Math Skills

Scientific Notation

It is difficult to work with very large or very small numbers when they are written in common decimal notation. Usually it is possible to accommodate such numbers by changing the SI prefix so that the number falls between 0.1 and 1000; for example, 237 000 000 mm can be expressed as 237 km, and 0.000 000 895 kg can be expressed as 0.895 mg. However, this prefix change is not always possible, either because an appropriate prefix does not exist or because it is essential to use a particular unit of measurement. In these cases, the best method of dealing with very large and very small numbers is to write them using scientific notation. Scientific notation expresses a number by writing it in the form $a \times 10^n$, where $1 < |a| < 10$ and the digits in the coefficient a are all significant. **Table 1** shows situations where scientific notation would be used.

Table 1 Examples of Scientific Notation

Expression	Common decimal notation	Scientific notation
124.5 million kilometres	124 500 000 km	1.245×10^8 km
154 thousand picometres	154 000 pm	1.54×10^5 pm
602 sextillion/mol	602 000 000 000 000 000 000 000/mol	6.02×10^{23}/mol

To multiply numbers in scientific notation, multiply the coefficients and add the exponents; the answer is expressed in scientific notation. Note that when writing a number in scientific notation, the coefficient should be between 1 and 10 and should be rounded to the same certainty (number of significant digits) as the measurement with the least certainty (fewest number of significant digits). Look at the following examples:

$$(4.73 \times 10^5 \text{ m})(5.82 \times 10^7 \text{ m}) = 27.5 \times 10^{12} \text{ m}^2 = 2.75 \times 10^{13} \text{ m}^2$$

$$(3.9 \times 10^4 \text{ N}) \div (5.3 \times 10^{-3} \text{ m}) = 0.74 \times 10^7 \text{ N/m} = 7.4 \times 10^6 \text{ N/m}$$

On many calculators, scientific notation is entered using a special key, labelled EXP or EE. This key includes "× 10" from the scientific notation; you need to enter only the exponent.

For example, to enter

7.5×10^4	press	7.5 EXP 4
3.6×10^{-3}	press	3.6 EXP +/–3

Uncertainty in Measurements

Two types of quantities are used in science: exact values and measurements. Exact values include defined quantities (1 m = 100 cm) and counted values (5 cars in a parking lot). Measurements, however, are not exact because there is always some uncertainty or error.

There are two types of measurement error. *Random error* results when an estimate is made to obtain the last significant digit for any measurement. The size of the random error is determined by the precision of the measuring instrument. For example, when measuring length, it is necessary to estimate between the marks on the measuring tape. If these marks are 1 cm apart, the random error will be greater and the precision will be less than if the marks are 1 mm apart.

Systematic error is associated with an inherent problem with the measuring system, such as the presence of an interfering substance, incorrect calibration, or room conditions. For example, if the balance is not zeroed at the beginning, all measurements will have a systematic error; using a slightly worn metre stick will also introduce error.

The precision of measurements depends on the gradations of the measuring device. *Precision* is the place value of the last measurable digit. For example, a measurement of 12.74 cm is more precise than one of 127.4 cm because the first value was measured to hundredths of a centimetre whereas the latter was measured to tenths of a centimetre.

When adding or subtracting measurements of different precision, the answer is rounded to the same precision as the least precise measurement. For example, using a calculator,

$$11.7 \text{ cm} + 3.29 \text{ cm} + 0.542 \text{ cm} = 15.532 \text{ cm}$$

The answer must be rounded to 15.5 cm because the first measurement limits the precision to a tenth of a centimetre.

No matter how precise a measurement is, it still may not be accurate. Accuracy refers to how close a value is to its accepted value. The *percentage error* is the absolute value of the difference between experimental and accepted values expressed as a percentage of the accepted value.

$$\% \text{ error} = \frac{|\text{experimental value} - \text{accepted value}|}{\text{accepted value}} \times 100\%$$

The percentage difference is the difference between a value determined by experiment and its predicted value. The *percentage difference* is calculated as

$$\% \text{ difference} = \frac{|\text{experimental value} - \text{predicted value}|}{\text{predicted value}} \times 100\%$$

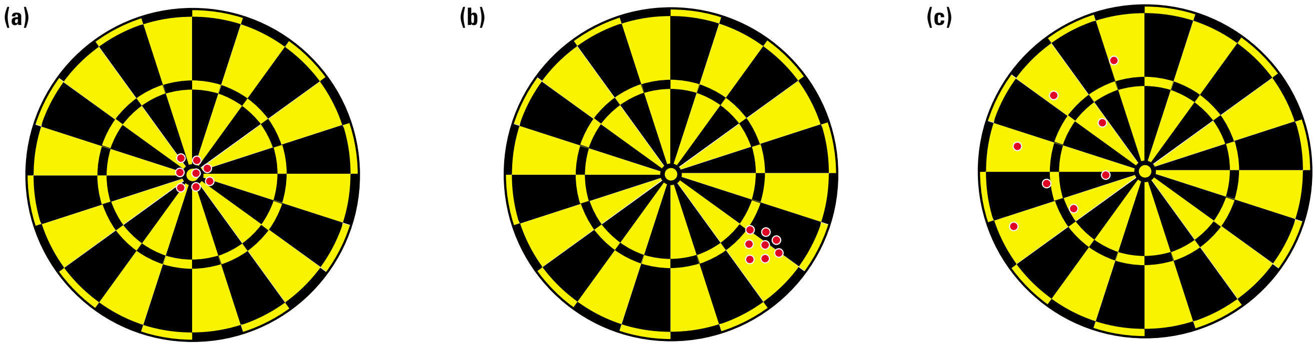


Figure 1
The positions of the darts in each of these figures are analogous to measured or calculated results in a laboratory setting. The results in **(a)** are precise and accurate, in **(b)** they are precise but not accurate, and in **(c)** they are neither precise nor accurate.

Figure 1 shows an analogy between precision and accuracy, and the positions of darts thrown at a dartboard.

How certain you are about a measurement depends on two factors: the precision of the instrument used and the size of the measured quantity. More precise instruments give more certain values. For example, a mass measurement of 13 g is less precise than a measurement of 12.76 g; you are more certain about the second measurement than the first. Certainty also depends on the measurement. For example, consider the measurements 0.4 cm and 15.9 cm; both have the same precision. However, if the measuring instrument is precise to ± 0.1 cm, the first measurement is 0.4 ± 0.1 cm (0.3 cm or 0.5 cm) or an error of ± 25%, whereas the second measurement could be 15.9 ± 0.1 cm (15.8 cm or 16.0 cm) for an error of ± 0.6%. For both factors—the precision of the instrument used and the value of the measured quantity—the more digits there are in a measurement, the more certain you are about the measurement.

Significant Digits

The certainty of any measurement is communicated by the number of significant digits in the measurement. In a measured or calculated value, significant digits are the digits that are certain plus one estimated (uncertain) digit. Significant digits include all digits correctly reported from a measurement.

Follow these rules to decide whether a digit is significant:

1. If a decimal point is present, zeros to the left of the first non-zero digit (leading zeros) are not significant.
2. If a decimal point is not present, zeros to the right of the last non-zero digit (trailing zeros) are not significant.
3. All other digits are significant.
4. When a measurement is written in scientific notation, all digits in the coefficient are significant.
5. Counted and defined values have infinite significant digits.

Table 2 shows some examples of significant digits.

Table 2 Certainty in Significant Digits

Measurement	Number of significant digits
32.07 m	4
0.0041 g	2
5×10^5 kg	1
6400 s	2
204.0 cm	4
10.0 kJ	3
100 people (counted)	infinite

An answer obtained by multiplying and/or dividing measurements is rounded to the same number of significant digits as the measurement with the fewest number of significant digits. For example, if we use a calculator to solve the following equation:

$$(77.8 \text{ km/h})(0.8967 \text{ h}) = 69.76326 \text{ km}$$

However, the certainty of the answer is limited to three significant digits, so the answer is rounded up to 69.8 km.

Rounding Off

Use these rules when rounding answers to calculations:

1. When the first digit discarded is less than five, the last digit retained should not be changed.
   3.141 326 rounded to 4 digits is 3.141
2. When the first digit discarded is greater than five, or if it is a five followed by at least one digit other than zero, the last digit retained is increased by 1 unit.
   2.221 372 rounded to 5 digits is 2.2214
   4.168 501 rounded to 4 digits is 4.169
3. When the first digit discarded is five followed by only zeros, the last digit retained is increased by 1 if it is odd, but not changed if it is even.
   2.35 rounded to 2 digits is 2.4
   2.45 rounded to 2 digits is 2.4
   –6.35 rounded to 2 digits is –6.4

Measuring and Estimating

Many people believe that all measurements are *reliable* (consistent over many trials), *precise* (to as many decimal places as possible), and *accurate* (representing the actual value). But many things can go wrong when measuring.

- There may be limitations that make the instrument or its use unreliable (inconsistent).
- The investigator may make a mistake or fail to follow the correct techniques when reading the measurement to the available precision (number of decimal places).
- The instrument may be faulty or inaccurate; a similar instrument may give different readings.

For example, when measuring the temperature of a liquid, it is important to keep the thermometer at the correct depth and the bulb of the thermometer away from the bottom and sides of the container. If you set a thermometer with its bulb on the bottom of a liquid-filled container, you will be measuring the temperature of the bottom of the container, and not the temperature of the liquid. There are similar concerns with other measurements.

To be sure that you have measured correctly, you should repeat your measurements at least three times. If your measurements appear to be reliable, calculate the mean and use that value. To be more certain about the accuracy, repeat the measurements with a different instrument.

Trigonometry

The word trigonometry comes from the Greek words *trigonon* and *metria*, meaning triangle measurement. The earliest use of trigonometry was for surveying. Today, trigonometry is used in navigation, electronics, music, and meteorology, to mention just a few. The first application of trigonometry was to solve right triangles. Trigonometry derives from the fact that for similar triangles, the ratio of corresponding sides will be equal.

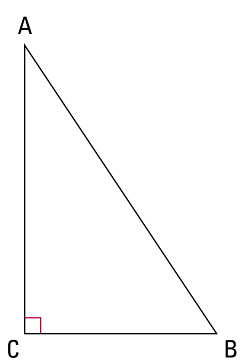


Figure 2
Right triangle

In the right triangle in **Figure 2**, consider the ratio $\frac{AC}{AB}$. In relation to the angle B, AC is the *opposite side* and AB is the *hypotenuse*. This ratio of $\frac{\textit{opposite side}}{\textit{hypotenuse}}$ is called the *sine ratio* (abbreviated *sin*). For the given triangle

$$\sin B = \frac{AC}{AB} \quad \text{and} \quad \sin A = \frac{BC}{AB}$$

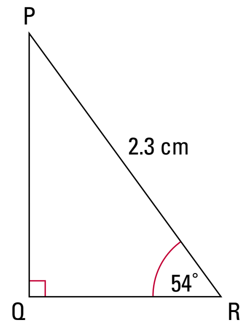


Figure 3
Right triangle

We can apply the sine ratio to determine the length of the two unknown sides of the triangle shown in **Figure 3**.

$$\sin R = \frac{\text{opp}}{\text{hyp}} = \frac{PQ}{PR}$$

$$\sin 54° = \frac{PQ}{2.3\text{ cm}}$$

To use your calculator to find sin 54°, make sure it is in degree mode, and enter sin 54. This should produce the answer 0.80901699.

To find QR, we can use the Pythagorean theorem or apply the sine ratio to angle P.

$$\sin P = \frac{QR}{PR}$$

$$\sin 36° = \frac{QR}{2.3\text{ cm}}$$

$$QR = (0.5878)(2.3\text{ cm})$$
$$QR = 1.4\text{ cm (to two significant digits)}$$

Two other trigonometric ratios that are frequently used when working with right triangles are the cosine and tangent ratios, abbreviated *cos* and *tan* respectively. They are defined as

$$\text{cosine } \theta = \frac{\text{adj}}{\text{hyp}} \qquad \text{tangent } \theta = \frac{\text{opp}}{\text{adj}}$$

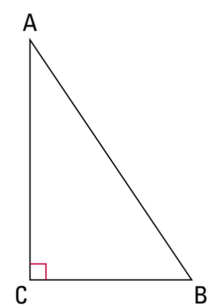


Figure 4
Right triangle

For the triangle shown in **Figure 4**

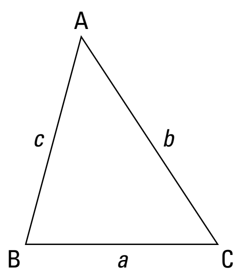
$$\cos B = \frac{BC}{AB} \qquad \tan B = \frac{AC}{BC}$$

$$\cos A = \frac{AC}{AB} \qquad \tan A = \frac{BC}{AC}$$

Trigonometry can also be used for triangles other than right triangles. The sine law and the cosine law can be useful when dealing with problems involving vectors.

The Sine Law

This law states that for any given triangle, *the ratio of the sine of an angle to the length of the opposite side is constant.* Thus, for **Figure 5**



$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Figure 5
Scalene triangle

The Cosine Law

This law states that for any given triangle, *the square of the length of any side is equal to the sum of the squares of the lengths of the other two sides, minus twice the product of the lengths of these two sides and the cosine of the angle between them (the included angle).* Thus, for **Figure 5**

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = b^2 + a^2 - 2ab \cos C$$

Note that the first part of the cosine law is the Pythagorean theorem; the last factor simply adjusts for the fact that the angle is not a right angle. If the angle is a right angle, the cosine is zero and the term disappears, leaving the Pythagorean theorem.

Example

Look at **Figure 6** and calculate the length of side AC.

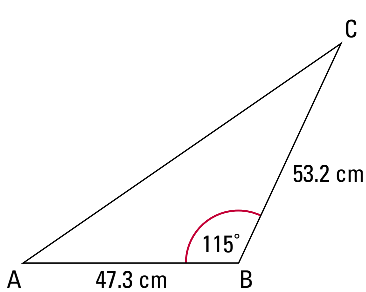


Figure 6

By the cosine law we have

$$\begin{aligned} AC^2 &= AB^2 + BC^2 - 2(AB)(BC) \cos B \\ &= (47.3 \text{ cm})^2 + (53.2 \text{ cm})^2 - 2(47.3 \text{ cm})(53.2 \text{ cm}) \cos 115° \\ &= 2237 \text{ cm}^2 + 2830 \text{ cm}^2 - (5033 \text{ cm}^2)(\cos 115°) \\ &= 5067 \text{ cm}^2 - (5033 \text{ cm}^2)(-0.4226) \\ AC^2 &= 7194 \text{ cm}^2 \\ AC &= 84.8 \text{ cm} \end{aligned}$$

Equations and Graphs

Linear Equations

Any equation that can be written in the form $Ax + By = C$ is called a *linear*, or first-degree equation in two variables. However, most often the equation is rearranged as $y = mx + b$, where y is the dependent variable (on the y-axis) and x is the independent variable (on the x-axis). This equation is known as the slope-intercept form because m is the slope of the line on the graph and b is the y-intercept.

Linear equations are encountered in many areas of science. For example, the equation for the velocity ($\vec{v}$) of an object at a given time (Δt) is given by the linear equation $\vec{v} = \vec{v}_i + \vec{a}\Delta t$, where $\vec{v}_i$ is the initial velocity and $\vec{a}$ is the acceleration. If we change this equation to the slope-intercept form ($y = mx + b$), it reads $\vec{v} = \vec{a}\Delta t + \vec{v}_i$, where $\vec{v}$ represents the y variable and Δt represents the x variable.

Example

Plot the graph of the equation for the velocity of an object with initial velocity 20.0 m/s [E] and acceleration 5.0 m/s^2 [E].

$$\vec{v} = 20.0 \text{ m/s} + (5.0 \text{ m/s}^2)\Delta t$$

The slope of the line can be calculated using the following equation:

$$m = \frac{\text{rise}}{\text{run}} = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$

where y_1 and x_1, and y_2 and x_2 are any two points on the line.

From **Figure 7**, we can choose two points (1, 25) and (5.5, 47.5); note that one is a data point and one is not. We can now calculate the slope.

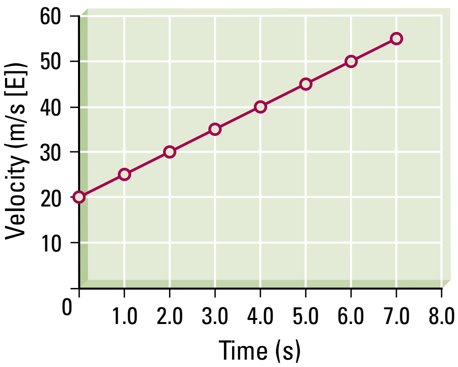


Figure 7
Velocity-time graph

$$m = \frac{v_2 - v_1}{t_2 - t_1}$$
$$= \frac{(47.5 - 25.0) \text{ m/s [E]}}{(5.5 - 1.0) \text{ s}}$$
$$= \frac{22.5 \text{ m/s [E]}}{4.5 \text{ s}}$$
$$m = 5.0 \text{ m/s}^2 \text{ [E]}$$

The slope of the line is 5.0 m/s^2 [E]. This is a positive value, indicating a positive slope. A negative slope (a line sloped the other way) would have a negative value.

If the value of one of the variables is known, the other value can be read from the graph or obtained by solving the equation using algebraic skills. For example, if $\Delta t = 2.0$ s you can see on the graph that the corresponding y coordinate is 30.0 m/s. Solving algebraically we get the same result:

$$\vec{v} = \vec{v}_i + \vec{a}\Delta t$$
$$= 20.0 \text{ m/s} + 5.0 \text{ m/s}^2 \text{ (2.0 s)}$$
$$= 20.0 \text{ m/s} + 10.0 \text{ m/s}$$
$$\vec{v} = 30.0 \text{ m/s}$$

Variation Equations

When y varies directly as x, written as $y \propto x$, it means that $y = kx$, where k is the constant of variation. When y varies inversely as x, written as $y \propto \frac{1}{x}$, it means that $y = \frac{k}{x}$ or $xy = k$.

In many situations there is a combination of direct and inverse variation, commonly referred to as joint variation. Problems dealing with joint variation are solved by substituting the values of the variables from a known experiment to calculate k and then using the value of k to determine the missing variables in another experiment.

Example

The electrical resistance of a wire (R) varies directly as its length and inversely as the square of its diameter. An investigation determines that 50.0 m of wire of diameter 3.0 mm has a resistance of 8.0 Ω. Determine the constant of variation for this type of wire. Without doing another investigation, determine the resistance of 40.0 m of the same type of wire if the diameter is 4.0 mm. The variation equation is

$$R = k\frac{l}{d^2}$$
$$8.0\ \Omega = k\frac{50.0 \text{ m}}{(3.0 \text{ mm})^2}$$
$$k = \frac{(8.0\ \Omega)(3.0 \text{ mm})^2}{50.0 \text{ m}}$$
$$= \left(\frac{1.44\ \Omega \cdot \text{mm}^2}{\text{m}}\right)\left(\frac{1 \text{ m}}{1000 \text{ mm}}\right)$$
$$k = 0.0014\ \Omega \cdot \text{mm}$$

Use this value of k to find R:

$$R = 0.0014\ \Omega \cdot \text{mm} \times \frac{40.0 \text{ m}}{(4.0 \text{ mm})^2} \times \left(\frac{1000 \text{ mm}}{1 \text{ m}}\right)$$
$$= 0.0014\ \Omega \cdot \text{mm} \times \left(\frac{4.00 \times 10^4}{16 \text{ mm}^2}\right)$$
$$= \frac{57.6\ \Omega \cdot \text{mm}^2}{16 \text{ mm}^2}$$
$$R = 3.6\ \Omega$$

Logarithms

Any positive number N can be expressed as a power of some base b where $b > 1$. Some obvious examples are

$16 = 2^4$	base 2, exponent 4
$25 = 5^2$	base 5, exponent 2
$27 = 3^3$	base 3, exponent 3
$0.001 = 10^{-3}$	base 10, exponent –3

In each example, the exponent is an integer. However, exponents may be any real number, not just an integer. If you use the x^y button on your calculator, you can experiment to get a better understanding of this concept.

The most common base is base 10. Some examples for base 10 are

$10^{0.5} = 3.162$
$10^{1.3} = 19.95$
$10^{-2.7} = 0.001995$

By definition, the exponent to which a base b must be raised to produce a given number N is called the *logarithm* of N to base b (abbreviated as $\log_b$). When the value of the base is not written it is assumed to be base 10. Logarithms to base 10 are called *common logarithms*. We can express the previous examples as logarithms:

$\log 3.162 = 0.5$
$\log 19.95 = 1.3$
$\log 0.001995 = -2.7$

Another base that is used extensively for logarithms is the base e (approximately 2.7183). Logarithms to base e are called *natural logarithms* (abbreviated as ln).

Most measurement scales are linear in nature. For example, a speed of 80 km/h is twice as fast as a speed of 40 km/h and four times as fast as a speed of 20 km/h. However, there are several examples in science where the range of values of the variable being measured is so great that it is more convenient to use a logarithmic scale to base 10.

One example of this is the scale for measuring the intensity level of sound. For example, a sound with an intensity level of 20 dB is 100 times (10^2) as loud as a sound with an intensity level of 0 dB, and 40 dB is 10 000 (10^4) times more intense than a sound of 0 dB. Other situations that use logarithmic scales are the acidity of a solution (the pH scale) and the intensity of earthquakes (the Richter scale).

Logarithmic Graphs

Quite often, graphing the results from experiments shows a logarithmic progression. For example, the series 1, 2, 3, 4, 5, 6 is a linear progression, whereas the series 10, 100, 1000, 10 000, 100 000 is a logarithmic progression. This means that the values increase exponentially. **Figure 8** is a graph of type $y = \log x$ to illustrate the sound intensity level scale, where sound intensity level in decibels (dB) is equal to the logarithm of intensity in watts per metre squared (W/m^2).

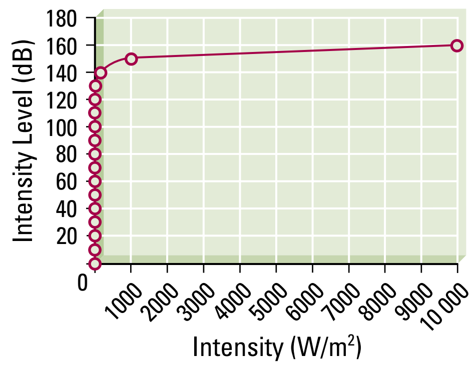


Figure 8
Graph of sound intensity level versus intensity

Notice that the scales on both axes are linear, so that we can see very little of the detail on the x-axis. Where the data range on one axis is extremely large and/or does not follow a linear progression, it is more convenient to change the scale (usually on the x-axis) so that we can "see" more detail of the entire range of values. Semi-log graph paper can be used to construct such graphs. If the scale on the x-axis is changed to a logarithmic scale, the graph of sound intensity level versus sound intensity on semi-log graph paper is shown in **Figure 9**.

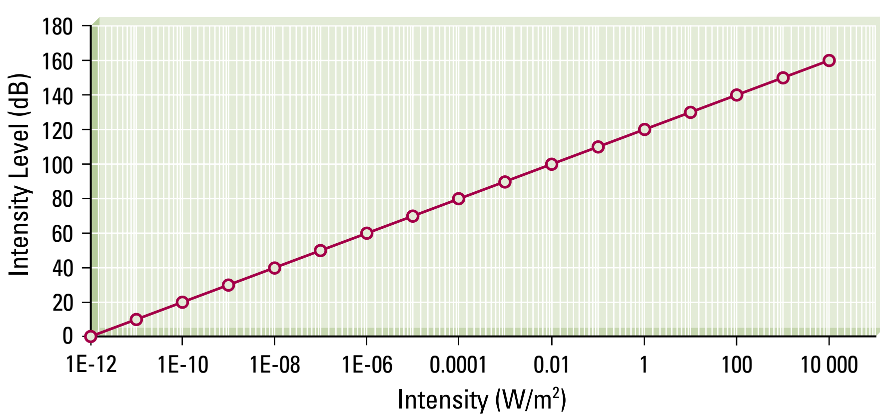


Figure 9
Semi-log graph of sound intensity level versus intensity

Dimensional Analysis

Dimensional analysis is a useful tool to determine whether an equation has been written correctly and to convert units. As is the case with many topics in physics, dimensional analysis can be "easy" or "hard" depending on the treatment we give it.

"Dimension" is a term that refers to quantities that we can measure in our universe. Three common dimensions are mass (m), length (l), and time (t). Note that the units of these dimensions are all *base units*—kilogram (kg), metre (m), and second (s). In dimensional analysis, all units are expressed as base units.

After a while, dimensional analysis becomes second nature. Suppose, for example, that after you solve an equation in which time Δt is the unknown, the final line in your solution is $\Delta t = 2.1$ kg. You know that something has gone seriously wrong on the right-hand side of the equation. It might be that care was not taken in cancelling certain units or that the equation was written incorrectly. For example, we can use dimensional analysis to determine if the following expression is valid:

$$\Delta d = v_{\text{i}} + \frac{1}{2}a\Delta t^2$$

One way to check is to insert the appropriate units. The usual technique when working with units is to put them in square brackets and to ignore numbers like the $\frac{1}{2}$ in the expression. The square brackets indicate that we are dealing with units only. The expression becomes

$$[\text{m}] = \left[\frac{\text{m}}{\text{s}}\right] + \left[\frac{\text{m}}{\text{s}^2}\right][\text{s}^2]$$

$$[\text{m}] = \left[\frac{\text{m}}{\text{s}}\right] + [\text{m}]$$

The expression is not valid because the units on the right-hand side of the equation do not equal the units on the left-hand side. The correct expression is

$$\Delta d = v_{\text{i}}\Delta t + \frac{1}{2}a\Delta t^2$$

You can check it out yourself by inserting the units in square brackets. If you wish, you can use the actual dimensions of length $[l]$ and time $[t]$ instead of substituting units. The dimensional analysis of the equation is

$$[l] = \left[\frac{l}{t}\right][t] + \left[\frac{l}{t^2}\right][t^2]$$

$$[l] = [l] + [l]$$

$$[l] = [l]$$

Remember that because we are dealing only with dimensions, there is no need to say $2l$ on the right-hand side.

You can also use dimensional analysis to change from one unit to another. For example, to convert 95 km/hr to m/s, kilometres must be changed to metres and hours to seconds. It helps to realize that 1 km = 1000 m and 1 hr = 3600 s. These two equivalencies allow the following two terms to be written:

$$\frac{1000\text{ km}}{1\text{ km}} = 1 \quad \text{and} \quad \frac{1\text{ hr}}{3600\text{ s}} = 1$$

Of course, the numerators and denominators could be switched (i.e., 1 km and 3600 s could be in the numerators) and the ratios would still be 1. However, as you will see, it is convenient to keep the ratios as they are for cancelling purposes. Because multiplying by 1 does not change the value of anything, we can write the following expression and cancel the units:

$$\frac{95\text{ km}}{\text{hr}} = \frac{95\text{ }\cancel{\text{km}}}{\cancel{\text{hr}}} \times \frac{1000\text{ m}}{1\text{ }\cancel{\text{km}}} \times \frac{1\text{ }\cancel{\text{hr}}}{3600\text{ s}}$$

Therefore,

$$\frac{95\text{ km}}{1\text{ hr}} = \frac{95\,000\text{ m}}{3600\text{ s}}$$

$$= 26.4\text{ m/s, or } 26\text{ m/s (to two significant digits)}$$

Example

What will be the magnitude of the acceleration of a 2100-g object that experiences a net force of magnitude 38.2 N?

First convert grams to kilograms:

$$\text{mass} = (2100\text{ g})\left(\frac{1\text{ kg}}{1000\text{ g}}\right)$$

$$\text{mass} = 2.1\text{ kg}$$

From Newton's second law,

$$F_{\text{net}} = ma$$

$$a = \frac{F_{\text{net}}}{m}$$

$$= \frac{38.2\text{ N}}{2.1\text{ kg}}$$

$$a = 18\text{ N/kg}$$

It is somewhat bothersome to leave acceleration with the units N/kg, so we will use dimensional analysis to change the units:

$$F_{\text{net}} = ma$$

$$[\text{N}] = [\text{kg}][\text{m/s}^2]$$

$$a = \frac{18\text{ N}}{\text{kg}} = \frac{18\text{ [kg][m/s}^2]}{[\text{kg}]}$$

$$a = 18\text{ m/s}^2$$