THE UNIVERSITY OF TEXAS AT AUSTIN
Department of Statistics and Data Sciences
College of Natural Sciences

SDS 321: Introduction to Probability and Statistics

Lecture 1: Axioms of Probability

Purnamrita Sarkar
Department of Statistics and Data Science
The University of Texas at Austin

www.cs.cmu.edu/∼psarkar/teaching

Getting Started

Your instructor:	Prof. Purna Sarkar
email:	`purna.sarkar@austin.utexas.edu`
Office Hours:	Tuesdays 11:30-12:30, GDC 7.306
Your TA:	Krishna Teja Rekapalli
email:	krrish1729@gmail.com
TA Office Hours:	Wednesdays 5-7pm

Course Overview

- This course provides an introduction to probability and statistics.
- The first section will be on fundamentals of probability, including:
  - Discrete and continuous random variables
  - Combinatorics
  - Multiple random variables
  - Functions of random variables
  - Limit theorems
- The second section will be on statistics, including:
  - Parameter estimation
  - Hypothesis testing
- We will consider mainly classical statistics. If time permits we will discuss Bayesian Statistics.

Course materials

- Course syllabus, slides and homework assignments will be posted at `www.cs.cmu.edu/~psarkar/teaching`
- Grades will be posted at `canvas.utexas.edu`
- The course text books are
  1. Introduction to Probability, by Dimitri P. Bertsekas and John N. Tsitsiklis.
  2. A First Course in Probability, by Sheldon Ross

  - Another good book that covers similar material is
    - Introduction to Probability, by Charles M. Grinstead and Laurie J. Snell

Assessment

- 4 exams. 2 midterms, and the final will consist of 2 midterm length exams.
- I will take the best 3 out of 4.
- The final grade will be 25% Homework, 25% from all 3 exams.
- Homework will be assigned (approximately) weekly, with roughly 10 homeworks in total.
- Homeworks should be submitted via Canvas by 5pm one week after it is assigned.

What is probability?

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- What does this mean?
- If I were to toss the coin 10 times, roughly 5 times I will see a head.
- A probability of 1 means it is certain, a probability of 0 means it is impossible.
- In general, you do an *experiment* many times, and you count how many times a particular *event* occurs. The proportion roughly gives you the probability of that particular event.

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- An event is a collection of possible outcomes.

Simple and compound events

- Simple event:
  - Your two coin tosses came up HH.
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- Compound event: can be decomposed into simple events
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    - You got $(1, 1)$, or $(1, 3)$ or $\ldots$.

Sets and sample spaces

We need to introduce some mathematical concepts to define probability more concretely:

- A **set** is a collection of objects, which are called **elements**
  - The natural numbers are a set, where the elements are individual numbers.
  - This class is the set, where the elements are the professor, the TA and the students.
- If an element x is in a set S, we write $x \in S$.
- If a set contains no elements, we call it the **empty set**, $\emptyset$.
- If a set contains every possible element, we call it the **universal set**, Ω.

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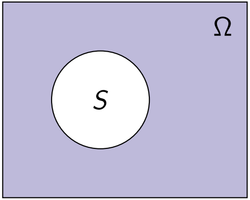
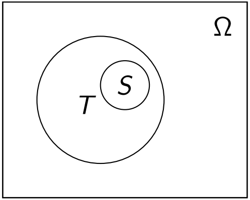
  - the set of all subsets of natural numbers, aka the power set
- We can use curly brackets to describe a set in terms of its elements:
  - Sample space of a die roll: $S = \{1, 2, 3, 4, 5, 6\}$
  - Arbitrary set where all the elements meet some criterion C:
    $S = \{x | x \text{ satisfies } C\}$

Operations on sets

- Let the **universal set** Ω be the set of all objects we might possibly be interested in.
- The **complement**, S^C, of a set S, w.r.t. Ω, is the set of all elements that are in Ω but not in S. So $\Omega^C = \emptyset$.
- We say $S \subseteq T$, if every element in S is also in T.
- $S \subseteq T$ and $T \subseteq S$ if and only if $S = T$.



S^C is the shaded region



$S \subset T \subset \Omega$

Operations on sets: Union, Intersection, Difference

- The **union**, $S \cup T$, of two sets S and T is the set of elements that are in either S or T (or both): $S \cup T = \{x | x \in S \text{ or } x \in T\}$.

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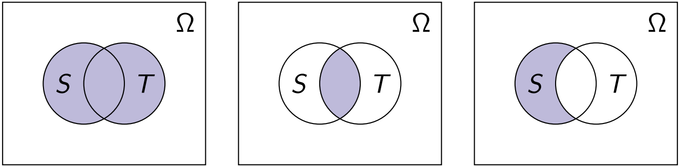

$S \cup T$ $\quad$ $S \cap T$ $\quad$ $S \setminus T = S \cap T^c$

Operations on sets

- We can extend the notions of union and intersection to multiple (even infinitely many!) sets:

$$\bigcup_{i=1}^{n} S_n = S_1 \cup S_2 \cup \cdots \cup S_n = \{x | x \in S_n \text{ for some } 1 \leq i \leq n\}$$

$$\bigcap_{i=1}^{n} S_n = S_1 \cap S_2 \cap \cdots \cap S_n = \{x | x \in S_n \text{ for all } 1 \leq i \leq n\}$$

- We say two sets are **disjoint** if their intersection is empty.
- We say a collection of sets are disjoint if no two sets have any common elements.
- If a collection of disjoint sets have union S, we call them a **partition** of S.

Probability laws

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- Axioms of probability:
  - **Nonnegativity**: $P(A) \geq 0$, for every event A.
  - **Additivity**: If A and B are two disjoint events, then the probability of their union satisfies $P(A \cup B) = P(A) + P(B)$.
    This extends to the union of infinitely many disjoint events:
    $$P(A_1 \cup A_2 \cup \dots) = P(A_1) + P(A_2) + \dots$$
  - **Normalization**: The probability of the entire sample space Ω is equal to 1, i.e. $P(\Omega) = 1$

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This is an example of an **uniform distribution**, where all outcomes are equally likely.

Properties of probability laws

All the following can be proven by decomposing a set into disjoint partitions and using the **additivity** and **non-negativity** rules.

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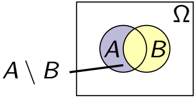
  - $A \cup B = (A \setminus B) \cup B$. So the additivity rule gives $P(A \cup B) = P(A \setminus B) + P(B)$. Can you finish the proof?



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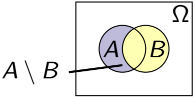

- $P(A \cup B) \leq P(A) + P(B)$.
- $P(A \cup B \cup C) = P(A) + P(A^c \cap B) + P(A^c \cap B^c \cap C)$.