sigma}}\sqrt{p.\overline{\sigma}} & \sqrt{p.\overline{\sigma}}\sqrt{p.\sigma} \end{pmatrix}$$

We can simplify $(p.\sigma)(p.\overline{\sigma})$ as
$(p.\sigma)(p.\overline{\sigma}) = p^2\sigma\overline{\sigma} = p^2 = m^2$

$$\sum_{s=1,2} u^s(p)\overline{u}^s(p) = \begin{pmatrix} m & p.\sigma \\ p.\overline{\sigma} & m \end{pmatrix}$$

Thus, we got our desired formula,

$$\sum_s u^s(p)\overline{u}^s(p) = \gamma^\mu p_\mu + m\mathbf{I} = \not{p} + m$$

Similarly, for antifermions we will get

$$\sum_s v^s(p)\overline{v}^s(p) = \gamma^\mu p_\mu - m\mathbf{I} = \not{p} - m$$

But for above formula, we used this identity

$$\sum_s \eta^s \eta^{s\dagger} = 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

4.2 Trace of odd Gamma matrices

Let's take an example $Tr\left(\gamma^\mu\gamma^\nu\gamma^\rho\right)$
To prove above statement, we will use gamma matrices identity as $[\gamma^\mu, \gamma^\nu] = 2g^{\mu\nu}$ Now,

$$Tr\left(\gamma^\mu\gamma^\nu\gamma^\rho\right); \qquad \gamma^\mu\gamma^\nu = 2g^{\mu\nu} - \gamma^\nu\gamma^\mu \tag{4.1}$$

Then, we can write

$$Tr\left(\gamma^\mu\gamma^\nu\gamma^\rho\right) = Tr\left[\left(2g^{\mu\nu}.\mathbf{I} - \gamma^\nu\gamma^\mu\right)\gamma^\rho\right] \tag{4.2}$$
$$Tr\left[2g^{\mu\nu}.\mathbf{I}\gamma^\rho - \gamma^\nu\gamma^\mu\gamma^\rho\right] \tag{4.3}$$
$$Since, \qquad Tr(A+B) = Tr(A) + Tr(B) \tag{4.4}$$
$$Tr\left(\gamma^\mu\gamma^\nu\gamma^\rho\right) = Tr\left(2g^{\mu\nu}.\mathbf{I}\gamma^\rho\right) - Tr\left(\gamma^\nu\gamma^\mu\gamma^\rho\right) \tag{4.5}$$

Using cyclic property of trace, we can write

$$Tr\left(\gamma^\nu\gamma^\mu\gamma^\rho\right) = Tr\left(\gamma^\mu\gamma^\nu\gamma^\rho\right) \tag{4.6}$$
$$2Tr\left(\gamma^\mu\gamma^\nu\gamma^\rho\right) = Tr\left(2g^{\mu\nu}.\mathbf{I}\gamma^\rho\right) \tag{4.7}$$
$$= 4g^{\mu\nu}Tr\left(\gamma^\rho\right) \tag{4.8}$$
$$Tr(\gamma^\rho) = Tr(\gamma^5\gamma^5\gamma^\rho) \quad , \quad (\gamma^5)^2 = 1 \tag{4.9}$$
$$= -Tr(\gamma^5\gamma^\rho\gamma^5) \quad , \quad [\gamma^\rho, \gamma^5] = 0 \tag{4.10}$$
$$= -Tr(\gamma^5\gamma^5\gamma^\rho) \tag{4.11}$$
$$= -Tr(\gamma^\rho) \tag{4.12}$$
$$= 0 \tag{4.13}$$

So $Tr(\gamma^\rho) = 0$ implies

$$Tr\left(\gamma^\nu \gamma^\mu \gamma^\rho\right) = 0$$

Hence, as we can see trace of odd gamma matrices becomes zero.

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