Solutions Manual

Klaus-Jürgen Bathe

Finite Element Procedures

Second Edition

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Contents

Preface

We present these exercise solutions to help you using my textbook *Finite Element Procedures*, 2nd edition, K.J. Bathe, Watertown, MA, 2014. The solutions have been largely prepared by P.-G. Lee, A. Iosilevich, D. Pantuso, X. Wang, K. T. Kim and L. Zhang in my finite element research group at M.I.T. I helped in giving guidance.

We give solutions to the exercises that do not require the use of a computer program. However, to indicate how the exercises in which a finite element program is to be used might be solved, we also include the solutions to three such exercises. For these studies, the computer programs ADINA (for structural analysis) and ADINA CFD (for fluid flow analysis) have been used. These finite element programs are part of the ADINA System, see www.adina.com.

I would like to express my great appreciation for the efforts made by my research group in the preparation of these solutions. While much effort was expended to compile this material, there is of course no certainty that there are no errors. I would be pleased to hear of any error corrections or important improvements to be made in these solutions.

My personal wish is that these exercise solutions will help in learning finite element procedures so that you will experience some of the joy and satisfaction that we have felt working in the field.

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Part A

Solutions to Exercises of Chapters 2 to 11

(no use or little use of a computer program)

2.1 $\underline{B}^T\underline{A}\, k\, \underline{C}\,\underline{B} = k(\underline{B}^T\underline{A}\,\underline{C}\,\underline{B}) = k[(\underline{B}^T\underline{A})(\underline{C}\,\underline{B})] = 2960$

(No. of multiplications) $= 3\cdot 3\cdot 2 + 3 + 1 = 22$

2.2 (a) 1st Case: $\underline{A}^{-1} = \begin{bmatrix} 2/5 & 1/5 \\ 1/5 & 3/5 \end{bmatrix}$, 2nd case: $\underline{A}^{-1} = \begin{bmatrix} 2/3 & 0 & -1/3 \\ 0 & 1/4 & 0 \\ -1/3 & 0 & 2/3 \end{bmatrix}$

(b) 1st case: $\det \underline{A} = 3\cdot 2 - (-1)\cdot(-1) = 5$

2nd case: $\det \underline{A} = 2\cdot\begin{vmatrix} 4 & 0 \\ 0 & 2 \end{vmatrix} - 0 + 1\cdot\begin{vmatrix} 0 & 4 \\ 1 & 0 \end{vmatrix} = 12$

2.3 Modify column by column

$$\underline{A} = \begin{bmatrix} 1 & 4 & -7 \\ 3 & 1 & 1 \\ 4 & -1 & 6 \\ -1 & 0 & -1 \\ 2 & 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 4 & -7 \\ 0 & -11 & 22 \\ 0 & -17 & 34 \\ 0 & 4 & -8 \\ 0 & -7 & 13 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

Hence rank $= 3$

kernel $\underline{\alpha} = \underline{0}$

2.4 $\det \underline{A} = k - 1 = 0 \quad \therefore k = 1$

$$\underline{A} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad \therefore (\text{rank}) = 2$$

Let $\underline{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\therefore \alpha_1 = \alpha_3, \quad \alpha_2 = \alpha_3$

Hence, kernel $\underline{\alpha}^T = [\, 1 \quad 1 \quad 1 \,]$

2.5 (a) $\cos\alpha = \dfrac{\underline{u}\cdot\underline{v}}{|\underline{u}||\underline{v}|} = \dfrac{2\cdot 1 + 3\cdot 2 + 4\cdot 3}{\sqrt{2^2+3^2+4^2}\sqrt{1^2+2^2+3^2}} = \dfrac{20}{\sqrt{29}\sqrt{14}} \qquad \therefore \alpha = 6.98°$

(b) rotation matrix $\underline{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 30° & \sin 30° \\ 0 & -\sin 30° & \cos 30° \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & 1/2 \\ 0 & -1/2 & \sqrt{3}/2 \end{bmatrix}$

$$\therefore \underline{u}' = \underline{P}\,\underline{u} = \begin{bmatrix} 2 \\ \dfrac{4+3\sqrt{3}}{2} \\ \dfrac{-3+4\sqrt{3}}{2} \end{bmatrix}, \quad \underline{v}' = \underline{P}\,\underline{v} = \begin{bmatrix} 1 \\ \dfrac{3+2\sqrt{3}}{2} \\ \dfrac{-2+3\sqrt{3}}{2} \end{bmatrix}$$

(c) $\cos\alpha' = \dfrac{\underline{u}'\cdot\underline{v}'}{|\underline{u}'||\underline{v}'|} = \dfrac{20}{\sqrt{29}\sqrt{14}}$

$\therefore \alpha' = \alpha = 6.98°$ (of course the same angle as in (a) for any angle θ.)

2.6 (a) $\underline{P}^T\underline{P} = (\underline{I} - \alpha \underline{v}\underline{v}^T)^T(\underline{I} - \alpha\underline{v}\underline{v}^T)$

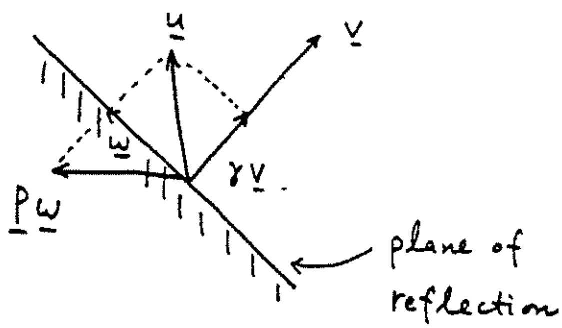
$$= \underline{I} - 2\alpha \underline{v}\underline{v}^T + \alpha^2 \underline{v}\underline{v}^T\underline{v}\,\underline{v}^T$$

$$= \underline{I} - 2\frac{2}{\underline{v}^T\underline{v}}\underline{v}\underline{v}^T + \frac{4}{(\underline{v}^T\underline{v})^2}\underline{v}(\underline{v}^T\underline{v})\underline{v}^T$$

$$= \underline{I} - \frac{4}{\underline{v}^T\underline{v}}\underline{v}\underline{v}^T + \frac{4}{\underline{v}^T\underline{v}}\underline{v}\underline{v}^T = \underline{I}$$

$\therefore$ $\underline{P}$ is an orthogonal matrix.

(b) Assume that $\underline{u}$ has components $\gamma\underline{v}$ into the direction of $\underline{v}$ and $\underline{w}$ perpendicular to $\underline{v}$.



For $\gamma\underline{v}$

$$\underline{P}\gamma\underline{v} = \gamma\underline{v} - \alpha\underline{v}\underline{v}^T\gamma\underline{v}$$

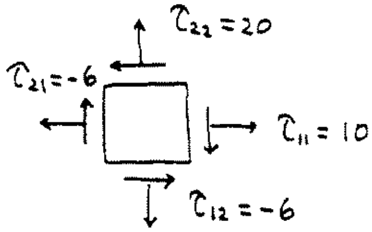
$$= \gamma\left[\underline{v} - \frac{2}{\underline{v}^T\underline{v}}\underline{v}(\underline{v}^T\underline{v})\right] = -\gamma\underline{v}$$

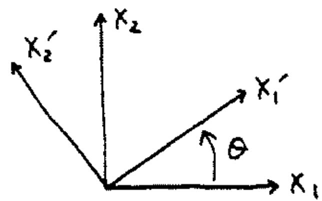
Therefore, the component into the direction of $\underline{v}$ has its direction reversed.

For $\underline{w}$, we have $\underline{P}\underline{w} = \underline{w} - \alpha\underline{v}\underline{v}^T\underline{w} = \underline{w}$ $\quad(\underline{v}^T\underline{w} = 0)$

Hence this component is not transformed.

2.7 (a)





Rotation matrix $\underset{\sim}{P} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$$\underset{\sim}{\tau}' = \underset{\sim}{P}\,\underset{\sim}{\tau}\,\underset{\sim}{P}^T$$

$$= \begin{bmatrix} 15-5\cos2\theta-6\sin2\theta & -6\cos2\theta+5\sin2\theta \\ -6\cos2\theta+5\sin2\theta & 15+5\cos2\theta+6\sin2\theta \end{bmatrix}$$

$$\therefore\ -6\cos2\theta+5\sin2\theta=0$$

i) $\cos2\theta=\dfrac{5}{\sqrt{61}}$, $\sin2\theta=\dfrac{6}{\sqrt{61}}$ $\longrightarrow$ $\underset{\sim}{\tau}'=\begin{bmatrix} 15-\sqrt{61} & 0 \\ 0 & 15+\sqrt{61} \end{bmatrix}$

ii) $\cos2\theta=-\dfrac{5}{\sqrt{61}}$, $\sin2\theta=-\dfrac{6}{\sqrt{61}}$ $\longrightarrow$ $\underset{\sim}{\tau}'=\begin{bmatrix} 15+\sqrt{61} & 0 \\ 0 & 15-\sqrt{61} \end{bmatrix}$

(b) $S_{ij}=\tau_{ij}-\tau_m\delta_{ij}$: 2nd-order tensor

(τ_{ij} and δ_{ij} : 2nd-order tensors, τ_m : scalar)

Let P_{ij} be the components of a transformation matrix to a new basis, then $S_{ij}'=P_{ik}P_{jl}S_{kl}$

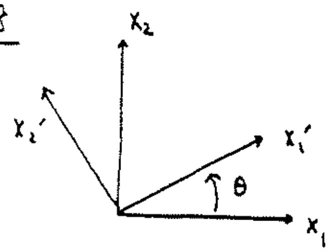
$$\therefore\ \bar{\sigma}'=\left[\tfrac{3}{2}S_{ij}'S_{ij}'\right]^{1/2}=\left[\tfrac{3}{2}(P_{ik}P_{jl}S_{kl})(P_{im}P_{jn}S_{mn})\right]^{1/2}$$

$$=\left[\tfrac{3}{2}\delta_{km}\delta_{ln}S_{kl}S_{mn}\right]^{1/2} \quad (\text{because } P_{ik}P_{im}=\delta_{km})$$

$$=\left[\tfrac{3}{2}S_{kl}S_{kl}\right]^{1/2}=\bar{\sigma}$$

$\therefore$ $\bar{\sigma}$ is a scalar.

2.8



rotation matrix $\underline{P} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \underline{P} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1\cos\theta + x_2\sin\theta \\ -x_1\sin\theta + x_2\cos\theta \end{bmatrix}$$

If g is a vector, it must obey $\underline{g}' = \underline{P}\,\underline{g}$

i.e., $$\begin{bmatrix} x_1' \\ x_1' + x_2' \end{bmatrix} = \begin{bmatrix} x_1\cos\theta + (x_1 + x_2)\sin\theta \\ -x_1\sin\theta + (x_1+x_2)\cos\theta \end{bmatrix} = \begin{bmatrix} x_1' + x_1\sin\theta \\ x_2' + x_1\cos\theta \end{bmatrix}$$

or $$\begin{bmatrix} 0 \\ x_1' \end{bmatrix} = \begin{bmatrix} x_1\sin\theta \\ x_1\cos\theta \end{bmatrix}$$

This equation can be satisfied only for particular values of θ. Hence g is not a vector.

2.9 $\underline{\epsilon}\big|_{ij} = \frac{1}{2}(\underline{X}^T\underline{X} - \underline{I})\big|_{ij} = \frac{1}{2}(X_{ki}X_{kj} - \delta_{ij})$

$$= \frac{1}{2}\left[(\delta_{ki} + u_{k,i})(\delta_{kj} + u_{k,j}) - \delta_{ij}\right]$$

$$= \frac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j})$$

Let $x_i' = P_{ij}x_j$, then $u_{i,j}' = \frac{\partial u_i'}{\partial x_j'} = P_{ip}P_{jq}u_{p,q}$

$$\epsilon_{ij}' = \frac{1}{2}(u_{i,j}' + u_{j,i}' + u_{k,i}'u_{k,j}')$$

$$= \frac{1}{2}\left[P_{ip}P_{jq}u_{p,q} + P_{jq}P_{ip}u_{q,p} + (P_{kr}P_{is}u_{r,s})(P_{km}P_{jn}u_{m,n})\right]$$

$$= \frac{1}{2}\left(P_{ip}P_{jq}u_{p,q} + P_{jq}P_{ip}u_{q,p} + P_{ip}P_{jq}u_{k,p}u_{k,q}\right)$$

$$= \frac{1}{2}P_{ip}P_{jq}\left(u_{p,q} + u_{q,p} + u_{k,p}u_{k,q}\right)$$

$$= P_{ip}P_{jq}\epsilon_{pq}$$

Hence ϵ_{ij} is a 2nd-order tensor.

2.10 (a) δ_{ij} : 2nd-order tensor

$$\delta'_{ij} = \frac{\partial x'_i}{\partial x'_j} = \frac{\partial (P_{ip} x_p)}{\partial x_q} \frac{\partial x_q}{\partial x'_j} = P_{ip} P_{jq} \delta_{pq}$$

Similarly, $\delta_{ij}\delta_{km}$ is a 4-th order tensor.

$$\delta'_{ij}\delta'_{km} = (P_{ip}P_{jq}\delta_{pq})(P_{kr}P_{ms}\delta_{rs})$$
$$= (P_{ip}P_{jq}P_{kr}P_{ms})(\delta_{pq}\delta_{rs})$$

We have $C'_{ijkl} = P_{im}P_{jn}P_{kr}P_{ls}C_{mnrs}$ and we see C_{ijkl} is a 4-th order tensor.

(b) $\sigma_{ij} = C_{ijkm}\epsilon_{km}$ where $C_{ijkm} = \lambda\delta_{ij}\delta_{rs} + \mu(\delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk})$

Due to the symmetry of both the stress and strain tensors, the double indexed system of stress and strain components is often replaced by a single indexed system having a range of 6. Thus in the notation

$$\sigma_{11} = \sigma_1, \quad \sigma_{22} = \sigma_2, \quad \sigma_{33} = \sigma_3,$$
$$\sigma_{12} = \sigma_{21} = \sigma_4, \quad \sigma_{23} = \sigma_{32} = \sigma_5, \quad \sigma_{31} = \sigma_{13} = \sigma_6$$

and

$$\epsilon_{11} = \epsilon_1, \quad \epsilon_{22} = \epsilon_2, \quad \epsilon_{33} = \epsilon_3,$$
$$\epsilon_{12} + \epsilon_{21} = \epsilon_4, \quad \epsilon_{23} + \epsilon_{32} = \epsilon_5, \quad \epsilon_{31} + \epsilon_{13} = \epsilon_6$$

Then, $\sigma_K = C_{KM}\epsilon_M \quad (K, M = 1, 2, 3, 4, 5, 6)$

with

$$[\underline{C}_{KM}] = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & & & \\ \lambda & \lambda+2\mu & \lambda+2\mu & & \underline{O} & \\ \lambda & \lambda & \lambda & & & \\ & & & \mu & 0 & 0 \\ & \underline{O} & & 0 & \mu & 0 \\ & & & 0 & 0 & \mu \end{bmatrix}$$

2.10

Now, from the plane stress condition

$$\sigma_3 = \lambda \epsilon_1 + \lambda \epsilon_2 + (\lambda + 2\mu)\,\epsilon_3 = 0$$

$$\sigma_4 = \mu \epsilon_4 = 0$$

$$\sigma_5 = \mu \epsilon_5 = 0 \qquad \therefore \quad \epsilon_3 = -\frac{\lambda}{\lambda+2\mu}(\epsilon_1 + \epsilon_2)\,, \quad \epsilon_4 = \epsilon_5 = 0$$

$$\begin{aligned} \sigma_1 &= (\lambda+2\mu)\,\epsilon_1 + \lambda \epsilon_2 + \lambda \epsilon_3 \\ &= (\lambda+2\mu)\,\epsilon_1 + \lambda \epsilon_2 - \frac{\lambda^2}{\lambda+2\mu}(\epsilon_1+\epsilon_2) \\ &= \frac{4\mu(\lambda+\mu)}{\lambda+2\mu}\,\epsilon_1 + \frac{2\mu\lambda}{\lambda+2\mu}\,\epsilon_2 \\ &= \frac{E}{1-\nu^2}(\epsilon_1 + \nu \epsilon_2) \quad \left(\text{because } \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}\,,\ \mu = \frac{E}{2(1+\nu)}\right) \end{aligned}$$

Similarly, $\sigma_2 = \dfrac{E}{1-\nu^2}(\nu \epsilon_1 + \epsilon_2)$

$$\sigma_6 = \mu \epsilon_6 = \frac{E}{1-\nu^2}\left(\frac{1-\nu}{2}\epsilon_6\right)$$

That is, $\underline{\sigma} = \underline{C}\,\underline{\epsilon}$ in Table 4.3.

where $\underline{\sigma}^T = [\sigma_1 \ \ \sigma_2 \ \ \sigma_6] = [\tau_{xx} \ \ \tau_{yy} \ \ \tau_{xy}]$

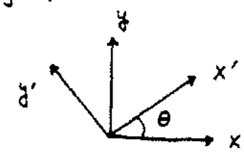
$\underline{\epsilon}^T = [\epsilon_1 \ \ \epsilon_2 \ \ \epsilon_6] = [\epsilon_{xx} \ \ \epsilon_{yy} \ \ \gamma_{xy}]$

$$\underline{C} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

2.10

(c) Let $\sigma = [\sigma_{ij}]$, $\epsilon = [\epsilon_{ij}]$, $\sigma' = [\sigma'_{ij}]$ and $\epsilon' = [\epsilon'_{ij}]$.

Also $P = \begin{bmatrix} \cos(x,x') & \cos(y,x') \\ \cos(x,y') & \cos(y,y') \end{bmatrix} = \begin{bmatrix} l_1 & l_2 \\ m_1 & m_2 \end{bmatrix}$



Then $\sigma' = P \sigma P^T = \begin{bmatrix} l_1^2\sigma_{11} + l_2^2\sigma_{22} + 2l_1l_2\sigma_{12} & l_1m_1\sigma_{11} + l_2m_2\sigma_{22} + (l_1m_2 + l_2m_1)\sigma_{12} \\ l_1m_1\sigma_{11} + l_2m_2\sigma_{22} + (l_1m_2 + l_2m_1)\sigma_{12} & m_1^2\sigma_{11} + m_2^2\sigma_{22} + 2m_1m_2\sigma_{12} \end{bmatrix}$

$\epsilon = P^T \epsilon' P = \begin{bmatrix} l_1^2\epsilon'_{11} + m_1^2\epsilon'_{22} + 2l_1m_1\epsilon'_{12} & l_1l_2\epsilon'_{11} + m_1m_2\epsilon'_{22} + (l_1m_2 + l_2m_1)\epsilon'_{12} \\ l_1l_2\epsilon'_{11} + m_1m_2\epsilon'_{22} + (l_1m_2 + l_2m_1)\epsilon'_{12} & l_2^2\epsilon'_{11} + m_2^2\epsilon'_{22} + 2l_2m_2\epsilon'_{12} \end{bmatrix}$

Now in order to represent σ and ϵ in a single indexed system, that is, in vector notation, we introduce τ and ε such that $\tau^T = [\sigma_{11} \;\; \sigma_{22} \;\; \sigma_{12}]$ and $\varepsilon^T = [\epsilon_{11} \;\; \epsilon_{22} \;\; 2\epsilon_{12}]$.

$\rightarrow$ $\tau' = T\tau$ and $\varepsilon = T^T\varepsilon'$

where $T = \begin{bmatrix} l_1^2 & l_2^2 & 2l_1l_2 \\ m_1^2 & m_2^2 & 2m_1m_2 \\ l_1m_1 & l_2m_2 & l_1m_2 + l_2m_1 \end{bmatrix}$

Hence $\tau' = T\tau = T(C\varepsilon) = (T C T^T)\varepsilon' = C'\varepsilon'$

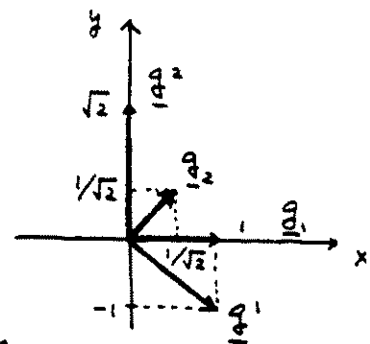
$\rightarrow$ $C' = T C T$

2.11 Let $\underline{g}^i = a^{ik}\underline{g}_k$

Then, $\underline{g}^i \cdot \underline{g}^j = (a^{ik}\underline{g}_k)\cdot \underline{g}^j = a^{ik}\delta_k^j = a^{ij}$

$\therefore \underline{g}^i = g^{ij}\underline{g}_j$ with $g^{ij} = \underline{g}^i \cdot \underline{g}^j$

2.12 $\underline{g}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \underline{g}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$



$\longrightarrow \quad \underline{g}^1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \underline{g}^2 = \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}$

(a) $\underline{R}\cdot\underline{u} = (3\underline{g}_1 + 4\underline{g}_2)\cdot(-2\underline{g}_1 + 3\underline{g}_2)$

$= -6\underline{g}_1\cdot\underline{g}_1 + 9\underline{g}_1\cdot\underline{g}_2 - 8\underline{g}_2\cdot\underline{g}_1 + 12\underline{g}_2\cdot\underline{g}_2$

$= 6 + \frac{1}{\sqrt{2}} \qquad \left(\underline{g}_1\cdot\underline{g}_1 = 1,\ \underline{g}_1\cdot\underline{g}_2 = \underline{g}_2\cdot\underline{g}_1 = \frac{1}{\sqrt{2}},\ \underline{g}_2\cdot\underline{g}_2 = 1\right)$

(b) $\underline{u} = u^i\underline{g}_i = u_i\underline{g}^i$

$-2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = u_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + u_2\begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}$

$\therefore u_1 = -2 + \frac{3}{\sqrt{2}}, \quad u_2 = 3 - \sqrt{2}$

(Here note that the same result is obtained formally using $u_i = g_{ij}u^j$.)

$\underline{R}\cdot\underline{u} = R^i\underline{g}_i \cdot u_j\underline{g}^j = R^i u_i$

$= 3\cdot\left(-2 + \frac{3}{\sqrt{2}}\right) + 4\cdot(3 - \sqrt{2}) = 6 + \frac{1}{\sqrt{2}}$

2.13 First evaluate $\tilde{\tau}^{mn}$ and $\tilde{\epsilon}_{mn}$.

$$\left.\begin{array}{l}\tilde{\tau}^{mn}\underline{g}_m\underline{g}_n = \tau_{ij}\underline{e}_i\underline{e}_j \\ \therefore \tilde{\tau}^{mn} = \tau_{ij}(\underline{e}_i\cdot\underline{g}^m)(\underline{e}_j\cdot\underline{g}^n)\end{array}\right\} \longrightarrow [\tilde{\tau}^{mn}] = \begin{bmatrix} 280 & -190\sqrt{2} \\ -190\sqrt{2} & 400 \end{bmatrix}$$

$$\left.\begin{array}{l}\tilde{\epsilon}_{mn}\underline{g}^m\underline{g}^n = \epsilon_{ij}\underline{e}_i\underline{e}_j \\ \therefore \tilde{\epsilon}_{mn} = \epsilon_{ij}(\underline{e}_i\cdot\underline{g}_m)(\underline{e}_j\cdot\underline{g}_n)\end{array}\right\} \longrightarrow [\tilde{\epsilon}_{mn}] = \begin{bmatrix} 0.01 & \frac{0.06}{\sqrt{2}} \\ \frac{0.06}{\sqrt{2}} & \frac{0.13}{2} \end{bmatrix}$$

Now calculate $\underline{\tau}\cdot\underline{\epsilon}$.

(Cartesian): $\tau_{ij}\epsilon_{ij} = (100)(0.01) + 2(10)(0.05) + (200)(0.02) = 6$

(Contravariant and covariant): $\tilde{\tau}^{mn}\tilde{\epsilon}_{mn} = (280)(0.01) + 2(-190\sqrt{2})\left(\frac{0.06}{\sqrt{2}}\right) + (400)\left(\frac{0.13}{2}\right) = 6$

Hence, the same results are obtained.

2.14 $\underline{a}\cdot(\underline{A}\,\underline{b}\,\underline{B}^T) = a_{ij}(A_{ik}b_{kl}B_{jl}) = \underbrace{(A_{ik}a_{ij}B_{jl})}_{kl \text{ element of } (\underline{A}^T\underline{a}\,\underline{B})}b_{kl}$

$\therefore \underline{a}\cdot(\underline{A}\,\underline{b}\,\underline{B}^T) = (\underline{A}^T\underline{a}\,\underline{B})\cdot\underline{b}$

2.15 (a) $p(\lambda) = \begin{vmatrix} 2-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 1 = 0$

$\therefore \lambda_1 = (3-\sqrt{5})/2 , \quad \lambda_2 = (3+\sqrt{5})/2$

For λ_1, $\begin{bmatrix} 2-(3-\sqrt{5})/2 & -1 \\ -1 & 1-(3-\sqrt{5})/2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$v_1^2 + v_2^2 = 1$

$\therefore \underline{v}_1 = \begin{bmatrix} \dfrac{\sqrt{5}-1}{\sqrt{2}\sqrt{5-\sqrt{5}}} \\ \dfrac{\sqrt{2}}{\sqrt{5-\sqrt{5}}} \end{bmatrix}$, Similarly $\underline{v}_2 = \begin{bmatrix} -\dfrac{\sqrt{5}+1}{\sqrt{2}\sqrt{5+\sqrt{5}}} \\ \dfrac{\sqrt{2}}{\sqrt{5+\sqrt{5}}} \end{bmatrix}$

Then, $\underline{A} = \sum_{i=1}^{2} \lambda_i \underline{v}_i \underline{v}_i^T$

$$= \frac{3-\sqrt{5}}{2} \begin{bmatrix} \dfrac{\sqrt{5}-1}{\sqrt{2}\sqrt{5-\sqrt{5}}} \\ \dfrac{\sqrt{2}}{\sqrt{5-\sqrt{5}}} \end{bmatrix} \Big[\quad '' \quad \Big] + \frac{3+\sqrt{5}}{2} \begin{bmatrix} -\dfrac{\sqrt{5}+1}{\sqrt{2}\sqrt{5+\sqrt{5}}} \\ \dfrac{\sqrt{2}}{\sqrt{5+\sqrt{5}}} \end{bmatrix} \Big[\quad '' \quad \Big]$$

(b) $\underline{A}^k = \sum_{i=1}^{2} \lambda_i^k \underline{v}_i \underline{v}_i^T$

Calculate first $\underline{A}^6$: $\lambda_1^6 = 161 - 72\sqrt{5}$, $\lambda_2^6 = 161 + 72\sqrt{5}$

$\underline{v}_1 \underline{v}_1^T = \begin{bmatrix} \dfrac{5-\sqrt{5}}{10} & \dfrac{\sqrt{5}}{5} \\ \dfrac{\sqrt{5}}{5} & \dfrac{5+\sqrt{5}}{10} \end{bmatrix}$, $\underline{v}_2 \underline{v}_2^T = \begin{bmatrix} \dfrac{5+\sqrt{5}}{10} & -\dfrac{\sqrt{5}}{5} \\ -\dfrac{\sqrt{5}}{5} & \dfrac{5-\sqrt{5}}{10} \end{bmatrix}$

$\therefore \underline{A}^6 = \lambda_1^6 \underline{v}_1 \underline{v}_1^T + \lambda_2^6 \underline{\lambda}_2 \underline{\lambda}_2^T = \begin{bmatrix} 233 & -144 \\ -144 & 89 \end{bmatrix}$

Similarly, $\underline{A}^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, $\underline{A}^{-2} = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$

2.16 Let $\underline{v} = \underline{v}_1 + \epsilon \underline{x}$, then $\epsilon = 0.1$ and $\underline{x}^T = [1 \quad 1 \quad 0]$

$$\rho(\underline{v}) = \frac{\underline{v}^T \underline{A}\, \underline{v}}{\underline{v}^T \underline{v}} = \frac{\underline{v}_1^T \underline{A}\, \underline{v}_1 + 2\epsilon \underline{v}_1^T \underline{A}\, \underline{x} + \epsilon^2 \underline{x}^T \underline{A}\, \underline{x}}{\underline{v}_1^T \underline{v}_1 + 2\epsilon \underline{x}^T \underline{v}_1 + \epsilon^2 \underline{x}^T \underline{x}}$$

$$= \frac{\lambda_1 + \epsilon^2 \underline{x}^T \underline{A}\, \underline{x}}{1 + \epsilon^2 \underline{x}^T \underline{x}} = \frac{\lambda_1 + 7\epsilon^2}{1 + 2\epsilon^2} = \lambda_1 + O(\epsilon^2)$$

$$\therefore \rho(\underline{v}) = 1 + (0.04902) \qquad = \lambda_1 + (0.04902)$$

Hence we see that when $\underline{v} = \underline{v}_1 + O(\epsilon)$, $\rho(\underline{v}) = \lambda_1 + O(\epsilon^2)$

2.17

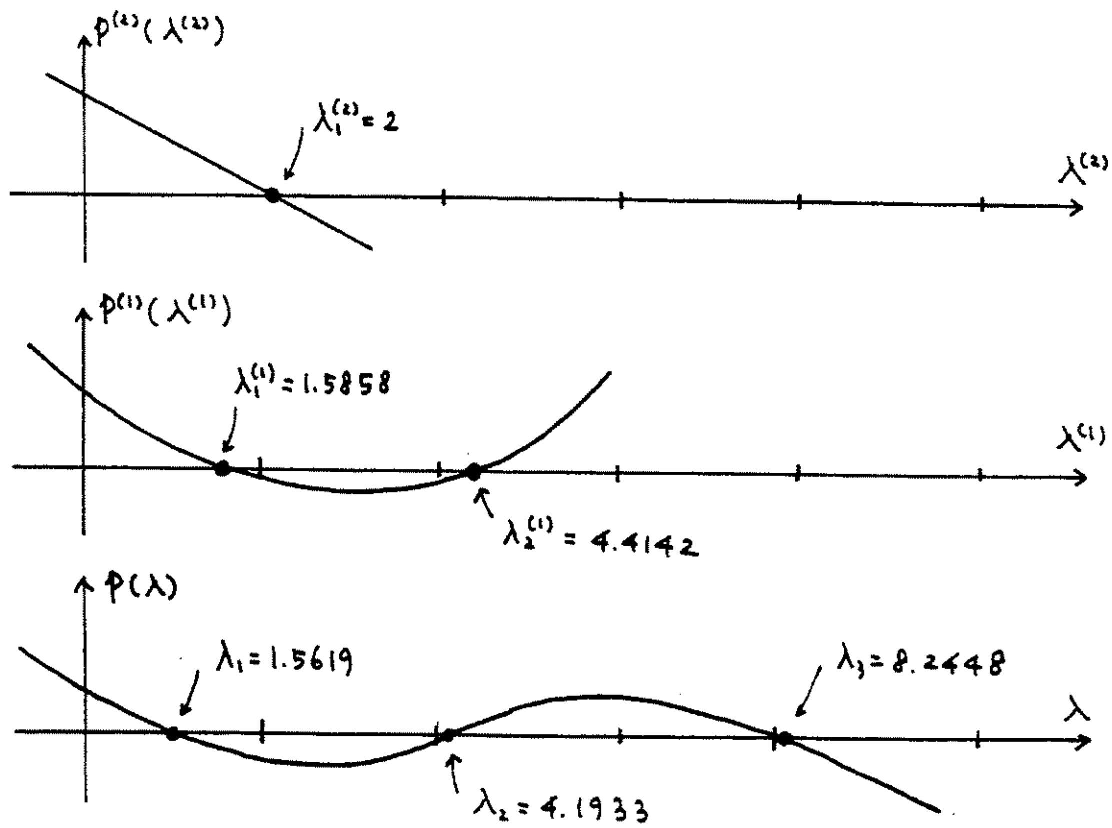
$$p(\lambda) = \begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 4-\lambda & -1 \\ 0 & -1 & 8-\lambda \end{vmatrix} = 54 - 54\lambda + 14\lambda^2 - \lambda^3$$

$\therefore \lambda_1 = 1.5619, \quad \lambda_2 = 4.1933, \quad \lambda_3 = 8.2448$

$$p^{(1)}(\lambda^{(1)}) = \begin{vmatrix} 2-\lambda^{(1)} & -1 \\ -1 & 4-\lambda^{(1)} \end{vmatrix} = 7 - 6\lambda^{(1)} + \lambda^{(1)2}$$

$\therefore \lambda_1^{(1)} = 1.5858, \quad \lambda_2^{(1)} = 4.4142$

$p^{(2)}(\lambda^{(2)}) = 2 - \lambda^{(2)} \qquad \therefore \lambda_1^{(2)} = 2$



2.18 $\|\underline{v}\|_2 = \left(|v_1|^2 + |v_2|^2 + \cdots + |v_n|^2\right)^{1/2} \le |v_1| + |v_2| + \cdots + |v_n| = \|\underline{v}\|_1$

$\therefore \quad \|\underline{v}\|_2 \le \|\underline{v}\|_1$ ——— ①

$$n\|\underline{v}\|_2 - \|\underline{v}\|_1 = n\left(|v_1|^2 + |v_2|^2 + \cdots + |v_n|^2\right)^{1/2} - \left(|v_1| + |v_2| + \cdots + |v_n|\right)$$

$$= \left\{\left(|v_1|^2 + |v_2|^2 + \cdots + |v_n|^2\right)^{1/2} - |v_1|\right\}$$

$$+ \cdots + \left\{\left(|v_1|^2 + |v_2|^2 + \cdots + |v_n|^2\right)^{1/2} - |v_n|\right\} \ge 0$$

$\therefore \quad n\|\underline{v}\|_2 \ge \|\underline{v}\|_1$ ——— ②

From ① and ②, $\left.\begin{array}{l} \|\underline{v}\|_2 \le \|\underline{v}\|_1 \le n\|\underline{v}\|_2 \\ \frac{1}{n}\|\underline{v}\|_1 \le \|\underline{v}\|_2 \le \|\underline{v}\|_1 \end{array}\right\}$ ——— ③

The relations in ③ read for $\underline{v}^T = [\,1 \quad 4 \quad -3\,]$

$\left(\|\underline{v}\|_1 = 1 + 4 + 3 = 8, \quad \|\underline{v}\|_2 = (1^2 + 4^2 + 3^2)^{1/2} = \sqrt{26} \right)$

$\sqrt{26} \le 8 \le 3\sqrt{26}$, $\frac{8}{3} \le \sqrt{26} \le 8$.

2.19 $\|\underset{\sim}{A}\underset{\sim}{B}\|_1 = \max_j \sum_{i=1}^n \left| \sum_{k=1}^n a_{ik} b_{kj} \right| \le \max_j \sum_{i=1}^n \sum_{k=1}^n |a_{ik}||b_{kj}|$

$$= \max_j \sum_{k=1}^n \left(\sum_{i=1}^n |a_{ik}| \right) |b_{kj}|$$

$$\le \max_j \sum_{k=1}^n \left[\left(\max_p \sum_{i=1}^n |a_{ip}| \right) |b_{kj}| \right]$$

$$= \left(\max_p \sum_{i=1}^n |a_{ip}| \right) \left(\max_j \sum_{k=1}^n |b_{kj}| \right) = \|\underset{\sim}{A}\|_1 \|\underset{\sim}{B}\|_1$$

$\therefore \|\underset{\sim}{A}\underset{\sim}{B}\|_1 \le \|\underset{\sim}{A}\|_1 \|\underset{\sim}{B}\|_1$

2.20 $\|\underset{\sim}{A}\underset{\sim}{v}\|_1 = \sum_{i=1}^n \left| \sum_{j=1}^n a_{ij} v_j \right| \le \sum_{i=1}^n \left(\sum_{j=1}^n |a_{ij}||v_j| \right)$

$$= \sum_{j=1}^n \left(\sum_{i=1}^n |a_{ij}| \right) |v_j| \le \sum_{j=1}^n \left(\max_k \sum_{i=1}^n |a_{ik}| \right) |v_j|$$

$$= \left(\max_k \sum_{i=1}^n |a_{ik}| \right) \left(\sum_{j=1}^n |v_j| \right) = \|\underset{\sim}{A}\|_1 \|\underset{\sim}{v}\|_1$$

$\therefore \|\underset{\sim}{A}\underset{\sim}{v}\|_1 \le \|\underset{\sim}{A}\|_1 \|\underset{\sim}{v}\|_1$

2.21 By definition, $\|\underset{\sim}{A}\|_2 = \sqrt{\tilde{\lambda}_n}$ where $\tilde{\lambda}_n$ = max. eigenvalue of $\underset{\sim}{A}^T\underset{\sim}{A}$

$\therefore \underset{\sim}{A}^T \underset{\sim}{A} = \underset{\sim}{A}^2 = \sum_{i=1}^n \lambda_i^2 \underset{\sim}{v}_i \underset{\sim}{v}_i^T \rightarrow \tilde{\lambda}_n$ is $\max_i \lambda_i^2$.

$\therefore \|\underset{\sim}{A}\|_2 = \sqrt{\tilde{\lambda}_n} = \max_{\text{all } i} |\lambda_i|$

2.22

$$\|A\|_{LR} = \sup_{y} \frac{\|Ay\|_R}{\|y\|_L} \qquad \left(\text{from } (2.170)\right)$$

$$= \sup_{y} \frac{\|Ay\|_{DL}}{\|y\|_L} \qquad \left(\text{use } \|\cdot\|_R = \|\cdot\|_{DL}\right)$$

$$= \sup_{y} \left\{ \frac{1}{\|y\|_L} \sup_{x} \frac{x^T A y}{\|x\|_L} \right\} \qquad \left(\text{use } (2.176)\right)$$

$$= \sup_{x,y} \frac{x^T A y}{\|x\|_L \|y\|_L} = k_A \qquad \leftarrow (2.177)$$

$$\left(\|A^{-1}\|_{RL}\right)^{-1} = \left(\sup_{z} \frac{\|A^{-1}z\|_L}{\|z\|_R}\right)^{-1} = \inf_{z} \frac{\|z\|_R}{\|A^{-1}z\|_L}$$

$$= \inf_{y} \frac{\|Ay\|_R}{\|y\|_L} \qquad (\text{set } z = Ay)$$

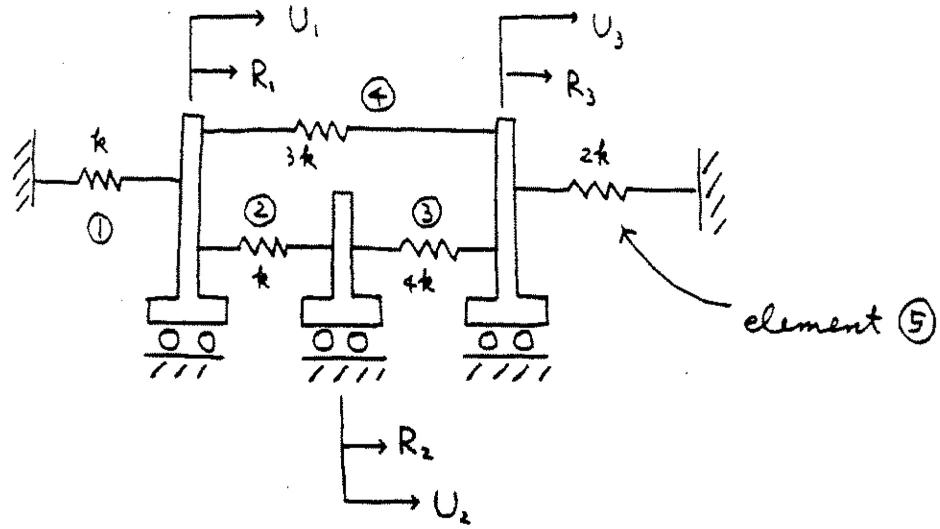
$$= \inf_{y} \frac{\|Ay\|_{DL}}{\|y\|_L} \qquad \left(\text{use } \|\cdot\|_R = \|\cdot\|_{DL}\right)$$

$$= \inf_{y} \left\{ \frac{1}{\|y\|_L} \sup_{x} \frac{x^T A y}{\|x\|_L} \right\}$$

$$= \inf_{y} \sup_{x} \frac{x^T A y}{\|x\|_L \|y\|_L} \qquad (\text{use } A = A^T)$$

$$= \inf_{x} \sup_{y} \frac{x^T A y}{\|x\|_L \|y\|_L} = \gamma_A \qquad \leftarrow (2.178)$$

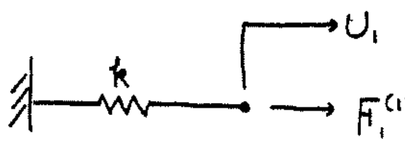
3.1



Consider the force balance in each element:

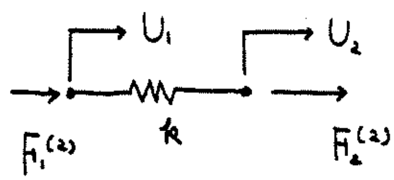
element ①

$$\begin{bmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} F_1^{(1)} \\ 0 \\ 0 \end{bmatrix}$$



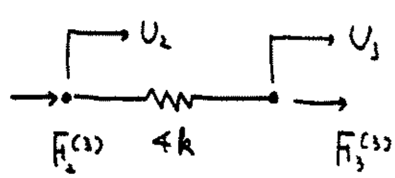
element ②

$$\begin{bmatrix} k & -k & 0 \\ -k & k & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} F_1^{(2)} \\ F_2^{(2)} \\ 0 \end{bmatrix}$$



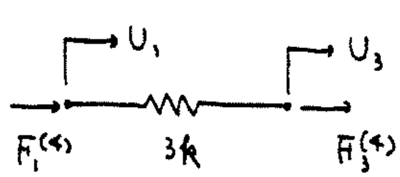
element ③

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 4k & -4k \\ 0 & -4k & 4k \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 0 \\ F_2^{(3)} \\ F_3^{(3)} \end{bmatrix}$$



element ④

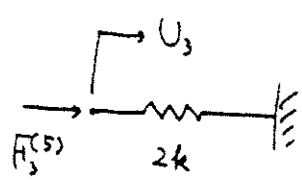
$$\begin{bmatrix} 3k & 0 & -3k \\ 0 & 0 & 0 \\ -3k & 0 & 3k \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} F_1^{(4)} \\ 0 \\ F_3^{(4)} \end{bmatrix}$$



3.1

element ⑤

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2k \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ F_3^{(5)} \end{bmatrix}$$



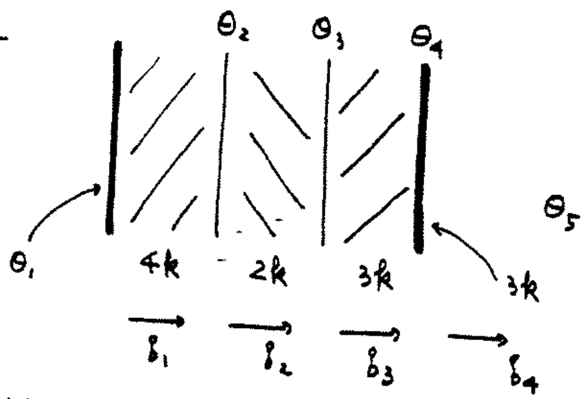
When all the matrices are summed up

$$\begin{bmatrix} 5k & -k & -3k \\ -k & 5k & -4k \\ -3k & -4k & 9k \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} F_1^{(1)} + F_1^{(2)} + F_1^{(4)} \\ F_2^{(2)} + F_2^{(3)} \\ F_3^{(3)} + F_3^{(4)} + F_3^{(5)} \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}$$

Hence, the governing equilibrium equation is

$$\begin{bmatrix} 5k & -k & -3k \\ -k & 5k & -4k \\ -3k & -4k & 9k \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \\ 0 \end{bmatrix}$$

3.2



$$\delta_1 = 4k(\theta_1 - \theta_2)$$

$$\delta_2 = 2k(\theta_2 - \theta_3)$$

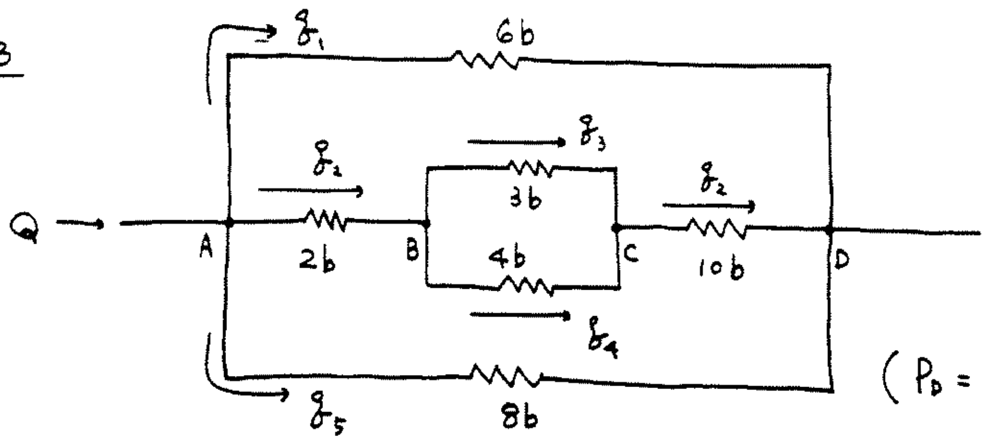
$$\delta_3 = 3k(\theta_3 - \theta_4)$$

$$\delta_4 = 3k(\theta_4 - \theta_5)$$

Using $\delta_1 = \delta_2 = \delta_3 = \delta_4$,

$$\begin{bmatrix} 6k & -2k & 0 \\ -2k & 5k & -3k \\ 0 & -3k & 6k \end{bmatrix} \begin{bmatrix} \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} = \begin{bmatrix} 4k\theta_1 \\ 0 \\ 3k\theta_5 \end{bmatrix}$$

3.3



$(P_D = 0)$

For each element:

$$\left.\begin{aligned} q_1 &= \frac{P_A}{6b}, \quad q_2\Big|_{AB} = \frac{P_A - P_B}{2b}, \quad q_3 = \frac{P_B - P_C}{3b}, \quad q_4 = \frac{P_B - P_C}{4b} \\ q_2\Big|_{CD} &= \frac{P_C}{10b}, \quad q_5 = \frac{P_A}{8b} \end{aligned}\right\} \text{— (i)}$$

From continuity

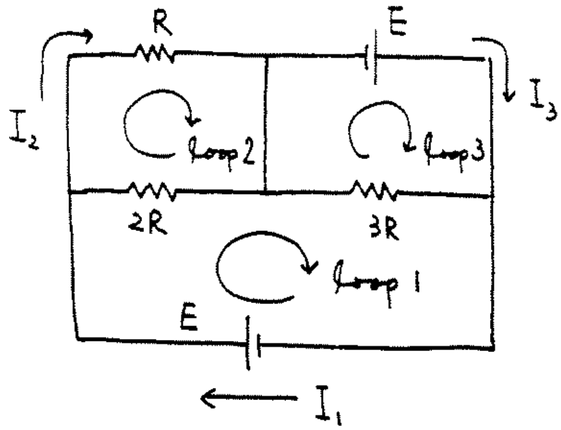
$$Q = q_1 + q_2\Big|_{AB} + q_5, \quad q_2\Big|_{AB} = q_2\Big|_{CD} = q_3 + q_4 \quad \Big\} \text{— (ii)}$$

From (i) and (ii),

$$\begin{bmatrix} 19 & -12 & 0 \\ -6 & 13 & -7 \\ 0 & -35 & 41 \end{bmatrix} \begin{bmatrix} P_A \\ P_B \\ P_C \end{bmatrix} = \begin{bmatrix} 24bQ \\ 0 \\ 0 \end{bmatrix}$$

$$\text{or} \quad \begin{bmatrix} 95 & -60 & 0 \\ -60 & 130 & -70 \\ 0 & -70 & 82 \end{bmatrix} \begin{bmatrix} P_A \\ P_B \\ P_C \end{bmatrix} = \begin{bmatrix} 120bQ \\ 0 \\ 0 \end{bmatrix}$$

3.4



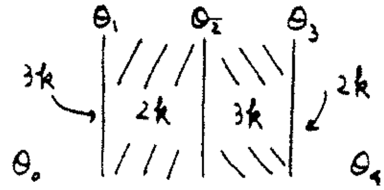
$$\text{loop 1}: \quad E = 2R(I_1 - I_2) + 3R(I_1 - I_3)$$
$$\text{loop 2}: \quad 0 = RI_2 + 2R(I_2 - I_1)$$
$$\text{loop 3}: \quad E = 3R(I_3 - I_1)$$

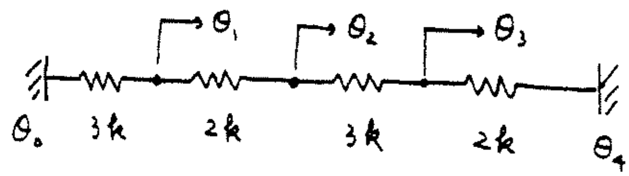
$$\longrightarrow \begin{bmatrix} 5R & -2R & -3R \\ -2R & 3R & 0 \\ -3R & 0 & 3R \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} E \\ 0 \\ E \end{bmatrix}$$

3.5 $\Pi = U - W$

$$= \left[\frac{1}{2}kU_1^2 + \frac{1}{2}k(U_2 - U_1)^2 + \frac{1}{2}(4k)(U_3 - U_2)^2 + \frac{1}{2}(3k)(U_3 - U_1)^2 + \frac{1}{2}(2k)U_3^2 \right]$$
$$- \left[R_1U_1 + R_2U_2 + R_3U_3 \right]$$

3.6





$$\Pi = U - W$$

$$= \left[\frac{1}{2}(2k)(\theta_2-\theta_1)^2 + \frac{1}{2}(3k)(\theta_3-\theta_2)^2\right] - \left[-\frac{1}{2}(3k)(\theta_1-\theta_0)^2 - \frac{1}{2}(2k)(\theta_4-\theta_3)^2\right]$$

Check: $\frac{\partial \Pi}{\partial \theta_1} = 0$, $2k\theta_1 - 2k\theta_2 = 3k(\theta_0 - \theta_1)$

$\frac{\partial \Pi}{\partial \theta_2} = 0$, $-2k\theta_1 + 5k\theta_2 - 3k\theta_3 = 0$

$\frac{\partial \Pi}{\partial \theta_3} = 0$, $-3k\theta_2 + 3k\theta_3 = 2k(\theta_4 - \theta_3)$

$$\begin{bmatrix} 2k & -2k & 0 \\ -2k & 5k & -3k \\ 0 & -3k & 3k \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} 3k(\theta_0-\theta_1) \\ 0 \\ 2k(\theta_4-\theta_3) \end{bmatrix}$$

Hence, the same governing eq'n is obtained as in Example :

3.7 From Exercise 3.1

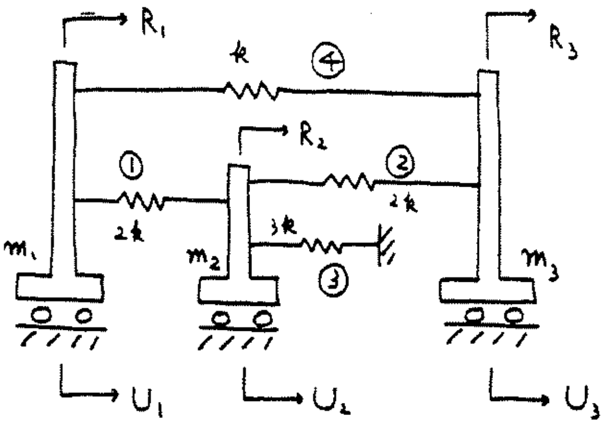
$$F_1^{(1)} + F_1^{(2)} + F_1^{(4)} = R_1 - m_1\ddot{U}_1$$

$$F_2^{(2)} + F_2^{(3)} = R_2 - m_2\ddot{U}_2$$

$$F_3^{(3)} + F_3^{(4)} + F_3^{(5)} = R_3 - m_3\ddot{U}_3$$

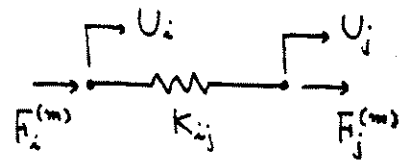
$$\rightarrow \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} \ddot{U}_1 \\ \ddot{U}_2 \\ \ddot{U}_3 \end{bmatrix} + \begin{bmatrix} 5k & -k & -3k \\ -k & 5k & -4k \\ -3k & -4k & 9k \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}$$

3.8



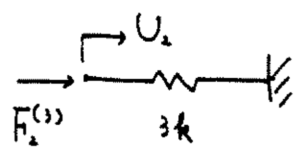
element ①, ② and ④ ;

$$\begin{bmatrix} K_{ij} & -K_{ij} \\ -K_{ij} & K_{ij} \end{bmatrix} \begin{bmatrix} U_i \\ U_j \end{bmatrix} = \begin{bmatrix} F_i^{(m)} \\ F_j^{(m)} \end{bmatrix}$$



element ③ ;

$$3k\,U_2 = F_2^{(3)}$$



at station i ($i = 1, 2, 3$)

$$F_1^{(1)} + F_1^{(4)} = R_1 - m_1 \ddot{U}_1$$

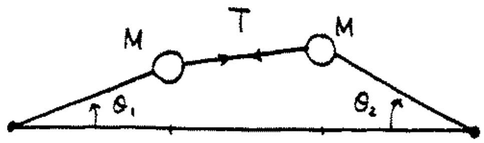
$$F_2^{(1)} + F_2^{(2)} + F_2^{(3)} = R_2 - m_2 \ddot{U}_2$$

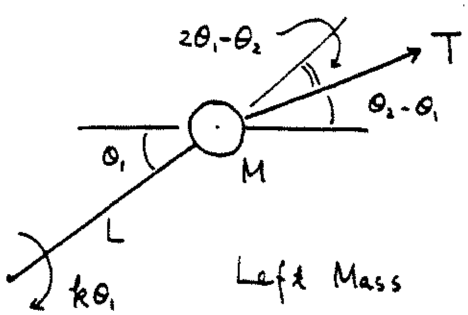
$$F_3^{(2)} + F_3^{(4)} = R_3 - m_3 \ddot{U}_3$$

Hence,

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} \ddot{U}_1 \\ \ddot{U}_2 \\ \ddot{U}_3 \end{bmatrix} + \begin{bmatrix} 3k & -2k & -k \\ -2k & 7k & -2k \\ -k & -2k & 3k \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 50 \\ 0 \\ 10 \end{bmatrix}$$

3.9





Left Mass

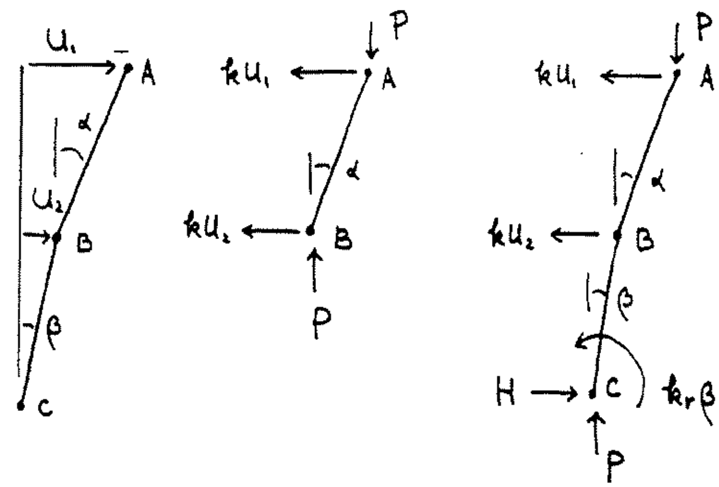
$$ML^2\ddot{\theta}_1 = -T(2\theta_1 - \theta_2)L - k\theta_1$$

Similarly, for right mass,

$$ML^2\ddot{\theta}_2 = -T(-\theta_1 + 2\theta_2)L - k\theta_2$$

$$\Rightarrow \begin{bmatrix} ML^2 & 0 \\ 0 & ML^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 2TL + k & -TL \\ -TL & 2TL + k \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

3.10



Direct approach

for bar AB, $PL\sin\alpha = ku_1 L\cos\alpha$

for bars ABC, $PL(\sin\alpha + \sin\beta) = ku_1 L(\cos\alpha + \cos\beta) + ku_2 L\cos\beta + k_r\beta$

assume small displacements:

$L\sin\alpha = u_1 - u_2$, $L\sin\beta = u_2$, $L\cos\alpha = L$,

$L\cos\beta = L$, $\alpha = (u_1 - u_2)/L$, $\beta = u_2/L$

$$\therefore \begin{bmatrix} P & -P \\ P & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} kL & 0 \\ 2kL & kL + \frac{k_r}{L} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} kL & 0 \\ 0 & 2kL \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} P & -P \\ -P & 2P \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Variational approach

$$\Pi = \frac{1}{2}ku_1^2 + \frac{1}{2}ku_2^2 + \frac{1}{2}k_r\beta^2 - PL(2 - \cos\alpha - \cos\beta)$$

3.10

approximating the trigonometric expressions to 2nd order

$$\cos\alpha = 1 - \frac{\alpha^2}{2}, \quad \cos\beta = 1 - \frac{\beta^2}{2} \quad \text{with} \quad \alpha = \frac{u_1 - u_2}{L}, \quad \beta = \frac{u_2}{L}$$

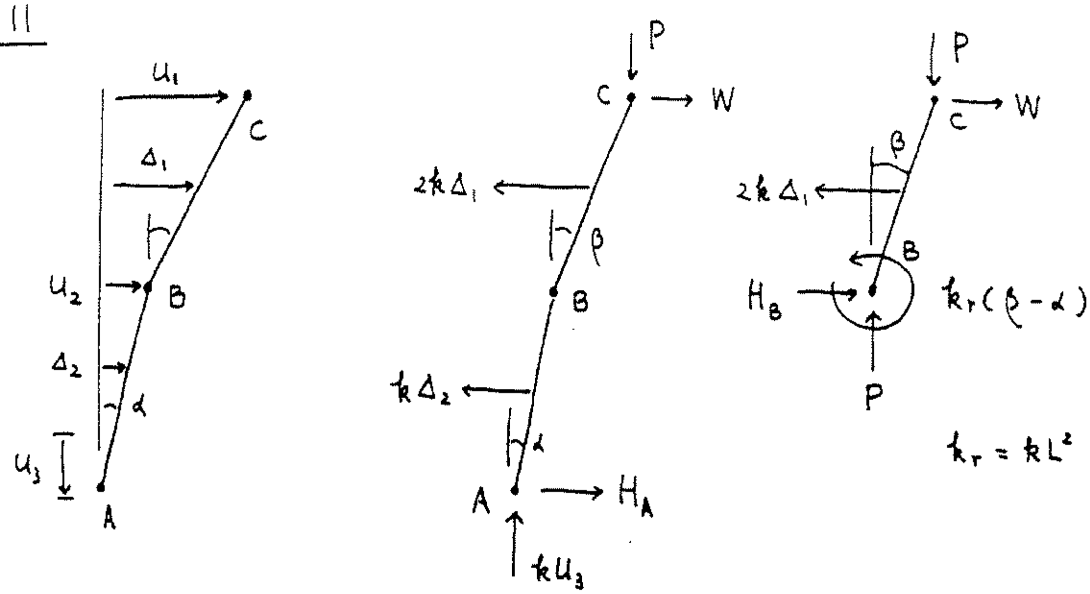
$$\therefore \quad \Pi = \frac{1}{2}ku_1^2 + \frac{1}{2}\left(k + \frac{k_r}{L^2}\right)u_2^2 - \frac{P}{2L}\left(u_1^2 - 2u_1u_2 + 2u_2^2\right)$$

$$\frac{\partial \Pi}{\partial u_1} = 0 \; ; \qquad kL\,u_1 = P(u_1 - u_2)$$

$$\frac{\partial \Pi}{\partial u_2} = 0 \; ; \qquad \left(kL + \frac{k_r}{L}\right)u_2 = P(-u_1 + 2u_2)$$

$$\longrightarrow \quad \begin{bmatrix} kL & 0 \\ 0 & 2kL \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} P & -P \\ -P & 2P \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

3.11



$k_r = kL^2$

For bar BC

$$-P(u_1-u_2) - WL\cos\beta + 2k\Delta_1 \frac{L}{2}\cos\beta + k_r(\beta-\alpha) = 0 \qquad \text{——①}$$

For bars ABC

$$-Pu_1 - WL(\cos\alpha+\cos\beta) + 2k\Delta_1 L\left(\cos\alpha + \frac{1}{2}\cos\beta\right) + k\Delta_2 \frac{L}{2}\cos\alpha = 0 \qquad \text{——②}$$

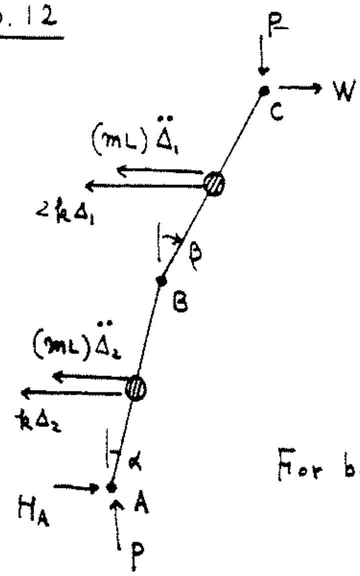
Assuming small displacements

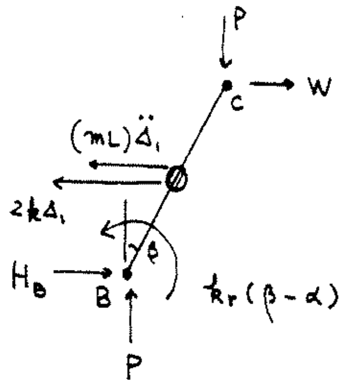
$$\left.\begin{aligned} &\beta = \frac{u_1-u_2}{L}, \quad \alpha = \frac{u_2}{L}, \quad \beta-\alpha = \frac{u_1-2u_2}{L} \\ &\Delta_1 = u_2 + \frac{L}{2}\sin\beta = u_2 + \frac{L}{2}\cdot\frac{u_1-u_2}{L} = \frac{1}{2}(u_1+u_2) \\ &\Delta_2 = \frac{L}{2}\sin\alpha = \frac{u_2}{2} \end{aligned}\right\} \qquad \text{——③}$$

Equations ① to ③ give the eigenproblem by setting $W=0$

$$\begin{bmatrix} \frac{3}{2}kL & -\frac{3}{2}kL \\ -\frac{3}{2}kL & \frac{19}{4}kL \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = P\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

3.12





We follow the same notation and sign convention as in exercise 3.11.

For bar BC,

$$\frac{mL^3}{12}\ddot{\beta} + (mL\ddot{\Delta}_1)\frac{L}{2} = -k_r(\beta-\alpha) - 2k\Delta_1\frac{L}{2} + P(u_1-u_2) + WL$$

$$\therefore \frac{mL^2}{3}\ddot{u}_1 + \frac{mL^2}{6}\ddot{u}_2 + \frac{3}{2}kL(u_1-u_2) = P(u_1-u_2) + WL \quad \text{—— ①}$$

For bars ABC,

$$\left[\frac{mL^3}{12}\ddot{\alpha} + (mL\ddot{\Delta}_2)\frac{L}{2}\right] + \left[\frac{mL^3}{12}\ddot{\beta} + (mL\ddot{\Delta}_1)\left(L+\frac{L}{2}\right)\right] = -2k\Delta_1\left(L+\frac{L}{2}\right) - k\Delta_2\frac{L}{2} + Pu_1 + 2WL$$

$$\therefore \frac{5}{6}mL^2\ddot{u}_1 + mL^2\ddot{u}_2 + \frac{3}{2}kLu_1 + \frac{7}{4}kLu_2 = Pu_1 + 2WL \quad \text{—— ②}$$

From eq. ① and ② we have

$$mL^2\begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{5}{6} & 1 \end{bmatrix}\begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} + kL\begin{bmatrix} \frac{3}{2} & -\frac{3}{2} \\ \frac{3}{2} & \frac{7}{4} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = P\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} WL \\ 2WL \end{bmatrix}$$

or

$$mL^2\begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} + kL\begin{bmatrix} \frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{19}{4} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = P\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} WL \\ 0 \end{bmatrix}$$

3.13 The result in Example 3.9: $\underset{\sim}{C}\dot{\underset{\sim}{\theta}} + \underset{\sim}{K}\underset{\sim}{\theta} = \underset{\sim}{Q}$

where $\underset{\sim}{C} = \begin{bmatrix} c_1 & \\ & c_2 \end{bmatrix}$, $\underset{\sim}{K} = \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}$, $\underset{\sim}{Q} = \begin{bmatrix} k_1\theta_5 \\ k_r(\theta_5^4 - \theta_2^4) + k_3\theta_a \end{bmatrix}$

Assume $\underset{\sim}{\theta} = \underset{\sim}{\phi} e^{-\lambda t}$, $\underset{\sim}{Q} = \underset{\sim}{0}$; $\dot{\underset{\sim}{\theta}} = -\lambda \underset{\sim}{\phi} e^{-\lambda t}$

$\longrightarrow \underset{\sim}{C}(-\lambda \underset{\sim}{\phi} e^{-\lambda t}) + \underset{\sim}{K}\underset{\sim}{\phi} e^{-\lambda t} = \underset{\sim}{0}$

$\therefore \quad \underset{\sim}{K}\underset{\sim}{\phi} = \lambda \underset{\sim}{C}\underset{\sim}{\phi}$

3.14 Invoke heat flow equilibrium in each slab.

$$c\dot{\theta}_2 = 4k(\theta_1^{new} - \theta_2) - 2k(\theta_2 - \theta_3)$$

$$c\dot{\theta}_3 = 2k(\theta_2 - \theta_3) - 3k(\theta_3 - \theta_4)$$

$$\frac{c}{2}\dot{\theta}_4 = 3k(\theta_3 - \theta_4) - 3k(\theta_4 - \theta_5)$$

$$\begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c/2 \end{bmatrix}\begin{bmatrix} \dot{\theta}_2 \\ \dot{\theta}_3 \\ \dot{\theta}_4 \end{bmatrix} + \begin{bmatrix} 6k & -2k & 0 \\ -2k & 5k & -3k \\ 0 & -3k & 6k \end{bmatrix}\begin{bmatrix} \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} = \begin{bmatrix} 4k\theta_1^{new} \\ 0 \\ 3k\theta_5 \end{bmatrix}$$

or $\underset{\sim}{C}\dot{\underset{\sim}{\theta}} + \underset{\sim}{K}\underset{\sim}{\theta} = \underset{\sim}{Q}$

With $\underset{\sim}{\theta} = \underset{\sim}{\phi} e^{-\lambda t}$ and $\underset{\sim}{Q} = \underset{\sim}{0}$, $\dot{\underset{\sim}{\theta}} = -\lambda e^{-\lambda t}\underset{\sim}{\phi}$

$\therefore \quad \underset{\sim}{K}\underset{\sim}{\phi} = \lambda \underset{\sim}{C}\underset{\sim}{\phi}$

where

$$\underset{\sim}{K} = \begin{bmatrix} 6k & -2k & 0 \\ -2k & 5k & -3k \\ 0 & -3k & 6k \end{bmatrix} \quad \text{and} \quad \underset{\sim}{C} = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c/2 \end{bmatrix}$$

3.15 $\Pi = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx - Ru\Big|_{x=L} + \frac{1}{2}ku^2\Big|_{x=0}$

$$\delta\Pi = \int_0^L EA\frac{du}{dx}\,\delta\frac{du}{dx}\,dx - R\delta u\Big|_{x=L} + ku\,\delta u\Big|_{x=0}$$

$$= \left(EA\frac{du}{dx} - R\right)\delta u\Big|_{x=L} - \left(EA\frac{du}{dx} - ku\right)\delta u\Big|_{x=0} - \int_0^L EA\frac{d^2u}{dx^2}\,\delta u\,dx = 0$$

$\therefore\ -EA\frac{d^2u}{dx^2} = 0$ in $0<x<L$ $\leftarrow$ D.E. of equilibrium

with $EA\frac{du}{dx} - R = 0$ at $x=L$ $\leftarrow$ natural B.C.

$EA\frac{du}{dx} - ku = 0$ at $x=0$ $\leftarrow$ natural B.C.

Now determine whether L_{2m} is symmetric and positive definite.

$$L_{2m} = -EA\frac{d^2u}{dx^2}$$

$$\int_0^L\left(-EA\frac{d^2u}{dx^2}\right)v\,dx = -EA\frac{du}{dx}v\Big|_0^L + \int_0^L EA\frac{du}{dx}\frac{dv}{dx}$$

(boundary term struck through)

$$= EAu\frac{dv}{dx}\Big|_0^L - \int_0^L EA\frac{d^2v}{dx^2}u\,dx \quad \therefore \text{ symmetric}$$

(boundary term struck through)

$$\int_0^L\left(-EA\frac{d^2u}{dx^2}\right)u\,dx = \int_0^L EA\left(\frac{du}{dx}\right)^2 dx > 0 \quad \therefore \text{ positive definite}$$

Note here that homogeneous boundary conditions are assumed.

3.16 $\Pi = \frac{1}{2}\int_0^L EI\left(\frac{d^2w}{dx^2}\right)^2 dx - M\frac{dw}{dx}\Big|_{x=L}$

with essential B.C. $w = \frac{dw}{dx} = 0$ at $x = 0$

$$\delta\Pi = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\left\{\frac{1}{2}\int_0^L EI\left(\frac{d^2w}{dx^2}+\varepsilon\frac{d^2\eta}{dx^2}\right)^2 dx - M(w'_L+\varepsilon\eta'_L)\right\} - \left\{\frac{1}{2}\int_0^L EI\left(\frac{d^2w}{dx^2}\right)^2 dx - Mw'_L\right\}\right]$$

$$= \int_0^L EI\frac{d^2w}{dx^2}\frac{d^2\eta}{dx^2}dx - M\eta'_L$$

Substitute δw for η

$$\therefore\ \delta\Pi = \int_0^L EI\frac{d^2w}{dx^2}\delta\frac{d^2w}{dx^2}dx - M\delta\frac{dw}{dx}\Big|_{x=L} = 0$$

Using integration by parts

$$\delta\Pi = \int_0^L EI w''\delta w'' dx - M\delta w'_L$$

$$= EIw''\delta w'\Big|_0^L - EIw'''\delta w\Big|_0^L + \int_0^L EIw^{(IV)}\delta w\,dx - M\delta w'_L = 0$$

$\therefore\ EIw^{(IV)} = 0 \quad 0<x<L \qquad \leftarrow$ D.E. of equilibrium

with $EIw'' - M = 0$ at $x=L$ and $EIw''' = 0$ at $x=L$ $\quad\leftarrow$ natural B.C.

Also $L_{2m} = EI\frac{d^4}{dx^4}$

We assume the boundary conditions are homogeneous.

3.16

For symmetry
$$\int_0^L EI u^{(IV)} v\,dx = EI u''' v \Big|_0^L - \int_0^L EI u''' v'\,dx$$
$$= -EI u'' v' \Big|_0^L + \int_0^L EI u'' v''\,dx$$
$$= EI u' v'' \Big|_0^L - EI u v''' \Big|_0^L + \int_0^L EI u v^{(IV)}\,dx$$
$$\therefore \int_0^L EI u^{(IV)} v\,dx = \int_0^L EI u v^{(IV)}\,dx$$

For positive definiteness
$$\int_0^L EI u^{(IV)} \cdot u\,dx = \int_0^L EI (u'')^2\,dx > 0$$

3.17 $\Pi = \frac{1}{2}\int_0^L k\left(\frac{d\theta}{dx}\right)^2 A\,dx - g^s A_L \theta_L$ with $\theta\big|_{x=0} = \theta_0$

$$\delta\Pi = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\left\{\int_0^L \frac{1}{2}k\left(\frac{d\theta}{dx}+\varepsilon\frac{d\eta}{dx}\right)^2 A\,dx - g^s A_L(\theta_L+\varepsilon\eta_L)\right\} - \left\{\int_0^L \frac{1}{2}k\left(\frac{d\theta}{dx}\right)^2 A\,dx - g^s A_L\theta_L\right\}\right]$$

$$= \int_0^L k\,\frac{d\theta}{dx}\frac{d\eta}{dx}A\,dx - g^s A_L\eta_L$$

Substitute $\delta\theta$ for η.

$$\delta\Pi = \int_0^L kA\,\frac{d\theta}{dx}\,\delta\frac{d\theta}{dx}\,dx - g^s A_L\,\delta\theta_L$$

Use integration by parts.

$$\delta\Pi = \left(kA\frac{d\theta}{dx} - g^s A\right)\delta\theta\Big|_{x=L} - kA\frac{d\theta}{dx}\,\delta\theta\Big|_{x=0} - \int_0^L \frac{d}{dx}\left(kA\frac{d\theta}{dx}\right)\delta\theta\,dx = 0$$

$\therefore$

$$-\frac{d}{dx}\left(kA\frac{d\theta}{dx}\right) = 0 \quad 0<x<L \quad \leftarrow \text{D.E. of equilibrium}$$

with $kA\frac{d\theta}{dx} - g^s A = 0$ at $x=L$ $\leftarrow$ natural b.c.

$\theta = \theta_0$ at $x=0$ $\leftarrow$ essential b.c.

Symmetry and positive definiteness

$$L_{2m} = -\frac{d}{dx}\left(kA\frac{d}{dx}\right)$$

$$-\int_0^L \frac{d}{dx}\left(kA\frac{du}{dx}\right)v\,dx = -\left(kA\frac{du}{dx}\right)v\Big|_0^L + \int_0^L\left(kA\frac{du}{dx}\right)\frac{dv}{dx}\,dx$$

(the boundary term $-\left(kA\frac{du}{dx}\right)v\Big|_0^L$ is struck through with an arrow)

3.17

$$= \int_0^L \frac{du}{dx}\left(kA\frac{dv}{dx}\right)dx$$

$$= u\left(kA\frac{dv}{dx}\right)\Big|_0^L - \int_0^L u\frac{d}{dx}\left(kA\frac{dv}{dx}\right)dx$$

$$\therefore \quad -\int_0^L \frac{d}{dx}\left(kA\frac{du}{dx}\right)v\,dx = -\int_0^L \frac{d}{dx}\left(kA\frac{dv}{dx}\right)u\,dx$$

and

$$-\int_0^L \frac{d}{dx}\left(kA\frac{du}{dx}\right)u\,dx = \int_0^L kA\left(\frac{du}{dx}\right)^2 dx > 0$$

3.18 $\Pi = \frac{1}{2}\int_0^L T\left(\frac{dw}{dx}\right)^2 dx + \frac{1}{2}\int_0^L k w^2 \, dx - P w_L \qquad \text{with } w_0 = 0$

$$\delta\Pi = 0 \;\rightarrow\; \int_0^L T \frac{dw}{dx}\,\delta\frac{dw}{dx}\,dx + \int_0^L k w\,\delta w\,dx - P\,\delta w_L = 0$$

Integration by parts :

$$\int_0^L \left(-T\frac{d^2w}{dx^2} + k w\right)\delta w\,dx + \left(T\frac{dw}{dx} - P\right)\delta w\Big|_L - T\frac{dw}{dx}\,\delta w\Big|_0 = 0$$

$$\therefore \quad -T\frac{d^2w}{dx^2} + k w = 0 \qquad 0 < x < L \qquad \leftarrow \text{D.E.}$$

with $T\frac{dw}{dx} - P = 0$ at $x = L$; $w = 0$ at $x = 0$ $\Big\}$ $\leftarrow$ B.C.'s

Considering symmetry with homogeneous b.c.'s ,

$$\int_0^L \left(-T\frac{d^2u}{dx^2} + k u\right) v\,dx = \left[-T\frac{du}{dx}v\right]_0^L + \int_0^L T\frac{du}{dx}\frac{dv}{dx}\,dx + \int_0^L k u v\,dx$$

$$= \left[T u \frac{dv}{dx}\right]_0^L - \int_0^L T u \frac{d^2v}{dx^2}\,dx + \int_0^L k u v\,dx$$

$$\therefore \quad \int_0^L \left(-T\frac{d^2u}{dx^2} + k u\right) v\,dx = \int_0^L \left(-T\frac{d^2v}{dx^2} + k v\right) u\,dx$$

and for positive definiteness

$$\int_0^L \left(-T\frac{d^2u}{dx^2} + k u\right) u\,dx = \int_0^L \left[T\left(\frac{du}{dx}\right)^2 + k u^2\right] dx > 0$$

3.19 (a) The suitable trial function should satisfy both the essential and natural b.c.'s. Try $w(x) = a_0 + a_1 x + a_2 x^2$

$$w\big|_0 = 0, \quad T\frac{dw}{dx}\Big|_L = P \qquad \therefore\ a_0 = 0, \quad a_1 = \frac{P}{T} - 2L$$

$$\longrightarrow \quad w(x) = \frac{P}{T}x + (x^2 - 2Lx)\cdot a_2 \qquad \text{with } f_2 = x^2 - 2Lx$$

(b) Classical Galerkin method:

$$\int_D R f_i \, dD = \int_0^L f_2 \left[-T\frac{d^2w}{dx^2} + kw\right] dx$$

$$= \int_0^L (x^2 - 2Lx)\left[-T(2a_2) + k\left\{\frac{P}{T}x + (x^2 - 2Lx)a_2\right\}\right] dx = 0$$

$$\therefore \left(\frac{8}{15}kL^5 + \frac{4}{3}TL^3\right) a_2 = \frac{5}{12}\,\frac{kPL^4}{T}$$

Least Squares method:

$$\frac{\partial}{\partial a_2}\left[\int_D R^2 dD\right] = \frac{\partial}{\partial a_2}\left[\int_0^L \left(-T\frac{d^2w}{dx^2} + kw\right)^2 dx\right] = 0$$

$$\therefore \left(8LT^2 + \frac{16}{3}kL^3T + \frac{16}{15}k^2L^5\right) a_2 = \frac{5}{6}\,\frac{k^2L^4P}{T}$$

3.20 $w(x) = a_0 + a_1 x + a_2 x^2$

From essential b.c., $w_0 = 0 = a_0 \quad \therefore \; w = a_1 x + a_2 x^2$

$$\Pi = \frac{1}{2}\int_0^L T(a_1 + 2a_2 x)^2 dx + \int_0^L \frac{1}{2}k(a_1 x + a_2 x^2)^2 dx - P(a_1 L + a_2 L^2)$$

$$\frac{\partial \Pi}{\partial a_i} = 0, \quad i = 1, 2$$

$$\longrightarrow \begin{bmatrix} \frac{kL^3}{3} + LT & \frac{kL^4}{4} + L^2 T \\ \frac{kL^4}{4} + L^2 T & \frac{kL^5}{5} + \frac{4L^3 T}{3} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} LP \\ L^2 P \end{bmatrix}$$

3.21 Essential b.c.'s are $w\big|_{x=0} = \frac{dw}{dx}\Big|_{x=0} = 0$

Let $w = cx^2$ ($\leftarrow$ satisfies essential b.c.'s)

$$\Pi = \frac{1}{2}\int_0^L EI\left(\frac{d^2w}{dx^2}\right)^2 dx - \frac{P}{2}\int_0^L \left(\frac{dw}{dx}\right)^2 dx + \frac{1}{2}k\, w_L^2$$

$$\frac{\partial \Pi}{\partial c} = 0; \quad (6EI_0 L + kL^4)c = \frac{4L^3}{3}Pc$$

$$\therefore P = \frac{9EI_0}{2L^2} + \frac{3}{4}kL$$

3.22 (a) $\Pi = \frac{1}{2}\int_0^L EI\left(\frac{d^2w}{dx^2}\right)^2 dx + \frac{1}{2}\int_0^L k w^2 dx - \frac{1}{2}\int_0^L P\left(\frac{dw}{dx}\right)^2 dx - W w_L$

with $w_0 = w'_0 = 0$ (essential b.c.)

(i) $w = a_1 x^2$

$$\frac{\partial \Pi}{\partial a_1} = 0 \rightarrow \left(3EI_0 L + \frac{kL^5}{5} - \frac{4PL^3}{3}\right) a_1 = WL^2$$

(ii) $w = b_1\left(1 - \cos\frac{\pi x}{2L}\right)$

$$\frac{\partial \Pi}{\partial b_1} = 0 \rightarrow \left[\frac{kL(3\pi-8)}{2\pi} + \frac{EI_0\pi^2(4+3\pi^2)}{128L^3} - \frac{P\pi^2}{8L}\right] b_1 = W$$

(b) In case of (i) with $W = 0$,

$$P = \frac{3}{4L^3}\left(3EI_0 L + \frac{kL^5}{5}\right) = 225.3$$

Similarly in case of (ii)

$$P = \frac{8L}{\pi^2}\left[\frac{EI_0\pi^2(3\pi^2+4)}{128L^3} + \frac{kL(3\pi-8)}{2\pi}\right] = 210.4$$

3.23 The boundary conditions are

$$k_2 \frac{d\theta}{dx}\Big|_{x=L} = 0 \quad \leftarrow \text{natural b.c.} \qquad , \qquad \theta_0 = 20 \quad \leftarrow \text{essential b.c.}$$

Now perform Ritz analysis with $\theta = a_0 + a_1 x + a_2 x^2$. As $\theta_0 = 20$
$\theta = 20 + a_1 x + a_2 x^2$. (2 unknown parameters)
Using $\frac{\partial \Pi}{\partial a_i} = 0$ for $i = 1, 2$,

$$\begin{bmatrix} \frac{(k_1+k_2)L}{2} & \frac{(k_1+3k_2)L^2}{4} \\ \frac{(k_1+3k_2)L^2}{4} & \frac{(k_1+7k_2)L^3}{6} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{L^2}{2} g^B \\ \frac{L^3}{3} g^B \end{bmatrix}$$

$$\text{with } \Pi = \int_0^{L/2} \frac{k_1}{2}\left(\frac{d\theta}{dx}\right)^2 dx + \int_{L/2}^{L} \frac{k_2}{2}\left(\frac{d\theta}{dx}\right)^2 dx - \int_0^L \theta g^B dx .$$

With $k_1 = 20$, $k_2 = 40$, $g^B = 100$ and $L = 10$ we have

$$a_1 = 1600/33 \ , \quad a_2 = -30/11$$

$$\therefore \quad \theta = 20 + \frac{1600}{33} x - \frac{30}{11} x^2$$

3.24 $\quad T\dfrac{\partial^2 w}{\partial x^2} = m\dfrac{\partial^2 w}{\partial t^2} - p$

(a)

$$w''\Big|_i = \frac{w_{i-1} - 2w_i + w_{i+1}}{(L/3)^2}$$

at station 1 $\qquad \dfrac{T(\not{w_0} - 2w_1 + w_2)}{(L/3)} = m\dfrac{L}{3}\ddot{w}_1 - p\dfrac{L}{3}$

at station 2 $\qquad \dfrac{T(w_1 - 2w_2 + \not{w_3})}{(L/3)} = m\dfrac{L}{3}\ddot{w}_2 - p\dfrac{L}{3}$

$$\therefore \begin{bmatrix} \frac{mL}{3} & 0 \\ 0 & \frac{mL}{3} \end{bmatrix}\begin{bmatrix} \ddot{w}_1 \\ \ddot{w}_2 \end{bmatrix} + \frac{3T}{L}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \frac{pL}{3} \\ \frac{pL}{3} \end{bmatrix}$$

(b) $\quad \Pi = \dfrac{1}{2}\displaystyle\int_0^L T\left(\frac{\partial w}{\partial x}\right)^2 dx - \int_0^L p\,w\,dx + \int_0^L \left(m\frac{\partial^2 w}{\partial t^2}\right) w\,dx$

$$= \frac{L}{3}\left[\frac{T}{2}\left(\frac{-w_0 + w_1}{L/3}\right)^2 + \frac{T}{2}\left(\frac{-w_1 + w_2}{L/3}\right)^2 + \frac{T}{2}\left(\frac{-w_2 + w_3}{L/3}\right)^2\right]$$

$$- \frac{L}{3}\left[p(w_1 + w_2)\right] + \frac{L}{3}\left[m(\ddot{w}_1 w_1 + \ddot{w}_2 w_2)\right]$$

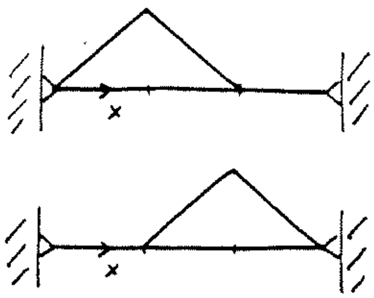
$\delta\Pi = 0$ yields

$$\begin{bmatrix} \frac{mL}{3} & 0 \\ 0 & \frac{mL}{3} \end{bmatrix}\begin{bmatrix} \ddot{w}_1 \\ \ddot{w}_2 \end{bmatrix} + \begin{bmatrix} \frac{6T}{L} & -\frac{3T}{L} \\ -\frac{3T}{L} & \frac{6T}{L} \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \frac{pL}{3} \\ \frac{pL}{3} \end{bmatrix}$$

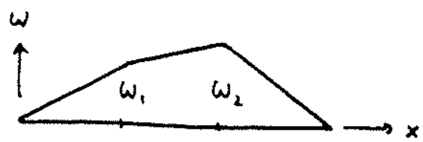
<u>Note</u> the same result is obtained as in (a).

3.24

(c)



$$
w = \begin{cases} \dfrac{3w_1}{L}x & 0 < x < \dfrac{L}{3} \\[2mm] \left(2 - \dfrac{3}{L}x\right)w_1 + \left(-1 + \dfrac{3}{L}x\right)w_2 & \dfrac{L}{3} < x < \dfrac{2L}{3} \\[2mm] 3w_2\left(1 - \dfrac{x}{L}\right) & \dfrac{2L}{3} < x < L \end{cases}
$$

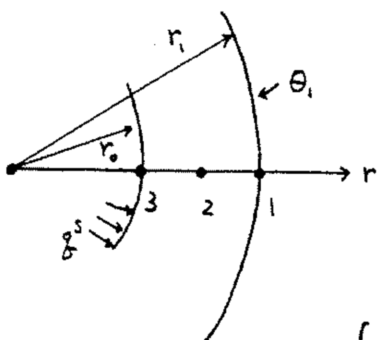


Using the function for w and invoking $\delta\Pi = 0$,

$$
\begin{bmatrix} \dfrac{2mL}{9} & \dfrac{mL}{18} \\[2mm] \dfrac{mL}{18} & \dfrac{2mL}{9} \end{bmatrix} \begin{bmatrix} \ddot{w}_1 \\[2mm] \ddot{w}_2 \end{bmatrix} + \begin{bmatrix} \dfrac{6T}{L} & -\dfrac{3T}{L} \\[2mm] -\dfrac{3T}{L} & \dfrac{6T}{L} \end{bmatrix} \begin{bmatrix} w_1 \\[2mm] w_2 \end{bmatrix} = \begin{bmatrix} \dfrac{PL}{3} \\[2mm] \dfrac{PL}{3} \end{bmatrix}
$$

3.25 $\Pi = U - W = \int_V \frac{1}{2} k \left(\frac{d\theta}{dr}\right)^2 dV - \int_S q^s \theta \, dS$

$$= \int_{r_0}^{r_1} k\pi h r \left(\frac{d\theta}{dr}\right)^2 dr - 2\pi q^s h r_0 \theta \Big|_{r_0}$$



$$\therefore \Pi = \int_{r_0}^{r_1} k\pi h r \left(\frac{d\theta}{dr}\right)^2 dr - 2\pi q^s h r_0 \theta_3$$

With the basis functions in Fig. 3.4. we have

$$\theta = \begin{cases} \frac{2}{r_1 - r_0}\left[\theta_3\left(\frac{r_0+r_1}{2} - r\right) + \theta_2 (r - r_0)\right] & \left(r_0 \le r \le \frac{r_0 + r_1}{2}\right) \\ \frac{2}{r_1 - r_0}\left[\theta_2 (r_1 - r) + \theta_1\left(r - \frac{r_0+r_1}{2}\right)\right] & \left(\frac{r_0+r_1}{2} \le r \le r_1\right) \end{cases}$$

From $\delta\Pi = 0$ we obtain

$$\frac{hk\pi}{r_1 - r_0}\begin{bmatrix} (3r_0 + r_1) & -(3r_0 + r_1) \\ -(3r_0 + r_1) & 4(r_0 + r_1) \end{bmatrix}\begin{bmatrix} \theta_3 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 2\pi h q^s r_0 \\ \frac{hk\pi}{r_1 - r_0}(r_0 + 3r_1)\theta_1 \end{bmatrix}$$

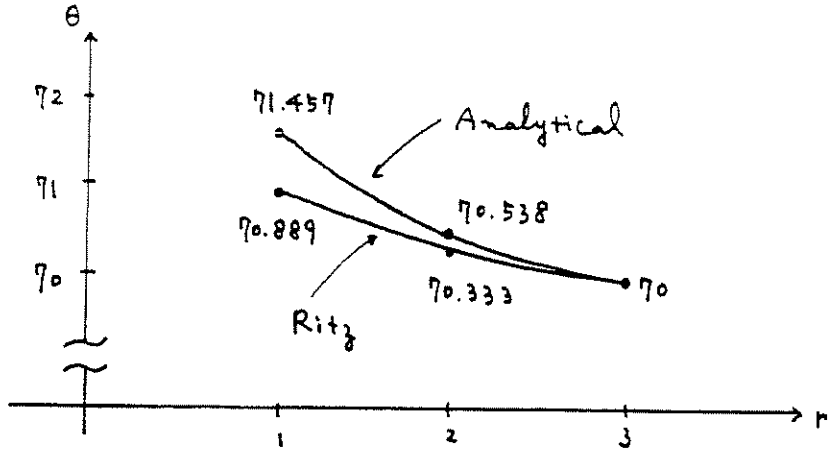
$$\therefore \begin{bmatrix} \theta_3 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} \frac{8 q^s r_0 (r_1^2 - r_0^2)}{k(3r_0 + r_1)(r_0 + 3r_1)} + \theta_1 \\ \frac{2 q^s r_0 (r_1 - r_0)}{k(r_0 + 3r_1)} + \theta_1 \end{bmatrix} = \begin{bmatrix} 70.889 \\ 70.333 \end{bmatrix}$$

$$\therefore \theta = \begin{cases} 71.445 - 0.556\, r & 1 \le r \le 2 \\ 71. - 0.333\, r & 2 \le r \le 3 \end{cases}$$

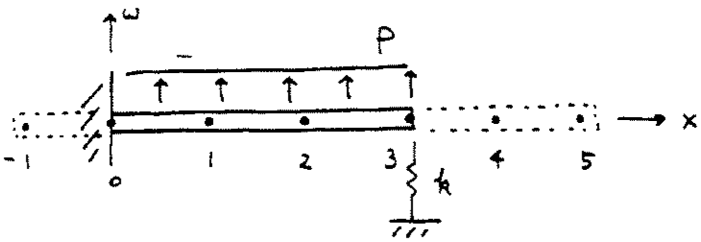
3.25

(Analytical sol.) $\Rightarrow \quad \theta = \theta_1 + \dfrac{g^s}{2k\pi h} \ln \dfrac{r_1}{r} = 70 + 15.915 \ln \dfrac{3}{r}$

$\therefore \quad \theta_3 = 71.457 \qquad \theta_2 = 70.538$



3.26



(a) D.E. for the system ; $EI\dfrac{d^4w}{dx^4} = \bar{p} \qquad 0<x<L$

with $w\big|_0 = w'\big|_0 = 0\,, \quad EIw_L'' = 0, \quad EIw_L''' = kw_L$

at station i $\qquad \dfrac{EI}{\left(\frac{L}{3}\right)^3}\left\{w_{i-2} - 4w_{i-1} + 6w_i - 4w_{i+1} + w_{i+2}\right\} = \bar{p}\left(\frac{L}{3}\right)$

$w_0 = w'\big|_0 = 0 \longrightarrow \quad w_{-1} = w_1$

$EIw_L'' = 0 \longrightarrow \dfrac{EI}{\left(\frac{L}{3}\right)^2}\left(w_2 - 2w_3 + w_4\right) = 0 \qquad \therefore\ w_4 = -w_2 + 2w_3$

$EIw_L''' = kw_L \longrightarrow \dfrac{EI}{2\cdot\left(\frac{L}{3}\right)^3}\left(-w_1 + 2w_2 - 2w_4 + w_5\right) = kw_3$

$\therefore\ w_5 = w_1 - 4w_2 + \left(4 + \dfrac{2kL^3}{27EI}\right)w_3$

Hence,

$$\frac{27EI}{L^3}\begin{bmatrix} 7 & -4 & 1 \\ -4 & 5 & -2 \\ 1 & -2 & 1 + \dfrac{kL^3}{27EI} \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} \dfrac{\bar{p}L}{3} \\ \dfrac{\bar{p}L}{3} \\ \dfrac{\bar{p}L}{6} \end{bmatrix}$$

(b) $\Pi = \displaystyle\int_0^L \frac{1}{2}EI\left(\frac{d^2w}{dx^2}\right)^2 dx + \frac{1}{2}kw_L^2 - \int_0^L \bar{p}\,w\,dx$

Use central differencing for station i :

3.26

$$w_i = \frac{1}{(L/3)^2}\left(w_{i-1} - 2w_i + w_{i+1} \right)$$

$$\Rightarrow \Pi = \left[\frac{L}{6}\Pi_0 + \frac{L}{3}(\Pi_1 + \Pi_2) + \frac{L}{6}\Pi_3 \right] + \frac{k}{2}w_3^2 - \left[p\left(\frac{L}{3}w_1 + \frac{L}{3}w_2 + \frac{L}{6}w_3 \right) \right]$$

$$\Pi_0 = \frac{EI}{2(L/3)^4}\left(w_{-1} - 2w_0 + w_1 \right)^2 = \frac{81EI}{2L^4}\left(4w_1^2 \right)$$

$$\Pi_1 = \frac{EI}{2(L/3)^4}\left(w_0 - 2w_1 + w_2 \right)^2 = \frac{81EI}{2L^4}\left(-2w_1 + w_2 \right)^2$$

$$\Pi_2 = \frac{EI}{2(L/3)^4}\left(w_1 - 2w_2 + w_3 \right)^2 = \frac{81EI}{2L^4}\left(w_1 - 2w_2 + w_3 \right)^2$$

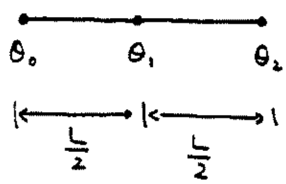
$$\Pi_3 = \frac{EI}{2(L/3)^4}\left(w_2 - 2w_3 + w_4 \right)^2 = 0$$

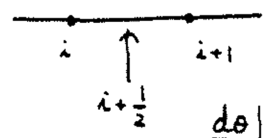
$$\therefore \quad \frac{EI}{L^3}\begin{bmatrix} 189 & -108 & 27 \\ -108 & 135 & -54 \\ 27 & -54 & 27 + \frac{kL^3}{EI} \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} \frac{Lp}{3} \\ \frac{Lp}{3} \\ \frac{Lp}{6} \end{bmatrix}$$

We note here that this is the same result as in (a).

3.27 $\Pi = \int_0^L \frac{k}{2}\left(\frac{d\theta}{dx}\right)^2 dx - \int_0^L \theta f^B dx$

with $\theta_0 = 20, \quad k_2 \frac{d\theta}{dx}\Big|_L = 0$





$$\frac{d\theta}{dx}\Big|_{i+\frac{1}{2}} = \frac{-\theta_i + \theta_{i+1}}{L/2}$$

$$\therefore \Pi = \frac{L}{2}\left\{ \Pi_{\frac{1}{2}} + \Pi_{\frac{3}{2}} \right\} - f^B\left(\frac{L}{4}\theta_0 + \frac{L}{2}\theta_1 + \frac{L}{4}\theta_2\right)$$

where $\Pi_{\frac{1}{2}} = \frac{k_1}{2}\left(\frac{-\theta_0+\theta_1}{L/2}\right)^2 = \frac{2k_1}{L^2}(-\theta_0+\theta_1)^2$

$$\Pi_{\frac{3}{2}} = \frac{k_2}{2}\left(\frac{-\theta_1+\theta_2}{L/2}\right)^2 = \frac{2k_2}{L^2}(-\theta_1+\theta_2)^2$$

$$\frac{\partial \Pi}{\partial \theta_1} = \frac{\partial \Pi}{\partial \theta_2} = 0 \qquad \therefore \begin{bmatrix} \frac{2(k_1+k_2)}{L} & -\frac{2k_2}{L} \\ -\frac{2k_2}{L} & \frac{2k_2}{L} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} \frac{f^B L}{2} + \frac{2k_1\theta_0}{L} \\ \frac{f^B L}{4} \end{bmatrix}$$

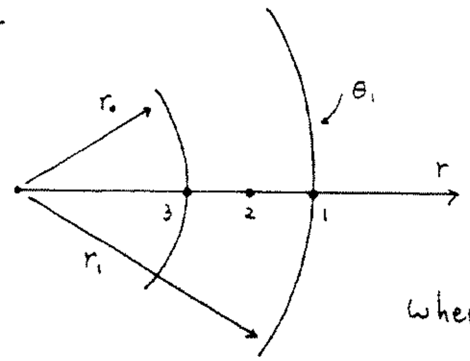
As $k_1 = 20$, $k_2 = 40$, $f^B = 100$ and $L = 10$

$$\begin{bmatrix} 12 & -8 \\ -8 & 8 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 580 \\ 250 \end{bmatrix} \qquad \therefore \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 207.5 \\ 238.75 \end{bmatrix}$$

The Ritz solution (Ex. 3.23) gave:

$\theta\left(\frac{L}{2}\right) = 194.2$, and $\theta(L) = 232.1$

3.28



$$\Pi = \int_{r_0}^{r_1} k\pi h r \left(\frac{d\theta}{dr}\right)^2 dr - 2\pi g^s h r_0 \theta \Big|_{r_0}$$

$$= \frac{r_1 - r_0}{2}\left[\Pi_{\frac{5}{2}} + \Pi_{\frac{3}{2}}\right] - 2\pi g^s h r_0 \theta_3$$

where
$$\Pi_{\frac{5}{2}} = k\pi h \left[\frac{-\theta_3 + \theta_2}{\left(\frac{r_1 - r_0}{2}\right)}\right]^2 \left(r_0 + \frac{r_1 - r_0}{4}\right)$$

$$\Pi_{\frac{3}{2}} = k\pi h \left[\frac{-\theta_2 + \theta_1}{\left(\frac{r_1 - r_0}{2}\right)}\right]^2 \left(r_0 + 3\,\frac{r_1 - r_0}{4}\right)$$

From $\frac{\partial \Pi}{\partial \theta_3} = 0$ and $\frac{\partial \Pi}{\partial \theta_2} = 0$,

$$\frac{hk\pi}{r_1 - r_0}\begin{bmatrix} 3r_0 + r_1 & -(3r_0 + r_1) \\ -(3r_0 + r_1) & 4(r_0 + r_1) \end{bmatrix}\begin{bmatrix} \theta_3 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 2\pi h g^s r_0 \\ \frac{hk\pi}{r_1 - r_0}(r_0 + 3r_1)\theta_1 \end{bmatrix}$$

$$\begin{bmatrix} 36\pi & -36\pi \\ -36\pi & 96\pi \end{bmatrix}\begin{bmatrix} \theta_3 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 20\pi \\ 4200\pi \end{bmatrix} \longrightarrow \begin{bmatrix} \theta_3 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 70.889 \\ 70.333 \end{bmatrix}$$

3.29

To apply the program, one should use the following analogies:

(i) Pipe networks:

element-end pressures (p_i) → nodal displacements (u_i),

fluid flows into the element (q_i) → nodal point forces $(F_i$

inverse of resistance $\left(\frac{1}{R_i}\right)$ → element stiffness $\left(\frac{EA}{L}\right)_i$;

(ii) dc networks:

element-end currents (I_i) → u_i,

voltage drops (E_i) → F_i,

resistance (R_i) → $\left(\frac{EA}{L}\right)_i$;

(iii) heat transfer:

surface temperatures (θ_i) → u_i,

heat flows into the element (q_i) → F_i,

conductivity coefficients (k_i) → $\left(\frac{EA}{L}\right)_i$.

3.31

System of equations corresponding to Lagrange multiplier method:

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} 10 \\ -1 \\ 0 \end{bmatrix}$$

gives the solution : $U_1 = 5$, and $U_2 = 0$ with $\lambda = 4$.

Using the penalty approach, we obtain:

$$\begin{bmatrix} 2 & -1 \\ -1 & 2+\alpha \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 10 \\ -1 \end{bmatrix}.$$

Taking $\alpha = 2 \cdot 10^4$, we find that $U_1 \approx 5.0001$, $U_2 = 0.000$

$$\underline{R}^* = \underline{K}\,\underline{U} = \begin{bmatrix} 10 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ -\lambda \end{bmatrix} + \underline{R}$$

3.32

$$\begin{bmatrix} 4k & -2k & -k \\ -2k & 3k & -k \\ -k & -k & 2k \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}, \quad R_1 = 1,\ R_2 = 0,\ R_3 = 1$$

(a) Lagrange multiplier method : $\underline{B} = [\, 0 \ \ 1 \ \ -1 \,]$, $\underline{V} = [\, 0 \,]$

$$\begin{bmatrix} 4k & -2k & -k & 0 \\ -2k & 3k & -k & 1 \\ -k & -k & 2k & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ \lambda \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\therefore \ U_1 = \frac{2}{k}, \quad U_2 = \frac{7}{3k}, \quad U_3 = \frac{7}{3k}, \quad \lambda = -\frac{2}{3}$$

(b) Penalty method :

$$(\underline{K} + \alpha \underline{B}^T \underline{B})\, \underline{U} = \underline{R} + \alpha \underline{B}^T \underline{V}$$

$$\underline{B}^T \underline{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \quad \underline{B}^T \underline{V} = \underline{0}$$

$$\begin{bmatrix} 4k & -2k & -k \\ -2k & 3k+\alpha & -k-\alpha \\ -k & -k-\alpha & 2k+\alpha \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

With $\alpha = (4k) \times 10^3$

$$U_1 = \frac{2}{k}, \quad U_2 = 2.3333\,\frac{1}{k}, \quad U_3 = 2.3334\,\frac{1}{k}$$

From Excercise 3.35 $\underline{\lambda} = \alpha\,(\underline{B}\,\underline{U} - \underline{V})$, and here with $\alpha = (4k) \times 10^3$, $\lambda = -0.6664$.

The additional forces to impose the constraint are then $\underline{B}^T \lambda$.

3.33 (i) Lagrange multiplier method :

$$\begin{bmatrix} 5 & -2 & 0 & 0 \\ -2 & 5 & -3 & -4 \\ 0 & -3 & 5 & 1 \\ 0 & -4 & 1 & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \lambda \end{bmatrix} = \begin{bmatrix} 30 \\ 0 \\ 40 \\ 0 \end{bmatrix} \leftarrow \left(\begin{array}{l} \theta_3 - 4\theta_2 = 0 \\ \therefore \underline{B} = [\,0 \;\; -4 \;\; 1\,] , \; \underline{V} = [\,0\,] \end{array} \right)$$

$$\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \lambda \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 50 \\ 20 \\ 80 \\ -60 \end{bmatrix} = \begin{bmatrix} 7.1429 \\ 2.8571 \\ 11.4286 \\ -8.5714 \end{bmatrix}$$

(ii) penalty method :

$$\underline{B}^T \underline{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 16 & -4 \\ 0 & -4 & 1 \end{bmatrix} , \quad \underline{B}^T \underline{V} = \underline{0}$$

$$\begin{bmatrix} 5 & -2 & 0 \\ -2 & 5+16\alpha & -3-4\alpha \\ 0 & -3-4\alpha & 5+\alpha \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} 30 \\ 0 \\ 40 \end{bmatrix}$$

Let $\alpha = (\max [k_{ii}])\beta$

	θ_1	θ_2	θ_3
For $\beta = 10$	7.1621	2.9054	11.4507
10^2	7.1448	2.8620	11.4308
10^3	7.1431	2.8576	11.4288
10^4	7.1429	2.8572	11.4286

It is sufficient to use $\alpha = 5\times10^4$ for the accurate response.

3.33

From equations (3.62) and (3.64) or from excercise 3.35

$$\underline{\lambda} = \alpha \, (\underline{B} \; \underline{U} \; - \underline{V}),$$

and hence here, for $\alpha = 5 \times 10^4$

$$\lambda = -8.5715$$

The additional heat fluxes to impose the constraint are then $\underline{B}^T \lambda$.

The same result is also obtained by calculating $\underline{R}^* = \underline{k} \, \underline{U}$. For $\alpha = 5 \times 10^4$,

$$\underline{R}^* = \begin{bmatrix} R_1^* \\ R_2^* \\ R_3^* \end{bmatrix} = \begin{bmatrix} 30 \\ -34.2856 \\ 48.5714 \end{bmatrix} = \begin{bmatrix} 30 \\ -34.2856 \\ 40 + (-\lambda) \end{bmatrix}$$

Hence we see that the heat flux of the amount $(-\lambda)$ should be applied at the face corresponding to station 3, and also the amount (-34.2856) at the face corresponding to station 2 in order to impose $\theta_3 = 4\theta_2$.

3.34 (i) Lagrange multiplier method : $\underline{B} = [\,0 \;\; 1 \;\; -2\,]$, $\underline{V} = [\,0\,]$

$$\begin{bmatrix} 9 & -6 & 0 & 0 \\ -6 & 31 & -25 & 1 \\ 0 & -25 & 31 & -2 \\ 0 & 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} P_A \\ P_C \\ P_D \\ \lambda \end{bmatrix} = \begin{bmatrix} R \\ 0 \\ 0 \\ 0 \end{bmatrix} , \quad R = 30bQ$$

$$\begin{bmatrix} P_A \\ P_C \\ P_D \\ \lambda \end{bmatrix} = R \begin{bmatrix} 55/351 \\ 8/117 \\ 4/117 \\ -38/117 \end{bmatrix} = R \begin{bmatrix} 0.15670 \\ 0.068376 \\ 0.034188 \\ -0.32479 \end{bmatrix}$$

(ii) Penalty method :

$$\underline{B}^T\underline{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & -2 & 4 \end{bmatrix} , \quad \underline{B}^T\underline{V} = \underline{0}$$

$$\begin{bmatrix} 9 & -6 & 0 \\ -6 & 31+\alpha & -25-2\alpha \\ 0 & -25-\alpha & 31+4\alpha \end{bmatrix} \begin{bmatrix} P_A \\ P_C \\ P_D \end{bmatrix} = \begin{bmatrix} R \\ 0 \\ 0 \end{bmatrix}$$

With $\alpha = 31 \times 10^4$ $\quad P_A = 0.15670R$, $P_C = 0.068377R$

$P_D = 0.034189R$.

Now calculating $\underline{R}^* = \underline{K}\,\underline{U}$ where $\underline{U}^T = [\,P_A \;\; P_C \;\; P_D\,]$

we have $\underline{R}^{*T} = [\,R \quad 0.32478R \quad -0.64956R\,]$

$= [\,R \quad -\lambda \quad 2\lambda\,]$

3.34

Hence to impose the condition of $P_c = 2P_D$ the flow of the amount of $(-\lambda)$ and (2λ) should be added to the stations C and D, respectively.

From eq. (3.62) and (3.64) or the solution of excercise 3.35, we have

$$\underline{\lambda} = \alpha \left(\underline{B}\,\underline{V} - \underline{V} \right)$$

Hence here, for $\alpha = 31 \times 10^4$,

$$\lambda = 0.32478\ R$$

which gives the same result for the Lagrange multiplier as we obtained in (i).

3.35 $\tilde{\Pi}^{**} = \frac{1}{2} \underline{U}^T \underline{K} \underline{U} - \underline{U}^T \underline{R} + \underline{\lambda}^T (\underline{B}\underline{U} - \underline{V}) - \frac{\underline{\lambda}^T \underline{\lambda}}{2\alpha}$

$$\delta\tilde{\Pi}^{**} = \delta\underline{U}^T \underline{K} \underline{U} - \delta\underline{U}^T \underline{R} + \delta\underline{\lambda}^T (\underline{B}\underline{U} - \underline{V}) + \delta\underline{U}^T \underline{B}^T \underline{\lambda} - \frac{\delta\underline{\lambda}^T \underline{\lambda}}{\alpha}$$

$$= \delta\underline{U}^T \left(\underline{K}\underline{U} - \underline{R} + \underline{B}^T \underline{\lambda} \right) + \delta\underline{\lambda}^T \left[(\underline{B}\underline{U} - \underline{V}) - \frac{\underline{\lambda}}{\alpha} \right] = 0$$

Since $\delta\underline{U}$ and $\delta\underline{\lambda}$ are arbitrary we obtain

$$\underline{K}\underline{U} - \underline{R} + \underline{B}^T \underline{\lambda} = \underline{0} \quad \text{—— ①}$$

$$(\underline{B}\underline{U} - \underline{V}) - \frac{\underline{\lambda}}{\alpha} = \underline{0} \quad \text{or} \quad \boxed{\underline{\lambda} = \alpha(\underline{B}\underline{U} - \underline{V})} \quad \text{—— ②}$$

From ~~eq.~~ ① and ②

$$(\underline{K} + \alpha \underline{B}^T \underline{B}) \underline{U} = \underline{R} + \alpha \underline{B}^T \underline{V} \quad (\leftarrow (3.64))$$

For the case considered in (3.60)

$$\underline{B} = \underline{e}_i^T, \quad \underline{V} = U_i^* \underline{e}_i$$

$$\therefore \quad \lambda = \alpha \left(\underline{e}_i^T \underline{U} - U_i^* \underline{e}_i \right) = \alpha \left(U_i - U_i^* \right)$$

3.36 $\tilde{\Pi}^* = \frac{1}{2}\underline{U}^T\underline{K}\,\underline{U} - \underline{U}^T\underline{R} + \frac{\alpha}{2}(\underline{B}\,\underline{U} - \underline{V})^T(\underline{B}\,\underline{U} - \underline{V}) + \underline{\lambda}^T(\underline{B}\,\underline{U} - \underline{V})$

(a) $\delta\tilde{\Pi}^* = \delta\underline{U}^T\underline{K}\,\underline{U} - \delta\underline{U}^T\underline{R} + \alpha\,\delta\underline{U}^T\underline{B}^T(\underline{B}\,\underline{U} - \underline{V})$

$\qquad + \delta\underline{\lambda}^T(\underline{B}\,\underline{U} - \underline{V}) + \delta\underline{U}^T\underline{B}^T\underline{\lambda}$

$= \delta\underline{U}^T\left[\underline{K}\,\underline{U} - \underline{R} + \alpha\,\underline{B}^T(\underline{B}\,\underline{U} - \underline{V}) + \underline{B}^T\underline{\lambda}\right] + \delta\underline{\lambda}^T(\underline{B}\,\underline{U} - \underline{V}) = 0$

Since $\delta\underline{U}$ and $\delta\underline{\lambda}$ are arbitrary, we obtain

$$\begin{bmatrix} \underline{K} + \alpha\underline{B}^T\underline{B} & \underline{B}^T \\ \underline{B} & \underline{0} \end{bmatrix}\begin{bmatrix} \underline{U} \\ \underline{\lambda} \end{bmatrix} = \begin{bmatrix} \underline{R} + \alpha\underline{B}^T\underline{V} \\ \underline{V} \end{bmatrix} \qquad \text{—— ①}$$

(b) $U_2 = \frac{1}{k} \quad \therefore \underline{B} = [\,0 \quad 1\,], \quad \underline{V} = [\,\frac{1}{k}\,]$

$\underline{B}^T\underline{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \underline{B}^T\underline{V} = \begin{bmatrix} 0 \\ \frac{1}{k} \end{bmatrix}$

$$\begin{bmatrix} 2k & -k & 0 \\ -k & k+\alpha & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} R_1 \\ \frac{\alpha}{k} \\ \frac{1}{k} \end{bmatrix}$$

$\therefore \quad U_1 = \frac{1+R_1}{2k}, \quad U_2 = \frac{1}{k}, \qquad \lambda = -\frac{1-R_1}{2}$

Note here that using the augmented Lagrangian method we obtain the exact solution for any value of α. For instance when $\alpha = 0$ eq. ① reduces to eq. (3.62) and we still have the exact solution.

4.1 Differential eq. throughout the body

$$E\frac{\partial}{\partial x}\left(A\frac{\partial u}{\partial x}\right) + f_x^B = 0$$

with natural b.c. $EA\frac{\partial u}{\partial x} = R$ at $x = L$

essential b.c. $u = 0$ at $x = 0$

Consider any arbitrary continuous displacement $\bar{u}$ with $\bar{u} = 0$ at $x = 0$. Then

$$\left[E\frac{\partial}{\partial x}\left(A\frac{\partial u}{\partial x}\right) + f_x^B\right]\bar{u} = 0$$

$$\therefore \int_0^L \left[E\frac{\partial}{\partial x}\left(A\frac{\partial u}{\partial x}\right) + f_x^B\right]\bar{u}\,dx = 0$$

Using $\frac{\partial}{\partial x}\left[E\left(A\frac{\partial u}{\partial x}\right)\bar{u}\right] = E\frac{\partial}{\partial x}\left(A\frac{\partial u}{\partial x}\right)\bar{u} + EA\frac{\partial u}{\partial x}\frac{\partial \bar{u}}{\partial x}$,

$$\int_0^L \left[\frac{\partial}{\partial x}\left\{EA\frac{\partial u}{\partial x}\bar{u}\right\} - EA\frac{\partial u}{\partial x}\frac{\partial \bar{u}}{\partial x} + f_x^B\bar{u}\right]dx = 0$$

Now from $\int_0^L \frac{\partial}{\partial x}\left\{EA\frac{\partial u}{\partial x}\bar{u}\right\}dx = \left[EA\frac{\partial u}{\partial x}\bar{u}\right]_{x=L} - \left[EA\frac{\partial u}{\partial x}\bar{u}\right]_{x=0}$

$$\int_0^L \left[-EA\frac{\partial u}{\partial x}\frac{\partial \bar{u}}{\partial x} + f_x^B\bar{u}\right]dx + \left[EA\frac{\partial u}{\partial x}\bar{u}\right]_{x=L} = 0$$

Hence we have

$$\int_0^L \frac{\partial \bar{u}}{\partial x}EA\frac{\partial u}{\partial x}\,dx = \int_0^L f_x^B\bar{u}\,dx + R\bar{U}_L$$

4.2 (a) $\int_V \bar{\underline{\epsilon}}^T \underline{\tau}\, dV = \int_0^L \left(\frac{d\bar{u}}{dx}\right) \tau \cdot A(x) \cdot dx$ where $\tau = E\frac{du}{dx}$

$\int_V \bar{\underline{U}}^T \underline{f}^B dV = 0$, $\int_S \bar{\underline{U}}^{S^T} \underline{f}^S dS = \bar{U}\big|_{x=L} \cdot F$

∴ Principle of virtual displacements gives

$$\int_0^L \left(\frac{d\bar{v}}{dx}\right) \tau A(x)\, dx = \bar{U}_L \cdot F \qquad \text{———} (*)$$

(b) (i) $\bar{U}(x) = a_0 x$ ∴ $\bar{\epsilon}(x) = a_0$

From equation (*),

$$(\text{L.H.S.}) = F \cdot \int_0^L a_0 \left(\frac{72}{73} + \frac{24x}{73L}\right)\left(1 - \frac{x}{4L}\right) dx = a_0 F L$$

$$(\text{R.H.S.}) = a_0 F L$$

'OK'

(ii) $\bar{U}(x) = a_0 x^2$ ∴ $\bar{\epsilon}(x) = 2a_0 x$

Similarly, $(\text{L.H.S.}) = 2F\int_0^L a_0 x \left(\frac{72}{73} + \frac{24x}{73L}\right)\left(1 - \frac{x}{4L}\right) dx = a_0 F L^2$

$$(\text{R.H.S.}) = a_0 F L^2$$

'OK'

(iii) $\bar{U}(x) = a_0 x^3$, $\bar{\epsilon}(x) = 3a_0 x^2$

Similarly, $(\text{L.H.S.}) = \frac{729}{730} a_0 F L^3$

$$(\text{R.H.S.}) = a_0 F L^3$$

∴ (L.H.S.) ≠ (R.H.S.) in this case.

Hence the given τ is not the exact solution of the mathematical model!

4.2

(c) $E\frac{\partial}{\partial x}\left(A\frac{\partial U}{\partial x}\right)=0$ with $EA\frac{\partial U}{\partial x}\Big|_{x=L}=F$

$\rightarrow$ $A\frac{\partial U}{\partial x}=C$ (constant). $C=\frac{F}{E}$ (from b.c.)

$\therefore$ $\frac{\partial U}{\partial x}=\frac{F}{EA_0}\left[1/(1-\frac{x}{4L})\right]$

$\therefore$ $\tau=E\frac{\partial U}{\partial x}=\frac{F}{A_0}\left[1/(1-\frac{x}{4L})\right]$

(d) Every given $\bar{U}(x)$ satisfies the essential b.c.

For any $\bar{\epsilon}(x)=\frac{\partial\bar{U}(x)}{\partial x}$

$$\int_0^L\bar{\epsilon}\left[\frac{F}{A_0}/(1-\frac{x}{4L})\right]\left[A_0(1-\frac{x}{4L})\right]dx=\int_0^L\bar{\epsilon}F\,dx=\bar{U}(L)\cdot F$$

That is, the equation (*) in part (a) holds.

Note that the principle of virtual displacements holds for any virtual displacements which satisfy the essential boundary conditions on Su. Hence using the three $\bar{U}(x)$ in part (b) we see the principle of virtual displacements is satisfied.

4.3 (a) $E\frac{\partial}{\partial x}(A\frac{\partial u}{\partial x}) + f^D = 0$ with $u|_{x=0} = 0$, $EA\frac{\partial u}{\partial x}|_{x=L} = 0$

$EA\frac{\partial u}{\partial x} = -f^D x + c_1$, $c_1 = f_x^D L$ (from b.c.)

$\therefore EA\frac{\partial u}{\partial x} = f^D(L-x)$, $\frac{\partial u}{\partial x} = \frac{f^D(L-x)}{EA} = \frac{f^D}{EA_o}\frac{L-x}{4-\frac{3x}{L}}$

$$\frac{L-x}{4-\frac{3x}{L}} = \frac{L}{3} - \frac{L}{3}\frac{1}{4-\frac{3x}{L}}$$

$\therefore u = \frac{f^D}{EA_o}\left[\frac{L}{3}x + \frac{L^2}{9}\ln\left(4-\frac{3x}{L}\right)\right] + c_2$, $C_2 = -\frac{f^D}{EA_o}\frac{L^2}{9}\ln 4$ (from b.c.

$$\longrightarrow \quad u = \frac{f^D}{EA_o}\left[\frac{L}{3}x + \frac{L^2}{9}\ln\left(1-\frac{3}{4L}x\right)\right]$$

(b) The exact response is $\tau_{xx} = E\frac{\partial u}{\partial x} = \frac{f^D(L-x)}{A}$

Principle of virtual work states $\int_0^L \tau_{xx}\bar{\epsilon}A\,dx = \int_0^L f^D\bar{u}\,dx$

(i) $\bar{u} = ax \rightarrow \bar{\epsilon} = a$

$(\text{l.h.s.}) = \int_0^L f^D(L-x)a\,dx = \frac{f^D aL^2}{2}$

$(\text{r.h.s.}) = \int_0^L f^D ax\,dx = \frac{f^D aL^2}{2}$ 'Satisfied'

(ii) $\bar{u} = ax^2 \rightarrow \bar{\epsilon} = 2ax$

Similarly, $(\text{l.h.s.}) = \frac{f^D aL^3}{3} = (\text{r.h.s.})$ 'satisfied'

(c) Let $\tau_{xx} = \frac{f^D L}{6A_o}$ which is the force applied devided by $6A|_{x}$

with $\bar{u} = ax^2$, $\bar{\epsilon} = 2ax$, $(\text{l.h.s.}) = \frac{f^D aL^3}{3} = (\text{r.h.s.})$

with $\bar{u} = ax$, $\bar{\epsilon} = a$, $(\text{l.h.S.}) = \frac{5f^D aL^2}{12}$ $(\text{r.h.s.}) = \frac{f^D aL^2}{2}$

Hence τ_{xx} is not the exact solution.

4.4 The Principle of virtual displacements states

$$\int_V \bar{\underline{\epsilon}}^T \underline{\tau}\, dV = \int_V \bar{\underline{U}}^T \underline{f}^B dV + \sum_i \bar{\underline{U}}^{i^T} \underline{R}_C^i \qquad (\text{because } \underline{f}^S = \underline{0}.)$$

As the exact solution for $\underline{\tau}$ is not known, consider a $\bar{\underline{u}}$ with $\bar{\underline{\epsilon}}$ equal to zero. Then the right-hand-side checks the force and moment balance between the body forces and applied concentrated loads.

(i) Vertical equilibrium: $\bar{\underline{U}}^T = [\,a \quad 0\,]$

$$\left(\int_0^1\int_0^2 [\,a \quad 0\,]\begin{bmatrix} 10(1+2x) \\ 20(1+y)\end{bmatrix} dx\,dy\right) + \left([\,a \quad 0\,]\begin{bmatrix} -60 \\ -45\end{bmatrix} + [\,a \quad 0\,]\begin{bmatrix} 0 \\ -15\end{bmatrix}\right)$$

$$= (60\,a) + (-60\,a) = 0 \qquad \text{"O.K."}$$

(ii) Horizontal equilibrium: $\bar{\underline{U}}^T = [\,0 \quad b\,]$

Similarly, $(60b) + (-60b) = 0$ "O.K."

(iii) Moment equilibrium: $\bar{\underline{U}}^T = [\,-y\theta \quad x\theta\,]$ which corresponds to a rigid body rotation about the lower left corner.

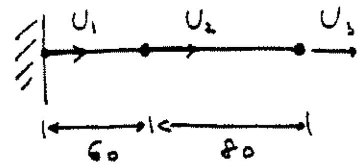
$$\left(\int_0^1\int_0^2 [\,-y\theta \quad x\theta\,]\begin{bmatrix} 10(2+2x) \\ 20(1+y)\end{bmatrix} dx\,dy\right)$$

$$+\left([\,-y\theta \quad x\theta\,]\Big|_{x=0,\,y=0}\begin{bmatrix} -60 \\ -45\end{bmatrix} + [\,-y\theta \quad x\theta\,]\Big|_{x=2,\,y=0}\begin{bmatrix} 0 \\ -15\end{bmatrix}\right)$$

$$= 30\theta + (-30\theta) = 0 \qquad \text{"O.K."}$$

Hence the body forces are in equilibrium with the applied concentrated nodal loads.

4.5 Let $\underline{U}^T = [\, U_1 \;\; U_2 \;\; U_3 \,]$



(a) Then,

$$\underline{H}^{(1)} = \left[\, 1-\frac{\eta}{60} \quad \frac{\eta}{60} \quad 0 \,\right]$$

$$\underline{H}^{(2)} = \left[\, 0 \quad 1-\frac{\eta}{80} \quad \frac{\eta}{80} \,\right]$$

$$\underline{B}^{(1)} = \left[\, -\frac{1}{60} \quad \frac{1}{60} \quad 0 \,\right] \qquad \underline{B}^{(2)} = \left[\, 0 \quad -\frac{1}{80} \quad \frac{1}{80} \,\right]$$

$$\underline{A}^{(1)} = A_0\left(1-\frac{\eta}{120}\right) \qquad A^{(2)} = \frac{A_0}{2}$$

Stiffness matrix $\underline{K}$ is given by

$$\underline{K} = E\int_0^{60} A_0\left(1-\frac{\eta}{120}\right)\begin{bmatrix} -\frac{1}{60} \\ \frac{1}{60} \\ 0 \end{bmatrix}\left[\, -\frac{1}{60} \quad \frac{1}{60} \quad 0 \,\right] d\eta$$

$$+ \; E\int_0^{80} \frac{A_0}{2}\begin{bmatrix} 0 \\ -\frac{1}{80} \\ \frac{1}{80} \end{bmatrix}\left[\, 0 \quad -\frac{1}{80} \quad \frac{1}{80} \,\right] d\eta$$

The force vector $\underline{R}$ is given by $\underline{R} = \underline{R}_B + \underline{R}_C$

where

$$\underline{R}_B = \left(\int_0^{60} A_0\left(1-\frac{\eta}{120}\right)\begin{bmatrix} 1-\frac{\eta}{60} \\ \frac{\eta}{60} \\ 0 \end{bmatrix} d\eta + \int_0^{80} \frac{A_0}{2}\begin{bmatrix} 0 \\ 1-\frac{\eta}{80} \\ \frac{\eta}{80} \end{bmatrix} d\eta\right) 0.1\, f_1(t)$$

$$\underline{R}_C = \begin{bmatrix} 0 \\ 0 \\ 20 \end{bmatrix} f_1(t)\, A_0$$

4.5

Finally, we obtain $\underset{\sim}{K}\,\underset{\sim}{U} = \underset{\sim}{R}$

where $\underset{\sim}{K} = EA_0 \begin{bmatrix} \frac{1}{80} & -\frac{1}{80} & 0 \\ -\frac{1}{80} & \frac{3}{160} & -\frac{1}{160} \\ 0 & -\frac{1}{160} & \frac{1}{160} \end{bmatrix}$, $\underset{\sim}{R} = \begin{bmatrix} 2.5 \\ 4 \\ 22 \end{bmatrix} f_1(t) A_0$

and $\underset{\sim}{U}^T = [U_1 \;\; U_2 \;\; U_3]$

(b) The mass matrix is given by

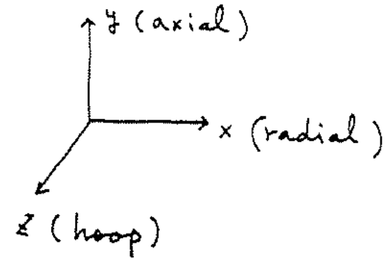
$$\underset{\sim}{M} = \rho \int_0^{60} A_0 \left(1 - \frac{\eta}{120}\right) \begin{bmatrix} 1 - \frac{\eta}{60} \\ \frac{\eta}{60} \\ 0 \end{bmatrix} \left[1 - \frac{\eta}{60} \quad \frac{\eta}{60} \quad 0\right] d\eta$$

$$+ \rho \frac{A_0}{2} \int_0^{80} \begin{bmatrix} 0 \\ 1 - \frac{\eta}{80} \\ \frac{\eta}{80} \end{bmatrix} \left[0 \quad 1 - \frac{\eta}{80} \quad \frac{\eta}{80}\right] d\eta$$

Hence, $\underset{\sim}{M} = \rho A_0 \begin{bmatrix} \frac{35}{2} & \frac{15}{2} & 0 \\ \frac{15}{2} & \frac{155}{6} & \frac{20}{3} \\ 0 & \frac{20}{3} & \frac{40}{3} \end{bmatrix}$

4.6 The strain components which contribute to the strain energy in this exercise are only ϵ_{xx} and ϵ_{zz}. Hence, the matrix $\underline{C}$ is given by

$$\underline{C} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix}.$$



And $$\underline{\epsilon} = \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{zz} \end{bmatrix} = \begin{bmatrix} \partial U/\partial x \\ U/x \end{bmatrix}$$

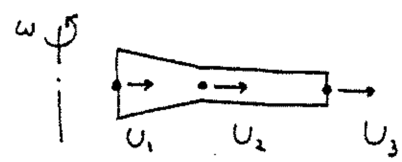
For each element

$$\underline{H}^{(1)} = \begin{bmatrix} 1-\frac{\eta}{60} & \frac{\eta}{60} & 0 \end{bmatrix}, \quad \underline{H}^{(2)} = \begin{bmatrix} 0 & 1-\frac{\eta}{80} & \frac{\eta}{80} \end{bmatrix}$$

$$\underline{B}^{(1)} = \begin{bmatrix} -\frac{1}{60} & \frac{1}{60} & 0 \\ \dfrac{1-\frac{\eta}{60}}{(\eta+20)} & \dfrac{\frac{\eta}{60}}{(\eta+20)} & 0 \end{bmatrix}, \quad \underline{B}^{(2)} = \begin{bmatrix} 0 & -\frac{1}{80} & \frac{1}{80} \\ 0 & \dfrac{1-\frac{\eta}{80}}{\eta+80} & \dfrac{\frac{\eta}{80}}{\eta+80} \end{bmatrix}$$

$$t^{(1)} = 3\left(1-\frac{\eta}{90}\right), \quad t^{(2)} = 1 \qquad (t = \text{thickness})$$

Here, $\underline{U}^T = \begin{bmatrix} U_1 & U_2 & U_3 \end{bmatrix}$



Then,
$$\underline{K} = \int_0^{60} \underline{B}^{(1)T}\underline{C}\,\underline{B}^{(1)} \cdot 1 \cdot 3\left(1-\frac{\eta}{90}\right)\cdot(\eta+20)\,d\eta + \int_0^{80} \underline{B}^{(2)T}\underline{C}\,\underline{B}^{(2)} \cdot 1 \cdot 1 \cdot (\eta+80)\,d\eta$$

$$\underline{R} = \underline{R}_B = \int_0^{60} \underline{H}^{(1)T}\cdot\rho\omega^2(\eta+20)\cdot 1 \cdot 3\left(1-\frac{\eta}{90}\right)\cdot(\eta+20)\,d\eta + \int_0^{80} \underline{H}^{(2)T}\cdot\rho\omega^2(\eta+80)\cdot 1 \cdot 1 \cdot (\eta+80)\,d\eta$$

Note that the matrices are obtained for a unit radian.

4.7 At time $t = 2.0$,

(a)

$$\underline{K} = \frac{E}{240}\begin{bmatrix} 2.4 & -2.4 & 0 \\ -2.4 & 15.4 & -13 \\ 0 & -13 & 13 \end{bmatrix}, \quad \underline{R} = \frac{1}{3}\begin{bmatrix} 150 \\ 186 \\ 68 \end{bmatrix} + \begin{bmatrix} R_r|_{U_1} \\ 0 \\ 0 \end{bmatrix}$$

Modifying the matrices by changing rows and columns we have

$$\begin{bmatrix} \underline{K}_{aa} & \underline{K}_{ab} \\ \underline{K}_{ba} & \underline{K}_{bb} \end{bmatrix}\begin{bmatrix} \underline{U}_a \\ \underline{U}_b \end{bmatrix} = \begin{bmatrix} \underline{R}_a \\ \underline{R}_b \end{bmatrix}$$ where $\underline{U}_a$ = unknown, $\underline{U}_b$ = known.

$$\underline{U}_a = \underline{K}_{aa}^{-1}(\underline{R}_a - \underline{K}_{ab}\underline{U}_b)$$

Hence $$\begin{bmatrix} U_2 \\ U_3 \end{bmatrix} = \left(\frac{E}{240}\begin{bmatrix} 15.4 & -13 \\ -13 & 13 \end{bmatrix}\right)^{-1}\left(\frac{1}{3}\begin{bmatrix} 186 \\ 68 \end{bmatrix} - \underline{0}\right) = \frac{1}{E}\begin{bmatrix} 8466.7 \\ 8885.1 \end{bmatrix}$$

And $$\tau^{(1)} = E\left[-\frac{1}{100} \quad \frac{1}{100} \quad 0\right]\left(\frac{1}{E}\begin{bmatrix} 0 \\ 8466.7 \\ 8885.1 \end{bmatrix}\right) = 84.7$$

$$\tau^{(2)} = E\left[0 \quad -\frac{1}{80} \quad \frac{1}{80}\right]\left(\quad '' \quad \right) = 5.23$$

(b) $$R_r|_{U_1} = \underline{K}_{ba}\underline{U}_a + \underline{K}_{bb}\underline{U}_b - \frac{1}{3}(150)$$

$$\therefore R_r|_{U_1} = -134.7$$

(c) $$u^{FE(1)} = \left[1-\frac{x}{100} \quad \frac{x}{100} \quad 0\right]\begin{bmatrix} 0 \\ 8466.7/E \\ 8885.1/E \end{bmatrix} = \frac{84.7}{E}x$$

$$u^{FE(2)} = \left[0 \quad 1-\frac{x}{80} \quad \frac{x}{80}\right]\left[\quad '' \quad \right] = \frac{1}{E}\left(8466.7 + 5.23\,x\right)$$

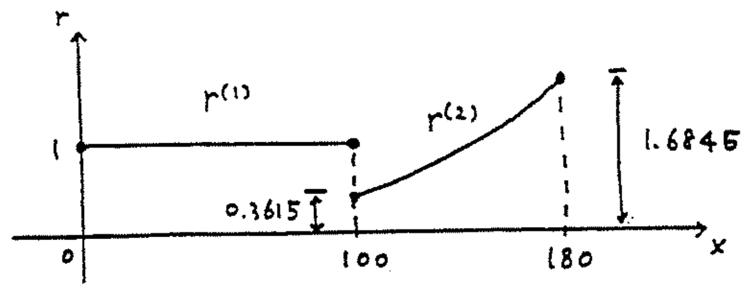
4.7

$$r = E\left[\frac{\partial}{\partial x}\left(A\frac{\partial u^{FE}}{\partial x}\right)\right] + f^B A$$

$$r^{(1)} = E\left[\frac{\partial}{\partial x}\left\{1\cdot\frac{\partial}{\partial x}\left(\frac{84.7}{E}x\right)\right\}\right] + 1.1 = 1$$

$$r^{(2)} = E\left[\frac{\partial}{\partial x}\left\{\left(1+\frac{x}{40}\right)^2\frac{\partial}{\partial x}\left(\frac{8466.7 + 5.23x}{E}\right)\right\}\right] + (0.1)\cdot 1\cdot\left(1+\frac{x}{40}\right)^2$$

$$= (0.3615 + 0.0025x)\left(1+\frac{x}{40}\right)$$



(d) $E^{FE} = \frac{1}{2}\underline{U}^T\underline{K}\,\underline{U} = \frac{1}{2}\underline{U}^T\underline{R} = 363165/E$

Now the exact strain energy is to be calculated.

for element ①

$$\tau^{(1)}(x) = \int_x^{100} 1.1\, d\eta + \int_0^{80}\left(1+\frac{\eta}{40}\right)^2(0.1)\,d\eta = 134.7 - x$$

$$E^{(1)} = \int_0^{100}\frac{1}{2E}\left[\tau^{(1)}(x)\right]^2 dx = \frac{400089}{E}$$

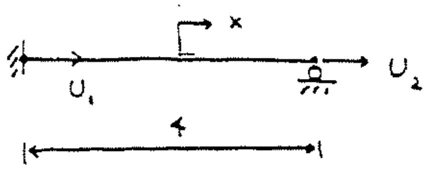
for element ②

$$\tau^{(2)}(x) = \frac{1}{\left(1+\frac{x}{40}\right)^2}\int_x^{80}\left(1+\frac{\eta}{40}\right)^2(0.1)\,d\eta$$

$$E^{(2)} = \int_0^{80}\frac{1}{2E}\left[\tau^{(2)}(x)\right]^2\left(1+\frac{x}{40}\right)^2 dx = \frac{11321}{E}$$

$$\therefore\ E^{exact} = E^{(1)} + E^{(2)} = \frac{411410}{E} > E^{FE} = \frac{363165}{E}$$

4.8 $\quad \epsilon^{th} = \alpha(\theta - \theta_{ref}) = \alpha\{(10x+20) - 20\} = 10\alpha x$



Analytical solution

$$\left.\begin{aligned} &\tau = E(\epsilon - \epsilon^{th}) = 0 \\ &\epsilon = \epsilon^{th} = 10\alpha x \\ &\epsilon = \frac{du}{dx}, \quad u = 5\alpha(x^2 - 4) \quad \left(\text{because } u\big|_{x=-2} = 0\right) \end{aligned}\right\} \text{—— ①}$$

Hence the exact response for displacement is parabolic, and the strain is linear while the stress is zero.

Finite element solution

$$\underline{H} = \left[\frac{2-x}{4} \quad \frac{2+x}{4}\right], \quad \underline{B} = \left[-\frac{1}{4} \quad \frac{1}{4}\right], \quad \underline{U}^T = [U_1 \;\; U_2]$$

$$\underline{K} = \int_{-2}^{2} \underline{B}^T E \underline{B}\, dx = E\begin{bmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

$$\underline{R}_I = \int_V \underline{B}^T \tau^I\, dV \quad \text{where } \tau^I = -E\left(\underline{H}\begin{bmatrix} 0 \\ 40\alpha \end{bmatrix} - 20\right) = -E(10\alpha x)$$

$$\therefore \quad \underline{R}_I = \int_{-2}^{2} \underline{B}^T[-E(10\alpha x)]\, dx = \underline{0}$$

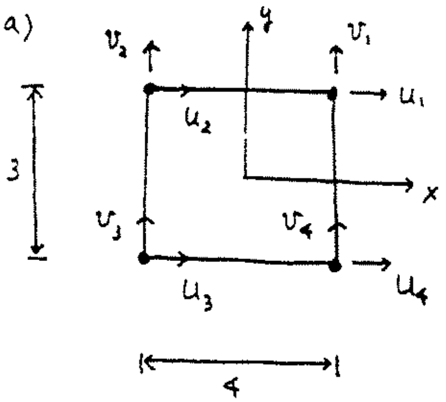
Hence

$$E\begin{bmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \underline{0} + \begin{bmatrix} R_1 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} &\therefore \quad U_1 = U_2 = 0, \quad R_1 = 0 \\ &\tau = E\underline{B}\underline{U} + \tau^I = -E(10\alpha x), \quad \epsilon = 0, \quad u = 0 \end{aligned}\right\} \text{—— ②}$$

Considering the responses in eq. ① and ②, we see that the finite element solution is far from the exact solution, because a single two-node element is used. A finite element based on a parabolic assumption is needed, in which case the exact solution is obtained with one element.

4.9 (a)



$$h_1 = \frac{1}{4}\left(1+\frac{x}{2}\right)\left(1+\frac{2}{3}y\right), \quad h_2 = \frac{1}{4}\left(1-\frac{x}{2}\right)\left(1+\frac{2}{3}y\right)$$

$$h_3 = \frac{1}{4}\left(1-\frac{x}{2}\right)\left(1-\frac{2}{3}y\right), \quad h_4 = \frac{1}{4}\left(1+\frac{x}{2}\right)\left(1-\frac{2}{3}y\right)$$

$$h_{1,x} = \frac{\partial h_1}{\partial x} = \frac{1}{8}\left(1+\frac{2}{3}y\right), \quad h_{2,x} = -\frac{1}{8}\left(1+\frac{2}{3}y\right)$$

$$h_{3,x} = -\frac{1}{8}\left(1-\frac{2}{3}y\right), \quad h_{4,x} = \frac{1}{8}\left(1-\frac{2}{3}y\right)$$

$$h_{1,y} = \frac{\partial h_1}{\partial y} = \frac{1}{6}\left(1+\frac{x}{2}\right), \quad h_{2,y} = \frac{1}{6}\left(1-\frac{x}{2}\right), \quad h_{3,y} = -\frac{1}{6}\left(1-\frac{x}{2}\right) \quad \text{and}$$

$$h_{4,y} = -\frac{1}{6}\left(1+\frac{x}{2}\right)$$

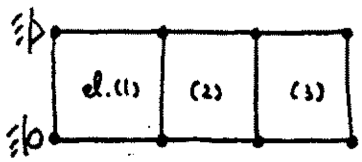
Then,

$$\underline{H} = \begin{bmatrix} h_1 & 0 & h_2 & 0 & h_3 & 0 & h_4 & 0 \\ 0 & h_1 & 0 & h_2 & 0 & h_3 & 0 & h_4 \end{bmatrix}$$

with $\hat{\underline{u}}^T = [u_1 \; v_1 \; u_2 \; v_2 \; u_3 \; v_3 \; u_4 \; v_4]$

$$\underline{B} = \begin{bmatrix} h_{1,x} & 0 & h_{2,x} & 0 & h_{3,x} & 0 & h_{4,x} & 0 \\ 0 & h_{1,y} & 0 & h_{2,y} & 0 & h_{3,y} & 0 & h_{4,y} \\ h_{1,y} & h_{1,x} & h_{2,y} & h_{2,x} & h_{3,y} & h_{3,x} & h_{4,y} & h_{4,x} \end{bmatrix}$$

(b)



$$K_{ij}^{(m)} = \int_V B_{ri}^{(m)} C_{rs}^{(m)} B_{sj}^{(m)} \, dV$$

$$K_{U_2U_2} = K_{u_4u_4}^{(1)} + K_{u_3u_3}^{(2)} = K_{77}^{(1)} + K_{55}^{(2)}$$

$$= \int_V B_{r7}^{(1)} C_{rs}^{(1)} B_{s7}^{(1)} \, dV + \int_V B_{r5}^{(2)} C_{rs}^{(2)} B_{s5}^{(2)} \, dV$$

4.9

$$\therefore K_{U_2U_2} = t\int_{-2}^{2}\int_{-3/2}^{3/2} [\, h_{4,x} \;\; 0 \;\; h_{4,y} \,] \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}\begin{bmatrix} h_{4,x} \\ 0 \\ h_{4,y} \end{bmatrix} dx\,dy$$

$$+ t\int_{-2}^{2}\int_{-3/2}^{3/2} [\, h_{3,x} \;\; 0 \;\; h_{3,y} \,] \frac{E}{1-\nu^2}\begin{bmatrix} & '' & \end{bmatrix}\begin{bmatrix} h_{3,x} \\ 0 \\ h_{3,y} \end{bmatrix} dx\,dy$$

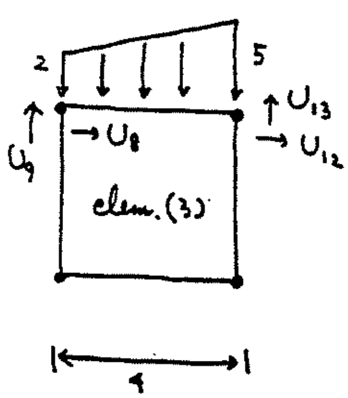
$$= \frac{Et}{1-\nu^2}\;\frac{17-8\nu}{18}$$

$\therefore$ $K_{U_6U_7}$, $K_{U_7U_6}$ and $K_{U_5U_{12}}$ are obtained similarly, i.e.,

$$K_{U_6U_7} = K^{(2)}_{u_4v_4} + K^{(3)}_{u_3v_3} = 0 = K_{U_7U_6}$$

$K_{U_5U_{12}} = 0$ (U_5 and U_{12} are not coupled together.)

(c)



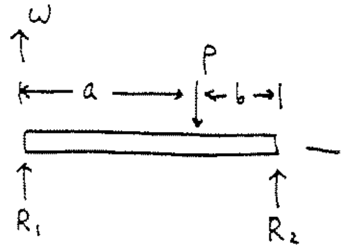
$$\underline{R}_S^{(3)} = t\int_{-2}^{2} \underline{H}^{S\,T}\underline{f}^S\,dx$$

$$\therefore\; R_9 = t\int_{-2}^{2}\left[\frac{1}{2}\left(1-\frac{x}{2}\right)\right]\left[\,2\,\frac{x-2}{4} - 5\,\frac{x+2}{4}\right]dx$$

$$= -6t$$

4.10 (a) $\int_V \bar{\underline{\epsilon}}^T \underline{\tau}\, dV = \int_V \bar{\underline{U}}^T \underline{f}^B dV + \int_{S_f} \bar{\underline{U}}^{S^T} \underline{f}^S dS + \sum_i \bar{\underline{U}}^{i^T} \underline{R}_c^i$

consider first $\bar{\underline{u}} = [\bar{\omega}] = \omega_0$

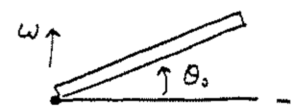


$(l.h.s.) = 0$

$(r.h.s.) = 0 + 0 + (R_1 + R_2)\omega_0 - P\omega_0$

$\therefore \quad R_1 + R_2 - P = 0 \qquad \text{— ①}$

Now for $\bar{\underline{u}} = \frac{x}{a+b}\theta_0$ (rotation about left end)



$(l.h.s.) = 0$

$(r.h.s.) = 0 + 0 + \frac{a\theta_0}{a+b}P - R_2\theta_0$

$\therefore \quad \frac{a}{a+b}P - R_2 = 0 \qquad \text{— ②}$

From ① and ②, $\quad R_2 = \frac{a}{a+b}P \quad , \quad R_1 = \frac{b}{a+b}P$

(b) P.V.D. states $\quad \int_V \bar{\underline{\epsilon}}^T \underline{\tau}\, dV = \sum_i \bar{\underline{U}}^{i^T} \underline{R}_c^i$

In order to make (l.h.s.) zero as done in part (a),

$$\bar{\underline{\epsilon}}^T = [\,\bar{\epsilon}_{xx} \;\; \bar{\epsilon}_{yy} \;\; \bar{\gamma}_{xy}\,] = \left[\frac{\partial \bar{u}}{\partial x} \quad \frac{\partial \bar{v}}{\partial y} \quad \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}\right] = \underline{0}$$

$$\frac{\partial \bar{u}}{\partial x} = 0 \longrightarrow \bar{u} = a(y), \qquad \frac{\partial \bar{v}}{\partial y} = 0 \longrightarrow \bar{v} = b(x)$$

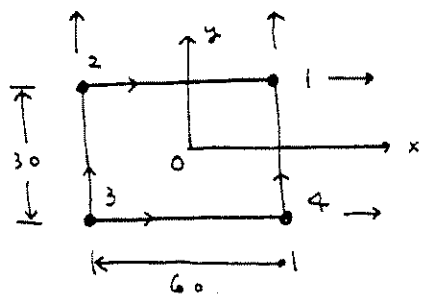
$$\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} = 0 \longrightarrow a(y) = a_0 + a_1 y\,, \quad b(x) = b_0 + b_1 x \qquad \therefore\; a_1 + b_1 = 0$$

Hence, $\bar{u} = a_0 + a_1 y$, $\bar{v} = b_0 - a_1 x$ should be used.

We note that these displacement fields represent the rigid body motions.

Hence it is necessary that the finite element displacement functions can represent the rigid body mode displacements.

4.11 (a) $\underline{\tau}^{I\,T} = [\ 0 \quad 10 \quad 20\ ]$



$$\begin{cases} h_1 = \frac{1}{4}\left(1+\frac{x}{30}\right)\left(1+\frac{y}{15}\right) \\ h_2 = \frac{1}{4}\left(1-\frac{x}{30}\right)\left(1+\frac{y}{15}\right) \\ h_3 = \frac{1}{4}\left(1-\frac{x}{30}\right)\left(1-\frac{y}{15}\right) \\ h_4 = \frac{1}{4}\left(1+\frac{x}{30}\right)\left(1-\frac{y}{15}\right) \end{cases}$$

$$\underline{u}^T = [\ u_1\ v_1\ u_2\ v_2\ u_3\ v_3\ u_4\ v_4\ ]$$

$$\rightarrow \underline{B} = \begin{bmatrix} h_{1,x} & 0 & h_{2,x} & 0 & h_{3,x} & 0 & h_{4,x} & 0 \\ 0 & h_{1,y} & 0 & h_{2,y} & 0 & h_{3,y} & 0 & h_{4,y} \\ h_{1,y} & h_{1,x} & h_{2,y} & h_{2,x} & h_{3,y} & h_{3,x} & h_{4,y} & h_{4,x} \end{bmatrix}$$

$$\therefore \underline{R}_I = \int_V \underline{B}^T \underline{\tau}^I dV = (0.5)\int_{-15}^{15}\int_{-30}^{30} \underline{B}^T \underline{\tau}^I \, dx\, dy$$

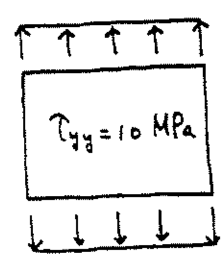
$$\therefore \underline{R}_I^T = [\ 300 \quad 300 \quad 300 \quad 0 \quad -300 \quad -300 \quad -300 \quad 0\ ]$$

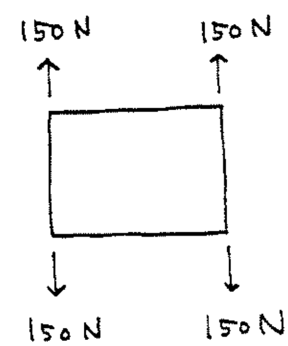
(b) We apply the superposition.

i) $\tau_{yy} = 10$, $\tau_{xx} = \tau_{xy} = 0$

$$R_y = (10\ \text{MPa})(60\ \text{mm})(0.5\ \text{mm})/2 = 150\ \text{N}$$

$$\rightarrow \underline{R}_S^{i)} = \begin{bmatrix} 0 \\ 150 \\ 0 \\ 150 \\ 0 \\ -150 \\ 0 \\ -150 \end{bmatrix}$$





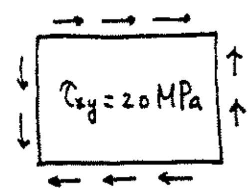
4.11

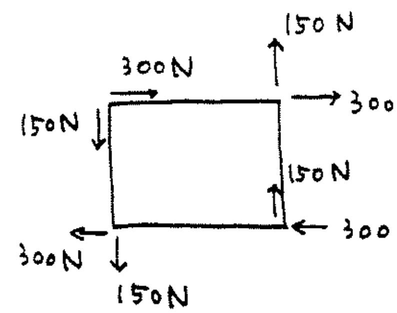
ii) $\tau_{xy} = 2\sigma, \quad \tau_{xx} = \tau_{yy} = 0$

$$P_x = (20\,MPa)(60\,mm)(0.5\,mm)/2 = 300\,N$$

$$P_y = (20\,MPa)(30\,mm)(0.5\,mm)/2 = 150\,N$$

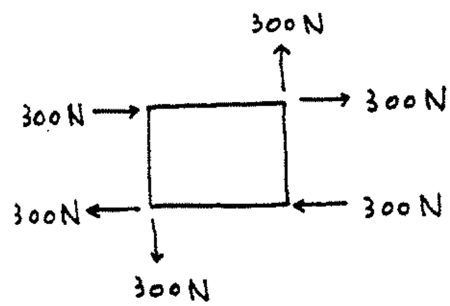
$$\rightarrow \underline{R}_s^{ii)} = \begin{bmatrix} 300 \\ 150 \\ 300 \\ -150 \\ -300 \\ -150 \\ -300 \\ 150 \end{bmatrix},$$





Hence,

$$\underline{R}_s = \underline{R}_s^{i)} + \underline{R}_s^{ii)} = \begin{bmatrix} 300 \\ 300 \\ 300 \\ 0 \\ -300 \\ -300 \\ -300 \\ 0 \end{bmatrix}$$



Now check the result using elementary statics.

$$\Sigma F_x = 300 + 300 - 300 - 300 = 0$$

$$\Sigma F_y = 300 + 0 - 300 + 0 = 0$$

$$\Sigma M_{\left(\substack{about \\ node\ 3}\right)} = -(300 \cdot 30) + (300 \cdot 60) - (300 \cdot 30) = 0 \quad (+\circlearrowleft)$$

Therefore, $\underline{R}_s$ is in equilibrium. And $\underline{R}_I$ in part (a) is equal to $\underline{R}_s$ in part (b). In fact, $\underline{R}_s = \underline{R}_I$ because $\tau_{ij,j} = 0$, see part (c)

4.11

(c) When the stresses are in equilibrium, we have $\tau_{ij,j} = 0$.

We want to prove:

$$\int_S f_i^S \bar{u}_i \, dS = \int_V \bar{\varepsilon}_{ij} \tau_{ij} \, dV \quad , \text{ or}$$

$$\int_S \underline{H}^{S^T} \underline{f}^S \, dS = \int_V \underline{B}^T \underline{\tau} \, dV \quad , \text{ i.e., } \underline{R}_S = \underline{R}_I$$

This follows because $\int_V \bar{\varepsilon}_{ij} \tau_{ij} \, dV = \int_V \bar{u}_{i,j} \tau_{ij} \, dV =$

$$= \int_V (\tau_{ij} \bar{u}_i)_{,j} \, dV - \int_V \underbrace{\tau_{ij,j}}_{0} \bar{u}_i \, dV \quad \overset{\text{use divergence theorem}}{=}$$

$$= \int_S (\tau_{ij} \bar{u}_i) n_j \, dS = \int_S f_i^S \bar{u}_i \, dS$$

Hence, the internal stresses must be in equilibrium, and we must use $f_i^S = \tau_{ij} n_j$, in order to have that $\underline{R}_S = \underline{R}_I$.

4.12

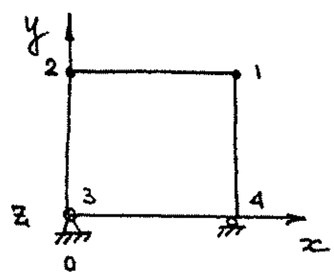
Since $\varepsilon_{zz} = 0$, we have:

$$\tau_{zz} = \nu(\tau_{xx} + \tau_{yy}) = 9 \text{ psi}$$

$$\varepsilon_{xx} = \frac{1}{E}(\tau_{xx} - \nu(\tau_{yy} + \tau_{zz})) = \frac{14.3}{E}$$

$$\varepsilon_{yy} = \frac{1}{E}(\tau_{yy} - \nu(\tau_{xx} + \tau_{zz})) = \frac{1.3}{E}$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy} = \frac{26}{E}$$



$$\begin{cases} u = x\varepsilon_{xx} + y\gamma_{xy} \\ v = y\varepsilon_{yy} \end{cases}$$

Node 1: $u_1 = 3.165 \cdot 10^{-6}$ in ;

$v_1 = 0.086 \cdot 10^{-6}$ in ;

Node 2: $u_2 = 1.734 \cdot 10^{-6}$ in ;

$v_2 = 0.086 \cdot 10^{-6}$ in ;

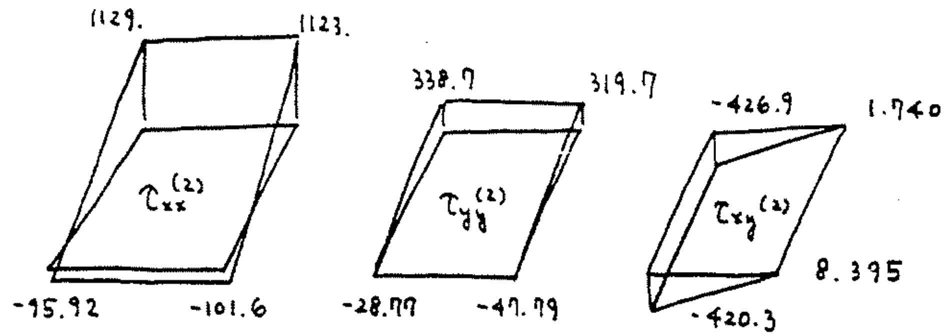
Node 3: $u_3 = 0$;

$v_3 = 0$;

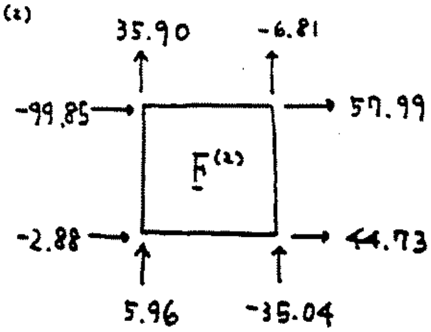
Node 4: $u_4 = 1.431 \cdot 10^{-6}$ in;

$v_4 = 0$.

4.13 (a)



In order to calculate $\int_{V^{(2)}} \underline{B}^{(2)T} \underline{\tau}^{(2)} dV^{(2)}$ the distribution of $\underline{\tau}^{(2)}$ must be known in advance. In a four-node element when displacements are interpolated by nodal values,



that is,

$$\underline{u} = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a_0 + a_1 x + a_2 y + a_3 xy \\ b_0 + b_1 x + b_2 y + b\ xy \end{bmatrix}$$

$$\therefore \underline{\tau} = \begin{bmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} a_1 + a_3 y \\ b_2 + b_3 x \\ (a_2 + b_1) + a_3 x + b_3 y \end{bmatrix}$$

$$= \frac{E}{1-\nu^2} \begin{bmatrix} (a_1 + \nu b_2) + \nu b_3 x + a_3 y \\ (\nu a_1 + b_2) + b_3 x + \nu a_3 y \\ \frac{1-\nu}{2} \{ (a_2 + b_1) + a_3 x + b_3 y \} \end{bmatrix}$$

From the given values of $\tau_{xx}^{(2)}$, $\tau_{yy}^{(2)}$ and $\tau_{xy}^{(2)}$ we can obtain by least squares all the constants needed and then

$$\tau_{xx} = 513.62 - 2.84x + 612.37y$$

4.13

$$\tau_{yy} = 145.54 - 9.46x + 183.71y$$
$$\tau_{xy} = -209.27 + 214.32x - 3.31y$$

$$\therefore \int_{V^{(2)}} \underline{B}^{(2)T} \underline{\tau}^{(2)} dV^{(2)} = \begin{bmatrix} 58.0 \\ -6.80 \\ -99.85 \\ 35.91 \\ -2.88 \\ 5.95 \\ 44.73 \\ -35.06 \end{bmatrix} \cong \underline{F}^{(2)} = \begin{bmatrix} 57.99 \\ -6.81 \\ -99.85 \\ 35.90 \\ -2.88 \\ 5.96 \\ 44.73 \\ -35.04 \end{bmatrix}$$

Note the error between $\underline{F}^{(2)}$ and $\int_{V^{(2)}} \underline{B}^{(2)T} \underline{\tau}^{(2)} dV$ stems from the fact that all the numerical values used have round-off error. Actually the two matrices are exactly the same.

(b) Check the balance.

$$\Sigma F_x = 57.99 - 99.85 - 2.88 + 44.73 = -0.01 \sim 0$$
$$\Sigma F_y = -6.81 + 35.90 + 5.96 - 35.04 = 0.01 \sim 0$$
$$\Sigma M_{\begin{pmatrix}\text{about}\\ \text{node 3}\end{pmatrix}} = -(57.99)(2) + (-6.81)(2) - (-99.85)(2) + (-35.04)(2)$$
$$= 0.02 \sim 0$$

The error similarly as in part (a) is due to round-off.

4.14 We have in this exercise 4 global unknown variables.

$$
\begin{array}{c}
\begin{array}{cccccccc} & U_2 & U_3 & & U_1 & & & \\ & u_1 & v_1 & & v_2 & & v_3 & \end{array} \\
\underline{K}_A = \left[\begin{array}{cccccc}
a_{11} & a_{12} & \cdot & a_{14} & \cdot & a_{16} \\
a_{21} & a_{22} & \cdot & a_{24} & \cdot & a_{26} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
a_{41} & a_{42} & \cdot & a_{44} & \cdot & a_{46} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
a_{61} & a_{62} & \cdot & a_{64} & \cdot & a_{66}
\end{array}\right]
\begin{array}{l} u_1 \; U_2 \\ v_1 \; U_3 \\ \\ v_2 \; U_1 \\ \\ v_3 \end{array}
\end{array}
$$

$$
\begin{array}{c}
\begin{array}{cccccc} & & & U_2 & U_3 & U_4 \\ u_1 & & & u_2 & v_2 & \theta_2 \end{array} \\
\underline{K}_B = \left[\begin{array}{cccccc}
b_{11} & \cdot & \cdot & \cdot & \cdot & b_{16} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & b_{44} & b_{45} & b_{46} \\
\cdot & \cdot & \cdot & b_{54} & b_{55} & b_{56} \\
b_{61} & \cdot & \cdot & b_{64} & b_{65} & b_{66}
\end{array}\right]
\begin{array}{l} u_1 \\ \\ \\ u_2 \; U_2 \\ v_2 \; U_3 \\ \theta_2 \; U_4 \end{array}
\end{array}
$$

Therefore the assemblage process gives

$$
\underline{K} = \left[\begin{array}{cccc}
a_{44} & a_{41} & a_{42} & 0 \\
a_{14} & (a_{11}+b_{44}) & (a_{12}+b_{45}) & b_{46} \\
a_{24} & (a_{21}+b_{54}) & (a_{22}+b_{55}) & b_{56} \\
0 & b_{64} & b_{65} & b_{66}
\end{array}\right]
\begin{array}{c} U_1 \\ U_2 \\ U_3 \\ U_4 \end{array}
$$

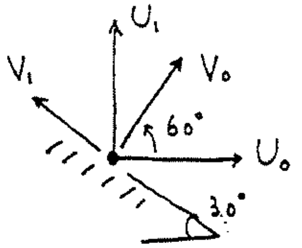
4.15 We have 5 global variables such as U_1, U_2, U_3, U_4 and U_5.

$$\underset{\sim}{K}_A = \begin{array}{c} \begin{array}{cccccc} U_1 & U_2 & & U_3 & U_4 & \end{array} \\ \left[\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1\eta} & a_{1\xi} \\ a_{21} & a_{22} & \cdots & a_{2\eta} & a_{2\xi} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{\eta 1} & a_{\eta 2} & \cdots & a_{\eta\eta} & a_{\eta\xi} \\ a_{\xi 1} & a_{\xi 2} & \cdots & a_{\xi\eta} & a_{\xi\xi} \end{array}\right] \end{array} \begin{array}{c} U_1 \\ U_2 \\ \\ U_3 \\ U_4 \end{array} , \quad \underset{\sim}{K}_B = \begin{array}{c} \begin{array}{ccc} U_3 & U_4 & U_5 \end{array} \\ \left[\begin{array}{cccc} b_{11} & b_{12} & b_{13} & \cdots \\ b_{21} & b_{22} & b_{23} & \cdots \\ b_{31} & b_{32} & b_{33} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{array}\right] \end{array} \begin{array}{c} U_3 \\ U_4 \\ U_5 \\ \\ \end{array}$$

And the assemblage process gives

$$\underset{\sim}{K} = \begin{array}{c} \begin{array}{ccccc} U_1 & U_2 & U_3 & U_4 & U_5 \end{array} \\ \left[\begin{array}{ccccc} a_{11} & a_{12} & a_{1\eta} & a_{1\xi} & 0 \\ a_{21} & a_{22} & a_{2\eta} & a_{2\xi} & 0 \\ a_{\eta 1} & a_{\eta 2} & a_{\eta\eta} + b_{11} & a_{\eta\xi} + b_{12} & b_{13} \\ a_{\xi 1} & a_{\xi 2} & a_{\xi\eta} + b_{21} & a_{\xi\xi} + b_{22} & b_{23} \\ 0 & 0 & b_{31} & b_{32} & b_{33} \end{array}\right] \end{array} \begin{array}{c} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{array}$$

4.16 The 1st step is to include the effects of d.o.f. U_0 in the stiffness matrix $\underline{K}$. The terms corresponding to U_0 are in the 5th row and column in $\underline{K}_A$. Since the roller allows a displacement only along the slope, we need to introduce the new variables V_0 and V_1 as shown below.



$$\begin{bmatrix} V_0 \\ V_1 \end{bmatrix} = \begin{bmatrix} \cos 60^\circ & \sin 60^\circ \\ -\sin 60^\circ & \cos 60^\circ \end{bmatrix} \begin{bmatrix} U_0 \\ U_1 \end{bmatrix}$$

↑ Let this matrix be $\underline{T}_{uv}$

and
$$\underline{T} = \begin{bmatrix} \underline{I}_1 & \underline{0} & \underline{0} \\ \underline{0} & \underline{T}_{uv} & \underline{0} \\ \underline{0} & \underline{0} & \underline{I}_2 \end{bmatrix}$$

where $\underline{I}_1 : 4 \times 4$, $\underline{I}_2 = 2 \times 2$

$\underline{I}_1, \underline{I}_2$: identity matrices

Then
$$\bar{\underline{K}}_A = \underline{T}^T \underline{K}_A \underline{T}$$

$$= \begin{bmatrix} \text{////} & a'_{15}\ a'_{16} & \text{////} \\ & \vdots & \\ a'_{51}\ a'_{52} & \cdots & a'_{57}\ a'_{58} \\ a'_{61}\ a'_{62} & \cdots & a'_{67}\ a'_{68} \\ & \vdots & \\ \text{////} & a'_{85}\ a'_{86} & \text{////} \end{bmatrix}$$

where in the hatched area the elements are the same as in $\underline{K}_A$.

4.16

The modified stiffness matrix has the form

$$\underline{K}^* = \begin{bmatrix} a'_{55} & a'_{56} & a'_{51} & a'_{52} & a'_{57} & a'_{58} & \cdots \\ a'_{65} & a'_{66} & a'_{61} & a'_{62} & a'_{67} & a'_{68} & \cdots \\ a'_{15} & a'_{16} & & & & & \\ a'_{25} & a'_{26} & & \multicolumn{4}{l}{\text{the elements corresponding}} \\ a'_{75} & a'_{76} & & \multicolumn{4}{l}{\text{to } U_2, U_3, \cdots, U_8 \text{ are}} \\ a'_{85} & a_{86} & & \multicolumn{4}{l}{\text{the same as in } \underline{K}} \\ \vdots & \vdots & & & & & \end{bmatrix} \begin{matrix} V_0 \\ V_1 \\ U_2 \\ U_3 \\ \vdots \\ U_8 \end{matrix}$$

(a) Now $V_0 = 0$ gives the modified stiffness $\underline{K}^{mod}$ as

$$\underline{K}^{mod} = \begin{bmatrix} \boxed{} \end{bmatrix}$$

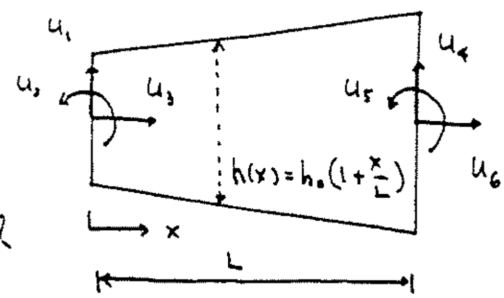
blocked elements in $\underline{K}^*$.

(b) Let $k \gg a'_{55}$ we have $\underline{K}^{mod}$ and $\underline{R}^{mod}$ as

$$\underline{K}^{mod} = \underline{K}^* + \begin{bmatrix} k & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix} = \begin{bmatrix} a'_{55} + k & a'_{56} & \cdots \\ a'_{65} & a'_{66} & \cdots \\ \vdots & & \end{bmatrix}$$

$$\underline{R}^{mod^T} = [\, k V_0 = 0 \underbrace{\cdots\cdots\cdots}_{\text{the same}} \,]$$

4.17 D.E. for the beam :



(a) $$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) = 0$$

Integrating the governing differential equation,

$$\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right) = C_1$$

$$EI\frac{d^2w}{dx^2} = C_1x + C_2 \quad \text{or} \quad \frac{d^2w}{dx^2} = \frac{12}{Eh_o^3}\left[\frac{C_1L}{(1+\frac{x}{L})^2} + \frac{C_2 - C_1L}{(1+\frac{x}{L})^3}\right]$$

$$\frac{dw}{dx} = \frac{12}{Eh_o^3}\left[\left\{-\frac{L^2}{(1+\frac{x}{L})} + \frac{L^2}{2}\frac{1}{(1+\frac{x}{L})^2}\right\}C_1 - \left\{\frac{L}{2}\frac{1}{(1+\frac{x}{L})^2}\right\}C_2 + C_3\right]$$

$$w = \frac{12}{Eh_o^3}\left[\left\{-L^3\ln(1+\frac{x}{L}) - \frac{L^3}{2}\frac{1}{(1+\frac{x}{L})}\right\}C_1 + \left\{\frac{L^2}{2}\frac{1}{(1+\frac{x}{L})}\right\}C_2 + C_3x + C_4\right]$$

$$\therefore \quad u_1 = w\big|_{x=0} = \frac{12}{Eh_o^3}\left(-\frac{L^3}{2}C_1 + \frac{L^2}{2}C_2 + C_4\right)$$

$$u_2 = \frac{dw}{dx}\bigg|_{x=0} = \frac{12}{Eh_o^3}\left(-\frac{L^2}{2}C_1 - \frac{L}{2}C_2 + C_3\right)$$

$$u_4 = w\big|_{x=L} = \frac{12}{Eh_o^3}\left\{-L^3(\ln 2 + \frac{1}{4})C_1 + \frac{L^2}{4}C_2 + C_3L + C_4\right\}$$

$$u_5 = \frac{dw}{dx}\bigg|_{x=L} = \frac{12}{Eh_o^3}\left\{-\frac{3}{8}L^2C_1 - \frac{L}{8}C_2 + C_3\right\}$$

$$\therefore \quad C_1 = \frac{Eh_o^3}{12}\frac{1}{(3\ln 2 - 2)L^2}\left(\frac{3}{L}u_1 + u_2 - \frac{3}{L}u_4 + 2u_5\right)$$

4.17

Let P the shear force acting over the cross section of the beam. Then

$$P = \frac{d}{dx}\left(EI\,\frac{d^2w}{dx^2}\right) = C_1$$

$$\therefore\; k_{11} = C_1 \Big|_{(u_1=1,\; u_2=u_4=u_5=0)} = \frac{Eh_0^3}{12}\,\frac{3}{(3\ln 2-2)L^3} = 3.15\,\frac{Eh_0^3}{L^3}$$

$$k_{12} = C_1 \Big|_{(u_1=u_4=u_5=0,\; u_2=1)} = \frac{Eh_0^3}{12}\,\frac{1}{(3\ln 2-2)L^2} = 1.05\,\frac{Eh_0^3}{L^2}$$

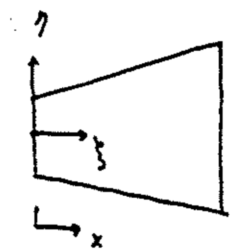
(b) $u = \underline{H}\,\hat{\underline{u}}$ where $\hat{\underline{u}}^T = [\,w_1\;\; \theta_1\;\; u_1\;\;\; w_2\;\; \theta_2\;\; u_2\,]$

$$\underline{H} = \left[\frac{6\eta}{L}\left(\frac{\xi}{L} - \frac{\xi^2}{L^2}\right) \vdots\; -\eta\left(1 - 4\frac{\xi}{L} + 3\frac{\xi^2}{L^2}\right) \vdots\; \cdots\right]$$

Neglecting the shear deformations,

$$\epsilon_{\xi\xi} = \frac{du}{d\xi}, \quad \tau_{\xi\xi} = E\,\epsilon_{\xi\xi}$$

$$\therefore\; \underline{B} = \left[\frac{6\eta}{L}\left(\frac{1}{L} - \frac{2\xi}{L^2}\right) \vdots\; -\eta\left(-\frac{4}{L} + \frac{6\xi}{L^2}\right) \vdots\; \cdots\right]$$



$$\underline{K} = E\int_0^L \int_{-h/2}^{h/2} \underline{B}^T\underline{B}\; d\eta\, d\xi \;, \quad h = h_0\left(1 + \frac{\xi}{L}\right)$$

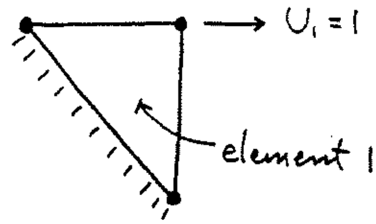
$$k_{11} = E\int_0^L \int_{-h/2}^{h/2} B_{11}^2\, d\eta\, d\xi = \frac{Eh_0^3}{12}\left(\frac{243}{5}\right)\frac{1}{L^3} = 4.05\,\frac{Eh_0^3}{L^3}$$

$$k_{12} = E\int_0^L \int_{-h/2}^{h/2} B_{11}B_{12}\, d\eta\, d\xi = \frac{Eh_0^3}{12}\left(\frac{87}{5}\right)\frac{1}{L^2} = 1.45\,\frac{Eh_0^3}{L^2}$$

Note that the difference between the two results is considerable because the finite element solution violates the internal equilibrium conditions.

4.18 One way to obtain K_{11} and K_{14} is to calculate the appropriate terms in the strain-displacement matrix and use them in evaluating the stiffnesses. Here we use a more physical approach. Note that in a 3-node triangular element the displacement has the form $a+bx+cy$, which is linear in x and y.

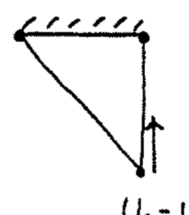
Consider K_{11}. We apply $U_1 = 1$ and calculate the corresponding force.



$$\underline{U}^T = [\, U_1 \; U_2 \; U_3 \; U_4 \; U_5 \,]$$
$$= [\, 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \,]$$
$$\underline{\epsilon}^T = [\epsilon_{xx} \;\; \epsilon_{yy} \;\; \gamma_{xy}] = [\tfrac{1}{4} \;\; 0 \;\; \tfrac{1}{4}]$$
$$\underline{\tau}^T = \frac{E}{1-\nu^2}[\, \epsilon_{xx} \;\; \nu\epsilon_{xx} \;\; \frac{1-\nu}{2}\gamma_{xy} \,]$$
$$\underline{\bar{U}}^T = [1 \;\; 0 \;\; 0 \;\; 0 \;\; 0] \;, \quad \underline{\bar{\epsilon}}^T = [\tfrac{1}{4} \;\; 0 \;\; \tfrac{1}{4}]$$

Then,
$$K_{11} = \int_A \underline{\bar{\epsilon}}^T \underline{\tau} \, (0.1)\, dA = 1.4835 \times 10^4$$

Similarly, for K_{14}, impose at node 2 the displacement $U_4 = 1$. That is,



$$\underline{U}^T = [\, 0 \;\; 0 \;\; 0 \;\; 1 \;\; 0 \,] \;, \quad \underline{\epsilon}^T = [\, 0 \;\; -\tfrac{1}{4} \;\; 0 \,] \;,$$
$$, \quad \underline{\tau}^T = \frac{E}{1-\nu^2}[\nu\epsilon_{yy} \;\; \epsilon_{yy} \;\; 0]$$

and $\underline{\bar{U}}^T = [\, 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \,] \;, \quad \underline{\bar{\epsilon}}^T = [\tfrac{1}{4} \;\; 0 \;\; \tfrac{1}{4}]$

4.18

Then, $K_{14} = \int_A \bar{\underline{\epsilon}}^T \underline{\tau} \, (0.1) \, dA = -3.2967 \times 10^3$

Note that the same results are obtained using $\int_V \underline{B}^T \underline{C} \, \underline{B} \, dV$.

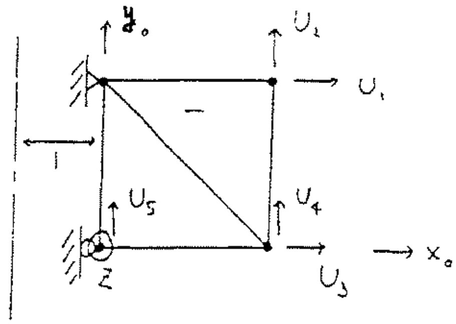
Now,

$$R_s\Big|_{U_4} = \int_0^4 P\left(1-\frac{x}{4}\right) \cdot \frac{x}{4} \, dx \, (0.1) = 0.067 \, P$$

$$R_s\Big|_{U_5} = \int_0^4 P\left(1-\frac{x}{4}\right) \cdot \left(1-\frac{x}{4}\right) dx \, (0.1) = 0.133 \, P.$$

and $R_s\Big|_{U_4} + R_s\Big|_{U_5} = 0.2 \, P$, which is the total applied force.

4.19

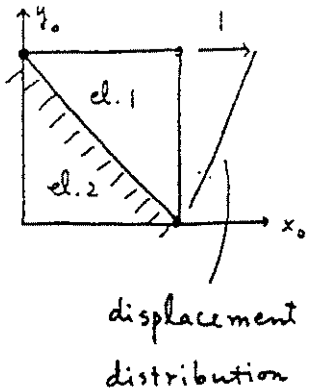


We again follow the method as in exercise 4.18.

Here $\underset{\sim}{\epsilon}^T = [\ \epsilon_{xx}\ \epsilon_{yy}\ \gamma_{xy}\ \epsilon_{zz}\ ]$

$$\underset{\sim}{C} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0 & \frac{\nu}{1-\nu} \\ \frac{\nu}{1-\nu} & 1 & 0 & \frac{\nu}{1-\nu} \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 1 \end{bmatrix}$$

For K_{11}, $\underset{\sim}{U}^T = [\ 1\ 0\ 0\ 0\ 0\ ]$



$$u = -1 + \frac{x_0}{4} + \frac{y_0}{4}$$

$$\therefore\ \underset{\sim}{\epsilon} = \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \\ \epsilon_{zz} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ 0 \\ \frac{1}{4} \\ \left(-1 + \frac{x_0}{4} + \frac{y_0}{4}\right) / (1+x_0) \end{bmatrix}$$

and $\bar{\underset{\sim}{U}}^T = [\ 1\ 0\ 0\ 0\ 0\ ] = \underset{\sim}{U}^T$

$$\therefore\ \underset{\sim}{K}_{11} = \int_V \bar{\underset{\sim}{\epsilon}}^T \underset{\sim}{C}\, dV = \int_0^4 \int_{4-y_0}^4 \left[\frac{1}{4}\ \ 0\ \ \frac{1}{4}\ \ \frac{-1+\frac{x_0}{4}+\frac{y_0}{4}}{1+x_0} \right] \underset{\sim}{C} \begin{bmatrix} \frac{1}{4} \\ 0 \\ \frac{1}{4} \\ \frac{-1+\frac{x_0}{4}+\frac{y_0}{4}}{1+x_0} \end{bmatrix} (1+x_0)\, dx_0\, dy_0$$

$$= 876656.$$

Similarly for K_{14}, consider $\underset{\sim}{U}^T = [\ 0\ 0\ 0\ 1\ 0\ ]$, $v = 1 - \frac{y_0}{4}$

Then $\underset{\sim}{\epsilon}^T = [\ 0\ \ -\frac{1}{4}\ \ 0\ \ 0\ ]$ and $\bar{\underset{\sim}{U}}^T = [\ 1\ 0\ 0\ 0\ 0\ ]$

4.19

$$\therefore K_{14} = \int_V \bar{\underline{\epsilon}}^T \underline{\tau}\, dV = \int_0^4 \int_{4-y_0}^4 \left[\frac{1}{4} \quad 0 \quad \frac{1}{4} \quad \frac{-1+\frac{x_0}{4}+\frac{y_0}{4}}{1+x} \right] \underline{C} \begin{bmatrix} 0 \\ -\frac{1}{4} \\ 0 \\ 0 \end{bmatrix} (1+x_0)\, dx_0\, dy_0$$

$$= -288462.$$

For the load vector,

$$R_s\Big|_{U_4} = \int_0^4 P\left(1-\frac{x_0}{4}\right)\cdot\frac{x_0}{4}\cdot(1+x_0)\,dx_0 = 2P$$

$$R_s\Big|_{U_5} = \int_0^4 P\left(1-\frac{x_0}{4}\right)\cdot\left(1-\frac{x_0}{4}\right)(1+x_0)\,dx_0 = \frac{8}{3}P$$

and $R_s\Big|_{U_4} + R_s\Big|_{U_5} = \frac{14}{3}P = \int_0^4 P\left(1-\frac{x_0}{4}\right)(1+x_0)\,dx_0$ ($\leftarrow$ total applied load)

4.20 (a) We consider the following displacement fields as the problem is a 3-dimensional one.

$$\left.\begin{aligned} u(x,y,z) &= \cos\theta\, \bar{H}\, \underline{u} \\ v(x,y,z) &= \cos\theta\, \bar{H}\, \underline{v} \\ w(x,y,z) &= \sin\theta\, \bar{H}\, \underline{w} \end{aligned}\right\} \text{ (a)} \quad \text{where } \bar{H} = [\,1 \quad x \quad y\,]\, \underline{A}^{-1}$$

and $\underline{A}^{-1}$ is given in example 4.17

$$\underline{A}^{-1} = \begin{bmatrix} x_2y_3 - x_3y_2 & x_3y_1 - x_1y_3 & x_1y_2 - x_2y_1 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix} \frac{1}{\det \underline{A}_1}$$

$$\det \underline{A}_1 = x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 1$$

then

$$\underline{A}^{-1} = \begin{bmatrix} 0 & 11 & -10 \\ 0 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

Eq. (a) can be written in the following form:

$$\begin{bmatrix} u(x,y,\theta) \\ v(x,y,\theta) \\ w(x,y,\theta) \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & & \\ & \cos\theta & \\ & & \sin\theta \end{bmatrix} \begin{bmatrix} 1\ x\ y & & \\ & 1\ x\ y & \\ & & 1\ x\ y \end{bmatrix} \begin{bmatrix} \underline{A}^{-1} & \underline{0} & \underline{0} \\ \underline{0} & \underline{A}^{-1} & \underline{0} \\ \underline{0} & \underline{0} & \underline{A}^{-1} \end{bmatrix}}_{= \underline{H}} \underline{U} \quad \text{—(b)}$$

with $\underline{U}^T = [\,u_1\ u_2\ u_3\ v_1\ v_2\ v_3\ w_1\ w_2\ w_3\,]$

or $\underline{u} = \underline{H}\,\underline{U}$

4.20

Considering six components of strain tensor,

$$\underline{\varepsilon}^T = \left[\frac{\partial u}{\partial x} \quad \frac{\partial v}{\partial y} \quad \frac{u}{x} + \frac{1}{x}\frac{\partial w}{\partial \theta} \quad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad \frac{\partial w}{\partial y} + \frac{1}{x}\frac{\partial v}{\partial \theta} \quad \frac{\partial w}{\partial x} + \frac{1}{x}\frac{\partial u}{\partial \theta} - \frac{w}{x} \right]$$

Then using the relation in (b),

$$\underline{\varepsilon} = \underline{B}\,\underline{U} \quad \text{where} \quad \underline{B} = \underline{\Theta}\,\underline{\bar{B}}\,\underline{\bar{A}}^{-1}.$$

$$\underline{\Theta} = \begin{bmatrix} \cos\theta & & & & & \\ & \cos\theta & & & \underline{0} & \\ & & \cos\theta & & & \\ & \underline{0} & & \cos\theta & & \\ & & & & \sin\theta & \\ & & & & & \sin\theta \end{bmatrix}, \quad \underline{\bar{A}}^{-1} = \begin{bmatrix} \underline{A}^{-1} & & \underline{0} \\ & \underline{A}^{-1} & \\ \underline{0} & & \underline{A}^{-1} \end{bmatrix}$$

$$\underline{\bar{B}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{x} & 1 & \frac{y}{x} & 0 & 0 & 0 & \frac{1}{x} & 1 & \frac{y}{x} \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{x} & -1 & -\frac{y}{x} & 0 & 0 & 1 \\ -\frac{1}{x} & -1 & -\frac{y}{x} & 0 & 0 & 0 & -\frac{1}{x} & 0 & -\frac{y}{x} \end{bmatrix}$$

Hence, the stiffness matrix $\underline{K}$ is

$$\underline{K} = \int_V \underline{B}^T \underline{C}\,\underline{B}\; x\,dx\,dy\,d\theta$$

where $\underline{C}$ is the 3-dimensional constitutive matrix.

$$\underline{C} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & & & \\ \frac{\nu}{1-\nu} & 1 & \frac{\nu}{1-\nu} & & \underline{0} & \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 1 & & & \\ & & & \frac{1-2\nu}{2(1-\nu)} & & \\ & \underline{0} & & & \frac{1-2\nu}{2(1-\nu)} & \\ & & & & & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix}$$

4.20

The mass matrix $\underline{M} = \rho \int_V \underline{H}^T \underline{H} \, x \, dx\, dy\, d\theta$ where $\underline{H}$ is given in (b)

And load vector

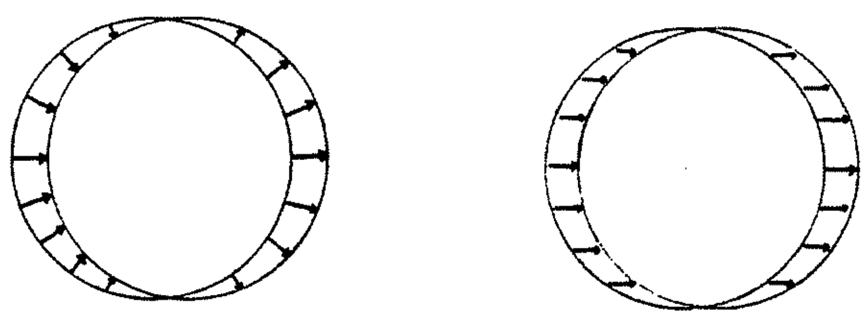
$$\underline{R}_S = \int_0^{2\pi} \int_0^1 \underline{H}^{S^T} \underline{f}^S \, 10 \, dy \, d\theta$$

where

$$\underline{H}^S = \begin{bmatrix} \cos\theta & & \\ & \cos\theta & \\ & & \sin\theta \end{bmatrix} \begin{bmatrix} 1 & 10 & y & & & & & & \\ & & & 1 & 10 & y & & & \\ & & & & & & 1 & 10 & y \end{bmatrix} \begin{bmatrix} \underline{A}^{-1} & & \\ & \underline{A}^{-1} & \\ & & \underline{A}^{-1} \end{bmatrix}$$

$$\underline{f}^S = \begin{bmatrix} f_1(t)\cos\theta \\ 0 \\ 0 \end{bmatrix}$$

(b)



applied load distribution

due to symmetry, the u-displacement will be as shown above.

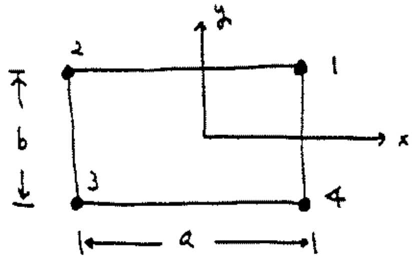
We see that at $\theta = 0$ and $\theta = \pi$ $\quad w = 0, \; u \neq 0$

at $\theta = \frac{\pi}{2}$ and $\theta = \frac{3}{2}\pi$ $\quad w \neq 0, \; u = 0$

The term $\cos\theta$ is a ~~good~~ choice for u and $\sin\theta$ for w.

4.21 $\Pi = U - W$ where $U = \frac{p^2}{2\beta}$

$$\therefore \delta U = \frac{p}{\beta}\,\delta p = \frac{p}{\beta}(-\beta\,\delta\epsilon_v) = -p\,\delta\epsilon_v = \beta\,\epsilon_v\,\delta\epsilon_v$$



$$h_1 = \frac{1}{4}\left(1+\frac{2x}{a}\right)\left(1+\frac{2y}{b}\right)$$

$$h_2 = \frac{1}{4}\left(1-\frac{2x}{a}\right)\left(1+\frac{2y}{b}\right)$$

$$h_3 = \frac{1}{4}\left(1-\frac{2x}{a}\right)\left(1-\frac{2y}{b}\right)$$

$$h_4 = \frac{1}{4}\left(1+\frac{2x}{a}\right)\left(1-\frac{2y}{b}\right)$$

$$u = [\,h_1 \;\; h_2 \;\; h_3 \;\; h_4\,]\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}, \quad v = [\,h_1 \;\; h_2 \;\; h_3 \;\; h_4\,]\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

$$\epsilon_{xx} = \frac{\partial u}{\partial x} = [\,h_{1,x} \;\; 0 \;\; h_{2,x} \;\; 0 \;\; h_{3,x} \;\; 0 \;\; h_{4,x} \;\; 0\,]\,\underline{U}$$

$$\epsilon_{yy} = \frac{\partial v}{\partial y} = [\,0 \;\; h_{1,y} \;\; 0 \;\; h_{2,y} \;\; 0 \;\; h_{3,y} \;\; 0 \;\; h_{4,y}\,]\,\underline{U}$$

where $\underline{U}^T = [\,u_1 \;\; v_1 \;\; u_2 \;\; v_2 \;\; u_3 \;\; v_3 \;\; u_4 \;\; v_4\,]$

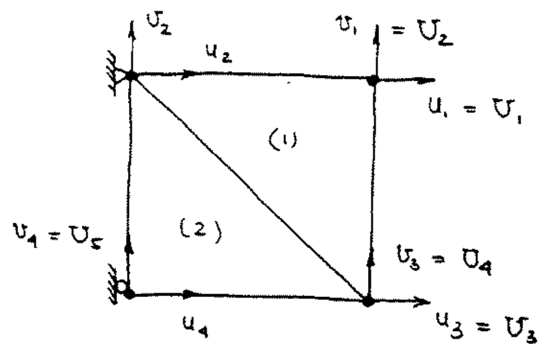
As $\epsilon_v = \epsilon_{xx} + \epsilon_{yy}$,

$$\therefore \epsilon_v \equiv B_{\epsilon_v}\underline{U} = [\,h_{1,x} \;\; h_{1,y} \;\; h_{2,x} \;\; h_{2,y} \;\; h_{3,x} \;\; h_{3,y} \;\; h_{4,x} \;\; h_{4,y}\,]\,\underline{U}$$

Hence

$$\underline{K} = t\int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{-\frac{a}{2}}^{\frac{a}{2}} \beta\,\underline{B}_{\epsilon_v}^T\,\underline{B}_{\epsilon_v}\;dx\,dy$$

4.22



Plane stress case:

element (1):

The total mass:

$$M^{(1)} = 0.1\rho \int_0^4 \int_{4-y}^4 dx\,dy = \frac{4}{5}\rho$$

Then the lumped nodal mass $m_n^{(1)} = \frac{1}{3} M^{(1)} = \frac{4}{15}\rho$, and the lumped mass matrix is

$$\underline{M}^{(1)} = \frac{4}{15}\rho\, \underline{I}_6 = \frac{4}{15}\rho \begin{bmatrix} 1 & & & & & \\ & 1 & & & \underline{0} & \\ & & 1 & & & \\ & & & 1 & & \\ & \underline{0} & & & 1 & \\ & & & & & 1 \end{bmatrix}$$

$$\begin{matrix} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 \\ | & | & & & | & | \\ V_1 & V_2 & & & V_3 & V_4 \end{matrix}$$

Similarly, the lumped mass matrix for the second element is:

$$\underline{M}^{(2)} = \frac{4}{15}\rho\, \underline{I}_6 = \frac{4}{15}\rho \begin{bmatrix} 1 & & & & & \\ & 1 & & & \underline{0} & \\ & & 1 & & & \\ & \underline{0} & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}$$

$$\begin{matrix} u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \\ & & | & | & & | \\ & & V_3 & V_4 & & V_5 \end{matrix}$$

4.22

The lumped mass matrix of the assemblage:

$$\underline{M} = \frac{4}{15}\rho \begin{bmatrix} 1 & & & & \\ & 1 & & \underline{0} & \\ & & 2 & & \\ & \underline{0} & & 2 & \\ & & & & 1 \end{bmatrix}$$

$U_1 \quad U_2 \quad U_3 \quad U_4 \quad U_5$

The lumped load vector: $\underline{R} = \underline{R}^{(2)} = 0.1\,p \int_0^4 \left(1 - \frac{x}{4}\right) \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} dx$:

$$\Rightarrow \quad \underline{R} = 0.1\,p \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \end{bmatrix}^T$$

$U_1 \quad U_2 \quad U_3 \quad U_4 \quad U_5$

Axisymmetric case

$$M^{(1)} = \rho \int_0^4 \int_{4-y}^4 (1+x)\, dx\, dy = \frac{88}{3}\rho \;; \quad m_n^{(1)} = \frac{88}{9}\rho$$

$$M^{(2)} = \rho \int_0^4 \int_0^{4-y} (1+x)\, dx\, dy = \frac{56}{3}\rho \;; \quad m_n^{(2)} = \frac{56}{9}\rho$$

$$\underline{M} = \frac{1}{9}\rho \begin{bmatrix} 88 & & & & \\ & 88 & & & \\ & & 144 & & \\ & & & 144 & \\ & & & & 56 \end{bmatrix}$$

$U_1 \quad U_2 \quad U_3 \quad U_4 \quad U_5$

4.22

The lumped load vector

$$\underline{R} = \underline{R}^{(2)} = p \int_0^4 \left(1 - \frac{x}{4}\right) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/2 \\ 1/2 \end{bmatrix} (1+x)\, dx = \frac{2}{3} p \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Note that all matrices are obtained per unit radian.

4.26

Let $\underline{e}_h = \underline{u} - \underline{u}_h$ and $a(\cdot\,,\cdot)$ be the bilinear form for the strain energy.

Then, $a(\underline{u},\underline{u}) = a(\underline{u}_h,\underline{u}_h) + 2a(\underline{u}_h,\underline{e}_h) + a(\underline{e}_h,\underline{e}_h)$

$\therefore\ a(\underline{e}_h,\underline{e}_h) = a(\underline{u},\underline{u}) - a(\underline{u}_h,\underline{u}_h)$

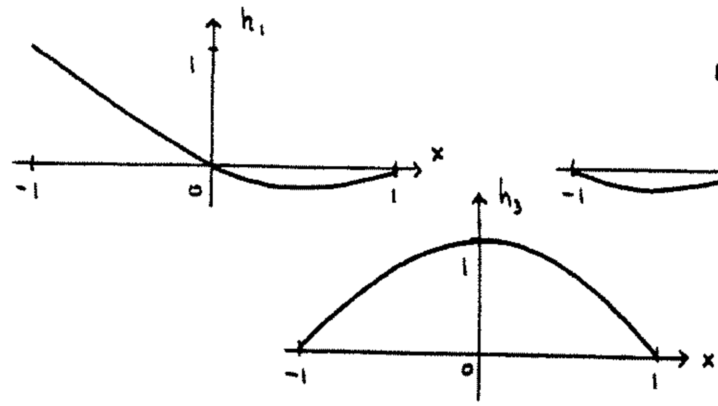
4.29 **Case i**

$$h_i = \begin{cases} 1 & \text{at node } i\,, \quad i = 1, 2, 3 \\ 0 & \text{at node } j \neq i \end{cases}$$

$$h_1 = \frac{1}{2}(1-x) - \frac{1}{2}(1-x^2) = -\frac{1}{2}(1-x)x$$

$$h_2 = \frac{1}{2}(1+x) - \frac{1}{2}(1-x^2) = \frac{1}{2}(1+x)x$$

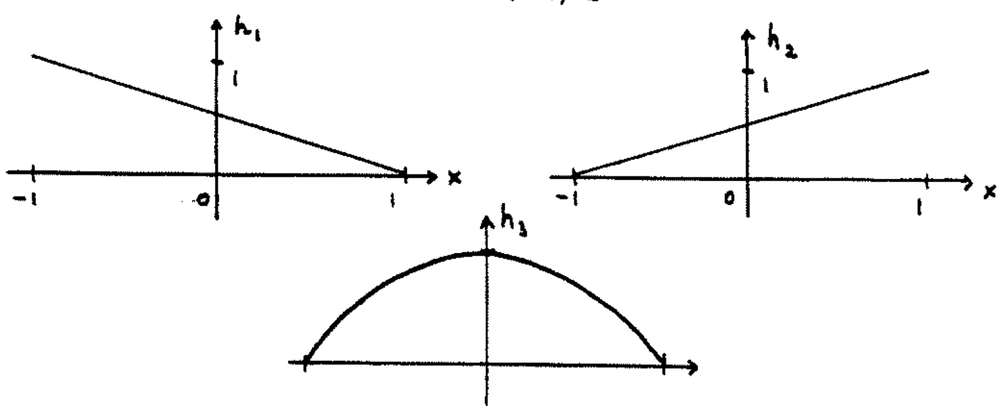
$$h_3 = 1 - x^2$$



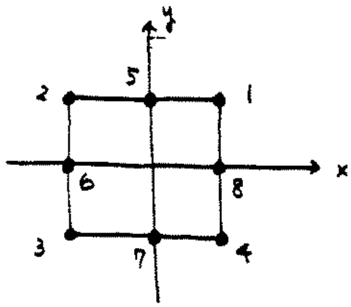
Case ii

$$h_i = \begin{cases} 1 & \text{at node } i\,, \quad i = 1, 2 \\ 0 & \text{at node } j \neq i\,, \quad j = 1, 2 \end{cases} \qquad \therefore \quad h_1 = \frac{1}{2}(1-x), \quad h_2 = \frac{1}{2}(1+x)$$

$$h_3 = \begin{cases} 1 & \text{at node } 3 \\ 0 & \text{at node } 1, 2 \end{cases} \qquad \therefore \quad h_3 = 1 - x^2$$



4.30



$$h_1 = \frac{1}{4}(1+x)(1+y), \quad h_2 = \frac{1}{4}(1-x)(1+y)$$

$$h_3 = \frac{1}{4}(1-x)(1-y), \quad h_4 = \frac{1}{4}(1+x)(1-y)$$

$$h_5 = \frac{1}{2}(1+y)\,\phi_2(x), \quad h_6 = \frac{1}{2}(1-x)\,\phi_2(y)$$

$$h_7 = \frac{1}{2}(1-y)\,\phi_2(x), \quad h_8 = \frac{1}{2}(1+x)\,\phi_2(y)$$

and $\phi_j = \dfrac{1}{\sqrt{2(2j-1)}}\left[\,P_j(x) - P_{j-2}(x)\,\right]$

(P_j : Legendre polynomials)

$$\therefore\ \phi_2 = \frac{1}{\sqrt{6}}\left[\frac{1}{2}(3x^2-1) - 1\right]$$

The displacement field u obtained by h_i's is

$$u = \sum_{i=1}^{8} h_i u_i = \left\{\frac{u_1+u_2+u_3+u_4}{4} - \sqrt{\frac{3}{32}}(u_5+u_6+u_7+u_8)\right\}\cdot 1$$

$$\left\{\frac{u_1-u_2-u_3+u_4}{4} + \sqrt{\frac{3}{32}}(u_6-u_8)\right\}\cdot x + \left\{\frac{u_1+u_2-u_3-u_4}{4} + \sqrt{\frac{3}{32}}(-u_5+u_7)\right\}\cdot y$$

$$+\sqrt{\frac{3}{32}}(u_5+u_7)\cdot x^2 \quad + \frac{u_1-u_2+u_3-u_4}{4}\,xy \; + \sqrt{\frac{3}{32}}(u_6+u_8)\cdot y^2$$

$$+ \sqrt{\frac{3}{32}}(u_5-u_7)\cdot x^2y + \sqrt{\frac{3}{32}}(-u_6+u_8)\cdot xy^2$$

Hence, the Pascal triangle is

$$\begin{array}{ccccc} & & 1 & & \\ & x & & y & \\ x^2 & & xy & & y^2 \\ & x^2y & & xy^2 & \end{array}$$

4.31 $\phi_j(x) = \frac{1}{[2(2j-1)]^{1/2}} \{ P_j(x) - P_{j-2}(x) \}$

side 1 $h_5 = \frac{1}{2}(1+y)\phi_2(x) = \frac{1}{2}(1+y)\left[\frac{1}{\sqrt{6}} \{ \frac{1}{2}(3x^2-1) - 1 \} \right]$

$h_9 = \frac{1}{2}(1+y)\phi_3(x) = \frac{1}{2}(1+y)\left[\frac{1}{\sqrt{10}} \{ \frac{1}{2}(5x^3-3x) - x \} \right]$

$h_{13} = \frac{1}{2}(1+y)\phi_4(x) = \frac{1}{2}(1+y)\left[\frac{1}{\sqrt{14}} \{ \frac{1}{8}(35x^4-30x^2+3) - \frac{1}{2}(3x^2-1) \} \right]$

side 2 $h_6 = \frac{1}{2}(1-x)\phi_2(y)$, $h_{10} = \frac{1}{2}(1-x)\phi_3(y)$, $h_{14} = \frac{1}{2}(1-x)\phi_4(y)$

side 3 $h_7 = \frac{1}{2}(1-y)\phi_2(x)$, $h_{11} = \frac{1}{2}(1-y)\phi_3(x)$, $h_{15} = \frac{1}{2}(1-y)\phi_4(x)$

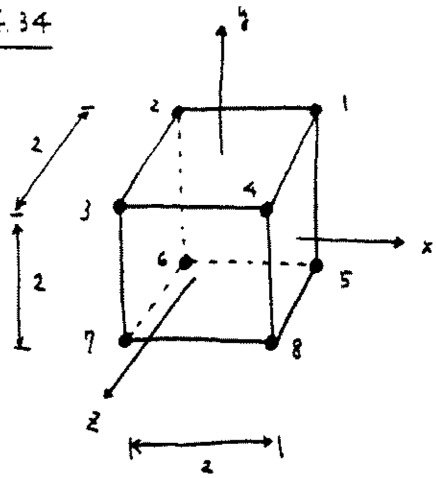
side 4 $h_8 = \frac{1}{2}(1+x)\phi_2(y)$, $h_{12} = \frac{1}{2}(1+x)\phi_3(y)$, $h_{16} = \frac{1}{2}(1+x)\phi_4(y)$

and also $h_{17} = (1-x^2)(1-y^2)$

Similarly as done in exercise 4.30, the Pascal triangle is given by

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & x & & y & & & \\
 & & x^2 & & xy & & y^2 & & \\
 & x^3 & & x^2y & & xy^2 & & y^3 & \\
x^4 & & x^3y & & x^2y^2 & & xy^3 & & y^4 \\
 & x^4y & & & & & & xy^4 &
\end{array}
$$

4.34



$$u = \sum_i h_i u_i + \alpha_1\phi_1 + \alpha_2\phi_2 + \alpha_3\phi_3$$

$$v = \sum_i h_i v_i + \alpha_4\phi_1 + \alpha_5\phi_2 + \alpha_6\phi_3$$

$$w = \sum_i h_i w_i + \alpha_7\phi_1 + \alpha_8\phi_2 + \alpha_9\phi_3$$

where $h_i = \frac{1}{8}(1+x_i x)(1+y_i y)(1+z_i z)$

$$\phi_1 = 1 - x^2, \quad \phi_2 = 1 - y^2, \quad \phi_3 = 1 - z^2$$

Let $\underline{u}^* = \begin{bmatrix} \hat{\underline{u}} \\ \underline{\alpha} \end{bmatrix}$ where $\hat{\underline{u}}^T = [\, u_1 \cdots u_8 \; v_1 \cdots v_8 \cdots w_1 \cdots w_8 \,]$

$\underline{\alpha}^T = [\, \alpha_1 \; \alpha_2 \; \cdots \; \alpha_9 \,]$

Then $\underline{\varepsilon} = [\, \underline{B} \; \vdots \; \underline{B}_{IC} \,] \begin{bmatrix} \hat{\underline{u}} \\ \underline{\alpha} \end{bmatrix}$

where $\underline{B}$ is the usual strain-displacement matrix of the 8-node element and $\underline{B}_{IC}$ is the contribution due to the incompatible modes.

Hence with our usual notation,

$$\begin{bmatrix} \int_V \underline{B}^T \underline{C} \underline{B} \, dV & \int \underline{B}^T \underline{C} \underline{B}_{IC} \, dV \\ \int_V \underline{B}_{IC}^T \underline{C} \underline{B} \, dV & \int_V \underline{B}_{IC}^T \underline{C} \underline{B}_{IC} \, dV \end{bmatrix} \begin{bmatrix} \hat{\underline{u}} \\ \underline{\alpha} \end{bmatrix} = \begin{bmatrix} \underline{R} \\ \underline{0} \end{bmatrix}$$

And the condition as in 2-D analysis is $\int_V \underline{B}_{IC}^T \, dV = 0$ —— (*)

We can easily check that the condition (*) is satified for the undistorted brick element, i.e.,

4.34

$$\int_V \begin{bmatrix} -2x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2z \\ 0 & -2y & 0 & -2x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2z & 0 & -2y & 0 \\ 0 & 0 & -2z & 0 & 0 & 0 & -2x & 0 & 0 \end{bmatrix} dV = \underline{0} \, .$$

However we can also check that the condition (*) is not satisfied for the general 3-D element (due to the Jacobian transformation). In order to satisfy the condition we modify $\underline{B}_{IC}$ by a correction $\underline{B}_{IC}^{c}$ as

$$\underline{B}_{IC}^{new} = \underline{B}_{IC} + \underline{B}_{IC}^{c}$$

$$\therefore \quad \underline{B}_{IC}^{c} = -\frac{1}{V}\int_V \underline{B}_{IC}\, dV$$

Note that the patch test is then passed using $\underline{B}_{IC}^{new}$ instead of $\underline{B}_{IC}$.

4.35 $\Pi_{HW} = \Pi - \int_V \underline{\tau}^T(\underline{\varepsilon} - \underline{\partial}_\varepsilon \underline{u})\, dV - \int_{S_u} \underline{f}^{S_u T}(\underline{u}^{S_u} - \underline{u}_p)\, dS$

where $\Pi = \frac{1}{2}\int_V \underline{\varepsilon}^T \underline{C}\, \underline{\varepsilon}\, dV - \int_V \underline{u}^T \underline{f}^B dV - \int_{S_f} \underline{u}^{S_f T} \underline{f}^{S_f} dS$

$\delta \Pi_{HW} = 0$ gives

$$\left[\int_V \delta\underline{\varepsilon}^T \underline{C}\,\underline{\varepsilon}\, dV - \int_V \delta\underline{u}^T \underline{f}^B dV - \int_{S_f} \delta\underline{u}^{S_f T}\underline{f}^{S_f} dS\right]$$
$$-\left[\int_V \delta\underline{\tau}^T(\underline{\varepsilon} - \underline{\partial}_\varepsilon \underline{u})dV + \int_V \underline{\tau}^T(\delta\underline{\varepsilon} - \underline{\partial}_\varepsilon \delta\underline{u})\, dV\right]$$
$$-\left[\int_{S_u} \delta\underline{f}^{S_u T}(\underline{u}^{S_u} - \underline{u}_p)dS + \int_{S_u} \underline{f}^{S_u T}\delta\underline{u}^{S_u} dS\right] = 0$$

$$\therefore \int_V \left[\delta\underline{\varepsilon}^T(\underline{C}\,\underline{\varepsilon} - \underline{\tau})\, dV - \delta\underline{\tau}^T(\underline{\varepsilon} - \underline{\partial}_\varepsilon \underline{u}) + \underline{\underline{\tau}^T \underline{\partial}_\varepsilon \delta\underline{u}} - \delta\underline{u}^T \underline{f}^B\right] dV$$
$$- \int_{S_f} \delta\underline{u}^{S_f T}\underline{f}^{S_f} dS$$
$$- \int_{S_u} \left[\delta\underline{f}^{S_u T}(\underline{u}^{S_u} - \underline{u}_p) + \underline{f}^{S_u T}\delta u^{S_u}\right] dS = 0$$

Using integration by parts and the divergence theorem (see also Example 4.2),

$$\int_V \underline{\tau}^T \underline{\partial}_\varepsilon \delta\underline{u}\, dV = -\int_V \delta\underline{u}^T(\underline{\partial}_\varepsilon^T \underline{\tau})\, dV + \int_{S_u + S_f} \delta\underline{u}^T(\underline{\tau}\,\underline{n})\, dS$$

Finally, we have

$$\int_V \left[\partial\underline{\varepsilon}^T(\underline{C}\,\underline{\varepsilon} - \underline{\tau})\, dV - \delta\underline{\tau}^T(\underline{\varepsilon} - \underline{\partial}_\varepsilon \underline{u}) - \delta\underline{u}^T\{(\underline{\partial}_\varepsilon^T \underline{\tau}) + \underline{f}^B\}\right] dV$$
$$- \int_{S_f} \delta\underline{u}^{S_f T}(f^{S_f} - \underline{\tau}\,\underline{n})\, dS$$
$$- \int_{S_u} \left[\delta\underline{f}^{S_u T}(\underline{u}^{S_u} - \underline{u}_p) + (\underline{f}^{S_u} - \underline{\tau}\,\underline{n})^T \delta\underline{u}^{S_n}\right] dS = 0$$

4.35

Hence for the volume of the body,

$$\underline{\tau} = \underline{C}\,\underline{\varepsilon} \quad \text{(stress-strain law)}$$

$$\underline{\varepsilon} = \underline{\partial}_{\varepsilon}\,\underline{u} \quad \text{(compatibility condition)}$$

$$\underline{\partial}_{\varepsilon}^{T}\,\underline{\tau} + \underline{f}^{B} = \underline{0} \quad \text{(equilibrium equation)}$$

and for the surface of the body,

$$\underline{f}^{S_f} = \bar{\underline{\tau}}\,\underline{n} \quad \text{on } S_f \quad \text{(prescribed tractions)}$$

$$\underline{f}^{S_u} = \bar{\underline{\tau}}\,\underline{n} \quad \text{on } S_u \quad \text{(boundary equilibrium condition)}$$

$$\underline{u}^{S_u} = \underline{u}_p \quad \text{on } S_u \quad \text{(prescribed displacements)}$$

4.36 $\Pi_{HW}(\underline{u}, \underline{\varepsilon}, \underline{\tau}) = \Pi - \int_V \underline{\tau}^T(\underline{\varepsilon} - \underline{\partial}_\varepsilon \underline{u})\, dV - \int_{S_u} \underline{f}^{S_u T}(\underline{u}^{S_u} - \underline{u}_p)\, dS$

where $\Pi = \frac{1}{2}\int_V \underline{\varepsilon}^T \underline{C}\, \underline{\varepsilon}\, dV - \int_V \underline{u}^T \underline{f}^B dV - \int_{S_f} \underline{u}^{S_f T} \underline{f}^{S_f} dS$

Let $\underline{\varepsilon} = \underline{C}^{-1}\underline{\tau}$, then

$$\Pi_{HW} = \left[\frac{1}{2}\int_V \underline{\tau}^T \underline{C}^{-1} \underline{\tau}\, dV - \int_V \underline{u}^T \underline{f}^B dV - \int_{S_f} \underline{u}^{S_f T} \underline{f}^{S_f} dS \right]$$
$$- \int_V \underline{\tau}^T(\underline{C}^{-1}\underline{\tau} - \underline{\partial}_\varepsilon \underline{u})\, dV - \int_{S_u} \underline{f}^{S_u T}(\underline{u}^{S_u} - \underline{u}_p)\, dS$$

$$\therefore \Pi_{HR}(\underline{u}, \underline{\tau}) = \int_V \left(-\frac{1}{2}\underline{\tau}^T \underline{C}^{-1}\underline{\tau}\right) dV + \int_V \underline{\tau}^T \underline{\partial}_\varepsilon \underline{u}\, dV - \int_V \underline{u}^T \underline{f}^B dV$$
$$- \int_{S_f} \underline{u}^{S_f T} \underline{f}^{S_f} dS - \int_{S_u} \underline{f}^{S_u T}(\underline{u}^{S_u} - \underline{u}_p)\, dS$$

Invoke the stationarity of Π_{HR} with the divergence theorem.

$$\delta\Pi_{HR} = \int_V \left[\delta\underline{\tau}^T(-\underline{C}^{-1}\underline{\tau} + \underline{\partial}_\varepsilon \underline{u}) - \delta\underline{u}^T(\underline{\partial}_\varepsilon^T \underline{\tau} + \underline{f}^B)\right] dV$$
$$- \int_{S_f} \delta\underline{u}^{S_f T}\{\underline{f}^{S_f} - (\underline{\tau}\,\underline{n})\}\, dS - \int_{S_u}\left[\delta\underline{f}^{S_u T}(\underline{u}^{S_u} - \underline{u}_p)\right.$$
$$\left. + \delta\underline{u}^{S_u T}\{\underline{f}^{S_u} - (\underline{\tau}\,\underline{n})\}\right].$$

Hence for the volume of the body,

Stress-displacement relation	$\underline{C}^{-1}\underline{\tau} = \underline{\partial}_\varepsilon \underline{u}$
equilibrium eq.	$\underline{\partial}_\varepsilon^T \underline{\tau} + \underline{f}^B = \underline{0}$

and for the surface of the body,

prescribed tractions	$\underline{f}^{S_f} = \underline{\bar{\tau}}\,\underline{n}$	on S_f
boundary equilibrium	$\underline{f}^{S_u} = \underline{\bar{\tau}}\,\underline{n}$	on S_u
prescribed displacements	$\underline{u}^{S_u} = \underline{u}_p$	on S_u

4.37 $\Pi_1 = \Pi - \int_{S_u} \underline{f}^{S_u T}(\underline{u}^{S_u} - \underline{u}_p)\, dS$

where $\Pi = \frac{1}{2}\int_V \underline{\varepsilon}^T \underline{C}\, \underline{\varepsilon}\, dV - \int_V \underline{u}^T \underline{f}^B dV - \int_{S_f} \underline{u}^{S_f T} \underline{f}^{S_f} dS$

$$\delta\Pi_1 = \delta\Pi - \left[\int_{S_u} \delta \underline{f}^{S_u T}(\underline{u}^{S_u} - \underline{u}_p)\, dS + \int_{S_u} \underline{f}^{S_u T} \delta \underline{u}^{S_u} dS \right] = 0$$

note $\delta\Pi = \int_V \delta\underline{\varepsilon}^T \underline{C}\, \underline{\varepsilon}\, dV - \int_V \delta\underline{u}^T \underline{f}^B dV - \int_{S_f} \delta \underline{u}^{S_f T} \underline{f}^{S_f} dS$

and with $\underline{\tau} = \underline{C}\,\underline{\varepsilon}$, $\underline{\varepsilon} = \underline{\partial}_\varepsilon \underline{u}$

$$\int_V \delta\underline{\varepsilon}^T \underline{C}\, \underline{\varepsilon}\, dV = \int_V \underline{\tau}^T \delta\, \underline{\partial}_\varepsilon \underline{u}\, dV$$

$$= - \int_V (\underline{\partial}_\varepsilon^T \underline{\tau})^T \delta\underline{u}\, dV + \int_{S_u + S_f} (\underline{\tau}\, \underline{n})^T \delta \underline{u}\, dS$$

$$\therefore \delta\Pi_1 = \int_V (\underline{\partial}_\varepsilon^T \underline{\tau} + \underline{f}^B)^T \delta \underline{u}\, dV - \int_{S_f} (\underline{\tau}\,\underline{n} - \underline{f}^{S_f})^T \delta \underline{u}^{S_f} dS$$

$$- \int_{S_u} \left[(\underline{\tau}\,\underline{n} - \underline{f}^{S_u})^T \delta\underline{u}^{S_u} + (\underline{u}^{S_u} - \underline{u}_p)^T \delta \underline{f}^{S_u} \right] dS = 0$$

$\Longrightarrow$ $\underline{\partial}_\varepsilon^T \underline{\tau} + \underline{f}^B = 0$ in V

$\bar{\underline{\tau}}\, \underline{n} = \underline{f}^{S_f}$ on S_f

$\bar{\underline{\tau}}\, \underline{n} = \underline{f}^{S_u}$ on S_u

and $\underline{u}^{S_u} - \underline{u}_p = 0$ on S_u

Hence, the vector $\underline{f}^{S_u}$ enforces the surface displacement conditions on S_u.

4.38 (a)

$$\left.\begin{array}{l} u \rightarrow \text{parabolic} \\ \varepsilon \rightarrow \text{linear} \\ \tau \rightarrow \text{constant} \end{array}\right\} \Longrightarrow \left\{\begin{array}{ll} u = \underline{H}\,\hat{\underline{u}}, & \hat{\underline{u}}^T = [u_1 \;\; u_2 \;\; u_3] \\ \varepsilon = \underline{E}\,\hat{\underline{\varepsilon}}, & \hat{\underline{\varepsilon}}^T = [\varepsilon_1 \;\; \varepsilon_2] \\ \tau = \underline{T}\,\hat{\underline{\tau}}, & \hat{\underline{\tau}}^T = [\tau_0] \end{array}\right.$$

$$\underline{H} = \left[\frac{(1+x)x}{2} \quad -\frac{(1-x)x}{2} \quad 1-x^2 \right]$$

$$\underline{B} = \frac{\partial \underline{H}}{\partial x} = \left[\tfrac{1}{2}+x \quad -\tfrac{1}{2}+x \quad -2x \right]$$

$$\underline{E} = \left[\frac{1+x}{2} \quad \frac{1-x}{2} \right] \quad \text{and} \quad \underline{T} = [\,1\,]$$

The Stiffness matrix

$$\underline{K}_1 = \begin{bmatrix} \underline{0} & \underline{0} & \underline{K}_{u\tau} \\ \underline{0} & \underline{K}_{\varepsilon\varepsilon} & \underline{K}_{\varepsilon\tau} \\ \underline{K}_{u\tau}^T & \underline{K}_{\varepsilon\tau}^T & \underline{0} \end{bmatrix} \begin{matrix} \hat{\underline{u}} \\ \hat{\underline{\varepsilon}} \\ \hat{\underline{\tau}} \end{matrix}$$

$$\underline{K}_{u\tau} = \int_V \underline{B}^T \underline{T}\, dV = \begin{bmatrix} A \\ -A \\ 0 \end{bmatrix}, \quad \underline{K}_{\varepsilon\varepsilon} = \int_V \underline{E}^T \underline{C}\, \underline{E}\, dV = \begin{bmatrix} \frac{2EA}{3} & \frac{EA}{3} \\ \frac{EA}{3} & \frac{2EA}{3} \end{bmatrix}$$

$$\text{and} \quad \underline{K}_{\varepsilon\tau} = -\int_V \underline{E}^T \underline{T}\, dV = \begin{bmatrix} -A \\ -A \end{bmatrix}$$

From $\underline{K}_1$,

$$\left[\underline{K}_{u\tau} \left(\underline{K}_{\varepsilon\tau}^T \underline{K}_{\varepsilon\varepsilon}^{-1} \underline{K}_{\varepsilon\tau} \right)^{-1} \underline{K}_{u\tau}^T \right] \hat{\underline{u}} = \underline{R}$$

$$\therefore \quad \underline{K} = \frac{EA}{2} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} u_1 \\ u_2 \\ u_3 \end{matrix} \qquad \text{——— (1)}$$

(b) Similarly,

$$\underline{H} = \left[\frac{(1+x)x}{2} \quad -\frac{(1-x)x}{2} \quad 1-x^2 \right]$$

$$\underline{B} = \left[\tfrac{1}{2}+x \quad -\tfrac{1}{2}+x \quad -2x \right]$$

$$\underline{E} = \underline{T} = [\,1\,]$$

4.38

and $\underline{K}_{u\varepsilon} = \begin{bmatrix} A \\ -A \\ 0 \end{bmatrix}$, $\underline{K}_{\varepsilon\varepsilon} = [2EA]$, $\underline{K}_{\varepsilon\tau} = [-2A]$

Hence,
$$\underline{K} = \frac{EA}{2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} u_1 \\ u_2 \\ u_3 \end{matrix} \qquad \text{———} \quad ②$$

The stiffness matrices $\underline{k}$ in eq. ① and ② are identical and equal to the matrix of the two-node displacement-based truss element. This is due to the fact that a constant stress is assumed in each formulation. Hence, the formulations in (a) and (b) are not particularly sensible.

4.39

$$\tau_{ij} = K\varepsilon_v \delta_{ij} + 2G\varepsilon'_{ij} \quad \text{—— (a)}$$

$$\tau_{ij} = C_{ijrs}\varepsilon_{rs} \quad \text{—— (b)}$$

$$\underline{\tau} = \underline{C}\,\underline{\varepsilon} \quad \text{—— (c)}$$

Let's start from eq.(b) to show that the three equations are representing the same stress-strain law for an isotropic material.

Due to symmetry of both stress and strain tensors the double indexed system of stress and strain components is often replaced by a single system having a range of 6. Let us define

$$\tau_{11} = \sigma_1, \quad \tau_{22} = \sigma_2, \quad \tau_{33} = \sigma_3,$$

$$\tau_{12} = \tau_{21} = \sigma_4, \quad \tau_{23} = \tau_{32} = \sigma_5, \quad \tau_{31} = \tau_{13} = \sigma_6$$

and $\varepsilon_{11} = \varepsilon_1, \quad \varepsilon_{22} = \varepsilon_2, \quad \varepsilon_{33} = \varepsilon_3$

$$\varepsilon_{12} + \varepsilon_{21} = \gamma_{12}, \quad \varepsilon_{23} + \varepsilon_{32} = \gamma_{23}, \quad \varepsilon_{31} + \varepsilon_{13} = \gamma_{31}$$

Then we have $\sigma_K = C_{KM}\varepsilon_M \quad (K, M = 1, 2, \cdots, 6)$

Using $C_{ijrs} = \lambda\delta_{ij}\delta_{rs} + \mu(\delta_{ir}\delta_{js} + \delta_{is}\delta_{jr})$

$$[C_{KM}] = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & & & \\ \lambda & \lambda+2\mu & \lambda & & \underline{0} & \\ \lambda & \lambda & \lambda+2\mu & & & \\ & & & \mu & 0 & 0 \\ & \underline{0} & & 0 & \mu & 0 \\ & & & 0 & 0 & \mu \end{bmatrix}$$

Substituting $\lambda = \dfrac{E\nu}{(1+\nu)(1-2\nu)}$ and $\mu = \dfrac{E}{2(1+\nu)}$, we obtain $\underline{C}$ in Table 4.3. In matrix form eq.(b) is eq.(c), $\underline{\tau} = \underline{C}\,\underline{\varepsilon}$, where $\underline{\tau}^T = [\sigma_1\ \sigma_2\ \sigma_3\ \sigma_4\ \sigma_5\ \sigma_6]$ and $\underline{\varepsilon}^T = [\varepsilon_1\ \varepsilon_2\ \varepsilon_3\ \gamma_{12}\ \gamma_{23}\ \gamma_{31}]$

4.39

Now derive (a), using (b).

$$\tau_{kk} = (3\lambda + 2\mu)\,\varepsilon_{ii} \quad \text{(sum on } i \text{ and } k\text{)}$$

$$\therefore -3p = (3\lambda + 2\mu)\,\varepsilon_v \quad (\because \tau_{kk} = -3p,\ \varepsilon_{ii} = \varepsilon_v)$$

$$p = -\frac{3\lambda + 2\mu}{3}\,\varepsilon_v$$

As the bulk modulus is given by $p = -K\varepsilon_v$

$$K = \frac{3\lambda + 2\mu}{3} = \lambda + \frac{2}{3}\mu = \frac{E}{3(1-2\nu)}$$

$$\tau_{ij} = C_{ijrs}\varepsilon_{rs} = \lambda\delta_{ij}\delta_{rs}\varepsilon_{rs} + \mu(\delta_{ir}\delta_{js} + \delta_{is}\delta_{jr})\varepsilon_{rs}$$

$$= \lambda\delta_{ij}\varepsilon_{kk} + \mu(\varepsilon_{ij} + \varepsilon_{ji}) = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij} \quad (\varepsilon_{ij} = \varepsilon_{ji})$$

$$= \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\left(\varepsilon'_{ij} + \frac{\varepsilon_{kk}}{3}\delta_{ij}\right) \quad \left(\leftarrow \text{use } \varepsilon'_{ij} = \varepsilon_{ij} - \frac{\varepsilon_v}{3}\delta_{ij}\right)$$

$$= \left(\frac{3\lambda + 2\mu}{3}\right)\delta_{ij}\varepsilon_v + 2\mu\varepsilon'_{ij}$$

Hence $\underline{\tau_{ij} = K\varepsilon_v\delta_{ij} + 2G\varepsilon'_{ij}}$ $\left(K = \frac{3\lambda + 2\mu}{3},\ \mu = G \right)$

4.40 In order to obtain in the u/p formulation the same stiffness matrix as in the pure displacement formulation, we first calculate the pressure $p(r,s)$ from the pure displacement formulation. Then we identify which terms are present in it, and determine what interpolation should be used in the u/p formulation for the pressure interpolation.

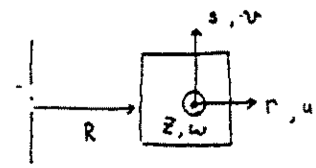
(a) 4-node, plane strain

$$p \sim \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \; ; \qquad \frac{\partial u}{\partial r} \text{ contains } 1,\ s \text{ as bases}$$

$$\frac{\partial v}{\partial s} \quad \text{"} \quad 1,\ r$$

$$\therefore\ p = p(1, r, s)$$

(b) 4-node, axisymmetric



$$p \sim \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{u}{x} \; ; \qquad \frac{\partial u}{\partial r} \text{ contains } 1,\ s$$

$$\frac{\partial v}{\partial s} \quad \text{"} \quad 1,\ r$$

$$u \quad \text{"} \quad 1,\ r,\ s,\ rs$$

$$\therefore\ p = p(1, r, s, rs)$$

(c) 9-node, plane strain

$$p \sim \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \; : \qquad \frac{\partial u}{\partial r} \text{ contains } 1,\ r,\ s,\ rs,\ s^2,\ rs^2$$

$$\frac{\partial v}{\partial s} \quad \text{"} \quad 1,\ r,\ s,\ r^2,\ rs,\ r^2 s$$

$$\therefore\ p = p(1, r, s, r^2, rs, s^2, r^2 s, rs^2)$$

4.41

$$\begin{bmatrix} \underline{K}_{uu} & \underline{K}_{up} \\ \underline{K}_{up}^T & \underline{K}_{pp} \end{bmatrix} \begin{bmatrix} \hat{\underline{u}} \\ \hat{\underline{p}} \end{bmatrix} = \begin{bmatrix} \underline{R} \\ \underline{0} \end{bmatrix} \qquad \leftarrow (4.147)$$

Now partitioning each submatrix to impose $u_1 = \bar{u}$,

$$\underline{K}_{uu} = \begin{bmatrix} K_{uu}^A & \underline{K}_{uu}^B \\ (\underline{K}_{uu}^B)^T & \underline{K}_{uu}^C \end{bmatrix} \begin{matrix} \leftarrow u_1 \\ \leftarrow \underline{U}_r \end{matrix} , \quad \underline{K}_{up} = \begin{bmatrix} \underline{K}_{up}^A \\ \underline{K}_{up}^B \end{bmatrix} , \quad \underline{R} = \begin{bmatrix} R^A \\ \underline{R}^B \end{bmatrix}$$

$$\hat{\underline{u}}^T = [\, u_1 \quad \underline{U}_r^T \,] \quad \text{with} \quad \underline{U}_r^T = [\, u_2 \; u_3 \; u_4 \; v_1 \; v_2 \; v_3 \; v_4 \,]$$

Then we have

$$K_{uu}^A u_1 + \underline{K}_{uu}^B \underline{U}_r + \underline{K}_{up}^A \hat{\underline{p}} = R^A \qquad — ①$$

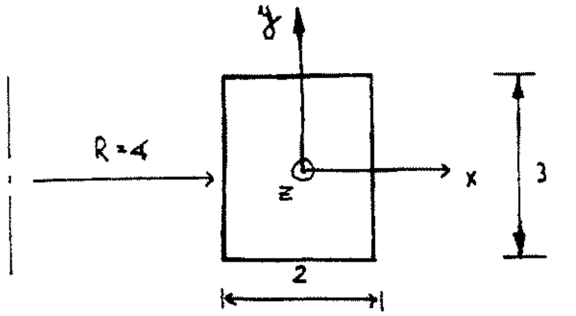
$$(\underline{K}_{uu}^B)^T u_1 + \underline{K}_{uu}^C \underline{U}_r + \underline{K}_{up}^B \hat{\underline{p}} = \underline{R}^B \qquad — ②$$

$$(\underline{K}_{up}^A)^T u_1 + (\underline{K}_{up}^B)^T \underline{U}_r + \underline{K}_{pp} \hat{\underline{p}} = \underline{0} \qquad — ③$$

Now we can obtain the element contribution of $\underline{U}_r$ to the stiffness matrix of the assemblage by eliminating u_1 and statically condensing out the pressure degree of freedom.

Imposing the boundary condition $u_1 = \bar{u}$ prior to or after the static condensation of $\hat{\underline{p}}$ yields the same element contribution to the stiffness of the assemblage, because u_1, $\underline{U}_r$ and $\hat{\underline{p}}$ satisfy equations ①, ② and ③

4.42



$$h_1 = \frac{1}{4}(1+x)(1+\frac{2}{3}y)$$
$$h_2 = \frac{1}{4}(1-x)(1+\frac{2}{3}y)$$
$$h_3 = \frac{1}{4}(1-x)(1-\frac{2}{3}y)$$
$$h_4 = \frac{1}{4}(1+x)(1-\frac{2}{3}y)$$

$$\hat{\underline{u}}^T = [\, u_1 \;\; u_2 \;\; u_3 \;\; u_4 \;\; v_1 \;\; v_2 \;\; v_3 \;\; v_4 \,]$$

$$\underline{u}(x,y) = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \underline{H}\,\hat{\underline{u}} \quad \text{where} \quad \underline{H} = \begin{bmatrix} h_1 & h_2 & h_3 & h_4 & & \underline{0} & & \\ & \underline{0} & & & h_1 & h_2 & h_3 & h_4 \end{bmatrix}$$

$$\underline{\varepsilon}' = \begin{bmatrix} \varepsilon_{xx} - \frac{1}{3}(\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz}) \\ \varepsilon_{yy} - \frac{1}{3}(\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz}) \\ \gamma_{xy} \\ \varepsilon_{zz} - \frac{1}{3}(\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz}) \end{bmatrix} = \begin{bmatrix} \frac{2}{3}\varepsilon_{xx} - \frac{1}{3}\varepsilon_{yy} - \frac{1}{3}\varepsilon_{zz} \\ -\frac{1}{3}\varepsilon_{xx} + \frac{2}{3}\varepsilon_{yy} - \frac{1}{3}\varepsilon_{zz} \\ \gamma_{xy} \\ -\frac{1}{3}\varepsilon_{xx} - \frac{1}{3}\varepsilon_{yy} + \frac{2}{3}\varepsilon_{zz} \end{bmatrix}$$

where $\varepsilon_{xx} = \frac{\partial u}{\partial x}$, $\varepsilon_{yy} = \frac{\partial v}{\partial y}$, $\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$ and $\varepsilon_{zz} = \frac{u}{x+5}$

$$\therefore \quad \underline{\varepsilon}' = \underline{B}_D\,\hat{\underline{u}}$$

$$\underline{B}_D = \begin{bmatrix} \frac{2}{3}h_{1,x} - \frac{1}{3}\frac{h_1}{x+5} & \frac{2}{3}h_{2,x} - \frac{1}{3}\frac{h_2}{x+5} & \cdots & -\frac{1}{3}h_{1,y} & -\frac{1}{3}h_{2,y} & \cdots \\ -\frac{1}{3}h_{1,x} - \frac{1}{3}\frac{h_1}{x+5} & -\frac{1}{3}h_{2,x} - \frac{1}{3}\frac{h_2}{x+5} & \cdots & \frac{2}{3}h_{1,y} & \frac{2}{3}h_{2,y} & \cdots \\ h_{1,y} & h_{2,y} & \cdots & h_{1,x} & h_{2,x} & \cdots \\ -\frac{1}{3}h_{1,x} + \frac{2}{3}\frac{h_1}{x+5} & -\frac{1}{3}h_{2,x} + \frac{2}{3}\frac{h_2}{x+5} & \cdots & -\frac{1}{3}h_{1,y} & -\frac{1}{3}h_{2,y} & \cdots \end{bmatrix}$$

4.42

$$\varepsilon_v = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} \qquad \therefore \; \varepsilon_v = \underline{B}_v \, \hat{\underline{u}}$$

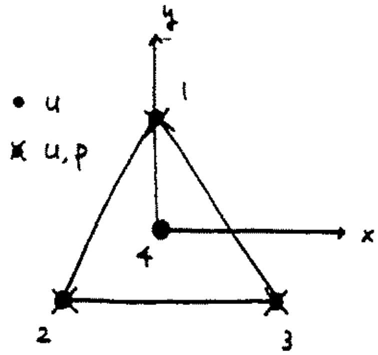
$$\underline{B}_v = \left[h_{1,x} + \frac{h_1}{x+5} \quad h_{2,x} + \frac{h_2}{x+5} \quad \cdots \quad h_{1,y} \quad h_{2,y} \quad \cdots \right]$$

$$\underline{p} = \underline{H}_p \, \hat{\underline{p}} \quad \text{where} \quad \hat{\underline{p}} = [p_o] \quad \text{and} \quad \underline{H}_p = [1]$$

And

$$\underline{C}' = \begin{bmatrix} 2G & & & \\ & 2G & & \\ & & G & \\ & & & 2G \end{bmatrix}$$ (terms not shown are zero.)

4.43



Let $h_1 = \frac{1}{3}(1+\sqrt{3}\,y)$

$h_2 = \frac{1}{6}(2 - 3x - \sqrt{3}\,y)$

$h_3 = \frac{1}{6}(2 + 3x - \sqrt{3}\,y)$

$h_4 = \frac{1}{4}(1+\sqrt{3}\,y)(2-3x-\sqrt{3}\,y)(2+3x-\sqrt{3}\,y)$

And $h_1' = h_1 - \frac{1}{3}h_4$, $h_2' = h_2 - \frac{1}{3}h_4$, $h_3' = h_3 - \frac{1}{3}h_4$

Then, $\underline{u}(x,y) = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \underline{H}\,\hat{\underline{u}}$, $\hat{\underline{u}}^T = [u_1 \; u_2 \; u_3 \; u_4 \; v_1 \; v_2 \; v_3 \; v_4]$

$$\underline{H} = \begin{bmatrix} h_1' & h_2' & h_3' & h_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_1' & h_2' & h_3' & h_4 \end{bmatrix}$$

$p = \underline{H}_p \hat{\underline{p}}$, $\hat{\underline{p}}^T = [p_1 \;\; p_2 \;\; p_3]$, $\underline{H}_p = [h_1 \;\; h_2 \;\; h_3]$

$\underline{\varepsilon}' = \underline{B}_D \hat{\underline{u}}$

$$\underline{B}_D = \begin{bmatrix} \frac{2}{3}h_{1,x}' & \frac{2}{3}h_{2,x}' & \frac{2}{3}h_{3,x}' & \frac{2}{3}h_{4,x} & -\frac{1}{3}h_{1,y}' & -\frac{1}{3}h_{2,y}' & -\frac{1}{3}h_{3,y}' & -\frac{1}{3}h_{4,y} \\ -\frac{1}{3}h_{1,x}' & -\frac{1}{3}h_{2,x}' & -\frac{1}{3}h_{3,x}' & -\frac{1}{3}h_{4,x} & \frac{2}{3}h_{1,y}' & \frac{2}{3}h_{2,y}' & \frac{2}{3}h_{3,y}' & \frac{2}{3}h_{4,y} \\ h_{1,y}' & h_{2,y}' & h_{3,y}' & h_{4,y} & h_{1,x}' & h_{2,x}' & h_{3,x}' & h_{4,x} \\ -\frac{1}{3}h_{1,x}' & -\frac{1}{3}h_{2,x}' & -\frac{1}{3}h_{3,x}' & -\frac{1}{3}h_{4,x} & -\frac{1}{3}h_{1,y}' & -\frac{1}{3}h_{2,y}' & -\frac{1}{3}h_{3,y}' & -\frac{1}{3}h_{4,y} \end{bmatrix}$$

$\varepsilon_V = \underline{B}_V \hat{\underline{u}}$

$$\underline{B}_V = [h_{1,x}' \;\; h_{2,x}' \;\; h_{3,x}' \;\; h_{4,x} \;\; h_{1,y}' \;\; h_{2,y}' \;\; h_{3,y}' \;\; h_{4,y}]$$

4.44 $p = p_0 + p_1 x + p_2 y = \underline{H}_p \hat{\underline{P}}$

where $\underline{H}_p = [\,1 \quad x \quad y\,]$, $\hat{\underline{P}}^T = [\,p_0 \quad p_1 \quad p_2\,]$

$$\therefore \; K_{pp} = -\int_V \underline{H}_p^T \frac{1}{K} \underline{H}_p \, dV = -\frac{3(1-2\nu)}{E} \int_{-1}^{1}\int_{-1}^{1} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix} [\,1 \;\; x \;\; y\,] \, dx\, dy$$

$$= -\frac{1}{E} \begin{bmatrix} 0.24 & 0 & 0 \\ 0 & 0.08 & 0 \\ 0 & 0 & 0.08 \end{bmatrix}$$

4.47

The continuity condition is:

$\exists\, M > 0$ <u>independent</u> of the bulk modulus, such that $\forall\ \underline{v}_{h_1}, \underline{v}_{h_2} \in V_h \quad |a(\underline{v}_{h_1}, \underline{v}_{h_2})| \le M \|\underline{v}_{h_1}\| \|\underline{v}_{h_2}\|$.

Clearly, this condition also holds if $\underline{v}_{h_1}, \underline{v}_{h_2} \in K_h(0)$ because $K_h(0) \subset V_h$.

The ellipticity condition is:

$\exists\, \alpha > 0$ such that $\forall\ \underline{v}_h \in K_h(0)\ a(\underline{v}_h, \underline{v}_h) \geq \alpha \|\underline{v}_h\|^2$, where constant α does not depend on the bulk modulus. Note that the bilinear form $a(\cdot,\cdot)$ does not depend on the bulk modulus as per Eq. (4.151)

Note that because $K_h(0) \subset V_h$, properties 1-3 given in sec. 4.3.4. still hold $\forall\ \underline{v}_h \in K_h(0)$.

Therefore, we have

$$\alpha \|\underline{u} - \underline{u}_h\|^2 \underset{\substack{\uparrow \\ \text{use ellipticity}}}{\leq} a(\underline{u} - \underline{u}_h, \underline{u} - \underline{u}_h) \underset{\text{use prop. 3}}{=} \inf_{\underline{v}_h \in K_h(0)} a(\underline{u} - \underline{v}_h, \underline{u} - \underline{v}_h) \underset{\text{use continuity}}{\leq}$$

$$\leq M \inf_{\underline{v}_h \in K_h(0)} \|\underline{u} - \underline{v}_h\|^2.$$

Denoting $d(\underline{u}, K_h(0)) = \inf_{\underline{v}_h \in K_h(0)} \|\underline{u} - \underline{v}_h\|$, we have

$\|\underline{u} - \underline{u}_h\| \le \tilde{c}\, d(\underline{u}, K_h(0))$, where $\tilde{c} = \sqrt{\frac{M}{\alpha}}$ is the constant independent of the bulk modulus.

4.48 We prove the relation for the 2D-case without loss of generality

Let $\underline{v}^T = (\underline{v}_1 - \underline{v}_2)^T = [\,u \quad v\,]$ and $a = \frac{\partial u}{\partial x}$, $b = \frac{\partial v}{\partial y}$.

Then from $0 \le (a-b)^2$, $2ab \le a^2 + b^2$

$$(a^2 + b^2) + 2ab \le 2(a^2+b^2) \qquad \therefore \; (a+b)^2 \le 2(a^2+b^2)$$

$$\Rightarrow \|\operatorname{div} \underline{v}\|_0^2 = \int_{Vol} \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right]^2 dVol \le 2 \int_{Vol} \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] dVol$$

$$\le c \int_{Vol} \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] dVol =$$

$$= c\, \|\underline{v}\|_V^2$$

Hence we have

$$\|\operatorname{div}(\underline{v}_1 - \underline{v}_2)\|_0 \le c\, \|\underline{v}_1 - \underline{v}_2\|_V$$

where c is a constant and $\underline{v}_1, \underline{v}_2 \in V_h$.

4.49 $\displaystyle\int_{Vol} \left[P_h (\operatorname{div} \underline{v}_h) - \operatorname{div} \underline{v}_h \right] q_h \, dVol = 0 , \qquad q_h = 1$

$$\text{Let } \underline{v}_h = \begin{bmatrix} u_h \\ v_h \end{bmatrix} = \begin{bmatrix} h_1 & 0 & h_2 & 0 & \cdots & h_8 & 0 \\ 0 & h_1 & 0 & h_2 & \cdots & 0 & h_8 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ \vdots \\ u_8 \\ v_8 \end{bmatrix} = \underline{H} \, \hat{\underline{v}}_h$$

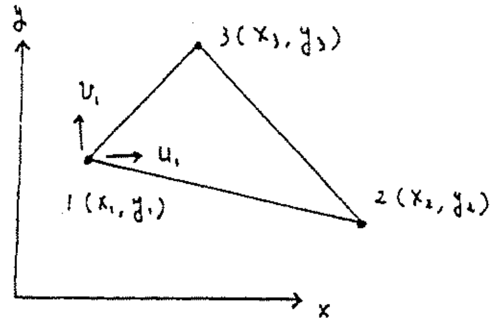
$$\operatorname{div} \underline{v}_h = \underline{B} \, \hat{\underline{v}}_h \quad , \quad \underline{B} = \left[h_{1,x} \;\; h_{1,y} \;\; h_{2,x} \;\; h_{2,y} \;\; \cdots \;\; h_{8,x} \;\; h_{8,y} \right]$$

$$\int_{Vol} P_h (\operatorname{div} \underline{v}_h) \, q_h \, dVol = \int_{Vol} \operatorname{div} \underline{v}_h \, q_h \, dVol$$

$$\rightarrow \; 4 P_0 = \int_{-1}^{1} \int_{-1}^{1} \underline{B} \, \hat{\underline{v}}_h \, dx\, dy \qquad \left(\because \; P_h(\operatorname{div} \underline{v}_h) = P_0 \cdot 1 \;\; (P_0 : \text{const.}) \right)$$

$$\therefore \; P_0 = \frac{1}{12} \left[u_1 - u_2 - u_3 + u_4 - 4u_6 + 4u_8 + v_1 + v_2 - v_3 - v_4 + 4v_5 - 4v_7 \right]$$

4.50



(a) $$\underline{u}(x,y) = \begin{bmatrix} \underline{u}(x,y) \\ \underline{v}(x,y) \end{bmatrix} = \underline{H}\,\hat{\underline{u}}$$

where $\hat{\underline{u}}^T = [u_1 \; u_2 \; u_3 \; v_1 \; v_2 \; v_3]$

$$\underline{H} = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} \underline{A}^{-1}$$

$\underline{A}$: defined in Example 4.17.

Axisymmetric condition:

$$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \\ \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \\ u/x \end{bmatrix},$$

Plane stress/strain cond:

$$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{bmatrix}$$

$\underline{\varepsilon} = \underline{B}\,\hat{\underline{u}}$, where

$$\underline{B} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ \frac{1}{x} & 1 & \frac{y}{x} & 0 & 0 & 0 \end{bmatrix} \underline{A}^{-1}$$

$$\underline{B} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \underline{A}^{-1}$$

With a general 3/1 u/p element we obtain the stiffness matrix as in Example 4.32,

$$\begin{bmatrix} \underline{K}_{uu} & \underline{K}_{up} \\ \underline{K}_{up}^T & \underline{K}_{pp} \end{bmatrix} \begin{bmatrix} \hat{\underline{u}} \\ \hat{\underline{p}} \end{bmatrix} = \begin{bmatrix} \underline{R} \\ \underline{0} \end{bmatrix} \qquad \text{——— ①}$$

where $\underline{K}_{uu} = \int_V \underline{B}_D^T \underline{C}' \underline{B}_D \, dV$, $\underline{K}_{up} = -\int_V \underline{B}_V^T \underline{H}_p \, dV$, $\underline{K}_{pp} = -\int_V \underline{H}_p^T \frac{1}{\kappa} \underline{H}_p \, dV$

4.50

$\underline{B}_D$, $\underline{B}_V$, $\underline{H}_p$ and $\hat{\underline{p}}$ are given as follows:

$\underline{\varepsilon}' = \underline{B}_D \hat{\underline{u}}$, $\varepsilon_V = \underline{B}_V \hat{\underline{u}}$, $p = \underline{H}_p \hat{\underline{p}}$ with $\underline{H}_p = [1]$, $\hat{\underline{p}} = [p_0]$

(i) plane strain condition

$\underline{B}_D$, $\underline{B}_V$: defined in Example 4.32

$$\underline{B}_D = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{3} & -\frac{1}{3} & 0 \end{bmatrix} \underline{B} \quad , \quad \underline{B}_V = [1 \quad 1 \quad 0]\, \underline{B}$$

(ii) plane stress condition

$$\underline{B}_D = \frac{1}{1-\nu} \begin{bmatrix} \frac{2-\nu}{3} & -\frac{1-2\nu}{3} & 0 \\ -\frac{1-2\nu}{3} & \frac{2-\nu}{3} & 0 \\ 0 & 0 & 1-\nu \\ -\frac{1+\nu}{3} & -\frac{1+\nu}{3} & 0 \end{bmatrix} \underline{B} \quad , \quad \underline{B}_V = \left[\frac{1-2\nu}{1-\nu} \quad \frac{1-2\nu}{1-\nu} \quad 0 \right] \underline{B}$$

(iii) axisymmetric condition

$$\underline{B}_D = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & 0 & \frac{2}{3} \end{bmatrix} \underline{B} \quad , \quad \underline{B}_V = [1 \quad 1 \quad 0 \quad 1]\, \underline{B}$$

4.50

(b) Let $\underline{B}_D = \underline{C}_D \underline{B}$, $\underline{B}_V = \underline{C}_V \underline{B}$ in each case.

Then using eq. ① we have

$$\underline{K}\,\hat{\underline{u}} = \underline{R} \quad \text{where} \quad \underline{K} = \underline{K}_{uu} - \underline{K}_{up}\underline{K}_{pp}^{-1}\underline{K}_{up}^T \quad \text{—— ②}$$

Hence
$$\underline{K} = \int_V \underline{B}_D^T \underline{C}' \underline{B}_D \, dV - \left[\int_V \underline{B}_V^T \underline{H}_p \, dV\right]\left[-\frac{1}{\kappa}\int_V dV\right]^{-1}\left[\int_V \underline{B}_V^T \underline{H}_p \, dV\right]^T$$

$$= \int_V \underline{B}^T(\underline{C}_D^T \underline{C}' \underline{C}_D)\underline{B}\, dV - \left[\int_V \underline{B}^T \underline{C}_V^T dV\right]\left[\frac{-\kappa}{\int_V dV}\right]\left[\int_V \underline{C}_V \underline{B}\, dV\right]$$

For the plane stress and plane strain cases we have that $\underline{K} = \int_V \underline{B}^T(\underline{C}_D^T \underline{C}' \underline{C}_D + \kappa\, \underline{C}_V^T \underline{C}_V)\underline{B}\, dV$,

and calculating $\underline{C}_D^T \underline{C}' \underline{C}_D + \kappa\, \underline{C}_V^T \underline{C}_V$ in each case one can check that the corresponding stress-strain matrices $\underline{C}$ defined in Table 4.3 are obtained. Therefore, the u/p formulation gives the same result as the displacement-based method.

For the axisymmetric case, the $\underline{B}$ matrix depends on x and y. Therefore, in general, results different from the displacement-based solution are obtained

(c) When total incompressibility is considered, $\kappa \to \infty$.

$$\therefore \underline{K}_{pp} = \underline{0} \qquad \therefore \begin{bmatrix} \underline{K}_{uu} & \underline{K}_{up} \\ \underline{K}_{up}^T & \underline{0} \end{bmatrix}\begin{bmatrix} \hat{\underline{u}} \\ \hat{\underline{p}} \end{bmatrix} = \begin{bmatrix} \underline{R} \\ \underline{0} \end{bmatrix}$$

Note that the static condensation in eq. ② cannot be performed here.

4.51 In example 4.32 for a 4/1 u/p element,

$$\begin{bmatrix} \underline{K}_{uu} & \underline{K}_{up} \\ \underline{K}_{pu} & \underline{K}_{pp} \end{bmatrix} \begin{bmatrix} \hat{\underline{u}} \\ \hat{\underline{p}} \end{bmatrix} = \begin{bmatrix} \underline{R} \\ \underline{0} \end{bmatrix}$$

where $\underline{K}_{uu} = \int \underline{B}_D^T \underline{C}' \underline{B}_D \, dV$, $\underline{K}_{up} = \underline{K}_{pu}^T = -\int_V \underline{B}_V^T \underline{H}_p \, dV$,

$$\underline{K}_{pp} = -\int \underline{H}_p^T \frac{1}{\kappa} \underline{H}_p \, dV$$

$$\underline{B}_D = \begin{bmatrix} \frac{2}{3} h_{1,x} & \frac{2}{3} h_{2,x} & \cdots & -\frac{1}{3} h_{1,y} & -\frac{2}{3} h_{2,y} & \cdots \\ -\frac{1}{3} h_{1,x} & -\frac{1}{3} h_{2,x} & \cdots & \frac{2}{3} h_{1,y} & \frac{2}{3} h_{2,y} & \cdots \\ h_{1,y} & h_{2,y} & \cdots & h_{1,x} & h_{2,x} & \cdots \\ -\frac{1}{3} h_{1,x} & -\frac{1}{3} h_{2,x} & \cdots & -\frac{1}{3} h_{1,y} & -\frac{1}{3} h_{2,y} & \cdots \end{bmatrix}$$

$$\underline{B}_V = [\, h_{1,x} \;\; h_{2,x} \;\; \cdots \;\; h_{1,y} \;\; h_{2,y} \;\; \cdots \,]$$

$$\underline{H}_p = [\,1\,], \quad \hat{\underline{u}}^T = [\, u_1 \; u_2 \; u_3 \; u_4 \;\; v_1 \; v_2 \; v_3 \; v_4 \,], \; \hat{\underline{p}} = [p_0]$$

$$\therefore \underline{K}_{pp} = -\int \frac{1}{\kappa} dV = -\frac{4}{\kappa}[1], \quad \underline{K}_{pp}^{-1} = -\frac{\kappa}{4}[1]$$

$$\underline{K}_{up} = \underline{K}_{pu}^T = -[\, 1 \; -1 \; -1 \; 1 \; \vdots \; 1 \; 1 \; -1 \; -1 \,]^T = -4\underline{D}^T$$

where $\underline{D} = \frac{1}{4}[\, 1 \; -1 \; -1 \; 1 \; \vdots \; 1 \; 1 \; -1 \; -1 \,]$

$$\therefore \underline{K} = \underline{K}_{uu} - \underline{K}_{up} \underline{K}_{pp}^{-1} \underline{K}_{pu} = \underline{K}_{uu} + 4\kappa \underline{D}^T \underline{D}$$

Now from the results of example 4.34,

$$\frac{\kappa}{2} \int_{Vol} \left(P_h (\operatorname{div} \underline{v}_h) \right)^2 dVol = \frac{\kappa}{2} \hat{\underline{u}}^T \underline{G}_h \hat{\underline{u}} = \frac{\kappa}{2} \hat{\underline{u}}^T 4 \underline{D}^T \underline{D} \hat{\underline{u}}$$

4.51

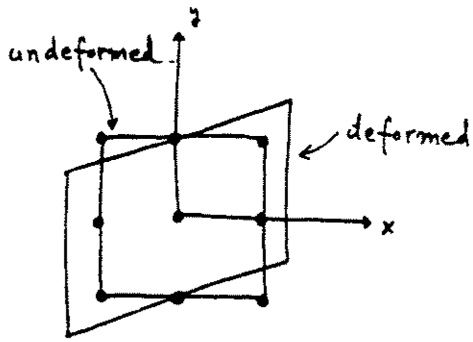
If we let $\Pi = \frac{1}{2} a(\underline{v}_h, \underline{v}_h) + \frac{\kappa}{2} \int_{Vol} \left(P_h (\mathrm{div}\, \underline{v}_h) \right)^2 dVol - \int_{Vol} \underline{f}^B \underline{v}_h \, dVol$

Using $\delta \Pi = 0$, we see

$$\underline{K} = \underline{K}_{uu} + 4\kappa \underline{D}^T \underline{D}$$

Hence we obtain the same element stiffness matrix as in example 4.32.

4.52



$Q_h = \{1, x, y\}$

$u_1 = 1 \quad u_2 = -1 \quad u_3 = -1 \quad u_4 = 1$

$u_6 = -1 \quad u_8 = 1$

$v_1 = 1 \quad v_2 = -1 \quad v_3 = -1 \quad v_4 = 1$

$v_6 = -1 \quad v_8 = 1$

$$\int_{Vol} \left[P_h(\mathrm{div}\, \underline{v}_h) - \mathrm{div}\, \underline{v}_h \right] q_h \, dVol = 0, \ \forall q_h \in Q_h$$

Using the given nodal displacements and $\mathrm{div}\, \underline{v}_h = \frac{\partial u_h}{\partial x} + \frac{\partial v_h}{\partial y}$,

$$u_h = x, \ v_h = x \qquad \therefore \ \mathrm{div}\, \underline{v}_h = 1$$

Let $P_h(\mathrm{div}\, \underline{v}_h) = p_0 + p_1 x + p_2 y$

Then $\int_{Vol} [(p_0 + p_1 x + p_2 y) - 1] \, q_h \, dx dy = 0$ for $q_h = 1$, x and y

$$\begin{bmatrix} \int_{Vol} dx dy & \int_{Vol} x \, dx dy & \int_{Vol} y \, dx dy \\ & \int_{Vol} x^2 dx dy & \int_{Vol} xy \, dx dy \\ \text{symmetric} & & \int_{Vol} y^2 dx dy \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \int_{Vol} dx dy \\ \int_{Vol} x \, dx dy \\ \int_{Vol} y \, dx dy \end{bmatrix}$$

$\therefore \ p_0 = 1, \ p_1 = p_2 = 0$

$P_h(\mathrm{div}\, \underline{v}_h) = 1$

4.53 Use the form of inf-sup condition given in eq. (4.176)

$$\forall \underline{u} \; \exists \underline{u}_I \in V_h \text{ such that}$$

$$\int_{Vol} \text{div}(\underline{u} - \underline{u}_I)\, q_h \, dVol = 0 \qquad \forall q_h \in D_h \qquad (*)$$

$$\| \underline{u}_I \| \leq c \| \underline{u} \|$$

with C independent of $\underline{u}$, $\underline{u}_I$ and h.

Given $\underline{u}$ smooth we find $\underline{u}_I \in V_h$ for each element (m),

$$\int_{Vol^{(m)}} (\text{div}\, \underline{u}_I - \text{div}\, \underline{u})\, q_h \, dVol^{(m)} = 0 \quad \text{for all } q_h \text{ with } Q_h = \{1\} \qquad (a)$$

To define $\underline{u}_I$, we prescribe for nodes $i = 1, \cdots, 4$

$$\underline{u}_I|_i = \underline{u}|_i \qquad \text{—— (b)}$$

Then we adjust the values at nodes $j = 5, 6, 7, 8$ such that

$$\int_{S_j} (\underline{u} - \underline{u}_I) \cdot \underline{n} \, dS = \int_{S_j} (\underline{u} - \underline{u}_I) \cdot \underline{\tau} \, ds = 0 \qquad \text{—— (c)}$$

for every edge $S_1, \cdots, S_4$ of the element with $\underline{n}$ the normal vector and $\underline{\tau}$ the tangential vector to the edge.

As $Q_h = \{1\}$, for every constant q_h, in (a),

$$\int_{Vol^{(m)}} \text{div}\, (\underline{u} - \underline{u}_I)\, q_h \, dVol^{(m)} = q_h \sum_{S_1, \cdots, S_4} \int_{S_j} (\underline{u} - \underline{u}_I) \cdot \underline{n} \, ds \qquad \text{—(d)}$$

Hence (*) is satisfied because of (d) and (c), also, $\underline{u}_I$ constructed element by element through (b) and (c) will be continuous from element to element; and note that clearly if $\underline{u}$ is a (vector) polynomial of degree ≤ 2 on the element, we obtain $\underline{u}_I \equiv \underline{u}$ and this ensures optimal bounds for $\| \underline{u} - \underline{u}_I \|$, and implies the condition $\| \underline{u}_I \| \leq c \| \underline{u} \|$ for all $\underline{u}$.

4.54 $(\underline{K}_{uu})_h \underline{U}_h + (\underline{K}_{up})_h \underline{P}_h = \underline{R}_h$ —— ①

$(\underline{K}_{pu})_h \underline{U}_h = \underline{0}$ —— ②

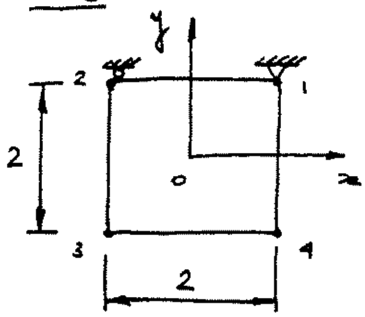
If condition ii is not satisfied we could add to any solution of eq. ① the nonzero pressure Q_h obtained in (4.189). Hence, clearly, for a unique solution it's necessary to have ii satisfied.

Moreover, the unique solution must satisfy eq. ② Hence, for a vector $\underline{V}_h$ satisfying ② the strain energy calculated in (4.188) must be positive. This implies that $(\underline{K}_{uu})_h$ must be positive definite on the space of vectors satisfying ②.

These conditions are also sufficient to yield a unique solution because a quadratic functional (i.e., the strain energy) has a minimum if its matrix is positive definite.

For a more strict and mathematical proof we refer to Brezzi, F., and Bathe, K.J. [B]

4.55



Consider a properly supported 2×2 element. For the plane strain condition we have:

$$(\underline{K}_{uu})_h = \int_{-1}^{1}\int_{-1}^{1} \underline{B}_D^T \underline{C}' \underline{B}_D \, dx \, dy,$$

$$(\underline{K}_{up})_h = (\underline{K}_{pu}^T)_h = -\int_{-1}^{1}\int_{-1}^{1} \underline{B}_V^T \underline{H}_p \, dx \, dy, \quad \text{where}$$

matrices $\underline{B}_D$, $\underline{C}'$, $\underline{B}_V$, $\underline{H}_p$ are given in example 4.32.

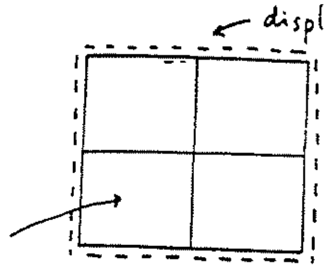
The ellipticity condition: $\underline{V}_h^T (\underline{K}_{uu})_h \underline{V}_h \geq \alpha \, \| \underline{u}_h \|_V^2$ $\forall \; \underline{V}_h \in \ker[(\underline{K}_{pu})_h]$.

This implies that $(\underline{K}_{uu})_h$ is a positive definite matrix on the subspace $\ker[(\underline{K}_{pu})_h] \subset V_h$.

One can easily check that for a properly supported element $(\underline{K}_{uu})_h$ is a positive definite matrix, i.e., $\forall \; \underline{V}_h \in \mathbb{R}^n$ $\underline{V}_h \neq \underline{0}$ $\underline{V}_h^T (\underline{K}_{uu})_h \underline{V}_h >$

Therefore, this holds for any subspace $V \in \mathbb{R}^n \Rightarrow$ the ellipticity condition is satisfied.

4.56



Let $p = \text{const} = p_0$

In this case we have

$$p_0 \int_{Vol} \operatorname{div} \underline{v}_h \, dVol = 0 \qquad \forall \underline{v}_h \in V_h$$

First, we know that

$$\int_{Vol} p_0 \operatorname{div} \underline{v}_h \, dVol = \int_S p_0 \, \underline{v}_h \cdot \underline{n} \, dS = 0 \qquad \forall \underline{v}_h \in V_h$$

Since $\displaystyle\int_{Vol} p_0 \operatorname{div} \underline{v}_h \, dVol = \int_{Vol} p_0 P_h (\operatorname{div} \underline{v}_h) \, dVol$

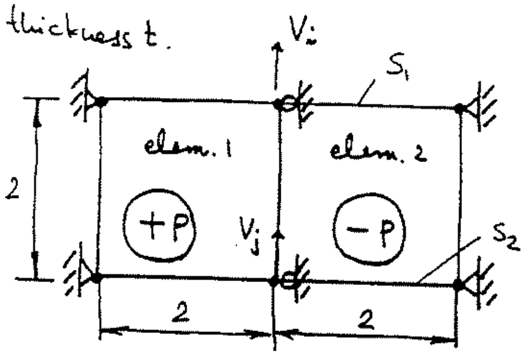
we have $\displaystyle\int_{Vol} p_0 P_h (\operatorname{div} \underline{v}_h) \, dVol = 0 \qquad \forall \underline{v}_h \in V_h$

Hence $p_0 \notin P_h (\operatorname{div} \underline{v}_h)$. Namely, otherwise we could choose $P_h (\operatorname{div} \underline{v}_h) = p_0$ and

$$\int_{Vol} p_0 P_h (\operatorname{div} \underline{v}_h) \, dVol > 0 \quad !$$

Hence, there is no $p = -P_h \operatorname{div} \underline{v}_h$, that is, p cannot be obtained from "a" div $\underline{v}_h$ projection.

4.57 thickness t.



$$\int_V p \operatorname{div} \underline{v}_h \, dV = \int_{V_1} P_1 \operatorname{div} \underline{v}_h \, dV + \int_{V_2} P_2 \operatorname{div} \underline{v}_h \, dV$$

$$= P \int_{S_1} \underline{v}_h \cdot \underline{n} \, dS - P \int_{S_2} \underline{v}_h \cdot \underline{n} \, dS$$

$$\int_{S_1} \underline{v}_h \cdot \underline{n} \, ds = \left[\int_{-1}^{1} V_i \frac{1+r}{2} dr - \int_{-1}^{1} V_j \frac{1+r}{2} dr \right] t = \left[V_i - V_j \right] t$$

$$\int_{S_2} \underline{v}_h \cdot \underline{n} \, ds = \left[\int_{-1}^{1} V_i \frac{1-r}{2} dr - \int_{-1}^{1} V_j \frac{1-r}{2} dr \right] t = \left[V_i - V_j \right] t$$

$\therefore \int_V p \operatorname{div} \underline{v}_h \, dV = 0$. And we see it is a spurious mode.

Also, consider

$$(\underline{K}_{up})_h = \left[(\underline{K}_{up})_h^{(1)} \;\vdots\; (\underline{K}_{up})_h^{(2)} \right] \quad \text{, where}$$

$$(\underline{K}_{up})_h^{(1)} = - \int_{Vol^{(1)}} \underline{B}_v^{T\,(1)} \underline{H}_p^{(1)} \, dVol = - t \int_{-1}^{1} \int_{-1}^{1} \left[h_{1,y} \quad h_{4,y} \right]^T [1] \, dx\,dy$$

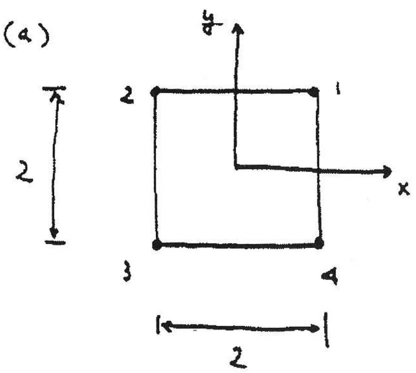
$$= -t \left[\tfrac{1}{2} \quad -\tfrac{1}{2} \right]^T$$

Similarly, $(\underline{K}_{up})_h^{(2)} = -t \left[\tfrac{1}{2} \quad -\tfrac{1}{2} \right]^T$

$$\therefore \quad (\underline{K}_{up})_h = \frac{t}{2} \begin{bmatrix} -1 & \vdots & -1 \\ 1 & \vdots & 1 \end{bmatrix}, \quad \text{and clearly}$$

$\exists \; \underline{P}_s : (\underline{K}_{up})_h \underline{P}_s = 0$, $\underline{P}_s \neq \underline{0} \Rightarrow$ we have a spurious mode.

4.58 (a)



For a single element,

$$\operatorname{div} \underline{v}_h = \frac{1}{4}\Big[(1+y) \;\; -(1+y) \;\; -(1-y) \;\; (1-y) \;\vdots\; (1+x) \;\; (1-x) \;\; -(1-x) \;\; -(1+x)\Big]\hat{\underline{u}}$$

where $\hat{\underline{u}}^T = [\,u_1 \; u_2 \; u_3 \; u_4 \;\vdots\; v_1 \; v_2 \; v_3 \; v_4\,]$

Choose $\hat{u}^T = [\,-1 \;\; 0 \;\; 0 \;\; 0 \;\; -1 \;\; 0 \;\; 0 \;\; 0\,]$ and the pressure distribution $\hat{P}_h$ as shown below.

P_0	0	0
0	0	0
0	0	0

$(P_0 > 0)$

Then, $\displaystyle\int_{Vol} \hat{P}_h \operatorname{div} \underline{v}_h = P_0 \int_{Vol^{e_1}} \operatorname{div} \underline{v}_h \, dVol^{e_1}$

$$= P_0 \int_{Vol^{e_1}} \frac{1}{4}\{(1+y)+(1+x)\}\, dVol^{e_1}$$

$$= P_0 > 0$$

(b) Choose a checker board pressure distribution as $\hat{P}_h$

+	−	+
−	+	−
+	−	+

$+ \rightarrow +\Delta p$

$- \rightarrow -\Delta p$

For a single element

$$\int_{Vol^i} p^i \operatorname{div} \underline{v}_h \, dVol^i$$

$$= p^i\,[\,1 \;\; -1 \;\; -1 \;\; 1 \;\vdots\; 1 \;\; 1 \;\; -1 \;\; -1\,]\,\hat{\underline{u}}$$

where $\hat{\underline{u}}^T = [\,u_1 \; u_2 \; u_3 \; u_4 \;\vdots\; v_1 \; v_2 \; v_3 \; v_4\,]$

The rest of the solution follows the development in example 4.38.

4.59 (a) In the u/p-c formulation we have

$$\begin{bmatrix} (\underline{K}_{uu})_h & (\underline{K}_{up})_h \\ (\underline{K}_{pu})_h & -\frac{1}{\kappa}\underline{T}_h \end{bmatrix} \begin{bmatrix} \underline{U}_h \\ \underline{P}_h \end{bmatrix} = \begin{bmatrix} \underline{R}_h \\ \underline{0} \end{bmatrix} \qquad \text{—— ①}$$

The inf-sup condition is written as

$$\inf_{\underline{w}_h \in V_h} \sup_{\underline{v}_h \in V_h} \frac{\int_{Vol} P_h(\text{div}\,\underline{w}_h)\, \text{div}\,\underline{v}_h \, dVol}{\| P_h(\text{div}\,\underline{w}_h)\| \, \| \underline{v}_h \|} \geq \beta > 0$$

and in the matrix form

$$\inf_{\underline{W}_h} \sup_{\underline{V}_h} \frac{\int_{Vol} \underline{W}_h^T \underline{G}_h \underline{V}_h \, dVol}{\| \underline{W}_h^T \underline{G}_h \underline{W}_h \| \, \| \underline{V}_h^T \underline{S}_h \underline{V}_h \|} \geq \beta > 0$$

where $\underline{W}_h^T \underline{G}_h \underline{V}_h$ is the matrix form of $P_h(\text{div}\,\underline{v}_h)\, \text{div}\,\underline{v}_h$

From eq.① $(\underline{K}_{pu})_h \underline{U}_h - \frac{1}{\kappa}\underline{T}_h \underline{P}_h = 0$

$\underline{T}_h^{-1}(\underline{K}_{pu})_h$: the matrix form associated with $P_h(\text{div}\,\underline{v}_h)$

$(\underline{K}_{pu})_h$: the one associated with $\text{div}\,\underline{u}_h$

$$\therefore \quad \underline{G}_h = (\underline{K}_{up})_h \underline{T}_h^{-1} (\underline{K}_{pu})_h$$

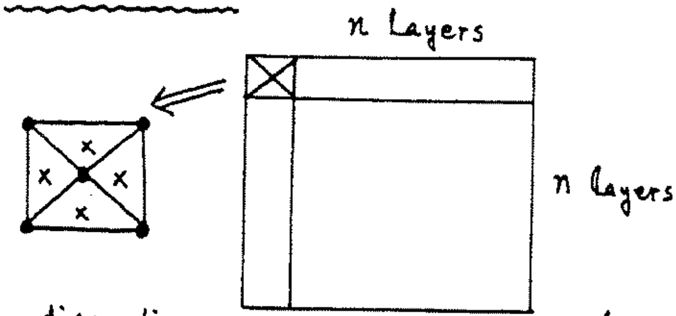
(b) $\underline{G}_h' \underline{Q}_h = \lambda' \underline{T}_h \underline{Q}_h$ where $\underline{G}_h' = (\underline{K}_{pu})_h \underline{S}_h^{-1} (\underline{K}_{up})_h$ —— ②

Considering $\underline{G}_h \underline{V} = \lambda \underline{S}_h \underline{V}$, —— ③

$$(\underline{K}_{up})_h \left(\underline{T}_h^{-1} (\underline{K}_{pu})_h \underline{V} \right) = \lambda \underline{S}_h \underline{V}$$

$$\underbrace{(\underline{K}_{pu})_h \underline{S}_h^{-1} (\underline{K}_{up})_h}_{\underline{G}_h'} \underbrace{\underline{T}_h^{-1} (\underline{K}_{pu})_h \underline{V}}_{\underline{Q}_h} = \lambda \underbrace{(\underline{K}_{pu})_h \underline{V}}_{\underline{T}_h \underline{Q}_h}$$

4.60 3/1 element



x discontinuous pressure d.o.f.

n_u = total d.o.f.'s in u

n_p = total d.o.f.'s in p

$$n_u = 2(n+1)^2 + 2n^2$$

$$n_p = 4n^2$$

$$\lim_{n\to\infty} \frac{n_u}{n_p} = \frac{4}{4} = 1$$

4.61 In the mixed formulated 2-, 3- and 4-node elements

$$EI\int_0^L (\beta_h')^2\,dx \geq \alpha \,\|\underline{v}_h\|^2 \quad \forall\, \underline{v}_h \in K_h(0)$$

because $\forall\, \underline{v}_h \in K_h(0)$ we have

use Poincaré - Friedrichs inequality

$$\int_0^L \left(\frac{\partial w_h}{\partial x}\right)^2 dx = \int_0^L (\beta_h)^2 dx \leq \int_0^L L^2\left(\frac{\partial \beta_h}{\partial x}\right)^2 dx$$

$$\|\underline{v}_h\|^2 \leq 2L^2 \int_0^L \left(\frac{\partial \beta_h}{\partial x}\right)^2 dx \qquad \text{i.e. } \alpha = \frac{EI}{2L^2}$$

$\underline{4.62}$ inf-sup condition : $\inf_{\gamma_h \in D_h} \sup_{\underline{v}_h \in V_h} \frac{\int_{Vol} \gamma_h \left[\frac{\partial w_h}{\partial x} - \beta_h \right]}{\| \gamma_h \| \| \underline{v}_h \|} \geq c > 0$

(i) displacement-based elements:

Define the following spaces:

$$K_h(0) = \{ \underline{v}_h = (w_h, \beta_h) \in V_h : \gamma_h (\underline{v}_h) = 0 \}, \text{ where}$$

$$\gamma_h (\underline{v}_h) = \frac{\partial w_h}{\partial x} - \beta_h .$$

This choice implies that $K_h(0) = \{ \underline{v}_h \in V_h : \frac{\partial w_h}{\partial x} = \beta_h \}$.

The inf-sup condition for the limit case (thickness $t = 0$) can be written as:

$$\inf_{\gamma_h \in K_h(0)} \sup_{\underline{v}_h \in V_h} \frac{\int_{Vol} \gamma_h \left[\frac{\partial w_h}{\partial x} - \beta_h \right] dVol}{\| \gamma_h \| \; \| \underline{v}_h \|} \equiv 0 .$$

Therefore, clearly the inf-sup condition is not satisfied.

(ii) mixed-interpolated elements:

As $K_h(0) \neq \{0\}$, for a typical γ_h given we choose

$$\hat{\underline{v}}_h = \begin{bmatrix} \hat{w}_h \\ \hat{\beta}_h \end{bmatrix} \quad \text{with } \hat{\beta}_h = 0 \text{ and } \frac{\partial \hat{w}_h}{\partial x} = \gamma_h$$

4.62

Now consider

$$\int_{Vol} \frac{\gamma_h \left[\frac{\partial \hat{w}_h}{\partial x} - \hat{\beta}_h \right] dVol}{\| \hat{\underline{v}}_h \|} = \sqrt{\int_{Vol} (\gamma_h)^2 dVol}$$

Hence we have

$$\sup_{\underline{v}_h \in V_h} \frac{\int_{Vol} \gamma_h \left[\frac{\partial w_h}{\partial x} - \beta_h \right] dVol}{\| \underline{v}_h \|} \geq \frac{\int_{Vol} \gamma_h \left[\frac{\partial \hat{w}_h}{\partial x} - \hat{\beta}_h \right] dVol}{\| \hat{\underline{v}}_h \|}$$

$$= \sqrt{\int_{Vol} (\gamma_h)^2 dVol}$$

Therefore

$$\inf_{\gamma_h \in P_h(D_h)} \sup_{\underline{v}_h \in V_h} \frac{\int_{Vol} \gamma_h \left[\frac{\partial w_h}{\partial x} - \beta_h \right] dVol}{\| \gamma_h \| \, \| \underline{v}_h \|} \geq 1$$

And the inf-sup condition is satisfied.

4.63

The inf-sup condition in the matrix form:

$$\inf_{\underline{W}_h} \sup_{\underline{V}_h} \frac{\int_L \underline{W}_h^T \underline{G}_h \underline{V}_h \, dx}{\| \underline{W}_h^T \underline{G}_h \underline{W}_h \| \; \| \underline{V}_h^T \underline{S}_h \underline{V}_h \|} = C_h > C > 0$$

where S_h is the matrix that represents

$$\| \underline{v}_h \|^2 = \int_L \left[\left(\frac{\partial w_h}{\partial x} \right)^2 + L^2 \left(\frac{\partial \beta_h}{\partial x} \right)^2 \right] dx = \underline{V}_h^T \underline{S}_h \underline{V}_h$$

For the two-node displacement-based element we have:

$\underline{V}_h = [\, w_1 \quad \theta_1 \;\vdots\; w_2 \quad \theta_2 \,]^T$,

$\underline{G}_h = \underline{B}_\gamma^T \underline{B}_\gamma$, where $\underline{B}_\gamma = [\, h_{1,x} \quad -h_1 \;\vdots\; h_{2,x} \quad -h_2 \,]$

$$\underline{S}_h = \begin{bmatrix} h_{1,x}^2 & 0 & h_{1,x} h_{2,x} & 0 \\ 0 & L^2 h_{1,x}^2 & 0 & L^2 h_{1,x} h_{2,x} \\ h_{1,x} h_{2,x} & 0 & h_{2,x}^2 & 0 \\ 0 & L^2 h_{1,x} h_{2,x} & 0 & L^2 h_{2,x}^2 \end{bmatrix}$$

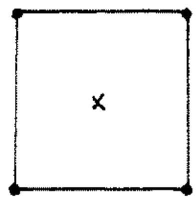
Then the inf-sup value C_h is given by the smallest nonzero eigenvalue of the eigenproblem $\underline{G}_h \underline{\phi} = \lambda \underline{S}_h \underline{\phi}$.

For the mixed-interpolated two-node element we have: $\underline{B}_\gamma^A = [\, h_{1,x} \quad -h_1\big|_{x=0} \;\vdots\; h_{2,x} \quad -h_2\big|_{x=0} \,]$,

and $\underline{G}_h = \underline{B}_\gamma^{A\,T} \underline{B}_\gamma$ (note that this new G_h is still symmetric)

4.64 n_u = number of displacement d.o.f. n_p = number of pressure d.o.f.

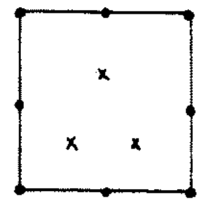
4/1



$n_u = 2(n+1)^2 \qquad n_p = n^2$

$\text{ratio} = \lim_{h\to 0} \frac{n_u}{n_p} = \frac{2}{1} = 2$

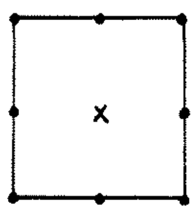
8/3



$n_u = \{(2n+1)^2 - n^2\}\cdot 2 \qquad n_p = 3n^2$

$\text{ratio} = \frac{6}{3} = 2$

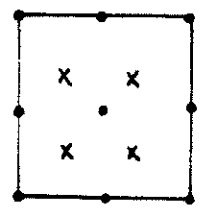
8/1



$n_u = \{(2n+1)^2 - n^2\}\cdot 2 \qquad n_p = n^2$

$\text{ratio} = \frac{6}{1} = 6$

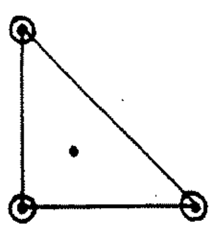
9/4



$n_u = (2n+1)^2\cdot 2 \qquad n_p = 4n^2$

$\text{ratio} = \frac{8}{4} = 2$

4/3-c



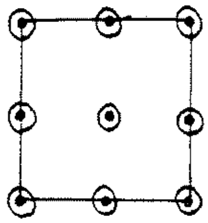
$n_u = \{(n+1)^2 + n^2\}\cdot 2 + (4n^2)\cdot 2$

$n_p = (n+1)^2 + n^2$

$\text{ratio} = \frac{12}{2} = 6$

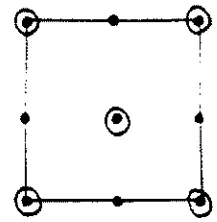
4.64

9/9-c



$n_u = 2 n_p$, ratio = 2

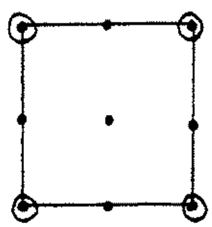
9/5-c



$n_u = 2(2n+1)^2$, $n_p = (n+1)^2 + n^2$

ratio = $\frac{8}{2} = 4$

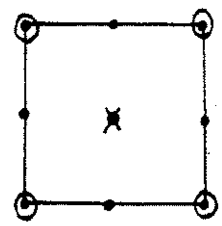
9/4-c



$n_u = 2(2n+1)^2$ $n_p = (n+1)^2$

ratio = $\frac{8}{1} = 8$

9/(4-c+1)



$n_u = 2(2n+1)^2$ $n_p = (n+1)^2 + n^2$

ratio = $\frac{8}{2} = 4$

5.1 For a 4-node truss element, a cubic interpolat polynomial is applicable.

Let $h_4 = a_0 + a_1 r + a_2 r^2 + a_3 r^3$

$\rightarrow h_4 = \begin{cases} 1 & \text{at } r = 1/3 \\ 0 & \text{at } r = -1, -\frac{1}{3} \text{ and } 1 \end{cases}$ $\rightarrow a_0 = 9/16,\ a_1 = 27/16,\ a_2 = -9/16,\ a_3 = -27/$

$\therefore\ h_4 = \frac{1}{16}(-27r^3 - 9r^2 + 27r + 9)$

For h_1, by superimposing a linear function $\frac{1}{2}(1-r)$, and als. $(1-r^2)$ and h_4 such that

$$h_1 = \begin{cases} 1 & \text{at } r = -1 \\ 0 & \text{at } r = -1/3,\ 1/3 \text{ and } 1 \end{cases}$$

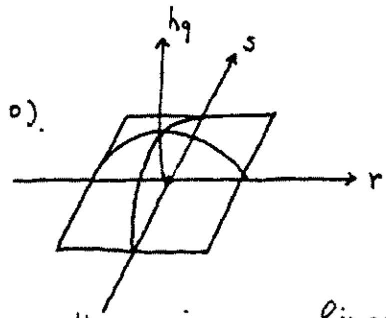
$$\therefore\ h_1 = \frac{1}{2}(1-r) - \frac{1}{2}(1-r^2) + \frac{1}{16}(-9r^3 + r^2 + 9r - 1)$$

Similarly, for h_2 and h_3, we obtain:

$$h_2 = \frac{1}{2}(1+r) - \frac{1}{2}(1-r^2) + \frac{1}{16}(9r^3 + r^2 - 9r - 1)$$

$$h_3 = (1-r^2) + \frac{1}{16}(27r^3 + 7r^2 - 27r - 7)$$

5.2 Construct first h_9 which contains r^2 and s^2 as the highest powers in r and s respectively. It is a parabola in r and s as shown.



Let $h_9 = a_9(1-r^2)(1-s^2)$ with $h_9 = 1$ at $(0,0)$.

$\therefore a_9 = 1 \longrightarrow h_9 = (1-r^2)(1-s^2)$

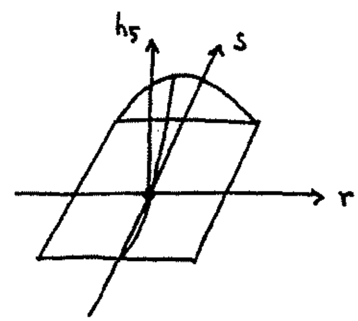
For h_5, the highest power in r is r^2 whereas there is a linear variation in the s direction. Let $g_5 = a_5(1-r^2)(1+s)$

$$g_5 = \begin{cases} 1 & \text{at } r=0,\ s=1 \\ 0 & \text{at other nodes, excluding node 9.} \end{cases}$$

$\therefore a_5 = \frac{1}{2} \rightarrow g_5 = \frac{1}{2}(1-r^2)(1+s)$

However when node 9 is present, h_5 is affected, and hence

$$\therefore\ h_5 = \frac{1}{2}(1-r^2)(1+s) \ \vdots\ -\frac{1}{2}h_9$$



Similarly,

$$h_6 = \frac{1}{2}(1-s^2)(1-r) \ \vdots\ -\frac{1}{2}h_9$$

$$h_7 = \frac{1}{2}(1-r^2)(1-s) \ \vdots\ -\frac{1}{2}h_9$$

$$h_8 = \frac{1}{2}(1-s^2)(1+r) \ \vdots\ -\frac{1}{2}h_9$$

From these h_i's $i = 5, 6, 7, 8, 9$ we can obtain the h_i's, $i = 1..4$ by superimposing the linear and parabolic functions. For h_1, which is affected by h_5 and h_8 on the sides of $r=1$ and $s=1$, and by h_9 at the center, let $h_1 = \frac{1}{4}(1+r)(1+s) + b_1 h_5 + b_2 h_8 + b_3 h_9$.

$h_1 = 0$ at nodes 5, 8 and 9 $\rightarrow b_1 = -\frac{1}{2} = b_2,\ b_3 = -\frac{1}{4}$

$$\therefore\ h_1 = \frac{1}{4}(1+r)(1+s) - \frac{1}{2}h_5 - \frac{1}{2}h_8 - \frac{1}{4}h_9$$

5.2

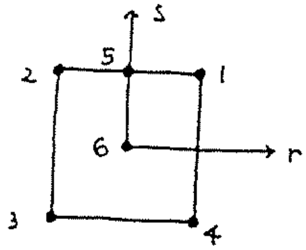
This procedure can be similarly applied to h_2, h_3 and h_4, which gives,

$$h_2 = \frac{1}{4}(1-r)(1+s) - \frac{1}{2}(h_5 + h_6) - \frac{1}{4}h_9$$

$$h_3 = \frac{1}{4}(1-r)(1-s) - \frac{1}{2}(h_6 + h_7) - \frac{1}{4}h_9$$

$$h_4 = \frac{1}{4}(1+r)(1-s) - \frac{1}{2}(h_7 + h_8) - \frac{1}{4}h_9$$

5.3 Using the interpolation functions in Fig. 5.4 we obtain



$$h_6 = (1-r^2)(1-s^2)$$

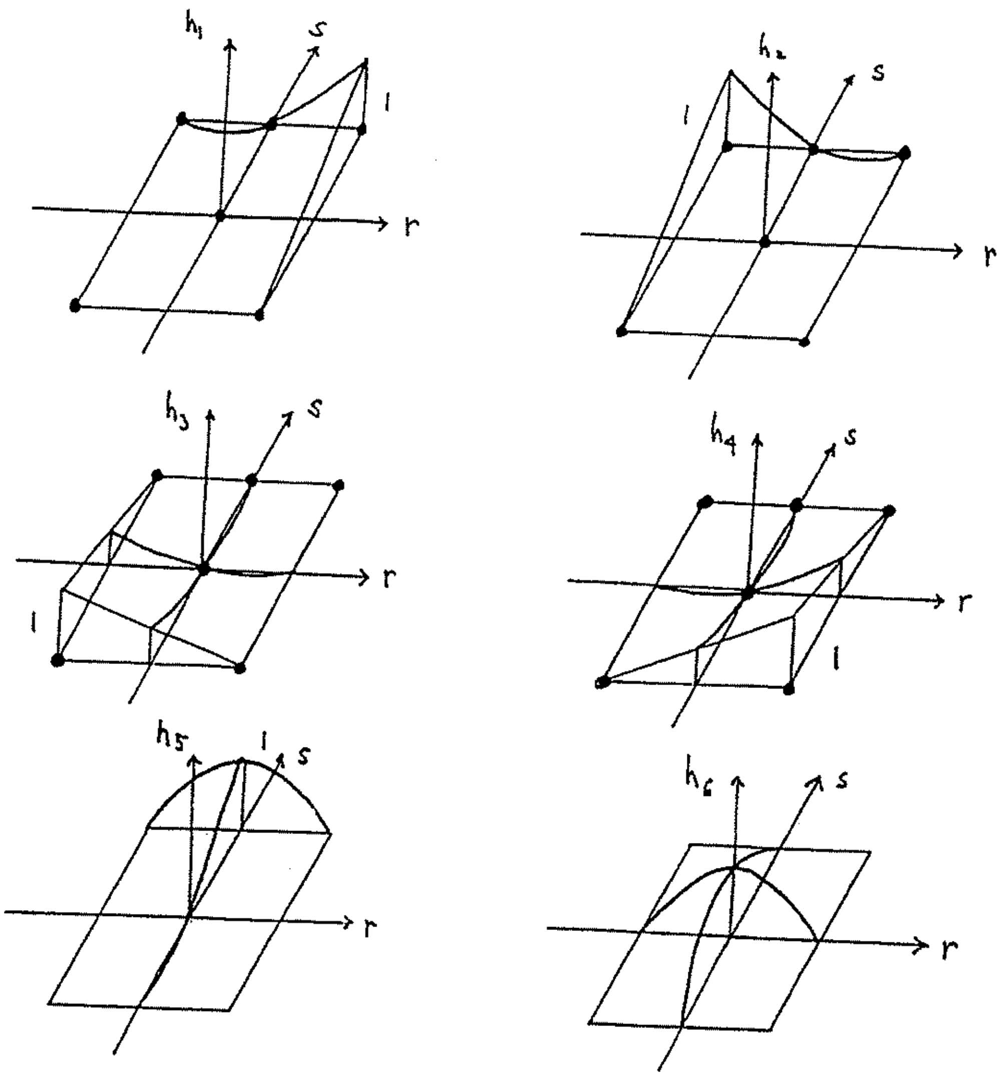
$$h_5 = \frac{1}{2}(1-r^2)(1+s) - \frac{1}{2}h_6$$

$$h_1 = \frac{1}{4}(1+r)(1+s) - \frac{1}{2}h_5 - \frac{1}{4}h_6$$

$$h_2 = \frac{1}{4}(1-r)(1+s) - \frac{1}{2}h_5 - \frac{1}{4}h_6$$

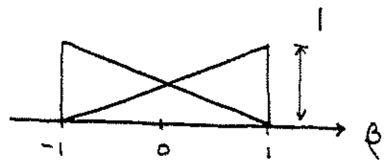
$$h_3 = \frac{1}{4}(1-r)(1-s) - \frac{1}{4}h_6$$

$$h_4 = \frac{1}{4}(1+r)(1-s) - \frac{1}{4}h_6$$

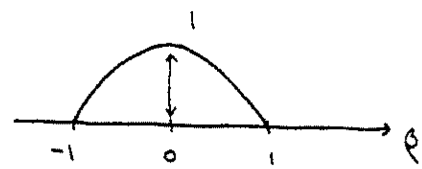


5.4 Let's construct first $g_i = G(r, r_i)\,G(s, s_i)\,G(t, t_i)$, $i = 1..8$

For $\beta_i = \pm 1$, For $\beta_i = 0$



$$G(\beta, \beta_i) = \frac{1}{2}(1 + \beta_i \beta)$$

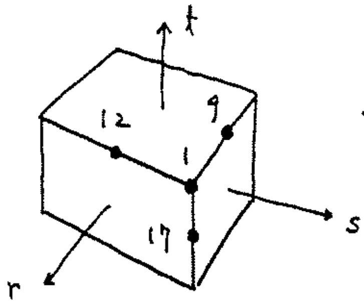


$$G(\beta, \beta_i) = 1 - \beta^2$$

Then, we easily get $g_i = G(r, r_i)\,G(s, s_i)\,G(t, t_i)$.

Now modify g_i's with the effect of g_j $(i \neq j)$, $j = 9..20$.

For node 1, let $h_1 = g_1 + a_1 g_9 + a_2 g_{12} + a_3 g_{17}$



$h_1 = 0$ at nodes 9, 12, and 17

that is, at $r = 0,\ s = 1,\ t = 1$

$r = 1,\ s = 0,\ t = 1$

$r = 1,\ s = 1,\ t = 0$

$$\therefore a_1 = a_2 = a_3 = \frac{1}{2}$$

$$\therefore h_1 = g_1 + \frac{1}{2} g_9 + \frac{1}{2} g_{12} + \frac{1}{2} g_{17}.$$

Similarly, for nodes 2, 3, .., 8 the interpolation functions can be modified to give those shown in Fig. 5.5.

5.5 The individual functions are obtained by combining the basic linear, parabolic, and cubic interpolations corresponding to the r- and s- directions. Thus using the functions in Fig. 5.3 we obtain

$$h_5 = \left[\frac{1}{16}\left(-27r^3 - 9r^2 + 27r + 9\right) \right]\left[\frac{1}{2}(1+s) \right]$$

$$h_6 = \left[(1-r^2) + \frac{1}{16}\left(27r^3 + 7r^2 - 27r - 7\right) \right]\left[\frac{1}{2}(1+s) \right]$$

$$h_2 = \left[\frac{1}{2}(1-r) - \frac{1}{2}(1-r^2) + \frac{1}{16}\left(-9r^3 + r^2 + 9r - 1\right) \right]\left[\frac{1}{2}(1+s) \right]$$

$$h_3 = \frac{1}{4}(1-r)(1-s)$$

$$h_7 = \left[\frac{1}{2}(1+r) \right]\left[\frac{1}{16}\left(-27s^3 - 9s^2 + 27s + 9\right) \right]$$

$$h_8 = \left[\frac{1}{2}(1+r) \right]\left[(1-s^2) + \frac{1}{16}\left(27s^3 + 7s^2 - 27s - 7\right) \right]$$

$$h_4 = \left[\frac{1}{2}(1+r) \right]\left[\frac{1}{2}(1-s) - \frac{1}{2}(1-s^2) + \frac{1}{16}\left(-9s^3 + s^2 + 9s - 1\right) \right]$$

$$h_1 = \frac{1}{4}(1+r)(1+s) - \frac{2}{3}h_5 - \frac{1}{3}h_6 - \frac{2}{3}h_7 - \frac{1}{3}h_8$$

5.6 Here $dV = (\underline{r}\,dr) \times (\underline{s}\,ds) \cdot (\underline{t}\,dt)$

where $\underline{r}^T = \left[\frac{\partial x}{\partial r} \;\; \frac{\partial y}{\partial r} \;\; \frac{\partial z}{\partial r} \right]$, $\underline{s}^T = \left[\frac{\partial x}{\partial s} \;\; \frac{\partial y}{\partial s} \;\; \frac{\partial z}{\partial s} \right]$

and $\underline{t}^T = \left[\frac{\partial x}{\partial t} \;\; \frac{\partial y}{\partial t} \;\; \frac{\partial z}{\partial t} \right]$

Then,

$$(\underline{r}\,dr) \times (\underline{s}\,ds) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial x}{\partial r}dr & \frac{\partial y}{\partial r}dr & \frac{\partial z}{\partial r}dr \\ \frac{\partial x}{\partial s}ds & \frac{\partial y}{\partial s}ds & \frac{\partial z}{\partial s}ds \end{vmatrix}$$

$$= \left(\frac{\partial y}{\partial r}\frac{\partial z}{\partial s} - \frac{\partial z}{\partial r}\frac{\partial y}{\partial s} \right) dr\,ds\,\hat{i} + \left(\frac{\partial z}{\partial r}\frac{\partial x}{\partial s} - \frac{\partial x}{\partial r}\frac{\partial z}{\partial s} \right) dr\,ds\,\hat{j}$$
$$+ \left(\frac{\partial x}{\partial r}\frac{\partial y}{\partial s} - \frac{\partial y}{\partial r}\frac{\partial x}{\partial s} \right) dr\,ds\,\hat{k}$$

Hence, $dV = (\underline{r}\,dr) \times (\underline{s}\,ds) \cdot (\underline{t}\,dt)$

$$= \left[\left(\frac{\partial y}{\partial r}\frac{\partial z}{\partial s} - \frac{\partial z}{\partial r}\frac{\partial y}{\partial s} \right) \frac{\partial x}{\partial t} + \left(\frac{\partial z}{\partial r}\frac{\partial x}{\partial s} - \frac{\partial x}{\partial r}\frac{\partial z}{\partial s} \right) \frac{\partial y}{\partial t} \right.$$
$$\left. + \left(\frac{\partial x}{\partial r}\frac{\partial y}{\partial s} - \frac{\partial y}{\partial r}\frac{\partial x}{\partial s} \right) \frac{\partial z}{\partial t} \right] dr\,ds$$

$\therefore dV = \det \underline{J}\; dr\,ds\,dt$

5.7 $x = \sum h_i x_i$, $y = \sum h_i y_i$, $\underline{J} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix}$ with $h_1 = (1+r)(1+s)/4$

$h_2 = (1-r)(1+s)/4$

$h_3 = (1-r)(1-s)/4$

$h_4 = (1+r)(1-s)/4$

element 1

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_3 + 3 + 3r \\ y_3 + 2 + 2s \end{bmatrix}, \quad \underline{J}_1 = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

element 2 $\underline{P}$: rotation matrix , $l = \cos 30°$, $m = \sin 30°$

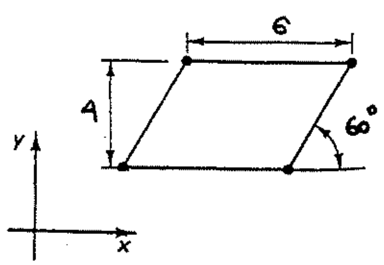
$$\underline{P} = \begin{bmatrix} l & m \\ -m & l \end{bmatrix}, \quad \therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l & -m \\ m & l \end{bmatrix} \begin{bmatrix} 3+3r \\ 2+2s \end{bmatrix} + \begin{bmatrix} x_3 \\ y_3 \end{bmatrix}$$

and $\underline{J}_2 = \begin{bmatrix} 3l & 3m \\ -2m & 2l \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} l & m \\ -m & l \end{bmatrix} = \underline{J}_1 \underline{P}$

Here we see that $\underline{J}$ of element 2 contains a rotation matrix $\underline{P}$ representing a 30-degree rotation.

element 3

First, consider an unrotated element.



$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 + \frac{2}{\sqrt{3}}(1+s) + 3r \\ 2 + 2s \end{bmatrix}$$

$$\underline{J}_0 = \begin{bmatrix} 3 & 0 \\ \frac{2}{\sqrt{3}} & 2 \end{bmatrix}$$

S.7

For element 3 we have:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{3}{2}\sqrt{3}\,(1+r) \\ \frac{3}{2}(1+r) + \frac{4}{\sqrt{3}}(1+s) \end{bmatrix}$$

$$\underline{J}_3 = \begin{bmatrix} \frac{3}{2}\sqrt{3} & \frac{3}{2} \\ 0 & \frac{4}{\sqrt{3}} \end{bmatrix} = \underline{J}_0\,\underline{P}$$

5.8 Let $h_1 = \frac{1}{4}(1+r)(1+s)$, $h_2 = \frac{1}{4}(1-r)(1+s)$, $h_3 = \frac{1}{4}(1-r)(1-s)$ and $h_4 = \frac{1}{4}(1+r)(1-s)$.

Then,

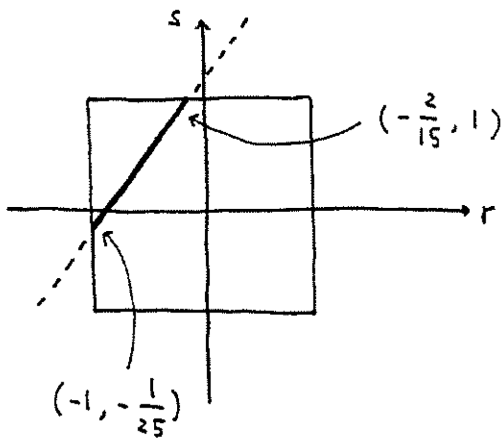
$$x = \sum h_i x_i = \frac{1}{4}(25 + 7r + s - 5rs)$$

$$y = \sum h_i y_i = \frac{1}{4}(17 + 5r + 9s + 5rs)$$

$$\therefore \quad \underline{J} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 7-5s & 5+5s \\ 1-5r & 9+5r \end{bmatrix}$$

The $\underline{J}$ is singular when its determinant vanishes.

$$\det \underline{J} = \frac{1}{8}(29 + 30r - 25s) = 0 \qquad \therefore \quad 29 + 30r - 25s = 0$$



5.9 $x = \sum h_i x_i$, $y = \sum h_i y_i$

$h_1 = \frac{1}{4}(1+r)(1+s)$, $h_2 = \frac{1}{4}(1-r)(1+s)$, $h_3 = \frac{1}{4}(1-r)(1-s)$, $h_4 = \frac{1}{4}(1+r)(1-s)$.

For element 1, replacing the element coordinates,

$$x = \frac{11+5r}{2}, \quad y = \frac{15+r+7s+rs}{4}$$

$$\therefore \underline{J}_1 = \frac{1}{4}\begin{bmatrix} 10 & 1+s \\ 0 & 7+r \end{bmatrix}$$

Similarly for element 2,

$$x = \frac{1}{8}\left\{17+10\sqrt{3} + (10\sqrt{3}-1)r - 7s - rs\right\}$$

$$y = \frac{1}{8}\left\{26+7\sqrt{3} + (10+\sqrt{3})r + 7\sqrt{3}s + \sqrt{3}rs\right\}$$

$$\therefore \underline{J}_2 = \frac{1}{8}\begin{bmatrix} (10\sqrt{3}-1)-s & (10+\sqrt{3})+\sqrt{3}s \\ -7-r & \sqrt{3}(7+r) \end{bmatrix}$$

Note that element 2 can be obtained by rotating element 1 around node 3 by $\theta = 30°$. Let

$$\underline{P} = \begin{bmatrix} \cos 30° & \sin 30° \\ -\sin 30° & \cos 30° \end{bmatrix}$$

Then $\underline{J}_1\underline{P} = \frac{1}{4}\begin{bmatrix} 10 & 1+s \\ 0 & 7+r \end{bmatrix}\frac{1}{2}\begin{bmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{bmatrix} = \frac{1}{8}\begin{bmatrix} (10\sqrt{3}-1)-s & (10+\sqrt{3})+\sqrt{3}s \\ -7-r & \sqrt{3}(7+r) \end{bmatrix}$

$$\therefore \underline{J}_2 = \underline{J}_1\underline{P}$$

5.10 Let $g_1 = \frac{1}{4}(1+r)(1+s)$, $g_2 = \frac{1}{4}(1-r)(1+s)$, $g_3 = \frac{1}{4}(1-r)(1-s)$,

$g_4 = \frac{1}{4}(1+r)(1-s)$, $g_5 = \frac{1}{2}(1-r^2)(1+s)$, $g_6 = \frac{1}{2}(1-s^2)(1-r)$,

$g_7 = \frac{1}{2}(1-r^2)(1-s)$ and $g_8 = \frac{1}{2}(1-s^2)(1+r)$.

(a) Here, $h_1 = g_1 - \frac{1}{2}(g_5 + g_8)$, $h_2 = g_2 - \frac{1}{2}(g_5 + g_6)$, $h_3 = g_3 - \frac{1}{2}(g_6 + g_7)$,

$h_4 = g_4 - \frac{1}{2}(g_7 + g_8)$, $h_5 = g_5$, $h_6 = g_6$, $h_7 = g_7$ and $h_8 = g_8$

$$x = \sum_{i=1}^{8} h_i x_i = \frac{1}{2}(17 + 6r + s - r^2 - r^2 s + 2rs^2)$$

$$y = \sum_{i=1}^{8} h_i y_i = \frac{1}{2}(9 + 2r + 7s + 3r^2 + 2rs + r^2 s)$$

when	x	y
$r = \frac{1}{2}$,	$x = \frac{1}{8}(79 + 3s + 4s^2)$,	$y = \frac{1}{8}(43 + 33s)$
$r = -\frac{1}{4}$	$\frac{1}{32}(247 + 15s - 8s^2)$	$\frac{1}{32}(139 + 105s)$
$s = \frac{3}{4}$	$\frac{1}{16}(142 + 57r - 14r^2)$	$\frac{1}{8}(57 + 19r + 15r^2)$
$s = -\frac{1}{3}$	$\frac{1}{9}(75 + 28r - 3r^2)$	$\frac{2}{3}(5 + r + 2r^2)$

(b) $h_1 = g_1 - \frac{1}{2}g_5$, $h_2 = g_2 - \frac{1}{2}(g_5 + g_6)$, $h_3 = g_3 - \frac{1}{2}g_6$, $h_4 = g_4$,

$h_5 = g_5$, and $h_6 = g_6$.

$$x = \sum_{i=1}^{6} h_i x_i = \frac{1}{4}(27 + 7r - 2rs - 7s^2 + 7rs^2)$$

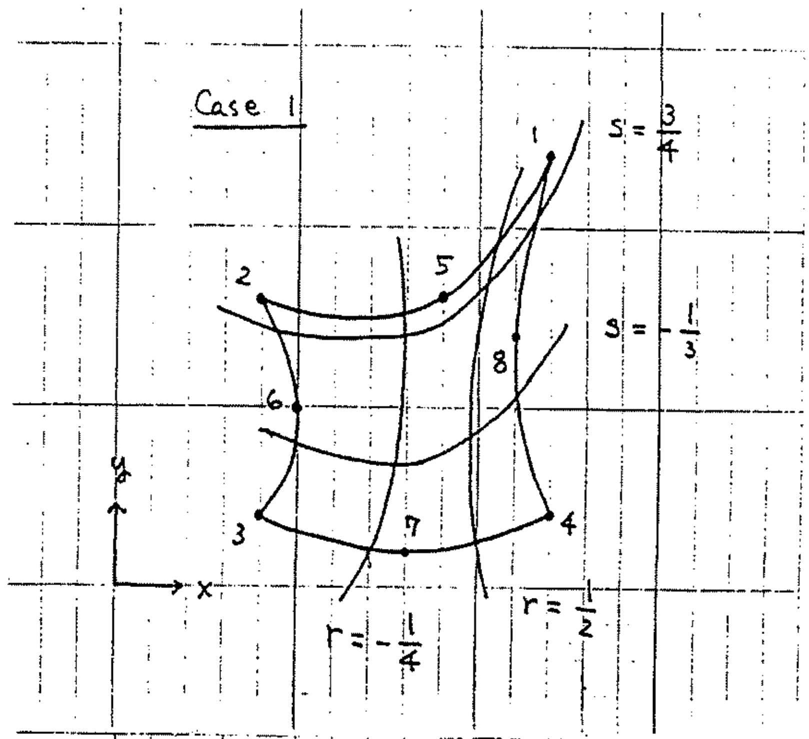
$$y = \sum_{i=1}^{6} h_i y_i = \frac{1}{4}(15 + 7r + 14s + 4rs + 7s^2 - 7rs^2)$$

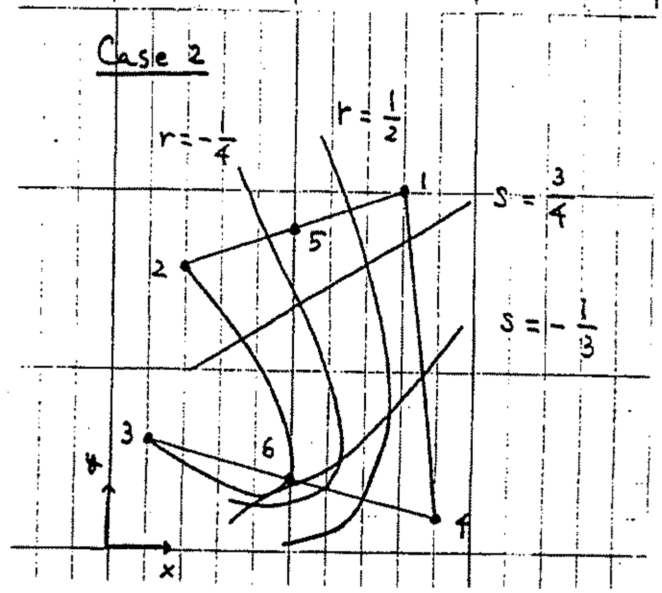
when	x	y
$r = \frac{1}{2}$,	$x = \frac{1}{8}(61 - 2s - 7s^2)$,	$y = \frac{1}{8}(37 + 32s + 7s^2)$
$r = -\frac{1}{4}$	$\frac{1}{16}(101 + 2s - 35s^2)$	$\frac{1}{16}(53 + 52s + 35s^2)$

5.10

when $s = \frac{3}{4}$, $\quad x = \frac{1}{64}(369 + 151r)$, $\quad y = \frac{1}{64}(471 + 97r)$

$s = -\frac{1}{3}$ $\quad \frac{1}{9}(59 + 19r)$ $\quad \frac{1}{9}(25 + 11r)$





5.11 $x = \sum h_i x_i \ , \quad y = \sum h_i y_i$

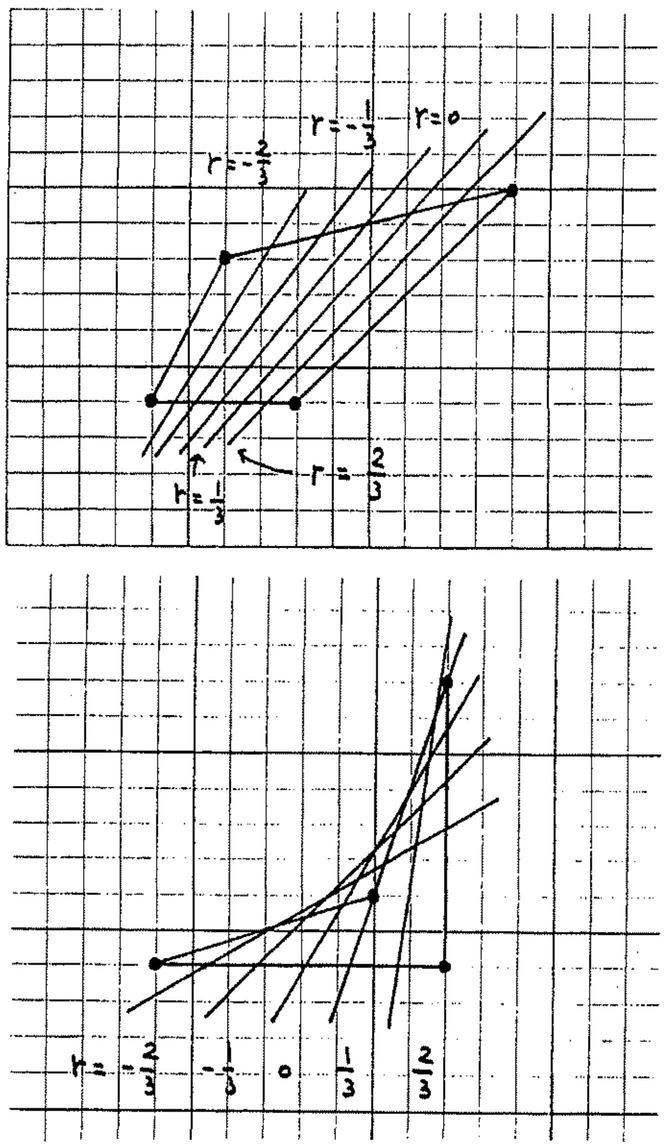
where $h_1 = \frac{1}{4}(1+r)(1+s)$, $h_2 = \frac{1}{4}(1-r)(1+s)$, $h_3 = \frac{1}{4}(1-r)(1-s)$

$h_4 = \frac{1}{4}(1+r)(1-s)$

For element 1,
$$\begin{cases} x = \frac{1}{2}(8+3r+2s+rs) \\ y = \frac{1}{4}(13+r+5s+rs) \end{cases}$$

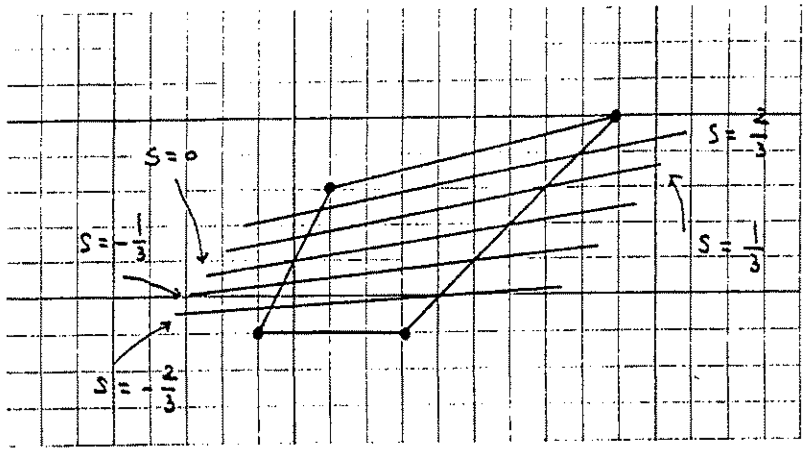
For element 2,
$$\begin{cases} x = \frac{1}{4}(19+5r+3s-3rs) \\ y = \frac{1}{4}(13+3r+5s+3rs) \end{cases}$$

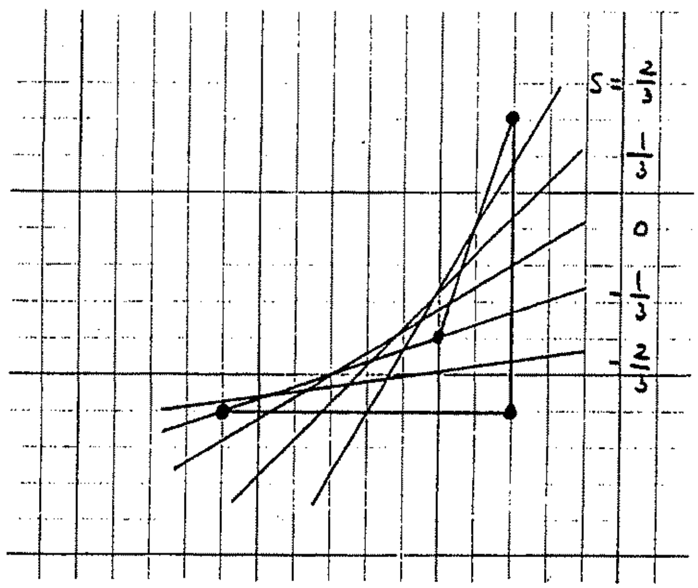
(a)



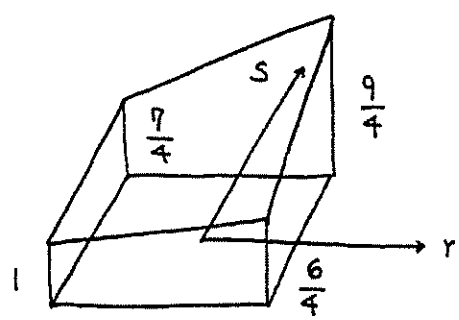
5.11

(b)





(c) $\det \underline{J} = \begin{vmatrix} \frac{3+s}{2} & \frac{1+s}{4} \\ \frac{2+r}{2} & \frac{5+r}{4} \end{vmatrix} = \frac{1}{8}(13 + 2r + 3s)$



5.12

For a square, rectangular, or parallelogram-shaped element we can write

$$x = a_1 + a_2 r + a_3 s$$

$$y = b_1 + b_2 r + b_3 s$$

Hence, $$\det \underline{J} = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = \text{const}$$

For all other cases we have in general

$$x = a_1 + a_2 r + a_3 s + a_4 rs \ldots$$

$$y = b_1 + b_2 r + b_3 s + b_4 rs \ldots$$

Therefore, $\det \underline{J} \neq \text{const.}$

5.13 $h_9 = (1-r^2)(1-s^2)$,

$h_5 = \frac{1}{2}(1-r^2)(1+s) - \frac{1}{2}h_9$, $h_6 = \frac{1}{2}(1-s^2)(1-r) - \frac{1}{2}h_9$,

$h_7 = \frac{1}{2}(1-r^2)(1-s) - \frac{1}{2}h_9$, $h_8 = \frac{1}{2}(1-s^2)(1+r) - \frac{1}{2}h_9$,

$$h_1 = \frac{1}{4}(1+r)(1+s) - \frac{1}{2}h_5 - \frac{1}{2}h_8 - \frac{1}{4}h_9$$

$$h_2 = \frac{1}{4}(1-r)(1+s) - \frac{1}{2}h_5 - \frac{1}{2}h_6 - \frac{1}{4}h_9$$

$$h_3 = \frac{1}{4}(1-r)(1-s) - \frac{1}{2}h_6 - \frac{1}{2}h_7 - \frac{1}{4}h_9$$

$$h_4 = \frac{1}{4}(1+r)(1-s) - \frac{1}{2}h_7 - \frac{1}{2}h_8 - \frac{1}{4}h_9$$

Then $x = \sum h_i x_i = 2 + 2r$

$$y = \sum h_i y_i = 2.5 + r^2\left(-0.5 + \frac{s}{2}\right) + \frac{3}{2}s$$

$$\therefore \underset{\sim}{J} = \begin{bmatrix} 2 & -r + rs \\ 0 & \frac{1}{2}(3 + r^2) \end{bmatrix}$$

5.14 The nodal point forces are given by

$$\underline{R}_S = \int_S \underline{H}^{S^T} \underline{f}^S \, dS \; , \quad dS = \det \underline{J}^S x \, ds \; , \quad \underline{f}^S = \begin{bmatrix} -\frac{P}{2}(3-s) \\ 0 \end{bmatrix}$$

where $\underline{H}^S$ is the matrix $\underline{H}$ evaluated at $r=1$.

$$h_1(1,s) = \tfrac{1}{2}(1+s) - \tfrac{1}{2}(1-s^2) \; , \quad h_2(1,s) = h_3(1,s) = 0 \; ,$$

$$h_4(1,s) = \tfrac{1}{2}(1-s) - \tfrac{1}{2}(1-s^2) \; , \quad h_5(1,s) = h_6(1,s) = h_7(1,s) = 0 \; ,$$

$$h_8(1,s) = 1 - s^2 \; , \quad h_9(1,s) = 0 \; .$$

$$\det \underline{J}^S = \frac{\partial y}{\partial s} = \frac{3}{2}$$

Thus,

$$\underline{R}_S = \int_{-1}^{1} \begin{bmatrix} h_1(1,s) & 0 \\ 0 & h_1(1,s) \\ h_2(1,s) & 0 \\ 0 & h_2(1,s) \\ & \vdots \\ h_9(1,s) & 0 \\ 0 & h_9(1,s) \end{bmatrix} \begin{bmatrix} -\frac{P}{2}(3-s) \\ 0 \end{bmatrix} (4)\left(\tfrac{3}{2}\right) ds$$

$$\therefore \; \underline{R}_S^T = [\, -2 \;\; 0 \;\vdots\; 0 \;\; 0 \;\vdots\; 0 \;\; 0 \;\vdots\; -4 \;\; 0 \;\vdots$$

$$0 \;\; 0 \;\vdots\; 0 \;\; 0 \;\vdots\; 0 \;\; 0 \;\vdots\; -12 \;\; 0 \;\vdots\; 0 \;\; 0 \,] \, P$$

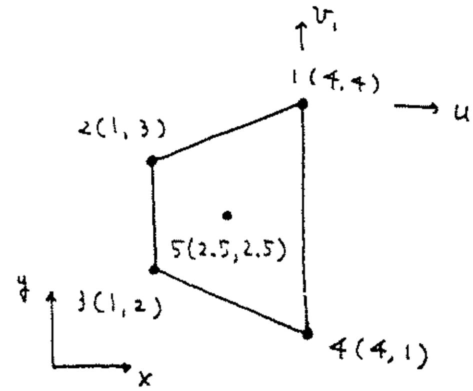
5.15 (a) $h_5 = (1-r^2)(1-s^2)$

$$h_1 = \frac{1}{4}(1+r)(1+s) - \frac{1}{4}h_5$$

$$h_2 = \frac{1}{4}(1-r)(1+s) - \frac{1}{4}h_5$$

$$h_3 = \frac{1}{4}(1-r)(1-s) - \frac{1}{4}h_5$$

$$h_4 = \frac{1}{4}(1+r)(1-s) - \frac{1}{4}h_5$$



(b) $x = \sum h_i x_i = 2.5 + 1.5r$

$y = \sum h_i y_i = 2.5 + s + 0.5rs$

The point $x = 2.5$, $y = 2.5$ corresponds to $r=0$, $s=0$.

$$\underline{J} = \begin{bmatrix} 1.5 & 0.5s \\ 0 & 1+0.5r \end{bmatrix}\Bigg|_{(0,0)} = \begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix}, \quad \underline{J}^{-1} = \begin{bmatrix} \frac{2}{3} & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underline{B}_{u_1}\Big|_{(0,0)} = \begin{bmatrix} h_{1,x} \\ 0 \\ h_{1,y} \end{bmatrix}\Bigg|_{(0,0)} = \begin{bmatrix} \frac{2}{3}h_{1,r} \\ 0 \\ h_{1,s} \end{bmatrix}\Bigg|_{(0,0)}$$

at $(r,s) = (0,0)$ $\quad h_{1,r} = \frac{1}{4} = h_{1,s}$

$$\therefore \quad \underline{B}_{u_1}\Big|_{(0,0)}^{T} = \begin{bmatrix} \frac{1}{6} & 0 & \frac{1}{4} \end{bmatrix}$$

5.16 (a) $x = \sum h_i x_i = \frac{1}{4}(3 + 13r + s + 3rs)$

$y = \sum h_i y_i = 1 + 3s + rs$

$$\therefore \quad \underline{J} = \frac{1}{4}\begin{bmatrix} 13+3s & 4s \\ 1+3r & 12+4r \end{bmatrix}$$

(b) $$\underline{J}^{-1} = \frac{1}{39 + 13r + 8s}\begin{bmatrix} 12+4r & -4s \\ -1-3r & 13+3s \end{bmatrix}, \quad h_{1,r} = \frac{1+s}{4}, \quad h_{1,s} = \frac{1+r}{4}$$

$$\therefore \underline{B}_{u_1} = \begin{bmatrix} h_{1,x} \\ 0 \\ h_{1,y} \\ \frac{h_1}{x} \end{bmatrix} = \begin{bmatrix} (3+2s+r)/(39+13r+8s) \\ 0 \\ (6+s+5r)/2(39+13r+8s) \\ (1+r)(1+s)/(3+13r+s+3rs) \end{bmatrix}$$

5.17 (a) $\underline{u} = \begin{bmatrix} u(r,s) \\ v(r,s) \end{bmatrix} = \begin{bmatrix} h_1 u_1 \\ 0 \end{bmatrix}$, $h_1 = \frac{1}{4}(1+r)(1+s) - \frac{1}{2}\left[\frac{1}{2}(1-r^2)(1+s)\right] - \frac{1}{2}\left[\frac{1}{2}(1-s^2)(1+r)\right]$

$x = 2r + C_1, \quad y = s + C_2 \qquad \therefore \underline{J} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad \underline{J}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}, \quad \det \underline{J} = 2$

$h_{1,r} = \frac{1}{4}(2r+s)(1+s), \quad h_{1,s} = \frac{1}{4}(r+2s)(1+r)$

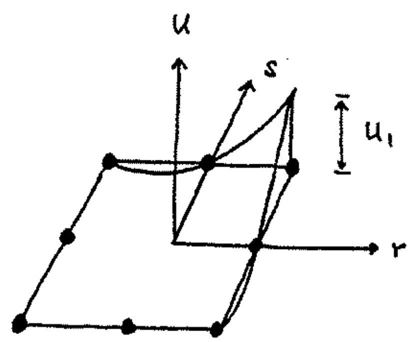
$$\therefore \underline{B} = \begin{bmatrix} h_{1,x} \\ 0 \\ h_{1,y} \end{bmatrix} = \begin{bmatrix} J_{11}^{-1} h_{1,r} + J_{12}^{-1} h_{1,s} \\ 0 \\ J_{21}^{-1} h_{1,r} + J_{22}^{-1} h_{1,s} \end{bmatrix} = \frac{1}{8} \begin{bmatrix} (2r+s)(1+s) \\ 0 \\ 2(r+2s)(1+r) \end{bmatrix}$$

$$\underline{C} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}, \quad \nu = 0.25$$

$$\longrightarrow \underline{K} = \int_{-1}^{1}\int_{-1}^{1} \underline{B}^T \underline{C}\, \underline{B}\, (2)(1)\, dr\, ds = [\, 0.77037\, E\,]$$

$$\underline{R} = [\, P\,] \qquad \therefore u_1 = \frac{P}{0.77037\, E} = (1.2981/E)P$$

$$\longrightarrow \underline{u} = \begin{bmatrix} h_1 \\ 0 \end{bmatrix} u_1$$

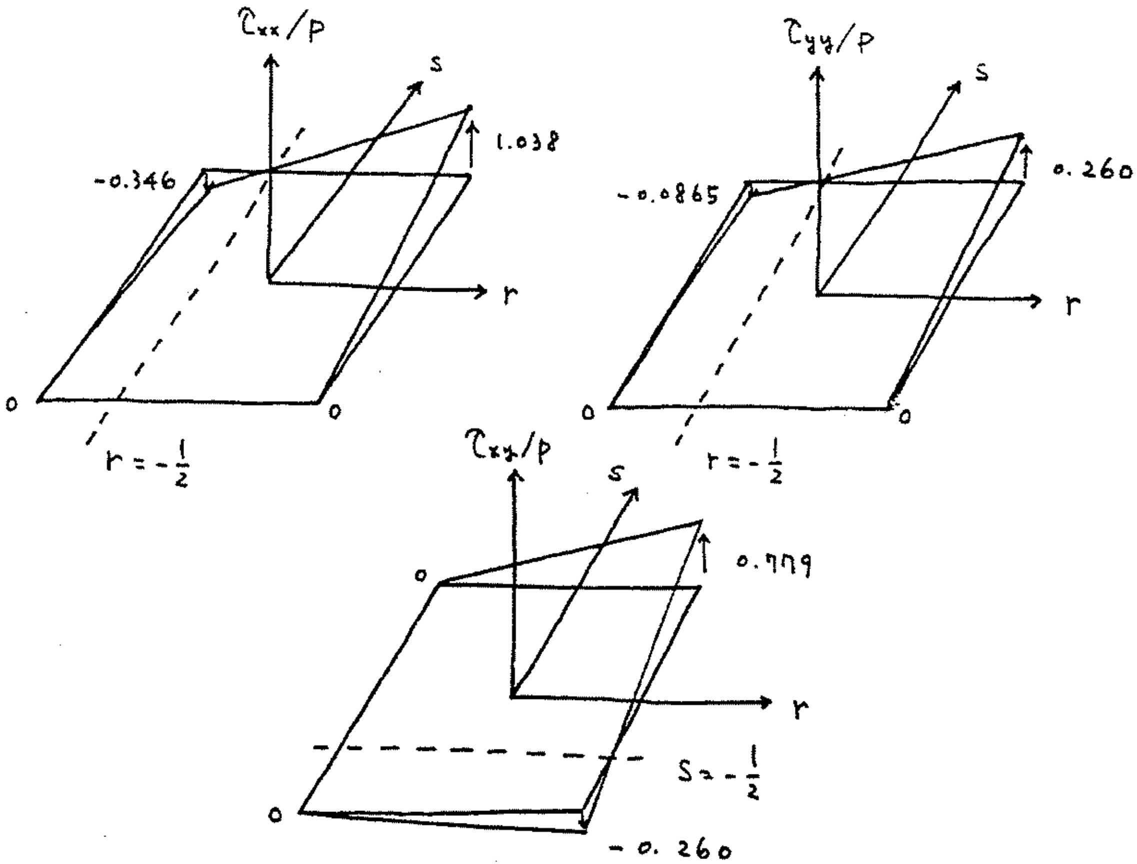


and $v = 0$ everywhere.

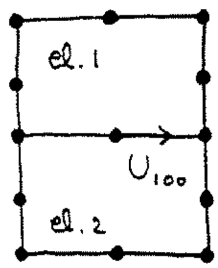
5.17

(b) $\underline{\tau} = \underline{C}\,\underline{\varepsilon}\;,\quad \underline{\varepsilon} = \underline{B}\,\hat{\underline{u}}\;,\quad \hat{\underline{u}} = [u_1]$

$$\therefore \begin{bmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{xy} \end{bmatrix} = \frac{E u_1}{1-\nu^2} \begin{bmatrix} \frac{1}{8}(2r+s)(1+s) \\ \nu\frac{1}{8}(2r+s)(1+s) \\ \frac{1-\nu}{2}\,\frac{1}{4}(r+2s)(1+r) \end{bmatrix} = \begin{bmatrix} 0.173\,(2r+s)(1+s) \\ 0.0433\,(2r+s)(1+s) \\ 0.130\,(r+2s)(1+r) \end{bmatrix} P$$



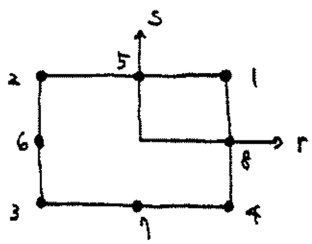
5.18



The d.o.f. U_{100} has an effect only on the element 1 and 2 shown on the left. As we consider the plane stress analysis,

$$\underline{C} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \quad , \quad \nu = 0.3 .$$

For element 1,



$$\underline{J} = \begin{bmatrix} 3 & 0 \\ 0 & \frac{3}{2} \end{bmatrix}, \ \det \underline{J} = \frac{9}{2}, \ \underline{J}^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{bmatrix}$$

$$\underline{B}_{U_7} = \begin{bmatrix} h_{7,x} \\ 0 \\ h_{7,y} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} h_{7,r} \\ 0 \\ \frac{2}{3} h_{7,s} \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} r(1-s) \\ 0 \\ -\frac{1}{3}(1-r^2) \end{bmatrix}$$

$$h_7 = \frac{1}{2}(1-r^2)(1-s)$$

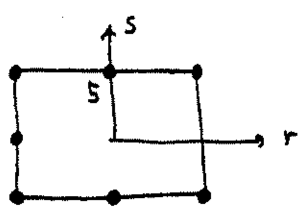
$$h_{7,r} = -r(1-s)$$

$$h_{7,s} = -\frac{1}{2}(1-r^2)$$

$$\therefore K_{77} = (0.5)\int_{-1}^{1}\int_{-1}^{1} \underline{B}_{U_7}^T \underline{C}\, \underline{B}_{U_7} \left(\frac{9}{2}\right) dr\, ds = 0.6935\, E$$

$$M_{77} = (0.5)\rho \int_{-1}^{1}\int_{-1}^{1} h_7 h_7 \left(\frac{9}{2}\right) dr\, ds = 1.60\, \rho$$

Similarly for element 2,

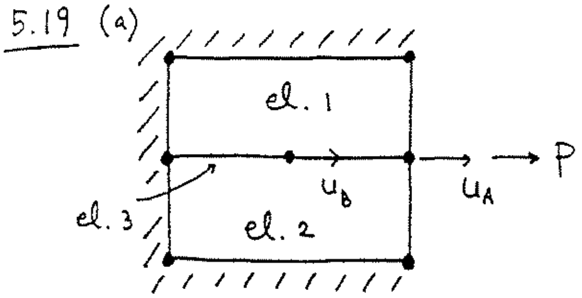


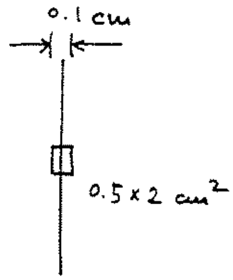
$$K_{55} = 0.6935\, E$$

$$M_{55} = 1.60\, \rho$$

Hence,
$$K_{U_{100}} = 2\,(0.6935\, E) = 1.387\ E$$
$$M_{U_{100}} = 2\,(1.60\, \rho) = 3.20\, \rho$$

5.19 (a)





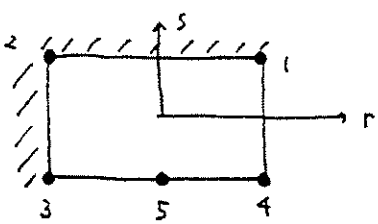
$$\underline{C} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

$E = 30 \times 10^6\ N/cm^2$, $\nu = 0.3$

$P = 6000\ N$

Because of geometric symmetries and boundary conditions, we need to evaluate only the stiffness coefficients corresponding to u_A and u_B.

For element 1,



$u = h_4 u_A + h_5 u_B$, $v = 0$

where $h_4 = \frac{1}{4}(1+r)(1-s) - \frac{1}{2}\left[\frac{1}{2}(1-r^2)(1-s)\right]$

$h_5 = \frac{1}{2}(1-r^2)(1-s)$

$$\underline{J}^{(1)} = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix},\quad \det \underline{J}^{(1)} = 12,\quad \underline{J}^{(1)-1} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

$$\underline{B}^{(1)} = \begin{bmatrix} h_{4,x} & h_{5,x} \\ 0 & 0 \\ h_{4,y} & h_{5,y} \end{bmatrix}\quad (\text{columns: } u_A,\ u_B),$$

$$h_{4,x} = J_{11}^{-1} h_{4,r} + J_{12}^{-1} h_{4,s} = \frac{1}{16}(1+2r)(1-s)$$

$$h_{4,y} = J_{21}^{-1} h_{4,r} + J_{22}^{-1} h_{4,s} = -\frac{1}{12}(1+r)r$$

Similarly, $h_{5,x} = -\dfrac{r(1-s)}{4}$, $h_{5,y} = -\dfrac{1-r^2}{6}$

$$\therefore\ \underline{K}^{(1)} = (0.1)\int_{-1}^{1}\int_{-1}^{1} \underline{B}^{(1)T}\underline{C}\ \underline{B}^{(1)}(12)\,dr\,ds$$

For element 2, because of symmetry, and using suitable nodal d.o.f. numbering, $\underline{K}^{(2)} = \underline{K}^{(1)}$.

5.19

For element 3,

$$u = h_2 u_A + h_3 u_B$$

$$h_2 = \frac{1}{2}(1+r) - \frac{1}{2}(1-r^2) \quad , \quad h_{2,r} = (1+2r)/2$$

$$h_3 = 1 - r^2 \quad , \quad h_{3,r} = -2r$$

$$\underline{J}^{(3)} = [4] \quad , \quad \underline{J}^{-1} = [1/4]$$

$$\underline{B}^{(3)} = [\, h_{2,x} \quad h_{3,x} \,] = [\, J^{-1} h_{2,r} \quad J^{-1} h_{3,r} \,] = [\, (1+2r)/8 \quad -r/2 \,]$$

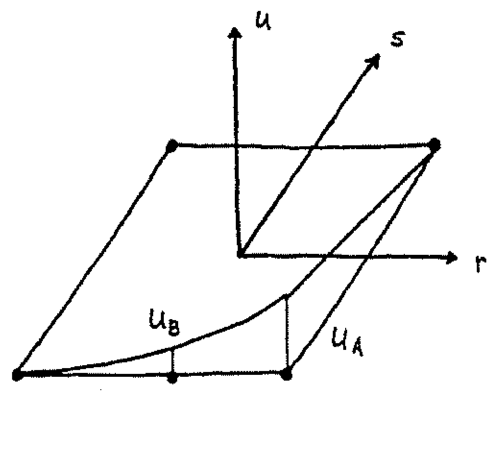
$$\therefore \underline{K}^{(3)} = EA \int_{-1}^{1} \underline{B}^T \underline{B} \, (4)\, dr \quad , \quad A = (0.5)(2)$$

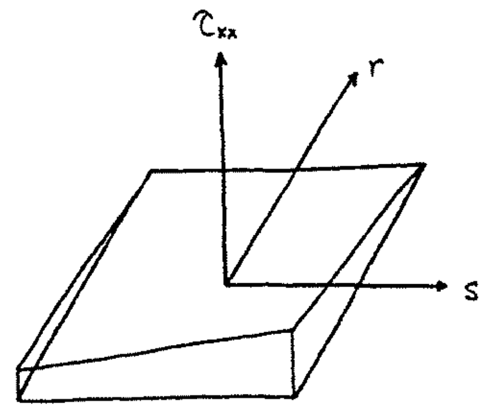
$$\Rightarrow \underline{K} = \underline{K}^{(1)} + \underline{K}^{(2)} + \underline{K}^{(3)} \quad , \quad \underline{U}^T = [\, u_A \quad u_B \,] \, , \quad \underline{R}^T = [\, P \quad 0 \,]$$

(b) For element 1

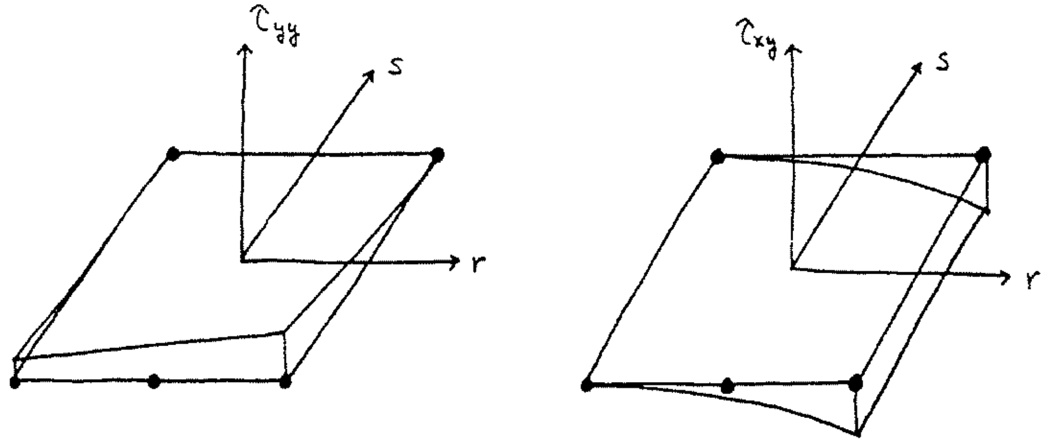
$$u = h_4 u_A + h_5 u_B$$

$$\underline{\tau} = \begin{bmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{xy} \end{bmatrix} = \underline{C}\,\underline{B}\,\underline{U} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \frac{1-s}{16} \{ (1+2r) u_A - 4r u_B \} \\ 0 \\ -\frac{r}{12}(1+r) u_A - \frac{1-r^2}{6} u_B \end{bmatrix}$$

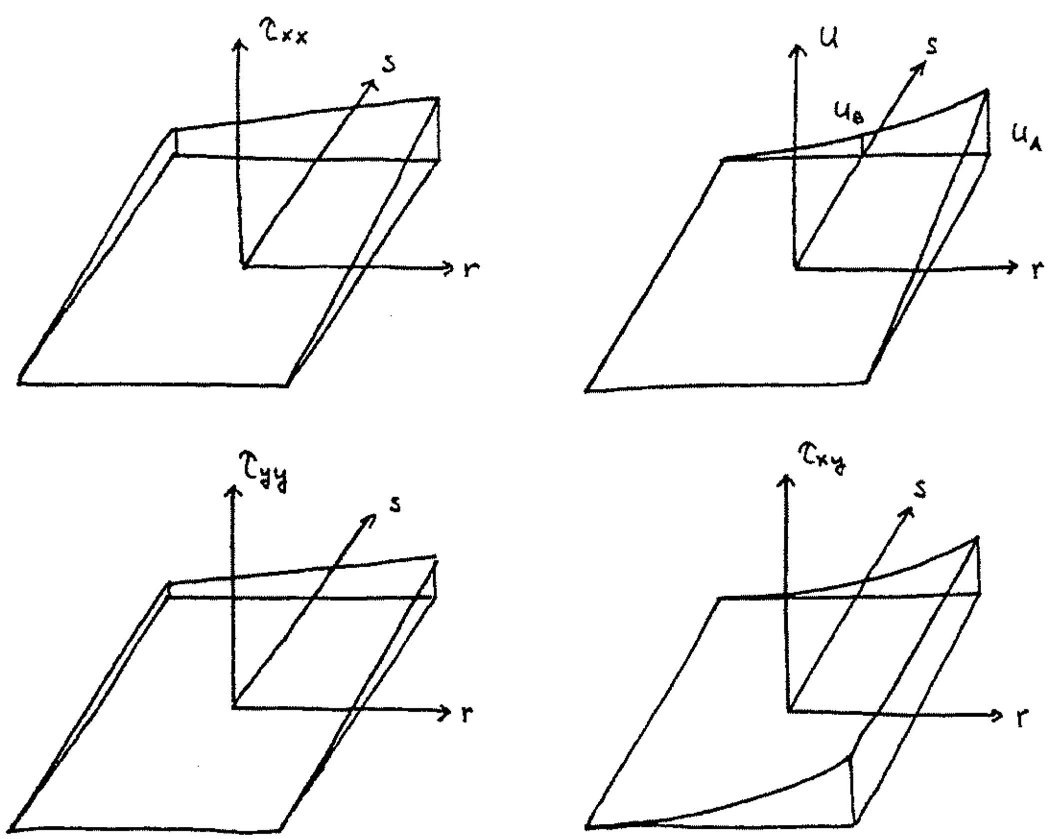




5.19



For element 2, by symmetry,

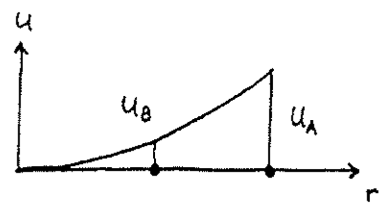


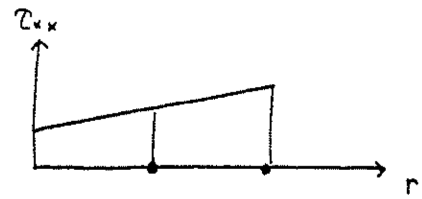
5.19

For element 3,

$$u = h_2 u_A + h_3 u_B \quad , \quad h_2 = \frac{1}{2}(1+r) - \frac{1}{2}(1-r^2), \quad h_3 = 1 - r^2$$

$$\tau_{xx} = E\left(\frac{1+2r}{8} u_A - \frac{r}{2} u_B\right)$$

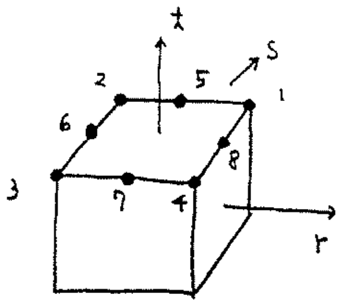




5.20 $\underset{\sim}{R}_s = \int_S \underset{\sim}{H}^{sT} \underset{\sim}{f}^s \, dS$

where $\underset{\sim}{H} = \begin{bmatrix} h_1 & 0 & 0 & h_2 & 0 & 0 & & & h_{20} & 0 & 0 \\ 0 & h_1 & 0 & 0 & h_2 & 0 & \cdots & & 0 & h_{20} & 0 \\ 0 & 0 & h_1 & 0 & 0 & h_2 & & & 0 & 0 & h_{20} \end{bmatrix}$

and $\underset{\sim}{H}^S = \underset{\sim}{H}\Big|_{t=1} = \begin{bmatrix} \underset{\sim}{A} & \underset{\sim}{O} \end{bmatrix}$, $\underset{\sim}{A} = \begin{bmatrix} h_1 & 0 & 0 & & h_8 & 0 & 0 \\ 0 & h_1 & 0 & \cdots & 0 & h_8 & 0 \\ 0 & 0 & h_1 & & 0 & 0 & h_8 \end{bmatrix}_{t=1}$



$h_1\big|_{t=1} = \frac{1}{4}(1+r)(1+s) - \frac{1}{4}(1-r^2)(1+s) - \frac{1}{4}(1-s^2)(1+r)$

$h_2\big|_{t=1} = \frac{1}{4}(1-r)(1+s) - \frac{1}{4}(1-r^2)(1+s) - \frac{1}{4}(1-s^2)(1-r)$

$h_3\big|_{t=1} = \frac{1}{4}(1-r)(1-s) - \frac{1}{4}(1-s^2)(1-r) - \frac{1}{4}(1-r^2)(1-s)$

$h_4\big|_{t=1} = \frac{1}{4}(1+r)(1-s) - \frac{1}{4}(1-r^2)(1-s) - \frac{1}{4}(1-s^2)(1+r)$

$h_5\big|_{t=1} = \frac{1}{2}(1-r^2)(1+s)$, $h_6\big|_{t=1} = \frac{1}{2}(1-s^2)(1-r)$,

$h_7\big|_{t=1} = \frac{1}{2}(1-r^2)(1-s)$, $h_8\big|_{t=1} = \frac{1}{2}(1-s^2)(1+r)$.

The coordinates of loading point is $(r, s, t) = \left(\frac{1}{3}, \frac{1}{2}, 1\right)$.

$\therefore \underset{\sim}{R}_s = \underset{\sim}{H}^{sT} \begin{bmatrix} 0 \\ 0 \\ -p\,\delta\left(\frac{1}{3}, \frac{1}{2}\right) \end{bmatrix} =$

$\therefore \underset{\sim}{R}_s^T = \frac{P}{72} \big[\, 0 \;\; 0 \;\; 6 \;\vdots\; 0 \;\; 0 \;\; 15 \;\vdots\; 0 \;\; 0 \;\; 11 \;\vdots\; 0 \;\; 0 \;\; 14 \;\vdots$

(↑ w_1, ↑ w_2, ↑ w_3, ↑ w_4)

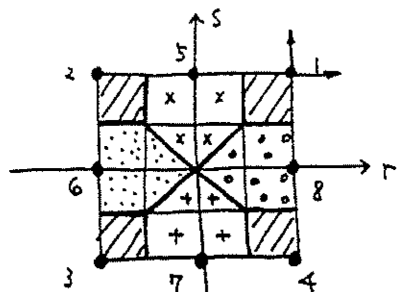
$0 \;\; 0 \;\; -48 \;\vdots\; 0 \;\; 0 \;\; -18 \;\vdots\; 0 \;\; 0 \;\; -16 \;\vdots\; 0 \;\; 0 \;\; -36 \;\vdots\; \underset{\sim}{O} \,\big]$

(↑ w_5, ↑ w_6, ↑ w_7, ↑ w_8)

5.21 When we are to obtain the consistent mass matrix $\underline{M}$, it is given by

$$\underline{M} = \int_V \rho \underline{H}^T \underline{H} \, dV .$$

On the other hand, a reasonable lumping of mass for instance for a 8-node element in a 2-D analysis is schematically shown in the following figure.



That is, we subdivide the area into 4×4 subdivisions, and

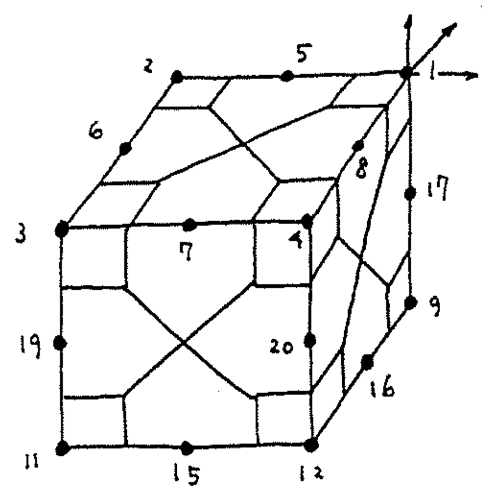
$$m_1 = \rho\left(\frac{A}{16}\right) = m_2 = m_3 = m_4$$

$$m_5 = \rho\left(A - 4\cdot\frac{A}{16}\right)\cdot\frac{1}{4} = \frac{3}{16}A\rho = m_6 = m_7 = m_8$$

$$\rightarrow \underline{M} = \begin{bmatrix} m_1 \underline{I}_2 & & & \\ & m_2 \underline{I}_2 & & \\ & & \ddots & \\ & & & m_8 \underline{I}_2 \end{bmatrix}$$

(m_i stands for two d.o.f. associated with node (i))

Similarly, in a 20-node element in 3-D analysis,



for corner nodes;

$$m_1 = m_2 = m_3 = m_4 = m_9 = m_{10} = m_{11} = m_{12}$$

$$= \rho\frac{V}{4^3} = \rho\,\frac{30\cdot 24\cdot 20}{4^3} = 1.755$$

for mid-side nodes,

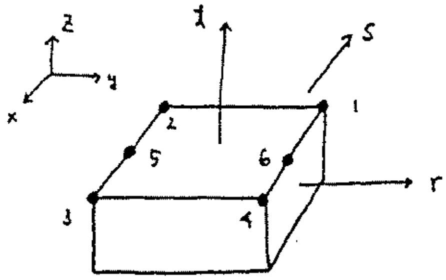
$$m_5 = m_6 = m_7 = m_8 = m_{13} = m_{14} = m_{15}$$

$$= m_{16} = m_{17} = m_{18} = m_{19} = m_{20}$$

$$= \frac{1}{12}\rho\left[V - 8\cdot\frac{V}{4^3}\right] = 8.190$$

$$\rightarrow \underline{M} = \begin{bmatrix} m_1 \underline{I}_3 & & \underline{0} \\ & \ddots & \\ \underline{0} & & m_{20} \underline{I}_3 \end{bmatrix}$$

5.22



On surface $t = 1$,

$$x = -\frac{3}{2}s + c_1, \quad y = 3r + c_2$$

$$\therefore \underline{J}^s = \begin{bmatrix} 0 & 3 \\ -\frac{3}{2} & 0 \end{bmatrix}, \quad \det \underline{J}^s = \frac{9}{2}$$

$$\underline{R}_s = \int_S \underline{H}^{s^T} \underline{f}^s \, dS, \quad \underline{f}^{s^T} = \left[\; 0 \quad 0 \quad -\frac{P_0(1-r)}{2} \; \right]$$

$$h_1\big|_{t=1} = \frac{1}{4}(1+r)(1+s) - \frac{1}{4}(1+r)(1-s^2)$$

$$h_2\big|_{t=1} = \frac{1}{4}(1-r)(1+s) - \frac{1}{4}(1-r)(1-s^2)$$

$$h_3\big|_{t=1} = \frac{1}{4}(1-r)(1-s) - \frac{1}{4}(1-r)(1-s^2)$$

$$h_4\big|_{t=1} = \frac{1}{4}(1+r)(1-s) - \frac{1}{4}(1+r)(1-s^2)$$

$$h_5\big|_{t=1} = \frac{1}{2}(1-r)(1-s^2), \quad h_6\big|_{t=1} = \frac{1}{2}(1+r)(1-s^2)$$

Hence,

$$\underline{R}_i = \int_{-1}^{1}\int_{-1}^{1} \begin{bmatrix} h_i\big|_{t=1} & & \\ & h_i\big|_{t=1} & \\ & & h_i\big|_{t=1} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -P_0 \frac{1}{2}(1-r) \end{bmatrix} \frac{9}{2} \, dr \, ds \, ;$$

$$\therefore \quad \underline{R}_1 = \left[\; 0 \quad 0 \quad -\frac{P_0}{2} \; \right]^T$$

$$\underline{R}_2 = \left[\; 0 \quad 0 \quad -P_0 \; \right]^T$$

$$\underline{R}_7 = \underline{R}_8 = \underline{0} \quad \left(\text{because} \quad h_7\big|_{t=1} = h_8\big|_{t=1} = 0 \right)$$

5.23 Let $g_1 = \frac{1}{4}(1+r)(1+s)$, $g_2 = \frac{1}{4}(1-r)(1+s)$, $g_3 = \frac{1}{4}(1-r)(1-s)$,

$g_4 = \frac{1}{4}(1+r)(1-s)$, $g_5 = \frac{1}{2}(1-r^2)(1+s)$, $g_6 = \frac{1}{2}(1-r)(1-s^2)$,

$g_7 = \frac{1}{2}(1-r^2)(1-s)$, $g_8 = \frac{1}{2}(1+r)(1-s^2)$

Then, $h_1 = g_1 - \frac{1}{2}(g_5 + g_8)$, $h_2 = g_2 - \frac{1}{2}(g_5 + g_6)$, $h_3 = g_3 - \frac{1}{2}(g_6 + g_7)$

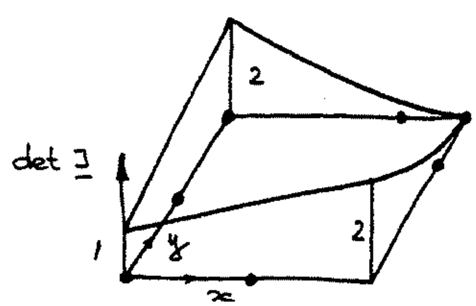
$h_4 = g_4 - \frac{1}{2}(g_7 + g_8)$, $h_5 = g_5$, $h_6 = g_6$, $h_7 = g_7$ and $h_8 = g_8$

$$x = \sum h_i x_i = x_3 + \frac{5}{4} + r + \frac{s}{4} - \frac{r^2}{4} - \frac{r^2 s}{4}$$

$$y = \sum h_i y_i = y_3 + \frac{5}{4} + \frac{r}{4} + s - \frac{s^2}{4} - \frac{r s^2}{4}$$

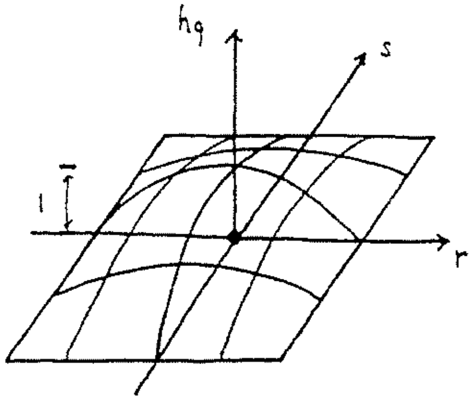
$$\therefore \underline{J} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix} = \begin{bmatrix} 1 - \frac{r}{2} - \frac{rs}{2} & \frac{1}{4} - \frac{s^2}{4} \\ \frac{1}{4} - \frac{r^2}{4} & 1 - \frac{s}{2} - \frac{rs}{2} \end{bmatrix}$$

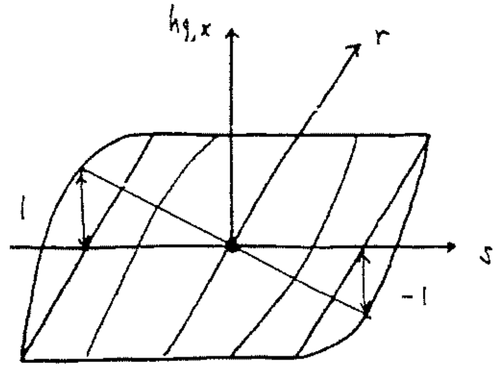
$$\rightarrow \det \underline{J} = \frac{1}{16}(3 - r - s - rs)(5 - r - s - 3rs)$$



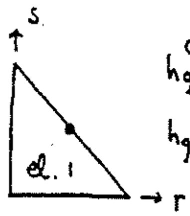
5.24 For node 9 in 1st element,

$$h_9 = (1-r^2)(1-s^2), \quad h_{9,x} = \frac{1}{2} h_{9,r} = -r(1-s^2)$$



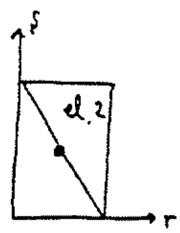


For node 9 in 2nd elements



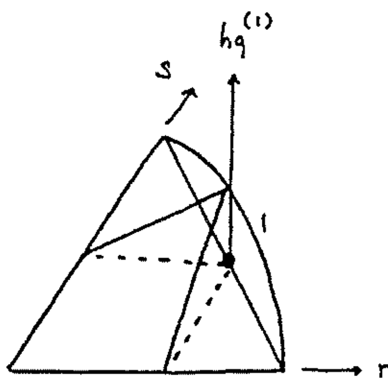
$$h_9^{(1)} = 4rs$$

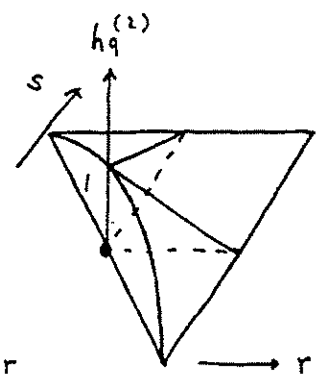
$$h_{9,x}^{(1)} = \frac{1}{4} h_{9,r}^{(1)} = s$$

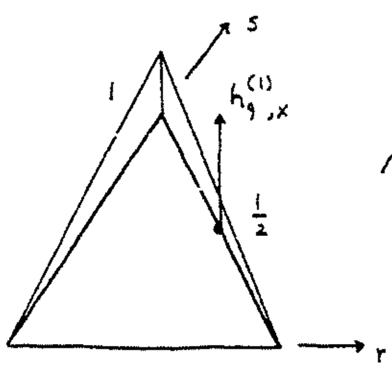


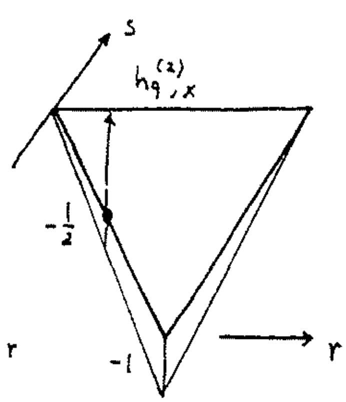
$$h_9^{(2)} = 4(1-r)(1-s)$$

$$h_{9,x}^{(2)} = -(1-s)$$

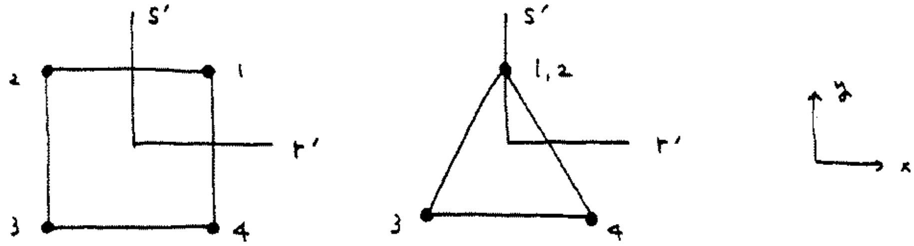








5.25 First, consider the collapsing as shown below.

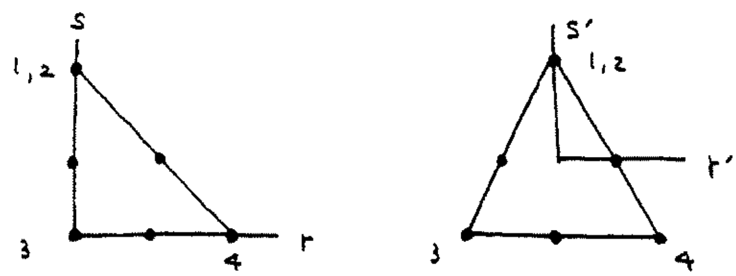


As $x_1 = x_2$, $y_1 = y_2$,

$$x = \frac{1}{2}(1+s')\,x_2 + \frac{1}{4}(1-r')(1-s')\,x_3 + \frac{1}{4}(1+r')(1-s')\,x_4$$

$$y = \quad '' \quad y_2 + \quad '' \quad y_3 + \quad '' \quad y_4$$

Now consider a mapping between the natural coordinates (r,s) and the collapsed coordinates (r', s')



$$r = \frac{1}{2}(1+s')r_2 + \frac{1}{4}(1-r')(1-s')\,r_3 + \frac{1}{4}(1+r')(1-s')\,r_4 = \frac{1}{4}(1+r')(1-s') \quad \text{— ①}$$

$$s = \frac{1}{2}(1+s') \quad \text{similarly.} \quad \text{— ②}$$

Finally by substituting ①, ② into the functions in Fig. 5.11, we check whether the functions given in (5.36) and (5.37) are obtained. Consider h_1 in Fig. 5.11,

$$h_1 = (1-r-s) - \frac{1}{2}h_4 - \frac{1}{2}h_6 = \frac{1}{4}(1-r')(1-s') - \frac{1}{4}(1-s'^2)(1-r')$$

$$- \frac{1}{4}(1-r'^2)(1-s') + \Delta h \qquad = h_3^*$$

5.25

with $\Delta h = \frac{1}{8}(1-r'^2)(1-s'^2)$,

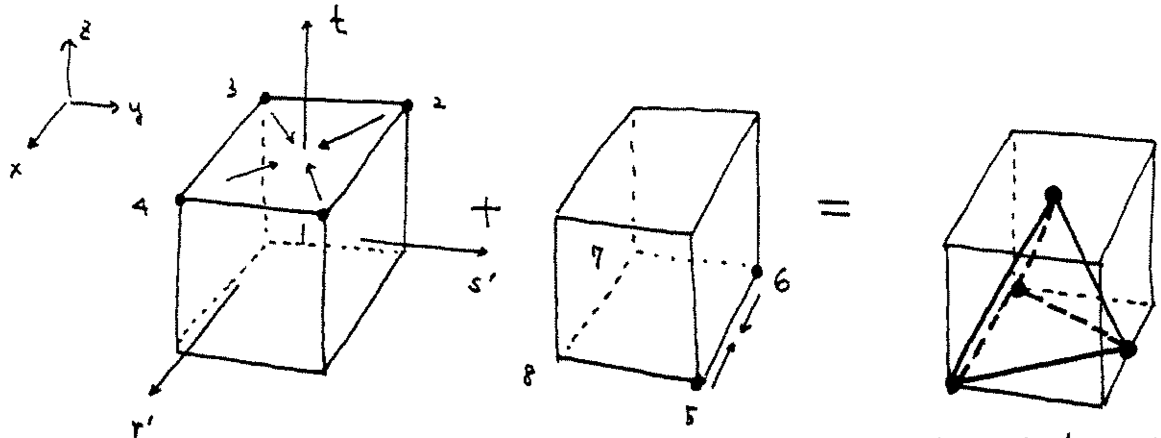
where we have substituted for r and s from

① and ②.

Note that in (5.36), r' is called r and s' is called s.

By similar procedures for each function, we see that the functions in (5.36) and (5.37) are obtained.

5.26

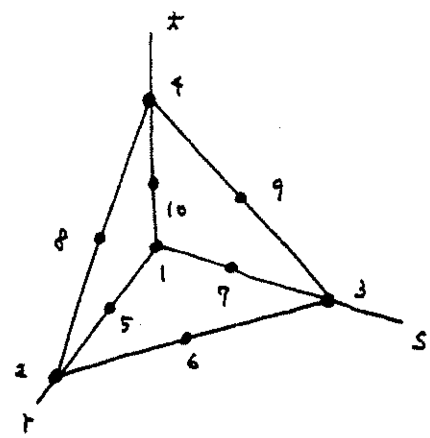


Collapsed element

In the collapsed element

$$x = \frac{1+t'}{2} x_1 + \frac{(1+s')(1-t')}{4} x_5 + \frac{1}{8}(1-r')(1-s')(1-t')\, x_7 + \frac{1}{8}(1+r')(1-s')(1-t')\, x_8$$

(Similarly for y and z)



Hence $r = \frac{1}{8}(1+r')(1-s')(1-t')$

$s = \frac{1}{4}(1+s')(1-t')$

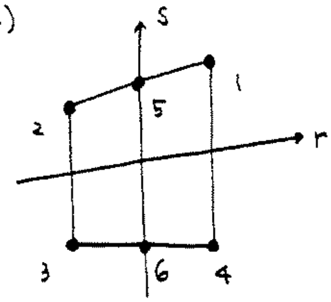
$t = \frac{1}{2}(1+t')$

Here $h_6 = 4rs = \frac{1}{8}(1+r')(1-s'^2)(1-t')^2$

$$= \underbrace{\frac{1}{4}(1+r')(1-s'^2)(1-t')}_{h_{16} \text{ in Fig. 5.5}} - \underbrace{\frac{1}{8}(1+r')(1-s'^2)(1-t'^2)}_{\Delta h_{16}}$$

$\therefore \quad h_{16}^{*} = \frac{1}{4}(1+r')(1-s'^2)(1-t') + \Delta h_{16} \quad$ where $\Delta h_{16} = -\frac{1}{8}(1+r')(1-s'^2)(1-t'^2)$

5.27 (a)



$$h_1 = \frac{1}{4}(1+r)(1+s) - \frac{1}{4}(1-r^2)(1+s),$$

$$h_2 = \frac{1}{4}(1-r)(1+s) - \frac{1}{4}(1-r^2)(1+s),$$

$$h_3 = \frac{1}{4}(1-r)(1-s) - \frac{1}{4}(1-r^2)(1-s),$$

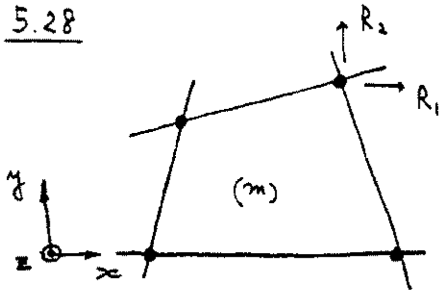
$$h_4 = \frac{1}{4}(1+r)(1-s) - \frac{1}{4}(1-r^2)(1-s),$$

$$h_5 = \frac{1}{2}(1-r^2)(1+s), \quad h_6 = \frac{1}{2}(1-r^2)(1-s)$$

(b) An element satisfies the convergence requirements, if the properties of compatibility and completeness are fulfilled As this element is used in a compatible mesh, we now check completeness.

$$\sum_{i=1}^{6} h_i = 1. \longrightarrow$$ the element satisfies the basic requirements for monotonic Convergence.

5.28



(a) $\underline{F}^{(m)} = \int_{V^{(m)}} \underline{B}^{(m)T} \underline{\tau}^{(m)} \, dV^{(m)}$

where $\underline{\tau}^{(m)} = \underline{C}\,\underline{B}^{(m)}\underline{U}$

$\underline{U}$: nodal displacement vector

In order to verify if the forces in $\underline{F}^{(m)}$ are in equilibrium, we give the element a rigid body translation in the x- and y- directions, and a rigid body rotation about the z - axis

i) Let $\bar{u}_i = u_0, \quad \bar{v}_i = v_0 \quad i = 1, 2, 3, 4.$

$$\bar{u} = \sum h_i \bar{u}_i = (\sum h_i) u_0 = u_0 , \quad \bar{v} = v_0$$

$$\therefore \bar{\epsilon}^T = \left[\frac{\partial \bar{u}}{\partial x} \quad \frac{\partial \bar{v}}{\partial y} \quad \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} \right] = \underline{0}$$

$$\therefore \underline{\bar{U}}^T \cdot \int_{V^{(m)}} \underline{B}^{(m)T} \underline{\tau}^{(m)} \, dV^{(m)} = \underline{\bar{U}}^T \underline{F}^{(m)} = 0$$

Hence, nodal forces in x- and y-directions are in equil.

ii) Let $\underline{u}_i = -(\underline{r}_i \times d\underline{\theta}) \qquad i = 1, 2, 3, 4$

where $\underline{r}_i^T = [x_i \;\; y_i \;\; 0]$, $d\underline{\theta}^T = [0 \;\; 0 \;\; 1]$

then, $\underline{u}_i^T = [-y_i \;\; x_i \;\; 0]$

$$\bar{u} = \sum h_i \bar{u}_i = \sum h_i (-y_i) = -\sum h_i y_i = -y$$

$$\bar{v} = \sum h_i v_i = \sum h_i x_i = x$$

$$\therefore \underline{\epsilon}^T = [0 \quad 0 \quad 0]$$

Hence, the nodal forces are also in moment equilibrium.

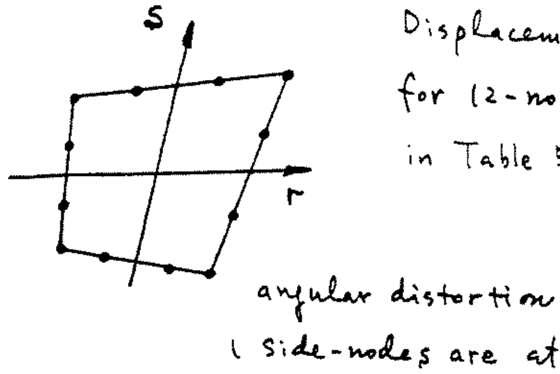
5.28

(b) Here $\underline{F}^{(m)} = \left[\int_{V^{(m)}} \underline{B}^{(m)T} \underline{C} \, \underline{B}^{(m)} \, dV^{(m)} \right] \underline{U} = \underline{K}^{(m)} \underline{U} = \underline{R}^{(m)}$

$\underline{F}^{(m)}$ is therefore in equilibrium at each node with the applied external loads including the reactions. That is

$$\sum_m \underline{F}^{(m)} = \underline{R} \, .$$

5.29



Displacement fields for 12-node element in Table 5.1

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & x & & y & & \\ & x^2 & & xy & & y^2 & \\ x^3 & & x^2y & & xy^2 & & y^3 \\ \hline & x^3y & & & & xy^3 & \end{array}$$

The physical coordinates are

$$\left.\begin{aligned} x &= \gamma_1 + \gamma_2 r + \gamma_3 s + \gamma_4 rs \\ y &= \delta_1 + \delta_2 r + \delta_3 s + \delta_4 rs \end{aligned}\right\} \text{ with } \gamma_1, \cdots, \delta_4 : \text{constants.} \quad \text{— (1)}$$

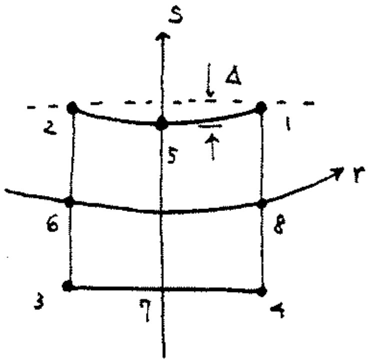
And the displacements are

$$u = \sum_{i=1}^{12} h_i u_i \quad \text{— (2)} \qquad v = \sum_{i=1}^{12} h_i v_i \quad \text{— (3)}$$

Considering the u-displacement interpolation, the constant and x- and y-terms given in (1) are clearly contained in (2) because eq. (2) interpolates u in terms of the functions ($1, r, s, r^2, rs, s^2, r^3, r^2s, rs^2, s^3, r^3s, rs^3$). Considering the x^2-term, we notice that the term r^2s^2 is not present in (2). Similarly, the other terms in the Pascal triangle are not present in the displacement interpolation (2). The same holds for the v-displ. Hence the terms $1, x, y$ in the 2nd column in table 5.1 are verified.

By the similar argument for the 16-node el. , it can be shown that the terms $1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3$ are contained in (2) and (3) whereas $x^3y, x^2y^2, xy^3, x^3y^2, x^2y^3, x^3y^3$ are not present.

5.30



The physical coordinates are obtained by locating nodes 6, 7 and 8 at the mid-side of corresponding edge as

$$x = a_0 + a_1 r \quad \text{——} \quad ①$$

$$y = b_0 + b_1 s + b_2 r^2 + b_3 r^2 s \quad \text{——} \quad ②$$

where $a_0, a_1, b_0, \cdots, b_3$ are constants.

The displacements are

$$u = \sum_{i=1}^{8} h_i u_i \quad \text{——} \quad ③ \qquad v = \sum_{i=1}^{8} h_i v_i \quad \text{——} \quad ④$$

Considering the u-displacement, the constant and x-and y- terms in ① and ② are clearly contained in ③ whereas the y^2-term is not present. When we have a curved edge distortion at node 8 rather than at node 5, the x^2-term is obviously not present. Similarly, the other terms in the Pascal triangle are not present in the displacement interpolation. This also applies to the v-displacement. Hence only the terms 1, x, y appear in Table 5.1.

By a similar argument, for the 9-node, 12-node and 16-node elements we only have the terms 1, x, and y for curved edge distortions.

5.31 We use a 4/1 isoparametric u/p element.

$$x = \sum h_i x_i = 2r + C_1 \ , \quad y = \sum h_i y_i = \frac{r}{2} + 2s + \frac{rs}{2} + C_2$$

$$\therefore \ \underline{J} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix} = \begin{bmatrix} 2 & \frac{1+s}{2} \\ 0 & \frac{4+r}{2} \end{bmatrix}, \quad \underline{J}^{-1} = \frac{1}{4+r} \begin{bmatrix} \frac{4+r}{2} & -\frac{1+s}{2} \\ 0 & 2 \end{bmatrix},$$

$$\det \underline{J} = 4 + r$$

$$\underline{u} = \begin{bmatrix} u \\ v \end{bmatrix} = \underline{H}\,\hat{\underline{u}} \quad \text{where} \quad \hat{\underline{u}}^T = [\,u_1 \ u_2 \ u_3 \ u_4 \ v_1 \ v_2 \ v_3 \ v_4\,]$$

$$\underline{H} = \begin{bmatrix} h_1 & h_2 & h_3 & h_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_1 & h_2 & h_3 & h_4 \end{bmatrix}$$

$$\underline{\varepsilon}' = \underline{B}_D\,\hat{\underline{u}} \ ,$$

$$\underline{B}_D = \begin{bmatrix} \frac{2}{3}h_{1,x} & \frac{2}{3}h_{2,x} & \cdots & -\frac{1}{3}h_{1,y} & -\frac{1}{3}h_{2,y} & \cdots \\ -\frac{1}{3}h_{1,x} & -\frac{1}{3}h_{2,x} & \cdots & \frac{2}{3}h_{1,y} & \frac{2}{3}h_{2,y} & \cdots \\ h_{1,y} & h_{2,y} & \cdots & h_{1,x} & h_{2,x} & \cdots \\ -\frac{1}{3}h_{1,x} & -\frac{1}{3}h_{2,x} & \cdots & -\frac{1}{3}h_{1,y} & -\frac{1}{3}h_{2,y} & \cdots \end{bmatrix}$$

where
$$h_{i,x} = J_{11}^{-1} h_{i,r} + J_{12}^{-1} h_{i,s}$$
$$h_{i,y} = J_{21}^{-1} h_{i,r} + J_{22}^{-1} h_{i,s}$$

$$\varepsilon_V = \varepsilon_{xx} + \varepsilon_{yy} = \underline{B}_V\,\hat{\underline{u}}$$

$$\underline{B}_V = [\,h_{1,x} \ \ h_{2,x} \ \cdots \ h_{1,y} \ \ h_{2,y} \ \cdots\,]$$

$$p = \underline{H}_p\,\hat{\underline{p}} \quad \text{where} \quad \underline{H}_p = [\,1\,]\ , \ \hat{\underline{p}} = [\,p_0\,]$$

$$\underline{C}' = \begin{bmatrix} 2G & & & \\ & 2G & & \\ & & G & \\ & & & 2G \end{bmatrix}$$

5.31

$$\therefore \underline{K}\,\underline{U} = \underline{R} \quad \text{where} \quad \underline{K} = \begin{bmatrix} \underline{K}_{uu} & \underline{K}_{up} \\ \underline{K}_{pu} & \underline{K}_{pp} \end{bmatrix}, \quad \underline{U} = \begin{bmatrix} \hat{\underline{u}} \\ \hat{\underline{p}} \end{bmatrix}, \quad \underline{R} = \begin{bmatrix} \underline{R}_u \\ \underline{0} \end{bmatrix}$$

and

$$\underline{K}_{uu} = \int_V \underline{B}_D^T \underline{C}' \underline{B}_D \, dV = \int_{-1}^{1}\int_{-1}^{1} \underline{B}_D^T \underline{C}' \underline{B}_D \det \underline{J} \, dr\,ds$$

$$\underline{K}_{up} = \underline{K}_{pu}^T = -\int_V \underline{B}_V^T \underline{H}_p \, dV = -\int_{-1}^{1}\int_{-1}^{1} \underline{B}_V^T \underline{H}_p \det \underline{J} \, dr\,ds$$

$$\underline{K}_{pp} = -\int \underline{H}_p^T \frac{1}{\kappa} \underline{H}_p \, dV = -\int_{-1}^{1}\int_{-1}^{1} \underline{H}_p^T \frac{1}{\kappa} \underline{H}_p \det \underline{J} \, dr\,ds$$

5.32 From the assumption in Fig. 5.18

$$u = -\beta(x)\, z \quad , \quad w = w(x) \quad \text{with} \quad \gamma = \frac{dw}{dx} - \beta$$

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{zz} \\ \gamma_{xz} \end{bmatrix} = \begin{bmatrix} \partial u/\partial x \\ \partial w/\partial z \\ \partial u/\partial z + \partial w/\partial x \end{bmatrix} = \begin{bmatrix} -\frac{d\beta}{dx} z \\ 0 \\ \gamma \end{bmatrix}$$

Let $\underline{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \gamma_{xz} \end{bmatrix}$

P.V.D in eq. (4.7) states that

$$\int_V \bar{\underline{\varepsilon}}^T \underline{\tau}\, dV = \int_V \bar{\underline{U}}^T \underline{f}^B\, dV + \int_{S_f} \bar{\underline{U}}^{S_f T} \underline{f}^{S_f}\, dS + \sum_i \bar{\underline{U}}_i^{\,T} \underline{R}_C^{\,i} \qquad (a)$$

$$(\text{l.h.s.}) = \int_V E\left(\frac{d\beta}{dx}\right)\left(\frac{d\bar\beta}{dx}\right) z^2 dV + \int_V Gk\left(\frac{dw}{dx} - \beta\right)\left(\frac{d\bar w}{dx} - \bar\beta\right) dV \qquad (b)$$

$$= E \int_0^L \left(\int_A z^2\, dA\right)\left(\frac{d\beta}{dx}\right)\left(\frac{d\bar\beta}{dx}\right) dx$$

$$+ Gk \int_0^L \left(\int_A dA\right)\left(\frac{dw}{dx} - \beta\right)\left(\frac{d\bar w}{dx} - \bar\beta\right) dx$$

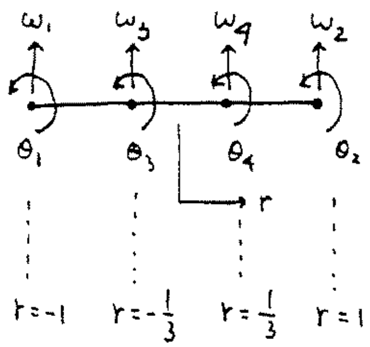
$$= EI \int_0^L \left(\frac{d\beta}{dx}\right)\left(\frac{d\bar\beta}{dx}\right) dx + GAk \int_0^L \left(\frac{dw}{dx} - \beta\right)\left(\frac{d\bar w}{dx} - \bar\beta\right) dx$$

$$(\text{r.h.s.}) = \int_0^L p\,\bar w\, dx + \int_0^L m\,\bar\beta\, dx$$

$$\therefore\; EI \int_0^L \left(\frac{d\beta}{dx}\right)\left(\frac{d\bar\beta}{dx}\right) dx + GAk \int_0^L \left(\frac{dw}{dx} - \beta\right)\left(\frac{d\bar w}{dx} - \bar\beta\right) dx = \int_0^L p\,\bar w\, dx + \int_0^L m\,\bar\beta\, dx$$

Note that in eq. (b) we introduced the shear correction factor k because the transverse shear strain is assumed to be constant through the thickness t (see Example 5.23).

5.33



$$h_1 = \frac{1}{2}(1-r) - \frac{1}{2}(1-r^2) + \frac{1}{16}(-9r^3 + r^2 + 9r - 1)$$

$$h_2 = \frac{1}{2}(1+r) - \frac{1}{2}(1-r^2) + \frac{1}{16}(9r^3 + r^2 - 9r - 1)$$

$$h_3 = (1-r^2) + \frac{1}{16}(27r^3 + 7r^2 - 27r - 7)$$

$$h_4 = \frac{1}{16}(-27r^3 - 9r^2 + 27r + 9)$$

$$h_{1,r} = \frac{1}{16}(1 + 18r - 27r^2), \quad h_{2,r} = \frac{1}{16}(-1 + 18r + 27r^2)$$

$$h_{3,r} = \frac{9}{16}(-3 - 2r + 9r^2), \quad h_{4,r} = \frac{9}{16}(3 - 2r - 9r^2)$$

Using these interpolation functions,

$$x = \frac{L}{2}(1+r) + x_1, \quad J = \frac{\partial x}{\partial r} = \frac{L}{2}, \quad \det J = \frac{L}{2}, \quad J^{-1} = \frac{2}{L}$$

$$w = \underline{H}_w \hat{\underline{u}}, \quad \beta = \underline{H}_\beta \hat{\underline{u}}$$

where $\hat{\underline{u}}^T = [\, w_1 \;\; w_2 \;\; w_3 \;\; w_4 \;\; \theta_1 \;\; \theta_2 \;\; \theta_3 \;\; \theta_4 \,]$

$$\underline{H}_w = [\, h_1 \;\; h_2 \;\; h_3 \;\; h_4 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \,]$$

$$\underline{H}_\beta = [\, 0 \;\; 0 \;\; 0 \;\; 0 \;\; h_1 \;\; h_2 \;\; h_3 \;\; h_4 \,]$$

Let $\underline{B}_w = \frac{d}{dx}\underline{H}_w = \frac{2}{L}[\, h_{1,r} \;\; h_{2,r} \;\; h_{3,r} \;\; h_{4,r} \;\; 0 \;\; 0 \;\; 0 \;\; 0 \,]$

$$\underline{B}_\beta = \frac{d}{dx}\underline{H}_\beta = \frac{2}{L}[\, 0 \;\; 0 \;\; 0 \;\; 0 \;\; h_{1,r} \;\; h_{2,r} \;\; h_{3,r} \;\; h_{4,r} \,]$$

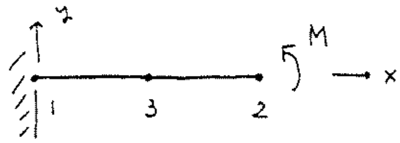
5.33

Hence, the stiffness $\underline{K}$ and mass matrix $\underline{M}$ are

$$\underline{K} = EI \int_{-1}^{1} \underline{B}_\beta^T \underline{B}_\beta \det J \, dr + GAk \int_{-1}^{1} (\underline{B}_w - H_\beta)^T (\underline{B}_w - \underline{H}_\beta) \det J \, dr$$

$$\underline{M} = \int_{-1}^{1} \begin{bmatrix} \underline{H}_w \\ \underline{H}_\beta \end{bmatrix}^T \rho \begin{bmatrix} bh & 0 \\ 0 & \frac{bh^3}{12} \end{bmatrix} \begin{bmatrix} \underline{H}_w \\ \underline{H}_\beta \end{bmatrix} \det J \, dr$$

5.34



Let the x-coordinate for node 3 be $\frac{L}{2}(1+\epsilon)$.

$$X = \sum h_i x_i = \frac{L}{2}(1+r)\{1+\epsilon(1-r)\}$$

$$J = \frac{L}{2}(1-2\epsilon r), \quad J^{-1} = \frac{2}{L(1-2\epsilon r)}$$

$$w = \sum h_i w_i = \frac{1}{2}(1+r)r\, w_2 + (1-r^2) w_3$$

$$\beta = \sum h_i \theta_i = \frac{1}{2}(1+r)r\, \theta_2 + (1-r^2)\theta_3$$

Then the curvature and shear strain are

$$\kappa = \frac{d\beta}{dx} = \frac{2}{L(1-2\epsilon r)}\left\{\left(\frac{1}{2}+r\right)\theta_2 - 2r\theta_3\right\} = \frac{2}{L(1-2\epsilon r)}\left\{\frac{\theta_2}{2} + r(\theta_2 - 2\theta_3)\right\}$$

$$\gamma = \frac{dw}{dx} - \beta = \frac{2}{L(1-2\epsilon r)}\left\{\frac{w_2}{2} + r(w_2 - 2w_3)\right\} - \left\{\frac{1}{2}\theta_2 + \frac{r^2}{2}(\theta_2 - 2\theta_3)\right\}$$

Now consider the case when $\epsilon = 0$, i.e.,

$$\kappa = \frac{2}{L}\left\{\frac{\theta_2}{2} + r(\theta_2 - 2\theta_3)\right\},$$

$$\gamma = \left(\frac{1}{L}w_2 - \theta_3\right) + \frac{2r}{L}\left(w_2 - 2w_3 - \frac{L}{4}\theta_2\right) + \frac{1}{2}r^2\left(2\theta_3 - \theta_2\right).$$

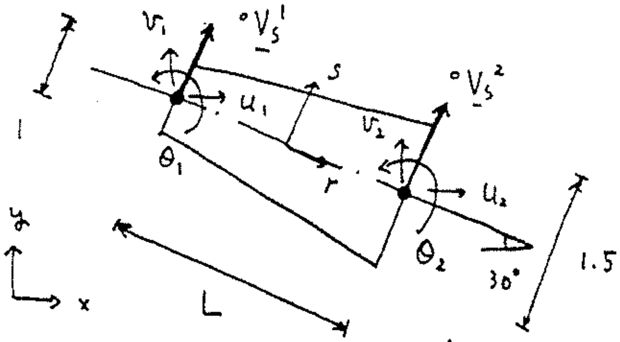
And the given loading condition states the curvature κ is constant everywhere in the beam, and the shear strain $\gamma = 0$. The exact Bernoulli beam theory solution is

$$\gamma = 0, \quad \text{and} \quad \theta_2 = \frac{ML}{EI}, \quad w_2 = \frac{ML^2}{2EI}, \quad \theta_3 = \frac{ML}{2EI}, \quad w_3 = \frac{ML^2}{8EI}$$

This solution is contained in the above equations, and therefore will be calculated with $\varepsilon = 0$.

However, when $\varepsilon \neq 0$, the Bernoulli beam theory solution cannot be reproduced by the finite element discretization. A spurious shear strain is calculated that makes the system very stiff, and gives locking.

5.35



$h_1 = \frac{1-r}{2}$, $h_2 = \frac{1+r}{2}$

$h_{1,r} = -\frac{1}{2}$, $h_{2,r} = \frac{1}{2}$

Note that all variables are constant in the t-direction.

$${}^{\circ}\underline{V}_s^1 = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} , \quad {}^{\circ}\underline{V}_s^2 = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} , \quad b_1 = 1 , \quad b_2 = 1.5$$

Hence,
$$ {}^{\circ}x = \sum_{k=1}^{2} h_k {}^{\circ}x_k + \frac{s}{2} \sum b_k h_k {}^{\circ}V_{sx}^k$$
$$= \frac{1}{2}(x_1 + x_2) + \frac{\sqrt{3}}{4} L r + \frac{s}{16}(5+r)$$
$$ {}^{\circ}y = \sum_{k=1}^{2} h_k {}^{\circ}y_k + \frac{s}{2} \sum b_k h_k {}^{\circ}V_{sy}^k$$
$$= \frac{1}{2}(y_1 + y_2) - \frac{L}{4} r + \frac{\sqrt{3}}{16} s(5+r)$$

Eq. (5.75) reduces to
$$\underline{V}_s^k = \theta_k \underline{e}_z \times {}^{\circ}\underline{V}_s^k = \begin{bmatrix} -\frac{\sqrt{3}}{2}\theta_k \\ \frac{1}{2}\theta_k \\ 0 \end{bmatrix}$$

$$\therefore \quad u(r,s) = \frac{u_2+u_1}{2} + \frac{u_2-u_1}{2} r - \frac{\sqrt{3}s}{16}\left\{ 2(1-r)\theta_1 + 3(1+r)\theta_2 \right\}$$
$$v(r,s) = \frac{v_2+v_1}{2} + \frac{v_2-v_1}{2} r + \frac{s}{16}\left\{ 2(1-r)\theta_1 + 3(1+r)\theta_2 \right\}$$

5.35

Now we consider the strains as

$$\begin{bmatrix} \epsilon_{rr} \\ \gamma_{rs} \end{bmatrix} = \sum_{k=1}^{2} \underline{B}_k \hat{\underline{u}}_k \quad , \quad \underline{u}_k^T = [\, u_k \ v_k \ \theta_k \,], \quad \underline{B} = [\, \underline{B}_1 \ \underline{B}_2 \,]$$

For eq. (5.73),

$$\begin{bmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial s} \end{bmatrix} = \sum_{k=1}^{2} \begin{bmatrix} \frac{\partial h_k}{\partial r} [\, 1 \ \ (\hat{g})_{31}^k \,] \\ h_k [\, 0 \ \ (\hat{g})_{31}^k \,] \end{bmatrix} \begin{bmatrix} u_k \\ \theta_k \end{bmatrix} = \begin{bmatrix} \frac{u_2 - u_1}{2} + \frac{\sqrt{3}}{16} s (2\theta_1 - 3\theta_2) \\ -\frac{\sqrt{3}}{16} \{ 2(1-r)\theta_1 + 3(1+r)\theta_2 \} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial v}{\partial r} \\ \frac{\partial v}{\partial s} \end{bmatrix} = \begin{bmatrix} \frac{v_2 - v_1}{2} + \frac{s}{16}(-2\theta_1 + 3\theta_2) \\ \frac{1}{16} \{ 2(1-r)\theta_1 + 3(1+r)\theta_2 \} \end{bmatrix}$$

The Jacobian matrix is

$$\underline{J} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{4} L + \frac{s}{16} & -\frac{L}{4} + \frac{\sqrt{3}}{16} s \\ \frac{5+r}{16} & \frac{\sqrt{3}(5+r)}{16} \end{bmatrix}$$

and

$$\underline{J}^{-1} = \begin{bmatrix} \frac{\sqrt{3}}{L} & -\frac{\sqrt{3}s - 4L}{(5+r)L} \\ -\frac{1}{L} & \frac{s + 4\sqrt{3}L}{(5+r)L} \end{bmatrix}$$

$$\therefore \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} = \sum_{k=1}^{2} \begin{bmatrix} J_{11}^{-1} \frac{\partial h_k}{\partial r} & (G3)_{11}^k \\ J_{21}^{-1} \frac{\partial h_k}{\partial r} & (G3)_{12}^k \end{bmatrix} \begin{bmatrix} u_k \\ \theta_k \end{bmatrix}$$

5.35

where $(G3)_{11}^{k} = J_{11}^{-1}\left(-s\,\frac{\sqrt{3}}{4}\,b_k\right)\frac{\partial h_k}{\partial r} + J_{12}^{-1}\left(-\frac{\sqrt{3}}{4}\,b_k\right)h_k$

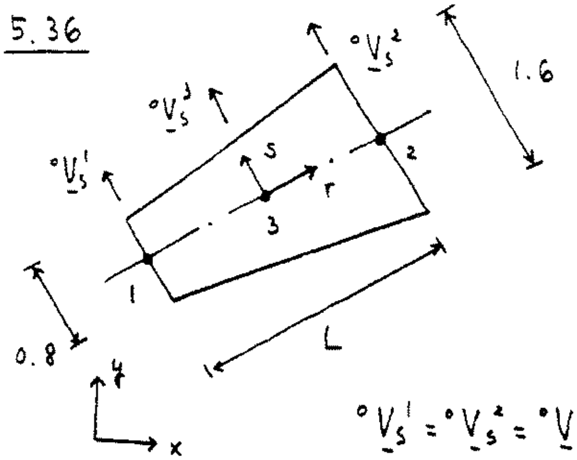
$$(G3)_{12}^{k} = J_{21}^{-1}\left(-s\,\frac{\sqrt{3}}{4}\,b_k\right)\frac{\partial h_k}{\partial r} + J_{22}^{-1}\left(-\frac{\sqrt{3}}{4}\,b_k\right)h_k$$

$$\begin{bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} = \sum_{k=1}^{2} \begin{bmatrix} J_{11}^{-1}\frac{\partial h_k}{\partial r} & (G3)_{21}^{k} \\ J_{21}^{-1}\frac{\partial h_k}{\partial r} & (G3)_{22}^{k} \end{bmatrix} \begin{bmatrix} v_k \\ \theta_k \end{bmatrix}$$

where $(G3)_{21}^{k} = J_{11}^{-1}\left(\frac{s}{4}\,b_k\right)\frac{\partial h_k}{\partial r} + J_{12}^{-1}\left(\frac{1}{4}\,b_k\right)h_k$

$$(G3)_{22}^{k} = J_{21}^{-1}\left(\frac{s}{4}\,b_k\right)\frac{\partial h_k}{\partial r} + J_{22}^{-1}\left(\frac{1}{4}\,b_k\right)h_k$$

5.36



$$h_1 = \frac{1}{2}(1-r) - \frac{1}{2}(1-r^2) = -\frac{r(1-r)}{2}$$

$$h_2 = \frac{1}{2}(1+r) - \frac{1}{2}(1-r^2) = \frac{r(1+r)}{2}$$

$$h_3 = 1 - r^2$$

$$h_{1,r} = -\frac{1}{2} + r, \quad h_{2,r} = \frac{1}{2} + r, \quad h_{3,r} = -2r$$

$${}^\circ \underline{V}_s^1 = {}^\circ \underline{V}_s^2 = {}^\circ \underline{V}_s^3 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad b_1 = 0.8, \; b_2 = 1.6, \; b_3 = 1.2$$

$${}^\circ x = \sum_{k=1}^{3} h_k {}^\circ x_k + \frac{s}{2} \sum_{k=1}^{3} b_k h_k {}^\circ V_{sx}^k$$

$$= x_3 + \frac{L}{2\sqrt{2}} r - \frac{3}{5\sqrt{2}} s - \frac{rs}{5\sqrt{2}}$$

Similarly, $\quad {}^\circ y = y_3 + \frac{L}{2\sqrt{2}} r + \frac{3}{5\sqrt{2}} s + \frac{rs}{5\sqrt{2}}$

(eq. 5.71)

$$\underline{V}_s^k = \theta_k \underline{e}_z \times {}^\circ \underline{V}_s^k = \begin{bmatrix} -\frac{1}{\sqrt{2}} \theta_k \\ -\frac{1}{\sqrt{2}} \theta_k \end{bmatrix}$$

(eq. 5.75)

$$u(r,s) = \sum_{k=1}^{3} h_k u_k + \frac{s}{2} \sum_{k=1}^{3} b_k h_k V_{sx}^k$$

$$= -\frac{r(1-r)}{2} u_1 + \frac{r(1+r)}{2} u_2 + (1-r^2) u_3$$

$$+ \frac{r(1-r)s}{5\sqrt{2}} \theta_1 - \frac{2r(1+r)s}{5\sqrt{2}} \theta_2 - \frac{3(1-r^2)s}{5\sqrt{2}} \theta_3$$

$$v(r,s) = \sum_{k=1}^{3} h_k v_k + \frac{s}{2} \sum_{k=1}^{3} b_k h_k V_{sy}^k$$

5.36

$$= -\frac{r(1-r)}{2}v_1 + \frac{r(1+r)}{2}v_2 + (1-r^2)v_3 + \frac{r(1-r)s}{5\sqrt{2}}\theta_1 - \frac{2r(1+r)s}{5\sqrt{2}}\theta_2 - \frac{3(1-r^2)s}{5\sqrt{2}}\theta_3 \Bigg)$$

(eq. 5.73)

$$\begin{bmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial s} \end{bmatrix} = \sum_{k=1}^{3} \begin{bmatrix} \frac{\partial h_k}{\partial r}\,[\,1 \quad (g)_{31}^{k}\,] \\ h_k\,[\,0 \quad (\hat{g})_{31}^{k}\,] \end{bmatrix} \begin{bmatrix} u_k \\ \theta_k \end{bmatrix}$$

$$= \begin{bmatrix} (-\frac{1}{2}+r)u_1 + (\frac{1}{2}+r)u_2 - 2ru_3 + \frac{s}{5\sqrt{2}}\{(1-2r)\theta_1 - 2(1+2r)\theta_2 + 3r\theta_3\} \\ \frac{1}{5\sqrt{2}}\{r(1-r)\theta_1 - 2r(1+r)\theta_2 - 3(1-r^2)\theta_3\} \end{bmatrix}$$

Similarly for v,

$$\begin{bmatrix} \frac{\partial v}{\partial r} \\ \frac{\partial v}{\partial s} \end{bmatrix} = \begin{bmatrix} (-\frac{1}{2}+r)v_1 + (\frac{1}{2}+r)v_2 - 2rv_3 + \frac{s}{5\sqrt{2}}\{(1-2r)\theta_1 - 2(1+2r)\theta_2 + 3r\theta_3\} \\ \frac{1}{5\sqrt{2}}\{r(1-r)\theta_1 - 2r(1+r)\theta_2 - 3(1-r^2)\theta_3\} \end{bmatrix}$$

(eq. 5.80)

$$\underline{J} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix} = \begin{bmatrix} -\frac{s}{5\sqrt{2}} + \frac{L}{2\sqrt{2}} & \frac{s}{5\sqrt{2}} + \frac{L}{2\sqrt{2}} \\ -\frac{3+r}{5\sqrt{2}} & \frac{3+r}{5\sqrt{2}} \end{bmatrix}$$

5.36

For (eq. 5.85)

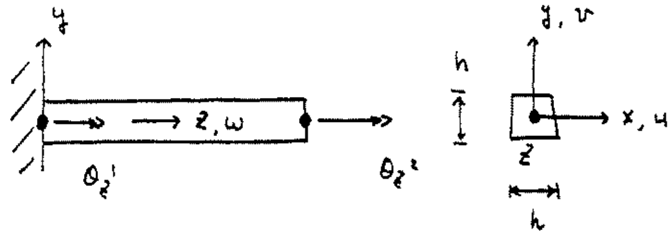
$$\begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{bmatrix} = \sum_{k=1}^{3} \begin{bmatrix} J_{11}^{-1} \dfrac{\partial h_k}{\partial r} & (G3)_{11}^{k} \\ J_{21}^{-1} \dfrac{\partial h_k}{\partial r} & (G3)_{12}^{k} \end{bmatrix} \begin{bmatrix} u_k \\ \theta_k \end{bmatrix}$$

where
$$(G3)_{11}^{k} = J_{11}^{-1}\left(-s\,\frac{b_k}{\sqrt{2}}\right)\frac{\partial h_k}{\partial r} + J_{12}^{-1}\left(-\frac{b_k}{2\sqrt{2}}\right)h_k$$
$$(G3)_{12}^{k} = J_{21}^{-1}\left(-s\,\frac{b_k}{2\sqrt{2}}\right)\frac{\partial h_k}{\partial r} + J_{22}^{-1}\left(-\frac{b_k}{2\sqrt{2}}\right)h_k$$

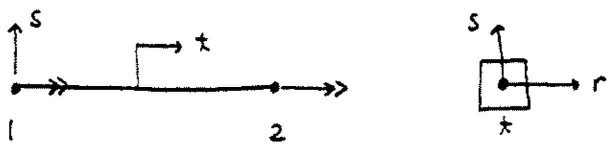
$$\begin{bmatrix} \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{bmatrix} = \sum_{k=1}^{3} \begin{bmatrix} J_{11}^{-1} \dfrac{\partial h_k}{\partial r} & (G3)_{21}^{k} \\ J_{21}^{-1} \dfrac{\partial h_k}{\partial r} & (G3)_{22}^{k} \end{bmatrix} \begin{bmatrix} v_k \\ \theta_k \end{bmatrix}$$

where
$$(G3)_{21}^{k} = J_{1}^{-1}\left(-s\,\frac{b_k}{2\sqrt{2}}\right)\frac{\partial h_k}{\partial r} + J_{12}^{-1}\left(-\frac{b_k}{2\sqrt{2}}\right)h_k$$
$$(G3)_{22}^{k} = J_{21}^{-1}\left(-s\,\frac{b_k}{2\sqrt{2}}\right)\frac{\partial h_k}{\partial r} + J_{22}^{-1}\left(-\frac{b_k}{2\sqrt{2}}\right)h_k$$

5.37 (i)



Neglecting warping effects, we only consider θ_z d.o.f.



$h_1 = \frac{1-t}{2}$

$h_2 = \frac{1+t}{2}$

$$u = -\frac{sh}{2}\frac{1+t}{2}\theta_z^2 \quad , \quad v = \frac{rh}{2}\frac{1+t}{2}\theta_z^2 \quad , \quad w = 0$$

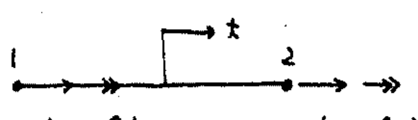
$$\underline{\gamma} = \begin{bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{bmatrix}_{t=0} = \underline{B}_\gamma \hat{\underline{u}}$$ (Note we are using the mixed element

where $\underline{B}_\gamma^T = \begin{bmatrix} \frac{rh}{2L} & -\frac{sh}{2L} \end{bmatrix}$, $\hat{\underline{u}} = [\theta_z^2]$

$$\underline{K} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \underline{B}_\gamma^T \underline{C}_\gamma \underline{B}_\gamma \det \underline{J}\, dr\, ds\, dt \qquad \left(\leftarrow \underline{C}_\gamma = \begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix} \right)$$

$$\therefore \underline{K} = \left[\frac{G\ h^4}{6L} \right] \quad , \quad \underline{R} = [\,T\,] \quad \Rightarrow \quad \theta_z^2 = \frac{6L}{G\ h^4} T$$

(ii) Now considering warping effects with the warping function given as $f = xy(x^2 - y^2)$,



$$u = -\frac{sh}{2}\frac{1+t}{2}\theta_z^2$$

$$v = \frac{rh}{2}\frac{1+t}{2}\theta_z^2$$

$$w = \left\{ xy(x^2 - y^2) \right\} \frac{1+t}{2} w_2$$

$$= \left\{ \frac{h^4}{16} rs(r^2 - s^2) \right\} \frac{1+t}{2} w_2$$

5.37

$$\underline{\epsilon} = \begin{bmatrix} \epsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \end{bmatrix} = \begin{bmatrix} \frac{2}{L}\frac{\partial w}{\partial t} \\ \left(\frac{2}{L}\frac{\partial v}{\partial t} + \frac{2}{h}\frac{\partial w}{\partial s}\right)\Big|_{t=0} \\ \left(\frac{2}{L}\frac{\partial u}{\partial t} + \frac{2}{h}\frac{\partial w}{\partial r}\right)\Big|_{t=0} \end{bmatrix} = \underline{B}\,\hat{\underline{u}}$$

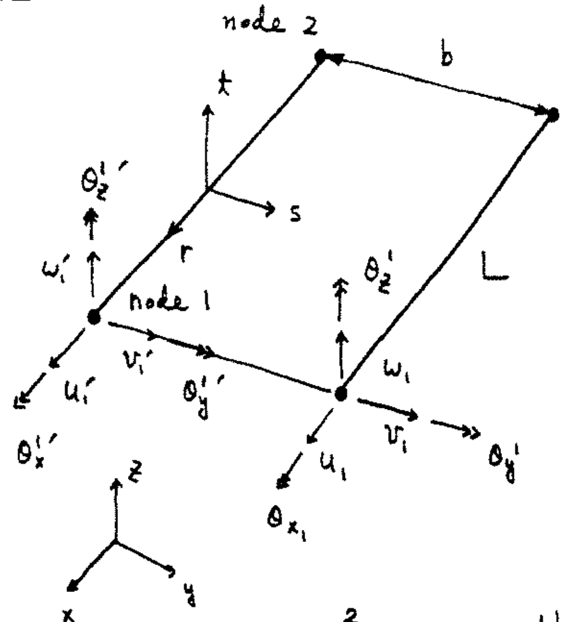
$$\text{where} \quad \underline{B} = \begin{bmatrix} 0 & h^4 rs(r^2 - s^2)/16L \\ rh/2L & h^3 r(r^2 - 3s^2)/16 \\ -sh/2L & h^3 s(3r^2 - s^2)/16 \end{bmatrix}, \quad \hat{\underline{u}} = \begin{bmatrix} \theta_z{}^2 \\ \omega_2 \end{bmatrix}$$

$$\underline{K} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \underline{B}^T \underline{C}\, \underline{B} \det \underline{J}\, dr\, ds\, dt \quad \left(\leftarrow \underline{C} = \begin{bmatrix} E & & \\ & G & \\ & & G \end{bmatrix} \right)$$

$$\therefore \underline{K} = \begin{bmatrix} \dfrac{G\ h^4}{6L} & -\dfrac{G\ h^6}{120} \\ -\dfrac{G\ h^6}{120} & \dfrac{h^8(Eh^2 + 45G\ L^2)}{16800\,L} \end{bmatrix}, \quad \underline{R} = \begin{bmatrix} T \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} \theta_z{}^2 \\ \\ \omega_2 \end{bmatrix} = \begin{bmatrix} \dfrac{6L(Eh^2 + 45G\ L^2)}{h^4 G\ (Eh^2 + 38G\ L^2)} \\ \dfrac{840L^2}{h^6(Eh^2 + 38G\ L^2)} \end{bmatrix} T$$

5.38



$$x = \frac{L}{2} r + c_1 , \quad y = \frac{h}{2} s + c_2 , \quad z = \frac{h}{2} t + c_3$$

$(c_1, c_2, c_3 :$ constants$)$

$$\underline{J} = \begin{bmatrix} \frac{L}{2} & & \\ & \frac{h}{2} & \\ & & \frac{h}{2} \end{bmatrix}, \quad \underline{J}^{-1} = \begin{bmatrix} \frac{2}{L} & & \\ & \frac{2}{h} & \\ & & \frac{2}{h} \end{bmatrix}$$

$$\det \underline{J} = \frac{Lh^2}{8}$$

$$h_1 = \frac{1+r}{2} , \quad h_2 = \frac{1-r}{2}$$

$$h_{1,r} = \frac{1}{2} , \quad h_{2,r} = -\frac{1}{2}$$

$$\left. \begin{aligned} u &= \sum_{k=1}^{2} h_k u_k' + \frac{th}{2} \sum_{k=1}^{2} h_k \theta_y'^k - \frac{sh}{2} \sum_{k=1}^{2} h_k \theta_z'^k \\ v &= \sum_{k=1}^{2} h_k v_k' - \frac{th}{2} \sum_{k=1}^{2} h_k \theta_x'^k \\ w &= \sum_{k=1}^{2} h_k w_k' + \frac{sh}{2} \sum_{k=1}^{2} h_k \theta_x'^k \end{aligned} \right\} \text{— ①}$$

The relation between variables in primed and unprimed systems is given by

$$\left. \begin{aligned} & u_k' = u_k + b\theta_z^k , \quad v_k' = v_k , \quad w_k' = w_k - b\theta_x^k , \\ & \theta_x'^k = \theta_x^k , \quad \theta_y'^k = \theta_y , \quad \theta_z'^k = \theta_z^k \end{aligned} \right\} \text{— ②}$$

From ① and ②,

$$u = \sum_{k=1}^{2} h_k u_k + \frac{th}{2} \sum_{k=1}^{2} h_k \theta_y^k + \left(b - \frac{sh}{2} \right) \sum_{k=1}^{2} h_k \theta_z^k$$

S.38

$$v = \sum_{k=1}^{2} h_k v_k - \frac{th}{2} \sum_{k=1}^{2} h_k \theta_x^k$$

$$w = \sum_{k=1}^{2} h_k w_k - \left(b - \frac{sh}{2}\right) \sum_{k=1}^{2} h_k \theta_x^k$$

Hence,

$$\underset{\sim}{\epsilon} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix} = \begin{bmatrix} \frac{2}{L}\frac{\partial u}{\partial r} \\ \left(\frac{2}{L}\frac{\partial w}{\partial r} + \frac{2}{h}\frac{\partial u}{\partial t}\right)\Big|_{r=0} \\ \left(\frac{2}{h}\frac{\partial u}{\partial s} + \frac{2}{L}\frac{\partial v}{\partial r}\right)\Big|_{r=0} \end{bmatrix} = \underset{\sim}{B}\,\hat{\underset{\sim}{u}}$$

where $\hat{\underset{\sim}{u}}^T = [\, u_1\ v_1\ w_1\ \theta_x^1\ \theta_y^1\ \theta_z^1 \ \vdots\ u_2\ v_2\ w_2\ \theta_x^2\ \theta_y^2\ \theta_z^2 \,]$

$$\underset{\sim}{B} = [\, \underset{\sim}{B}_1 \ \vdots\ \underset{\sim}{B}_2 \,]$$

and

$$\underset{\sim}{B}_k = \begin{bmatrix} \frac{2}{L}h_{k,r} & 0 & 0 & 0 & \frac{th}{L}h_{k,r} & \frac{2}{L}\left(b-\frac{sh}{2}\right)h_{k,r} \\ 0 & 0 & \frac{2}{L}h_{k,r} & -\frac{2}{L}\left(b-\frac{sh}{2}\right)h_{k,r} & \frac{1}{2} & 0 \\ 0 & \frac{2}{L}h_{k,r} & 0 & -\frac{th}{L}h_{k,r} & 0 & -\frac{1}{2} \end{bmatrix}$$

Then,

$$\underset{\sim}{K} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \underset{\sim}{B}^T \underset{\sim}{C}\, \underset{\sim}{B} \det \underset{\sim}{J}\, dr\, ds\, dt \ , \quad \underset{\sim}{C} = \begin{bmatrix} E & & \\ & Gk & \\ & & Gk \end{bmatrix}, \quad k = \frac{5}{6}$$

Now to evaluate the mass matrix and load vector, introduce the $\underset{\sim}{H}_u$ and $\underset{\sim}{H}_\beta$ matrices:

$$\underset{\sim}{H}_u = \begin{bmatrix} h_1 & 0 & 0 & 0 & 0 & bh_1 & \vdots & h_2 & 0 & 0 & 0 & 0 & bh_2 \\ 0 & h_1 & 0 & 0 & 0 & 0 & \vdots & 0 & h_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & h_1 & -bh_1 & 0 & 0 & \vdots & 0 & 0 & h_2 & -bh_2 & 0 & 0 \end{bmatrix}$$

5.38

$$\underline{H}_p = \left[\begin{array}{ccc|ccc} & h_1 & 0 \quad 0 & & h_2 & 0 \quad 0 \\ \underline{0} & 0 & h_1 \quad 0 & \underline{0} & 0 & h_2 \quad 0 \\ \uparrow & 0 & 0 \quad h_1 & \uparrow & 0 & 0 \quad h_2 \\ 3\times 3 & & & 3\times 3 & & \end{array} \right]$$

The mass matrix $\underline{M}$ is given by

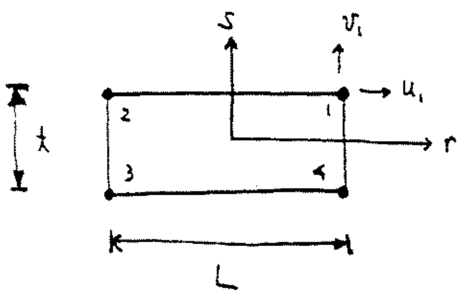
$$\underline{M} = \int_{-1}^{1} \begin{bmatrix} \underline{H}_u \\ \underline{H}_p \end{bmatrix}^T \underline{M}_o \begin{bmatrix} \underline{H}_u \\ \underline{H}_p \end{bmatrix} \left(\frac{L}{2}\right) dr$$

$$\text{where} \quad \underline{M}_o = \left[\begin{array}{ccc|ccc} \rho h^2 & & & & & \\ & \rho h^2 & & & \underline{0} & \\ & & \rho h^2 & & & \\ \hline & & & \rho \frac{h^4}{6} & & \\ & \underline{0} & & & \rho \frac{h^4}{12} & \\ & & & & & \rho \frac{h^4}{12} \end{array} \right] \begin{array}{l} \leftarrow u \\ \leftarrow v \\ \leftarrow w \\ \leftarrow \theta_x \\ \leftarrow \theta_y \\ \leftarrow \theta_z \end{array}$$

And the load vector is obtained by

$$\underline{R} = \int_{-1}^{1} \underline{H}_u^T \underline{f} \left(\frac{L}{2}\right) dr \qquad \text{where} \quad \underline{f}^T = [\, 0 \quad 0 \quad p \,]$$

5.39



(i) As a first step, evaluate the strain-displacement matrix $\underline{B}_{pl}$.

$$\underline{J} = \begin{bmatrix} \frac{L}{2} & 0 \\ 0 & \frac{t}{2} \end{bmatrix}, \quad \underline{J}^{-1} = \begin{bmatrix} \frac{2}{L} & 0 \\ 0 & \frac{2}{t} \end{bmatrix}, \quad \det \underline{J} = \frac{Lt}{4}$$

$$\underline{u} = \begin{bmatrix} u \\ v \end{bmatrix} = \underline{H}\,\hat{\underline{u}}, \quad \underline{H} = \begin{bmatrix} h_1 & 0 & h_2 & 0 & \cdots & h_4 & 0 \\ 0 & h_1 & 0 & h_2 & \cdots & 0 & h_4 \end{bmatrix}$$

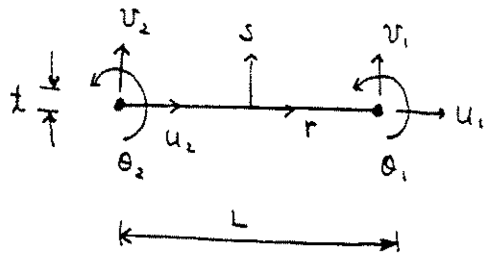
$$\underline{B}_{pl} = \begin{bmatrix} h_{1,x} & 0 & & h_{4,x} & 0 \\ 0 & h_{1,y} & \cdots & 0 & h_{4,y} \\ h_{1,y} & h_{1,x} & & h_{4,y} & h_{4,x} \end{bmatrix} = \begin{bmatrix} \frac{2}{L}h_{1,r} & 0 & & \frac{2}{L}h_{4,r} & 0 \\ 0 & \frac{2}{t}h_{1,s} & \cdots & 0 & \frac{2}{t}h_{4,s} \\ \frac{2}{t}h_{1,s} & \frac{2}{L}h_{1,r} & & \frac{2}{t}h_{4,s} & \frac{2}{L}h_{4,r} \end{bmatrix}$$

$$\therefore \underline{B}_{pl} = \left[\begin{array}{cc|cc|cc|cc} \frac{2}{L}\frac{1+s}{4} & 0 & -\frac{2}{L}\frac{1+s}{4} & 0 & -\frac{2}{L}\frac{1-s}{4} & 0 & \frac{2}{L}\frac{1-s}{4} & 0 \\ 0 & \frac{2}{t}\frac{1+r}{4} & 0 & \frac{2}{t}\frac{1-r}{4} & 0 & -\frac{2}{t}\frac{1-r}{4} & 0 & -\frac{2}{t}\frac{1+r}{4} \\ \frac{2}{t}\frac{1+r}{4} & \frac{2}{L}\frac{1+s}{4} & \frac{2}{t}\frac{1-r}{4} & -\frac{2}{L}\frac{1+s}{4} & -\frac{2}{t}\frac{1-r}{4} & -\frac{2}{L}\frac{1-s}{4} & -\frac{2}{t}\frac{1+r}{4} & \frac{2}{L}\frac{1-s}{4} \end{array}\right]$$

And
$$\underline{C}_{pl} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

S.39

(ii) Now construct $\underset{\sim}{B}_b$ of an isoparametric beam element.



$$\underset{\sim}{J} = \begin{bmatrix} \frac{L}{2} & 0 \\ 0 & \frac{t}{2} \end{bmatrix}$$

$$h_1 = \frac{1+r}{2}, \quad h_2 = \frac{1-r}{2}$$

$$h_{1,r} = \frac{1}{2}, \quad h_{2,r} = -\frac{1}{2}$$

$$\underset{\sim}{u} = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \sum h_k u_k - \frac{ts}{2} \sum h_k \theta_k \\ \sum h_k v_k \end{bmatrix}$$

$$\frac{\partial u}{\partial x} = \frac{2}{L}\left(\sum h_{k,r} u_k - \frac{ts}{2} \sum h_{k,r} \theta_k \right), \quad \frac{\partial u}{\partial y} = -\sum h_k \theta_k$$

$$\frac{\partial v}{\partial x} = \frac{2}{L} \sum h_{k,r} v_k, \quad \frac{\partial v}{\partial y} = 0$$

$$\therefore \underset{\sim}{B}_b = \begin{bmatrix} \frac{2}{L} h_{1,r} & 0 & -\frac{ts}{L} h_{1,r} & \frac{2}{L} h_{2,r} & 0 & -\frac{ts}{L} h_{2,r} \\ 0 & \frac{2}{L} h_{1,r} & -h_1 & 0 & \frac{2}{L} h_{2,r} & -h_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{L} & 0 & -\frac{ts}{2L} & -\frac{1}{L} & 0 & \frac{ts}{2L} \\ 0 & \frac{1}{L} & -\frac{1+r}{2} & 0 & -\frac{1}{L} & -\frac{1-r}{2} \end{bmatrix}$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ u_1 & v_1 & \theta_1 & u_2 & v_2 & \theta_2 \end{matrix}$$

(Note here $\underset{\sim}{\epsilon}^T = [\epsilon_{xx} \;\; \gamma_{xy}]$.)

And $\underset{\sim}{C}_b = \begin{bmatrix} E & 0 \\ 0 & Gk \end{bmatrix}$

5.39

(iii) Let the d.o.f. in the plane stress element be u_i^* and v_i^*, and apply the kinematic constraints given by

$$u_1^* = u_1 - \frac{t}{2}\theta_1 \ , \ u_2^* = u_2 - \frac{t}{2}\theta_2 \ , \ u_3^* = u_2 + \frac{t}{2}\theta_2 \ ,$$

$$u_4^* = u_1 + \frac{t}{2}\theta_1 \ , \ v_1^* = v_1 \ , \ v_2^* = v_2 \ , \ v_3^* = v_2 \ , \ v_4^* = v_1$$

Then $\underline{B}_{pl}$ becomes $\tilde{\underline{B}}_{pl}$ represented as

$$\tilde{\underline{B}}_{pl} = \left[\begin{array}{ccc:ccc} \frac{1}{L} & 0 & -\frac{ts}{2L} & -\frac{1}{L} & 0 & \frac{ts}{2L} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{L} & -\frac{1}{2}(1+r) & 0 & -\frac{1}{L} & -\frac{1}{2}(1-r) \end{array} \right]$$

That is, the first and third rows are identical to those in $\underline{B}_b$. The zeros in the 2nd row in $\tilde{\underline{B}}_{pl}$ only express the fact that the strain ϵ_{yy} is not included in the formulation. This strain is actually equal to $-\nu\epsilon_{xx}$ because the stress τ_{yy} is zero. Hence we now modify the $\underline{C}_{pl}$ matrix to give $\tilde{\underline{C}}$, i.e.,

$$\begin{bmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ -\nu\epsilon_{xx} \\ \gamma_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} (1-\nu^2)\,\epsilon_{xx} \\ 0 \\ \frac{1-\nu}{2}\,\gamma_{xy} \end{bmatrix}$$

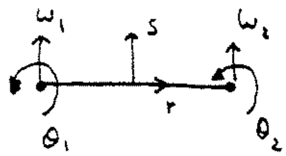
$$= \begin{bmatrix} E & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & G \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ 0 \\ \gamma_{xy} \end{bmatrix}$$

$$\therefore \ \tilde{\underline{C}} = \begin{bmatrix} E & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & Gk \end{bmatrix}$$ (k is included due to the shear assumption.)

Hence,

$$\int_V \underline{B}_b^T \underline{C}_b \underline{B}_b \, dV = \int_V \tilde{\underline{B}}_{pl} \tilde{\underline{C}} \, \underline{B}_{pl} \, dV$$

5.40



$$h_1 = \frac{1-r}{2}, \quad h_2 = \frac{1+r}{2}, \quad x = \frac{L}{2} r + c \quad (c: \text{const.})$$

$$J = \frac{\partial x}{\partial r} = \frac{L}{2}, \quad J^{-1} = \frac{2}{L}, \quad \det J = \frac{L}{2}$$

$$w = \underline{H}_w \hat{\underline{u}}, \quad \beta = \underline{H}_\beta \hat{\underline{u}}, \quad \hat{\underline{u}}^T = [\, w_1 \;\; w_2 \;\; \theta_1 \;\; \theta_2 \,]$$

where $\underline{H}_w = [\, h_1 \;\; h_2 \;\; 0 \;\; 0 \,]$, $\underline{H}_\beta = [\, 0 \;\; 0 \;\; h_1 \;\; h_2 \,]$

$$\underline{B}_w = \frac{d}{dx}\underline{H}_w = \left[-\frac{1}{L} \;\; \frac{1}{L} \;\; 0 \;\; 0 \right], \quad \underline{B}_\beta = \frac{d}{dx}\underline{H}_\beta = \left[0 \;\; 0 \;\; -\frac{1}{L} \;\; \frac{1}{L} \right]$$

$$\gamma = \underline{B}_\gamma \hat{\underline{u}} \quad \text{where} \quad \underline{B}_\gamma = (\underline{B}_w - \underline{H}_\beta)\big|_{r=0} = \left[-\frac{1}{L} \;\; \frac{1}{L} \;\; -\frac{1}{2} \;\; -\frac{1}{2} \right]$$

$$\therefore \; \underline{K} = \frac{EI}{(1-\nu^2)} \int_{-1}^{1} \underline{B}_\beta^T \underline{B}_\beta \det J \, dr + GAk \int_{-1}^{1} \underline{B}_\gamma^T \underline{B}_\gamma \det J \, dr$$

$$= \frac{Eh^3}{12(1-\nu^2)} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2L} & -\frac{1}{2L} \\ 0 & 0 & -\frac{1}{2L} & \frac{1}{2L} \end{bmatrix} + \frac{5Eh}{12(1+\nu)} \begin{bmatrix} \frac{1}{2L} & -\frac{1}{2L} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2L} & \frac{1}{2L} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{L}{8} & \frac{L}{8} \\ \frac{1}{4} & -\frac{1}{4} & \frac{L}{8} & \frac{L}{8} \end{bmatrix}$$

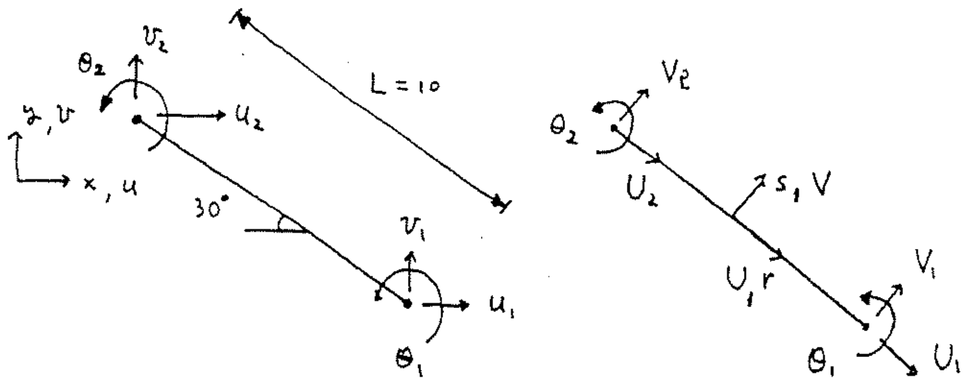
Note that in the term for the bending stiffness $E/(1-\nu^2)$ is used instead of E, because of the plane strain assumption.

That is, with $\varepsilon_{zz} = 0$ and $\tau_{yy} = 0$ we have

$$\varepsilon_{yy} = -\frac{\nu}{1-\nu} \varepsilon_{xx} \quad \text{and} \quad \tau_{xx} = \frac{E}{1-\nu^2} \varepsilon_{xx}.$$

In plane stress conditions of course $\tau_{xx} = E\,\varepsilon_{xx}$ in the beam.

5.41



$$\begin{bmatrix} U_k \\ V_k \end{bmatrix} = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix} \begin{bmatrix} u_k \\ v_k \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} u_k \\ v_k \end{bmatrix}$$

$$\therefore \begin{bmatrix} U \\ V \\ \beta \end{bmatrix} = \begin{bmatrix} h_1 & 0 & 0 & h_2 & 0 & 0 \\ 0 & h_1 & 0 & 0 & h_2 & 0 \\ 0 & 0 & h_1 & 0 & 0 & h_2 \end{bmatrix} \hat{\underline{U}} \quad , \quad \hat{\underline{U}}^T = [\, U_1 \; V_1 \; \theta_1 \; U_2 \; V_2 \; \theta_2 \,]$$

$$= \begin{bmatrix} \frac{\sqrt{3}}{2} h_1 & -\frac{1}{2} h_1 & 0 & \frac{\sqrt{3}}{2} h_2 & -\frac{1}{2} h_2 & 0 \\ \frac{1}{2} h_1 & \frac{\sqrt{3}}{2} h_1 & 0 & \frac{1}{2} h_2 & \frac{\sqrt{3}}{2} h_2 & 0 \\ 0 & 0 & h_1 & 0 & 0 & h_2 \end{bmatrix} \hat{\underline{u}}$$

$$\hat{\underline{u}}^T = [\, u_1 \; v_1 \; \theta_1 \; u_2 \; v_2 \; \theta_2 \,]$$

$u = h_1 u_1 + h_2 u_2$ ($\leftarrow$ displacement in the x-direction)

Let $\underline{H}_u = [\, h_1 \quad 0 \quad 0 \quad h_2 \quad 0 \quad 0 \,]$

$\underline{H}_V = [\, \frac{1}{2} h_1 \quad \frac{\sqrt{3}}{2} h_1 \quad 0 \quad \frac{1}{2} h_2 \quad \frac{\sqrt{3}}{2} h_2 \quad 0 \,]$

$\underline{H}_\beta = [\, 0 \quad 0 \quad h_1 \quad 0 \quad 0 \quad h_2 \,]$

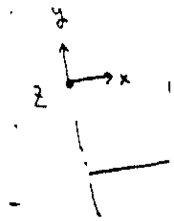
and $\underline{B}_\beta = \frac{2}{L} \frac{d\underline{H}_\beta}{dr} = [\, 0 \quad 0 \quad \frac{1}{10} \quad 0 \quad 0 \quad -\frac{1}{10} \,]$

$\underline{B}_\gamma = (\underline{B}_V - \underline{H}_\beta)\big|_{r=0} = [\, \frac{1}{20} \quad \frac{\sqrt{3}}{20} \quad -\frac{1}{2} \quad -\frac{1}{20} \quad -\frac{\sqrt{3}}{20} \quad -\frac{1}{2} \,]$

S.41

For x-coordinate, $x = \sum h_i x_i = \dfrac{40-5\sqrt{3}}{2} + \dfrac{5\sqrt{3}}{2} r$

$$\underset{\sim}{B} = \begin{bmatrix} \underset{\sim}{B}_{\beta} \\ \underset{\sim}{B}_{\gamma} \\ \underset{\sim}{H}_u / x \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{10} & 0 & 0 \\ \frac{1}{20} & \frac{\sqrt{3}}{20} & -\frac{1}{2} & -\frac{1}{20} & -\frac{\sqrt{3}}{20} & - \\ \frac{h_1}{x} & 0 & 0 & \frac{h_2}{x} & 0 \end{bmatrix}$$

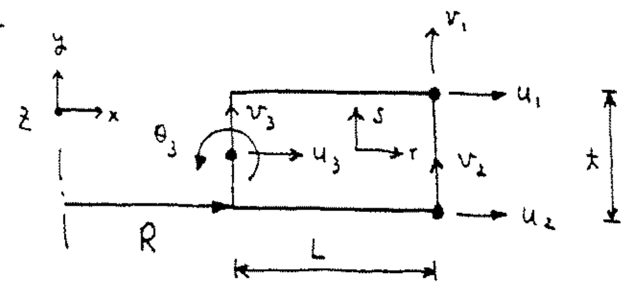


o Displacem

Applying static condensation for the strain ε_{yy} in Jacobi the matrix corresponding to the axisymmetric condition in Table 4.3, we obtain the stress-strain matrix: Henc

$$\underset{\sim}{C} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & 0 & \nu \\ 0 & \frac{1-\nu}{2} & 0 \\ \nu & 0 & 1 \end{bmatrix}$$

5.42



$h_1 = \frac{1}{4}(1+r)(1+s)$

$h_2 = \frac{1}{4}(1+r)(1-s)$

$h_3 = \frac{1}{2}(1-r)$

$x = \frac{L}{2}(1+r) + R$

Displacements: $u = h_1 u_1 + h_2 u_2 + h_3 u_3 - \frac{ts}{2} h_3 \theta_3$

$v = h_1 v_1 + h_2 v_2 + h_3 v_3$

Jacobian: $\underline{J} = \begin{bmatrix} L/2 & 0 \\ 0 & t/2 \end{bmatrix}$, $\underline{J}^{-1} = \begin{bmatrix} 2/L & 0 \\ 0 & 2/t \end{bmatrix}$

Hence,

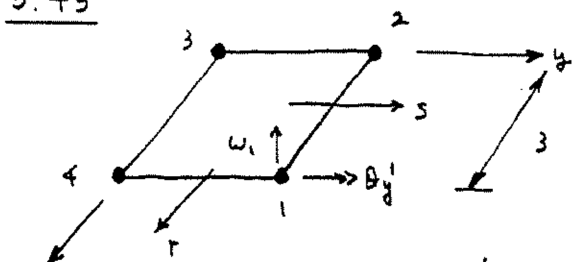
$$\underline{B} = \begin{bmatrix} \frac{1+s}{2L} & 0 & \frac{1-s}{2L} & 0 & -\frac{1}{L} & 0 & \frac{t}{2L}s \\ 0 & \frac{1+r}{2t} & 0 & -\frac{1+r}{2t} & 0 & 0 & 0 \\ \frac{1+r}{2t} & \frac{1+s}{2L} & -\frac{1+r}{2t} & \frac{1-s}{2L} & 0 & -\frac{1}{L} & -\frac{1-r}{2} \\ \frac{(1+r)(1+s)}{4x} & 0 & \frac{(1+r)(1-s)}{4x} & 0 & \frac{1-r}{2x} & 0 & -\frac{ts(1-r)}{4x} \end{bmatrix}$$

where $\underline{\epsilon} = \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \\ \epsilon_{zz} \end{bmatrix} = \begin{bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \\ u/x \end{bmatrix} = \underline{B}\,\hat{\underline{u}}$

$\hat{\underline{u}}^T = [\, u_1 \;\; v_1 \;\; u_2 \;\; v_2 \;\; u_3 \;\; v_3 \;\; \theta_3 \,]$

and $x = \frac{L}{2}(1+r) + R$

5.45

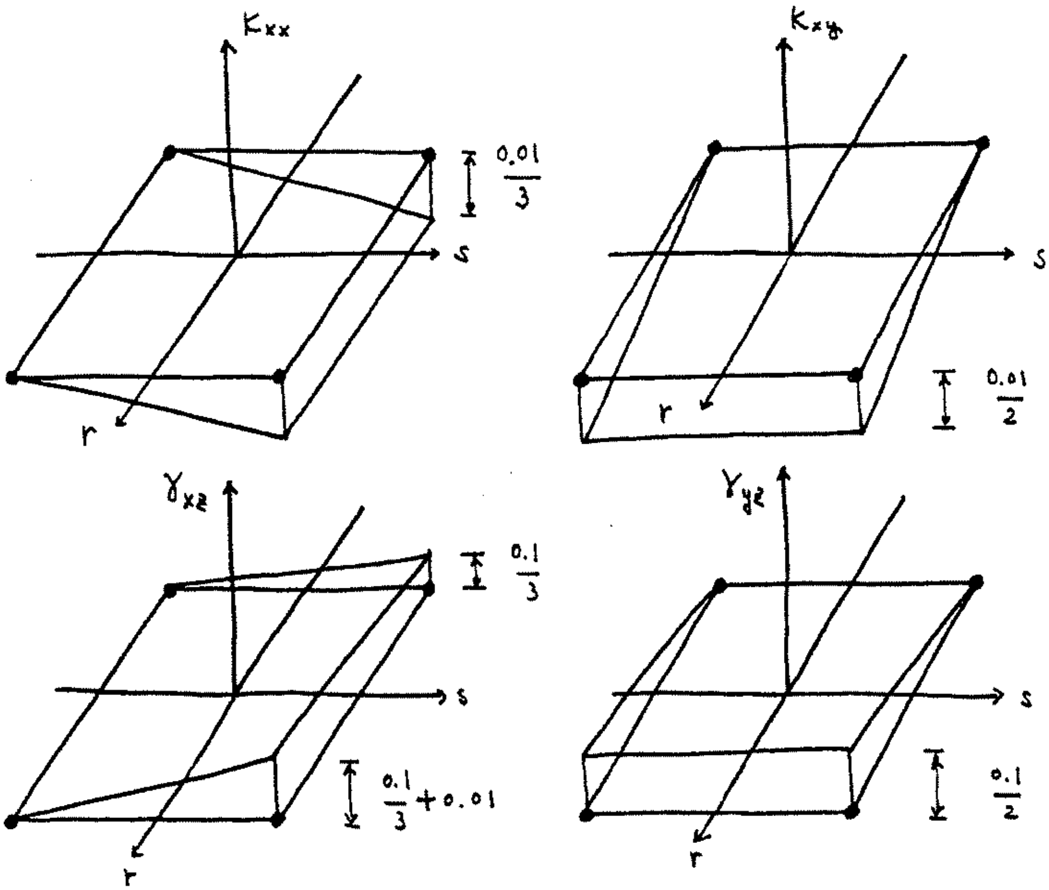


$\omega_1 = 0.1$, $\theta_y^1 = 0.01$
all others zero

From example 5.29,

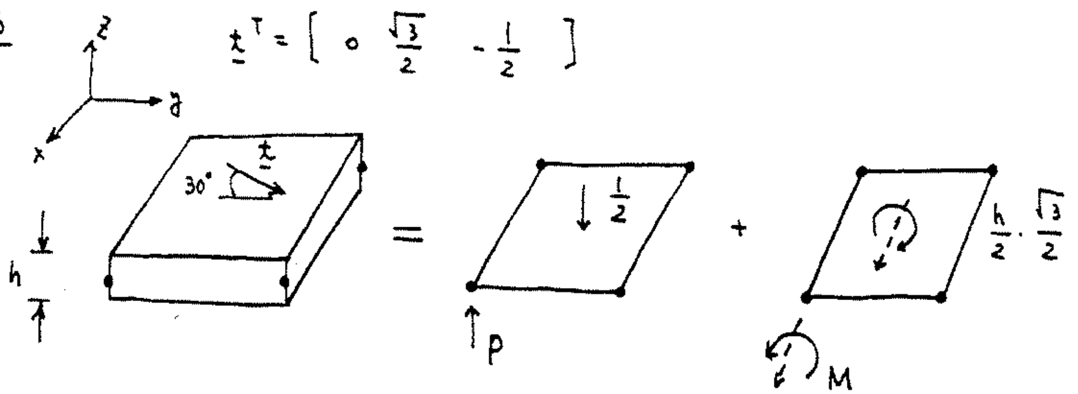
$$\underline{\kappa} = \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix} = \begin{bmatrix} -\frac{1}{6}(1+s)\,\theta_y^1 \\ 0 \\ -\frac{1}{4}(1+r)\,\theta_y^1 \end{bmatrix}$$

$$\underline{\gamma} = \begin{bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \begin{bmatrix} \frac{1}{6}(1+s)\omega_1 + \frac{1}{4}(1+r)(1+s)\,\theta_y^1 \\ \frac{1}{4}(1+r)\,\omega_1 \end{bmatrix}$$



5.46

$$\underline{t}^T = \begin{bmatrix} 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

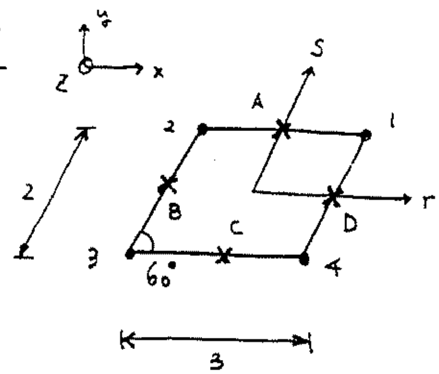


$$P = -\left(\frac{1}{2}\right)(3\cdot 2)/4 = -\frac{3}{4} \quad , \quad M = -\left(\frac{h}{2}\,\frac{\sqrt{3}}{2}\right)(3\cdot 2)/4 = -\frac{3\sqrt{3}\,h}{8}$$

$$\therefore \underline{R}^T = \left[\; -\frac{3}{4} \quad -\frac{3\sqrt{3}\,h}{8} \quad 0 \;\vdots\; \cdots \;\vdots\; -\frac{3}{4} \quad -\frac{3\sqrt{3}\,h}{8} \quad 0 \;\right]$$

(entries correspond to w_1, θ_x^1, θ_y^1, …, w_4, θ_x^4, θ_y^4)

5.47



$$x = \sum h_i x_i = \frac{3}{2} r + \frac{s}{2} + 2 + x_3$$

$$y = \sum h_i y_i = \frac{\sqrt{3}}{2} s + \frac{\sqrt{3}}{2} + y_3$$

$$\therefore \; \underline{J} = \begin{bmatrix} \frac{3}{2} & 0 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}, \quad \det \underline{J} = \sqrt{3}\,\frac{3}{4}$$

$$\begin{bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \begin{bmatrix} \sin\beta & -\sin\alpha \\ -\cos\beta & \cos\alpha \end{bmatrix} \begin{bmatrix} \gamma_{rz} \\ \gamma_{sz} \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & 0 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} \gamma_{rz} \\ \gamma_{sz} \end{bmatrix} \qquad (\alpha = 0^\circ,\ \beta = 60^\circ)$$

Now using eq. (5.103),

$$A_x = 6, \quad B_x = 0, \quad C_x = 2, \quad A_y = 0, \quad B_y = 0, \quad C_y = 2\sqrt{3}$$

$$\hat{\underline{u}}^T = [\, w_1 \;\; \theta_x^1 \;\; \theta_y^1 \;\; \cdots \;\; w_4 \;\; \theta_x^4 \;\; \theta_y^4 \,]$$

$$\gamma_{rz} = \frac{2}{3\sqrt{3}} \Big[\tfrac{1+s}{2} \;\; 0 \;\; \tfrac{3}{4}(1+s) \;\vdots\; -\tfrac{1+s}{2} \;\; 0 \;\; \tfrac{3}{4}(1+s) \;\vdots\; -\tfrac{1-s}{2} \;\; 0 \;\; \tfrac{3}{4}(1-s) \;\vdots\; \tfrac{1-s}{2} \;\; 0 \;\; \tfrac{3}{4}(1-s) \Big] \hat{\underline{u}}$$

$$\gamma_{sz} = \frac{1}{\sqrt{3}} \Big[\tfrac{1+r}{2} \;\; -\tfrac{\sqrt{3}(1+r)}{4} \;\; \tfrac{1+r}{4} \;\vdots\; \tfrac{1-r}{2} \;\; -\tfrac{\sqrt{3}(1-r)}{4} \;\; \tfrac{1-r}{4} \;\vdots\; -\tfrac{1-r}{2} \;\; -\tfrac{\sqrt{3}(1-r)}{4} \;\; \tfrac{1-r}{4} \;\vdots\; -\tfrac{1+r}{2} \;\; -\tfrac{\sqrt{3}(1+r)}{4} \;\; \tfrac{1+r}{4} \Big] \hat{\underline{u}}$$

$$\therefore \; \underline{\gamma} = \begin{bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \underline{B}\,\hat{\underline{u}}$$

where

$$\underline{B} = \left[\begin{array}{ccc:ccc:ccc:ccc} \frac{1+s}{6} & 0 & \frac{1+s}{4} & -\frac{1+s}{6} & 0 & \frac{1+s}{4} & -\frac{1-s}{6} & 0 & \frac{1-s}{4} & \frac{1-s}{6} & 0 & \frac{1-s}{4} \\ \frac{3r-s+2}{6\sqrt{3}} & -\frac{1+r}{4} & \frac{r-s}{4\sqrt{3}} & \frac{s-3r+4}{6\sqrt{3}} & -\frac{1-r}{4} & -\frac{r+s}{4\sqrt{3}} & \frac{3r-s-2}{6\sqrt{3}} & -\frac{1-r}{4} & -\frac{r-s}{4\sqrt{3}} & \frac{s-3r-4}{6\sqrt{3}} & -\frac{1+r}{4} & -\frac{r-s}{4\sqrt{3}} \end{array} \right]$$

We shall consider the 2x2 element only.

5.48 $$\Pi_{HW} = \Pi - \int_V \underline{\tau}^T(\underline{\varepsilon} - \underline{\partial}_\varepsilon \underline{u})\, dV - \int_{S_u} \underline{f}^{S_u T}(\underline{u}^{S_u} - \underline{u}_p)\, dS$$

where $$\Pi = \frac{1}{2}\int_V \underline{\varepsilon}^T \underline{C}\, \underline{\varepsilon}\, dV - \int_V \underline{u}^T \underline{f}^B dV - \int_{S_f} \underline{u}^{S_f T} \underline{f}^{S_f}\, dS$$

In the Hu-Washizu functional, the independent variables are stresses, strains, and displacements. Assume that the stresses are given by the strains, $\underline{\tau} = \underline{C}\,\underline{\varepsilon}$, then

$$\Pi^*_{HR} = \int_V \left(-\frac{1}{2}\underline{\varepsilon}^T\underline{C}\,\underline{\varepsilon} + \underline{\varepsilon}^T\underline{C}\,\underline{\partial}_\varepsilon \underline{u} - \underline{u}^T\underline{f}^B\right) dV + (\text{boundary terms}) \quad \text{— ①}$$

Note this variational indicator is also a Hellinger-Reissner functional, but we have here the strains and displacements as the independent variables.

In the plate formulation the variables are $w, \beta_x, \beta_y, \underline{\gamma}^{AS}$ (the superscript AS denotes the assumed shear strains). Hence the curvature $\underline{\kappa}$ is calculated from the displacements, and we can specialize ① further (see Example 4.30).

$$\tilde{\Pi}^*_{HR} = \int_A \left(\frac{1}{2}\underline{\kappa}^T\underline{C}_b\underline{\kappa} - \frac{1}{2}\underline{\gamma}^{AS^T}\underline{C}_s\,\underline{\gamma}^{AS} + \underline{\gamma}^{AS^T}\underline{C}_s\,\underline{\gamma} - \underline{u}^T\underline{f}^B\right) dA$$
$$+ (\text{boundary terms})$$

where $$\underline{\kappa} = \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix} = \begin{bmatrix} \beta_{x,x} \\ \beta_{y,y} \\ \beta_{x,y} + \beta_{y,x} \end{bmatrix}, \quad \underline{\gamma} = \begin{bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \begin{bmatrix} w_{,x} + \beta_x \\ w_{,y} + \beta_y \end{bmatrix}, \quad \underline{\gamma}^{AS} = \begin{bmatrix} \gamma_{xz}^{AS} \\ \gamma_{yz}^{AS} \end{bmatrix}$$

$$\underline{C}_b = \frac{Eh^3}{12(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \quad \text{and} \quad \underline{C}_s = \begin{bmatrix} Ghk & 0 \\ 0 & Ghk \end{bmatrix}$$

5.48

Now invoking $\delta \tilde{\Pi}^{*}_{HR} = 0$, we obtain the following

$$\left.\begin{aligned} &\text{regarding } \delta \underline{u}, \quad \int_A \left[\delta \underline{\kappa}^T \underline{C}_b \underline{\kappa} + \delta \underline{\gamma}^T \underline{C}_s \underline{\gamma}^{AS} \right] dA = \int_A \delta \underline{u}^T \underline{f}^B dA \\ &\text{regarding } \delta \underline{\gamma}^{AS}, \quad \int_A \delta \underline{\gamma}^{AS^T} \underline{C}_s \left(\underline{\gamma} - \underline{\gamma}^{AS} \right) dA = 0 \end{aligned}\right\} \quad (2)$$

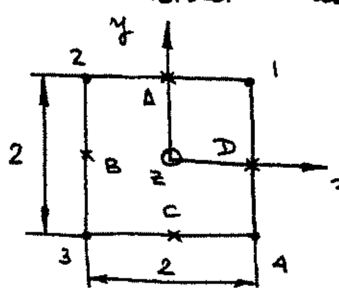
Let $\hat{\underline{u}}^T = [w_1 \ \theta_x^1 \ \theta_y^1 \ \cdots \ w_4 \ \theta_x^4 \ \theta_y^4]$, $\hat{\underline{\gamma}}^T = [\gamma_z^{AS} \ \gamma_z^{AS}]$

then, $\underline{u} = \underline{H} \hat{\underline{u}}$, $\underline{\kappa} = \underline{B}_\kappa \hat{\underline{u}}$, $\underline{\gamma} = \underline{B}_\gamma \hat{\underline{u}}$ and $\underline{\gamma}^{AS} = \underline{B}_\gamma^{AS} \hat{\underline{\gamma}}$ — (3)

Finally we obtain from (2) and (3),

$$\left.\begin{aligned} &\begin{bmatrix} \underline{K}_{uu} & \underline{K}_{u\gamma} \\ \underline{K}_{u\gamma}^T & \underline{K}_{\gamma\gamma} \end{bmatrix} \begin{bmatrix} \hat{\underline{u}} \\ \hat{\underline{\gamma}} \end{bmatrix} = \begin{bmatrix} \underline{R}_B \\ \underline{0} \end{bmatrix}, \quad \underline{K}_{uu} = \int_A \underline{B}_\kappa^T \underline{C}_b \underline{B}_\kappa \, dA \\ &\underline{K}_{u\gamma} = \int_A \underline{B}_\gamma^T \underline{C}_s \underline{B}_\gamma^{AS} dA, \quad \underline{K}_{\gamma\gamma} = -\int_A \underline{B}_\gamma^{AS^T} \underline{C}_s \underline{B}_\gamma^{AS} dA, \\ &\underline{R}_B = \int_A \underline{H}^T \underline{f}^B dA \end{aligned}\right\} \quad (4)$$

Consider now the 2×2 element. For this element we assume the following interpolation of shear strains:



$$\underline{\gamma}^{AS} = \begin{bmatrix} \gamma_{xz}^{AS} \\ \gamma_{yz}^{AS} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(1+s)\gamma_{xz}^{A} + \frac{1}{2}(1-s)\gamma_{xz}^{C} \\ \frac{1}{2}(1+r)\gamma_{yz}^{D} + \frac{1}{2}(1-r)\gamma_{yz}^{B} \end{bmatrix} =$$

$$= \underline{B}_\gamma^{AS} \hat{\underline{\gamma}}, \quad \text{where} \quad \hat{\underline{\gamma}} = [\gamma_{xz}^A \ \ \gamma_{yz}^B \ \ \gamma_{xz}^C \ \ \gamma_{yz}^D]^T.$$

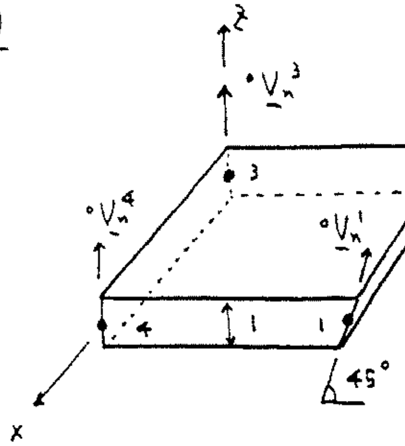
5.48

Matrices $\underline{C}_b$ and $\underline{C}_s$ are given by (5.97), and for derivation of matrices $\underline{B}_\gamma$, $\underline{B}_\kappa$, and $\underline{H}_w$ see Example 5.29.

Substituting these matrices into ④ and performing static condensation on $\hat{\underline{\gamma}}$, we obtain the final stiffness matrix as

$$\underline{K} = \underline{K}_{uu} - \underline{K}_{u\gamma} \left(\underline{K}_{\gamma\gamma} \right)^{-1} \underline{K}_{u\gamma}^T .$$

5.49



$${}^\circ V_n^1 = {}^\circ V_n^2 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$${}^\circ V_n^3 = {}^\circ V_n^4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$${}^\circ V_1^1 = {}^\circ V_1^2 = {}^\circ V_1^3 = {}^\circ V_1^4 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

From ${}^\circ V_2^k = {}^\circ V_n^k \times {}^\circ V_1^k$, $\quad {}^\circ V_2^1 = {}^\circ V_2^2 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$, $\quad {}^\circ V_2^3 = {}^\circ V_2^4 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$a_1 = a_2 = \sqrt{2}, \quad a_3 = a_4 = 1$

For the coordinates,

$${}^\circ x(r,s,t) = \sum h_k {}^\circ x_k + \frac{t}{2} \sum a_k h_k {}^\circ V_{nx}^k = 20(1+r)$$

$${}^\circ y(r,s,t) = \sum h_k {}^\circ y_k + \frac{t}{2} \sum a_k h_k {}^\circ V_{ny}^k = (1+s)\left(10 + \frac{t}{4}\right)$$

$${}^\circ z(r,s,t) = \sum h_k {}^\circ z_k + \frac{t}{2} \sum a_k h_k {}^\circ V_{nz}^k = \frac{t}{2}$$

For displacements,

$$V_n^k = -{}^\circ V_2^k \alpha_k + {}^\circ V_1^k \beta_k$$

when $k=1$ and 2, $\quad V_n^k = -\begin{bmatrix} 0 \\ \alpha_k/\sqrt{2} \\ -\alpha_k/\sqrt{2} \end{bmatrix} + \begin{bmatrix} \beta_k \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \beta_k \\ -\alpha_k/\sqrt{2} \\ \alpha_k/\sqrt{2} \end{bmatrix}$

when $k=3$ and 4, $\quad V_n^k = -\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \alpha_k + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \beta_k = \begin{bmatrix} \beta_k \\ -\alpha_k \\ 0 \end{bmatrix}$

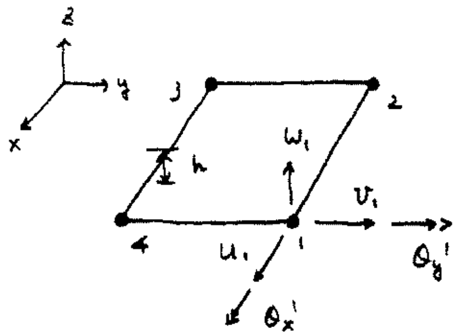
$$\therefore \quad u(r,s,t) = \sum h_k u_k + \frac{t}{2} \sum a_k h_k V_{nx}^k$$

$$v(r,s,t) = \sum h_k v_k + \frac{t}{2} \sum a_k h_k V_{ny}^k$$

$$w(r,s,t) = \sum h_k w_k + \frac{t}{2} \sum a_k h_k V_{nz}^k$$

5.50 For a flat element Eq. (5.107) and (5.118) reduce to

$$x = \Sigma h_k x_k \,, \quad y = \Sigma h_k y_k \quad \text{and} \quad z = \frac{t}{2}$$



$${}^\circ\underline{V}_1^k = \underline{e}_x \,, \quad {}^\circ\underline{V}_2^k = \underline{e}_y \,, \quad {}^\circ\underline{V}_n = \underline{e}_z$$

$$\underline{V}_n^k = -\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \alpha_k + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \beta_k$$

$$\therefore \underline{V}_n^k = \begin{bmatrix} \beta_k \\ -\alpha_k \\ 0 \end{bmatrix} = \begin{bmatrix} \theta_y^k \\ -\theta_x^k \\ 0 \end{bmatrix}$$

Hence,

$$u = \underbrace{\Sigma h_k u_k}_{\text{plane stress}} + \underbrace{\frac{t}{2} h \, \Sigma h_k \theta_y^k}_{\text{plate}}$$

$$v = \underbrace{\Sigma h_k v_k}_{\text{plane stress}} - \underbrace{\frac{t}{2} h \, \Sigma h_k \theta_x^k}_{\text{plate}}$$

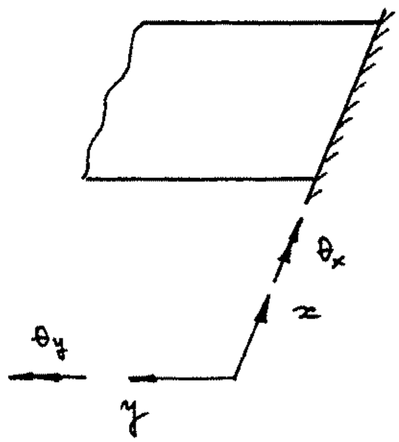
$$w = \underbrace{\Sigma h_k w_k}_{\text{plate}}$$

(wavy underline — plane stress; straight underline — plate)

From these displacements interpolations we see that the displacements in the shell can be obtained by adding the displacements in the plate element (Reissner - Mindlin type) and the plane stress membrane element. Looking at the $\underline{C}$ matrix, we know that the stress τ_{zz} is implied to be zero, which is also the case in the shell element.

5.52

The Kirchhoff plate boundary conditions correspond to a clamped plate as shown.



For the Reissner - Mindlin plate theory the following boundary conditions could be used:

case 1 : $w=0$, $\theta_x = \theta_y = 0$

case 2 : $w=0$, $\theta_x = 0$, θ_y left free

Case 1 corresponds to the "hard" b.c., whereas case 2 corresponds to the "soft" boundary conditions (see also page 449 of the textbook).

Usually case 2 is the appropriate choice, and if the thickness of the plate becomes very small, a boundary layer develops and the Kirchhoff solution is approached.

The b.c.'s in case 1 are constraining the plate particles further in that the sections normal to the x - axis can also not rotate.

5.53 $$\int_a^b F(r)\,dr = \int_a^b \Big[F_0 \frac{(r-r_1)(r-r_2)(r-r_3)}{(r_0-r_1)(r_0-r_2)(r_0-r_3)} + F_1 \frac{(r-r_0)(r-r_2)(r-r_3)}{(r_1-r_0)(r_1-r_2)(r_1-r_3)}$$

$$+ F_2 \frac{(r-r_0)(r-r_1)(r-r_3)}{(r_2-r_0)(r_2-r_1)(r_2-r_3)} + F_3 \frac{(r-r_0)(r-r_1)(r-r_2)}{(r_3-r_0)(r_3-r_1)(r_3-r_2)} \Big] dr$$

Using $r_0 = a$, $r_1 = a+h$, $r_2 = a+2h$, $r_3 = a+3h$ where $h = (b-a)/3$, the evaluation of the integral gives:

$$\int_a^b F(r)\,dr = \frac{b-a}{8}\left(F_0 + 3F_1 + 3F_2 + F_3 \right)$$

Hence, the Newton-Cotes constants are as given in Table 5.5 for the case $n = 3$.

5.54 Let $P(r) = (r-r_1)(r-r_2)(r-r_3)$

Then
$$\int_{-1}^{1}(r-r_1)(r-r_2)(r-r_3)\cdot 1\,dr = 0$$
$$\int_{-1}^{1}(r-r_1)(r-r_2)(r-r_3)\cdot r\,dr = 0$$
$$\int_{-1}^{1}(r-r_1)(r-r_2)(r-r_3)\cdot r^2\,dr = 0$$

Solving these, we obtain
$$-\frac{2}{3}(r_1+r_2+r_3) - 2r_1r_2r_3 = 0$$
$$-\frac{2}{5}(r_1+r_2+r_3) - \frac{2}{3}r_1r_2r_3 = 0$$
$$\frac{2}{3}(r_1r_2+r_2r_3+r_3r_1) + \frac{2}{5} = 0$$

$\therefore$ $r_1+r_2+r_3 = 0$

$r_1r_2+r_2r_3+r_3r_1 = -\frac{3}{5}$ $\qquad \therefore r_1 = -\sqrt{\frac{3}{5}},\ r_2 = 0,\ r_3 = \sqrt{\frac{3}{5}}$

$r_1r_2r_3 = 0$

The weights are respectively given by
$$\alpha_1 = \int_{-1}^{1}\frac{(r-r_2)(r-r_3)}{(r_1-r_2)(r_1-r_3)}\,dr = \frac{5}{9}$$
$$\alpha_2 = \int_{-1}^{1}\frac{(r-r_1)(r-r_3)}{(r_2-r_1)(r_2-r_3)}\,dr = \frac{8}{9}$$
$$\alpha_3 = \int_{-1}^{1}\frac{(r-r_1)(r-r_2)}{(r_3-r_1)(r_3-r_2)}\,dr = \frac{5}{9}$$

5.55 The integration order to be used depends on the order of r and s in $\underline{F}_K = \underline{B}^T \underline{C}\, \underline{B} \det \underline{J}$ and $\underline{F}_M = \rho \underline{H}^T \underline{H} \det \underline{J}$. For an undistorted element with sides 2a and 2b,

$$x = ar, \quad y = bs, \quad \underline{J} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \quad \det \underline{J} = ab$$

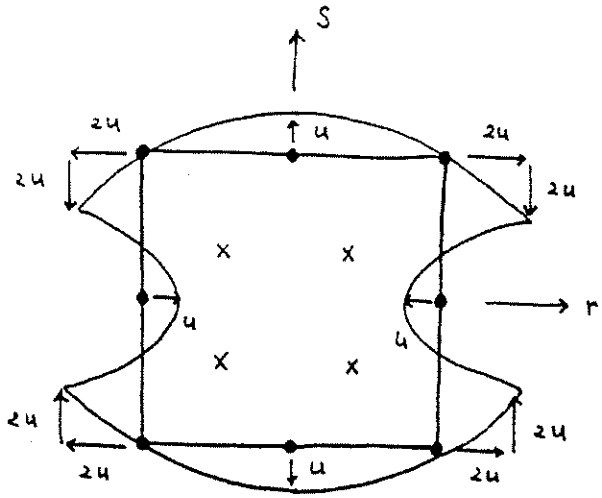
Since $\underline{J}$ is a constant matrix, the elements of $\underline{B}$ are therefore functions of r and s only.

For axisymmetric analysis, $\underline{F}_K = \underline{B}^T \underline{C}\, \underline{B} \det \underline{J}\, x$ and $\underline{F}_M = \rho \underline{H}^T \underline{H} \det \underline{J}\, x$.

Therefore, for the nine-node element we have a polynomial of maximal degree 4 in each variable. Therefore, 3×3 Gauss integration scheme is sufficient.

Note that in the axisymmetric analysis this scheme will not give the exact result due to the hoop strain.

5.56



Let $h_5 = \frac{1}{2}(1-r^2)(1+s)$

$h_6 = \frac{1}{2}(1-s^2)(1-r)$

$h_7 = \frac{1}{2}(1-r^2)(1-s)$

$h_8 = \frac{1}{2}(1-s^2)(1+r)$

Then, the interpolation functions are

$$h_1 = \frac{1}{4}(1+r)(1+s) - \frac{1}{2}(h_5 + h_8)$$

$$h_2 = \frac{1}{4}(1-r)(1+s) - \frac{1}{2}(h_5 + h_6)$$

$$h_3 = \frac{1}{4}(1-r)(1-s) - \frac{1}{2}(h_6 + h_7)$$

$$h_4 = \frac{1}{4}(1+r)(1-s) - \frac{1}{2}(h_7 + h_8)$$

With the length of side $2a$, $\underline{J} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, $\underline{J}^{-1} = \begin{bmatrix} 1/a & 0 \\ 0 & 1/a \end{bmatrix}$

Hence, $$\underline{\epsilon} = \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \frac{1}{a} \begin{bmatrix} \partial u/\partial r \\ \partial v/\partial s \\ \partial u/\partial s + \partial v/\partial r \end{bmatrix} = \underline{B}\,\hat{\underline{u}}$$

where $$\underline{B} = \frac{1}{a} \begin{bmatrix} h_{1,r} & 0 & h_{2,r} & 0 & & h_{8,r} & 0 \\ 0 & h_{1,s} & 0 & h_{2,s} & \cdots & 0 & h_{8,s} \\ h_{1,s} & h_{1,r} & h_{2,s} & h_{2,r} & & h_{8,s} & h_{8,r} \end{bmatrix}$$

5.56

Using the nodal displacements given, and simplifying,

$$\underline{B}\,\hat{\underline{u}} = \begin{bmatrix} -\frac{1}{3a}(1-3s^2) \\ \frac{1}{3a}(1-3r^2) \\ 0 \end{bmatrix} u$$

When $(r,s) = \left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$, $\underline{B}\,\hat{\underline{u}} = \underline{0}$

Therefore, if we apply 2×2 Gauss integration to evaluate the strain energy $E = \frac{1}{2}\,\hat{\underline{u}}^T\,\underline{k}\,\hat{\underline{u}}$, we obtain that

$$E = \frac{1}{2}\,\hat{\underline{u}}^T \sum_{i=1}^{4} \left\{ \underline{B}^T(r_i,s_i)\ \underline{C}\ \underline{B}(r_i,s_i)\ \hat{\underline{u}}\ \det \underline{J} \right\},$$

where $|r_i| = |s_i| = \frac{1}{\sqrt{3}}$

$\therefore\ E \equiv 0$. Hence, the 2×2 intagrated element contains the spurious mode shown.

5.57 We consider plane strain analysis for the 9/3 element with sides 2a and 2b.

$$\underline{J} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \quad \det \underline{J} = ab, \quad \underline{J}^{-1} = \begin{bmatrix} 1/a & 0 \\ 0 & 1/b \end{bmatrix}$$

Using the interpolation functions h_i, $i = 1, \ldots, 9$,

$$\underline{u} = \underline{H}\,\hat{\underline{u}}, \quad \hat{\underline{u}}^T = [\,u_1 \;\; v_1 \;\; u_2 \;\; v_2 \;\; \cdots \;\; u_9 \;\; v_9\,]$$

$$\underline{H} = \begin{bmatrix} h_1 & 0 & h_2 & 0 & \cdots & h_9 & 0 \\ 0 & h_1 & 0 & h_2 & \cdots & 0 & h_9 \end{bmatrix}$$

$$\underline{\varepsilon}' = \begin{bmatrix} \frac{2}{3}u_{,x} - \frac{1}{3}v_{,y} \\ -\frac{1}{3}u_{,x} + \frac{2}{3}v_{,y} \\ u_{,y} + v_{,x} \\ -\frac{1}{3}u_{,x} - \frac{1}{3}v_{,y} \end{bmatrix} = \underline{B}_D\,\hat{\underline{u}} \qquad \frac{\partial}{\partial x} = \frac{1}{a}\frac{\partial}{\partial r}, \quad \frac{\partial}{\partial y} = \frac{1}{b}\frac{\partial}{\partial s}$$

$$\therefore\ \underline{B}_D = \begin{bmatrix} \frac{2}{3a}h_{1,r} & -\frac{1}{3b}h_{1,s} & \frac{2}{3a}h_{2,r} & -\frac{1}{3b}h_{2,s} & \cdots & \frac{2}{3a}h_{9,r} & -\frac{1}{3b}h_{9,s} \\ -\frac{1}{3a}h_{1,r} & \frac{2}{3b}h_{1,s} & -\frac{1}{3a}h_{2,r} & \frac{2}{3b}h_{2,s} & \cdots & -\frac{1}{3a}h_{9,r} & \frac{2}{3b}h_{9,s} \\ \frac{1}{b}h_{1,s} & \frac{1}{a}h_{1,r} & \frac{1}{b}h_{2,s} & \frac{1}{a}h_{2,r} & \cdots & \frac{1}{b}h_{9,s} & \frac{1}{a}h_{9,r} \\ -\frac{1}{3a}h_{1,r} & -\frac{1}{3b}h_{1,s} & -\frac{1}{3a}h_{2,r} & -\frac{1}{3b}h_{2,s} & \cdots & -\frac{1}{3a}h_{9,r} & -\frac{1}{3b}h_{9,s} \end{bmatrix}$$

$$\varepsilon_V = \varepsilon_{xx} + \varepsilon_{yy} = \underline{B}_V\,\hat{\underline{u}}$$

$$\underline{B}_V = \begin{bmatrix} \frac{1}{a}h_{1,r} & \frac{1}{b}h_{1,s} & \frac{1}{a}h_{2,r} & \frac{1}{b}h_{2,s} & \cdots & \frac{1}{a}h_{9,r} & \frac{1}{b}h_{9,s} \end{bmatrix}$$

$$p = \underline{H}_p\,\hat{\underline{p}}, \quad \underline{H}_p = [\,1 \;\; r \;\; s\,], \quad \hat{\underline{p}}^T = [\,p_0 \;\; p_1 \;\; p_2\,]$$

5.57

Then the stiffness matrix $\underline{K}$ is given by

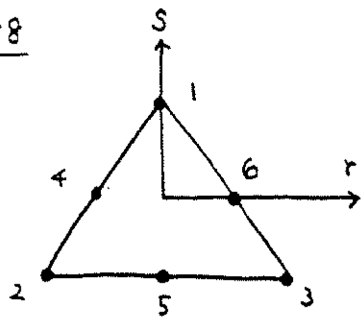
$$\underline{K} = \begin{bmatrix} \underline{K}_{uu} & \underline{K}_{up} \\ \underline{K}_{pu} & \underline{K}_{pp} \end{bmatrix} = \begin{bmatrix} \int_V \underline{B}_D^T \underline{C}' \underline{B}_D \, dV & -\int_V \underline{B}_V^T \underline{H}_p \, dV \\ -\left(\int_V \underline{B}_V^T \underline{H}_p \, dV \right)^T & -\int_V \underline{H}_p^T \frac{1}{\kappa} \underline{H}_p \, dV \end{bmatrix}$$

where $dV = \det \underline{J} \, dr\, ds = ab \, dr\, ds$

Hence, the highest order of r and s terms to be integrated in each submatrix is found in the $\underline{K}_{uu}$ matrix, and it has the form of $r^m s^n$ with $m, n \leq 4$. 2×2 Gauss integration is not adequate because the degree of precision is given by $2n-1$ with n equal to the no. of Gauss points used in each direction, which corresponds to the order 3 only. We have to therefore use 3×3 Gauss integration with the degree of precision $2 \cdot 3 - 1 = 5$.

This will give the exact stiffness matrix in a geometrically undistorted element, and is therefore "full" integration.

5.58



$$h_1 = \frac{1}{2}(1+s) - \frac{1}{2}(1-s^2)$$

$$h_2 = \frac{1}{4}(1-r)(1-s) - \frac{1}{4}(1-s^2)(1-r) - \frac{1}{4}(1-r^2)(1-s) + \Delta h$$

$$h_3 = \frac{1}{4}(1+r)(1-s) - \frac{1}{4}(1-r^2)(1-s) - \frac{1}{4}(1-s^2)(1+r) + \Delta h$$

$$h_4 = \frac{1}{2}(1-s^2)(1-r)$$

$$h_5 = \frac{1}{2}(1-r^2)(1-s) - 2\Delta h$$

$$h_6 = \frac{1}{2}(1-s^2)(1+r) \quad \text{with } \Delta h = \frac{(1-r^2)(1-s^2)}{8}$$

Using these interpolation functions,

$$\underline{u} = \begin{bmatrix} u \\ v \end{bmatrix} = \underline{H}\,\hat{\underline{u}} \quad , \quad \hat{\underline{u}}^T = [\, u_1 \ v_1 \ u_2 \ v_2 \ \cdots \ u_6 \ v_6 \,]$$

$$\underline{H} = \begin{bmatrix} h_1 & 0 & h_2 & 0 & \cdots & h_6 & 0 \\ 0 & h_1 & 0 & h_2 & \cdots & 0 & h_6 \end{bmatrix}$$

And the strain-displacement matrix is

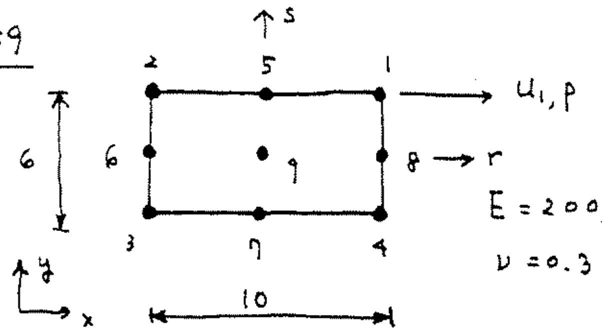
$$\underline{B} = \begin{bmatrix} h_{1,r} & 0 & h_{2,r} & 0 & & h_{6,r} & 0 \\ 0 & h_{1,s} & 0 & h_{2,s} & \cdots & 0 & h_{6,s} \\ h_{1,s} & h_{1,r} & h_{2,s} & h_{2,r} & & h_{6,s} & h_{6,r} \end{bmatrix}$$

The stiffness matrix is

$$\underline{K} = \int_V \underline{B}^T \underline{C}\, \underline{B}\, dV \quad \text{where } \underline{C} \text{ is for the plane stress case.}$$

Then the highest order terms to be integrated have the form of $r^m s^n$ with $m, n \le 4$. Hence, 3x3 Gauss integration is the full integration.

5.59



$E = 200{,}000$

$\nu = 0.3$

$$\underline{J} = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}, \quad \det \underline{J} = 15$$

$$\underline{J}^{-1} = \begin{bmatrix} 1/5 & 0 \\ 0 & 1/3 \end{bmatrix}$$

(a) $u = h_1 u_1$, $v = 0$ where $h_1 = \frac{1}{4}(1+r)(1+s) - \frac{1}{4}(1-r^2)(1+s)$
$\qquad - \frac{1}{4}(1-s^2)(1+r) + \frac{1}{4}(1-r^2)(1-s^2)$

$$\underline{\epsilon} = \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{5} h_{1,r} \\ 0 \\ \frac{1}{3} h_{1,s} \end{bmatrix} u_1, \quad h_{1,r} = \frac{1}{4}(1+2r)\, s(1+s)$$

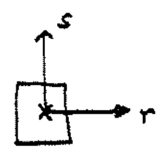
$$h_{1,s} = \frac{1}{4}\, r(1+r)(1+2s)$$

$$\therefore \underline{B}^T = \begin{bmatrix} \frac{1}{20}(1+2r)\, s\, (1+s) & 0 & \frac{1}{12} r(1+r)(1+2s) \end{bmatrix}$$

$$\rightarrow \underline{K} = \int_{-1}^{1}\int_{-1}^{1} \underline{B}^T \underline{C}\, \underline{B}\ (15)\, dr\, ds \qquad C = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

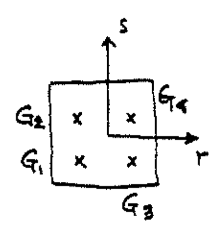
$$= [\,80911.7\,]$$

(b) Let $\underline{F} = 15\, \underline{B}^T \underline{C}\, \underline{B}$



1×1 integration $(r_0, s_0) = (0, 0)$, $\alpha_0 = 2$

$$\underline{K} = 2^2 \cdot \left[(15\, \underline{B}^T \underline{C}\, \underline{B}) \Big|_{r=0, s=0} \right] = [\,0\,]$$



2×2 integration $(r, s) = (\pm 1/\sqrt{3},\ \pm 1/\sqrt{3})$, $\alpha_i = 1$

$$\underline{K} = 1^2 \cdot \underline{F}|_{G_1} + 1^2 \cdot \underline{F}|_{G_2} + 1^2 \cdot \underline{F}|_{G_3} + 1^2 \cdot \underline{F}|_{G_4}$$

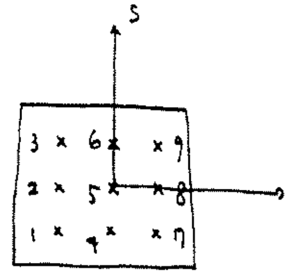
$$= 23.16 + 2378.71 + 2437.46 + 62587.1 = 67426.4$$

5.59

3×3 integration

$$r, s = -\sqrt{3/5} \quad 0 \quad \sqrt{3/5}$$

$$\alpha = 5/9 \quad 8/9 \quad 5/9$$



$$\underline{K} = \sum_{i,j} \alpha_i \alpha_j \underline{F}_{ij}$$

$$= \frac{5}{9}\left(\frac{5}{9} \underline{F}|_{G_1} + \frac{8}{9} \underline{F}|_{G_2} + \frac{5}{9} \underline{F}|_{G_3} \right)$$

$$+ \frac{8}{9}\left(\frac{5}{9} \underline{F}|_{G_4} + \frac{8}{9} \underline{F}|_{G_5} + \frac{5}{9} \underline{F}|_{G_6} \right)$$

$$+ \frac{5}{9}\left(\frac{5}{9} \underline{F}|_{G_7} + \frac{8}{9} \underline{F}|_{G_8} + \frac{5}{9} \underline{F}|_{G_9} \right)$$

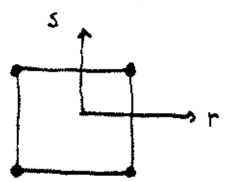
$$= \frac{5}{9}\left\{ \frac{5}{9}(149.45) + \frac{8}{9}(244.26) + \frac{5}{9}(6284.32) \right\}$$

$$+ \frac{8}{9}\left\{ \frac{5}{9}(251.24) + 0 + \frac{5}{9}(15572.9) \right\}$$

$$+ \frac{5}{9}\left\{ \frac{5}{9}(6199.2) + \frac{8}{9}(15140.4) + \frac{5}{9}(199587) \right\}$$

$$= 80911.7$$

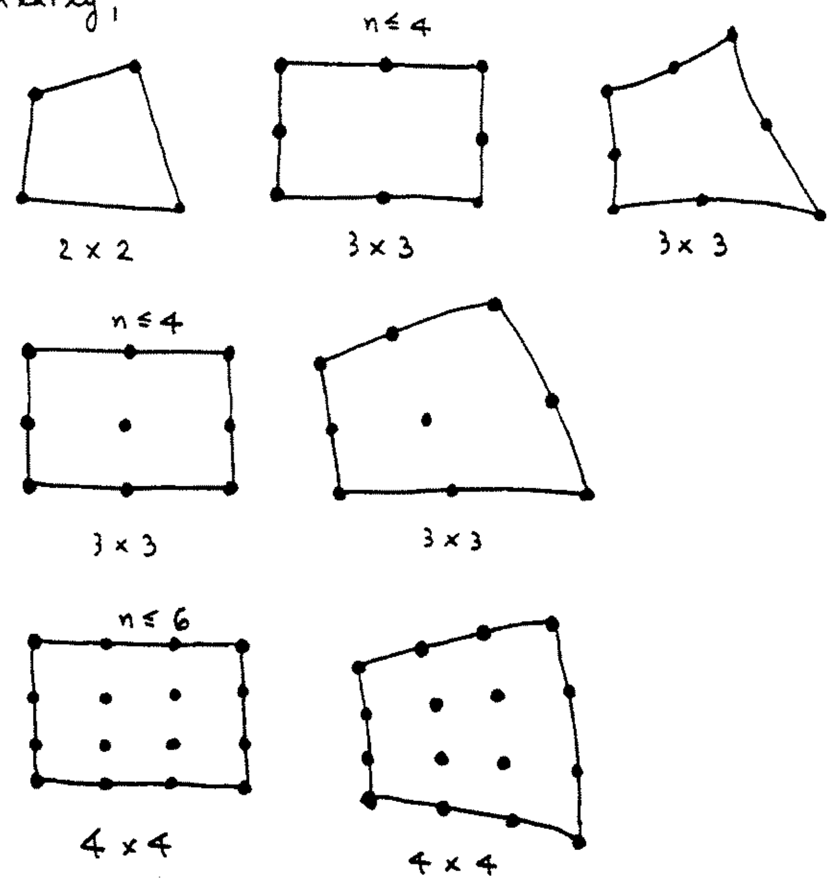
5.60 $\underline{M} = \int_V \rho \underline{H}^T \underline{H} \, dV$.



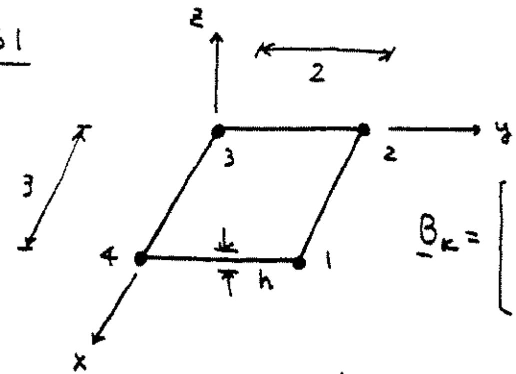
4-node

For the four-node element we have to integrate polynomials $Q_n(x, y)$ of order $n \le 2$ in <u>each</u> variable. Therefore, we use 2×2 integration.

Similarly,



5.61



$$\underline{J} = \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & 1 \end{bmatrix}, \quad \det \underline{J} = \frac{3}{2}, \quad \underline{J}^{-1} = \begin{bmatrix} \frac{2}{3} & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underline{B}_{\kappa} = \begin{bmatrix} 0 & 0 & -\frac{1}{6}(1+s) & \cdots & -\frac{1}{6}(1-s) \\ 0 & \frac{1}{4}(1+r) & 0 & \cdots & 0 \\ 0 & \frac{1}{6}(1+s) & -\frac{1}{4}(1+r) & & \frac{1}{4}(1+r) \end{bmatrix}$$

$$\underline{B}_{\gamma} = \begin{bmatrix} \frac{1}{6}(1+s) & 0 & \frac{1}{4}(1+r)(1+s) & \cdots & \frac{1}{4}(1+r)(1-s) \\ \frac{1}{4}(1+r) & -\frac{1}{4}(1+r)(1+s) & 0 & & 0 \end{bmatrix}$$

$$\underline{K} = \int_{-1}^{1}\int_{-1}^{1} \underline{B}_{\kappa}^{T} \underline{C}_{\kappa} \underline{B}_{\kappa} \left(\frac{3}{2}\right) dr\,ds + \int_{-1}^{1}\int_{-1}^{1} \underline{B}_{\gamma}^{T} \underline{C}_{\gamma} \underline{B}_{\gamma} \left(\frac{3}{2}\right) dr\,ds$$

$$\underline{C}_{\kappa} = \frac{E h^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}, \quad \underline{C}_{\gamma} = \begin{bmatrix} Ghk & 0 \\ 0 & Ghk \end{bmatrix}$$

Let $\hat{\underline{u}}^{T} = [\, w_1 \;\; \theta_x^1 \;\; \theta_y^1 \;\; w_2 \;\; \theta_x^2 \;\; \theta_y^2 \;\; w_3 \;\; \theta_x^3 \;\; \theta_y^3 \;\; w_4 \;\; \theta_x^4 \;\; \theta_y^4 \,]$

Then

$$\underline{B}_{\kappa}\Big|_{r=0,s=0} \hat{\underline{u}} = \begin{bmatrix} \frac{1}{6}(-\theta_y^1 + \theta_y^2 + \theta_y^3 - \theta_y^4) \\ \frac{1}{4}(\theta_x^1 + \theta_x^2 - \theta_x^3 - \theta_x^4) \\ \frac{1}{6}(\theta_x^1 - \theta_x^2 - \theta_x^3 + \theta_x^4) + \frac{1}{4}(-\theta_y^1 - \theta_y^2 + \theta_y^3 + \theta_y^4) \end{bmatrix} \quad (a)$$

$$\underline{B}_{\gamma}\Big|_{r=0,s=0} \hat{\underline{u}} = \begin{bmatrix} \frac{1}{6}(w_1 - w_2 - w_3 + w_4) + \frac{1}{4}(\theta_y^1 + \theta_y^2 + \theta_y^3 + \theta_y^4) \\ \frac{1}{4}(w_1 + w_2 - w_3 - w_4) + \frac{1}{4}(-\theta_x^1 - \theta_x^2 - \theta_x^3 - \theta_x^4) \end{bmatrix} \quad (b)$$

We now need to identify for what values of $\hat{\underline{u}}$ the columns in (a) and (b) are zero.

5.61

Consider the case $w_1 = w_2 = w_3 = w_4 = 0$ (although, the case $w_1 = w_2 = w_3 = 0$ with $w_4 \neq 0$ would need to be considered also in order to find all spurious modes).

$$-\theta_y^1 + \theta_y^2 + \theta_y^3 - \theta_y^4 = 0 \quad \text{—— (1)}$$

$$\theta_x^1 + \theta_x^2 - \theta_x^3 - \theta_x^4 = 0 \quad \text{—— (2)}$$

$$2(\theta_x^1 - \theta_x^2 - \theta_x^3 + \theta_x^4) + 3(-\theta_y^1 - \theta_y^2 + \theta_y^3 + \theta_y^4) = 0 \quad \text{—— (3)}$$

$$\theta_y^1 + \theta_y^2 + \theta_y^3 + \theta_y^4 = 0 \quad \text{—— (4)}$$

$$-\theta_x^1 - \theta_x^2 - \theta_x^3 - \theta_x^4 = 0 \quad \text{—— (5)}$$

The conditions are:

From (1) and (4), $\theta_y^1 + \theta_y^4 = 0$, $\theta_y^2 + \theta_y^3 = 0$

From (2) and (5), $\theta_x^1 + \theta_x^2 = 0$, $\theta_x^3 + \theta_x^4 = 0$

Let $\theta_x^1 = A_1$, $\theta_x^3 = A_3$, $\theta_y^1 = B_1$, $\theta_y^3 = B_3$

then $\theta_x^2 = -A_1$, $\theta_x^4 = -A_3$, $\theta_y^4 = -B_1$, $\theta_y^2 = -B_3$ } —— (6)

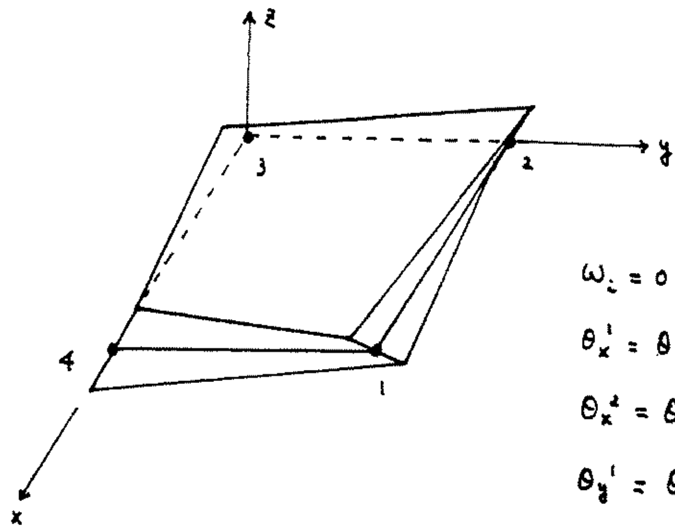
Eq. (6) and (3) give $4(A_1 - A_3) + 6(-B_1 + B_3) = 0$

which is satisfied, if

$$\left.\begin{array}{l} A_1 = A_3 \text{ and } B_1 = B_3 \\ \text{or } A_1 - A_3 = \frac{3}{2}(B_1 - B_3) \end{array}\right\} \text{—— (7)}$$

In these two cases in eq. (7), $\underline{B}_\kappa \hat{\underline{u}} = \underline{B}_\gamma \hat{\underline{u}} = \underline{0}$ and the element has these spurious zero energy modes. Note that these modes have no relation to the rigid body modes. The spurious mode with $A_1 = A_3$ and $B_1 = B_3$ is shown below.

5.61



$$\omega_i = 0 \qquad i = 1, 2, 3, 4$$

$$\theta_x^1 = \theta_x^3 = A_1$$

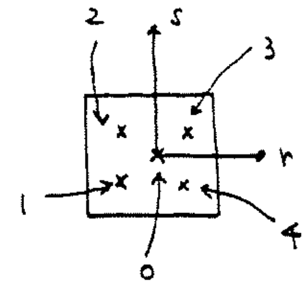
$$\theta_x^2 = \theta_x^4 = -A_1$$

$$\theta_y^1 = \theta_y^3 = B_1$$

$$\theta_y^2 = \theta_y^4 = -B_1$$

5.62 Using the strain-displacement matrices $\underline{B}_\kappa$ and $\underline{B}_\gamma$, find out what conditions are necessary for $\underline{B}_\kappa \hat{\underline{u}} = \underline{0}$, $\underline{B}_\gamma \hat{\underline{u}} = \underline{0}$ at the corresponding Gaussian points.

$$\underline{B}_\kappa\big|_{G_1} \hat{\underline{u}} = \underline{B}_\kappa\big|_{G_2} \hat{\underline{u}} = \underline{B}_\kappa\big|_{G_3} \hat{\underline{u}} = \underline{B}_\kappa\big|_{G_4} \hat{\underline{u}} = \underline{0}$$



where $G_1(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$, $G_2(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$,

$G_3(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$, $G_4(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$

$B_\gamma\big|_{G_0} \hat{\underline{u}} = \underline{0}$ where $G_0(0, 0)$

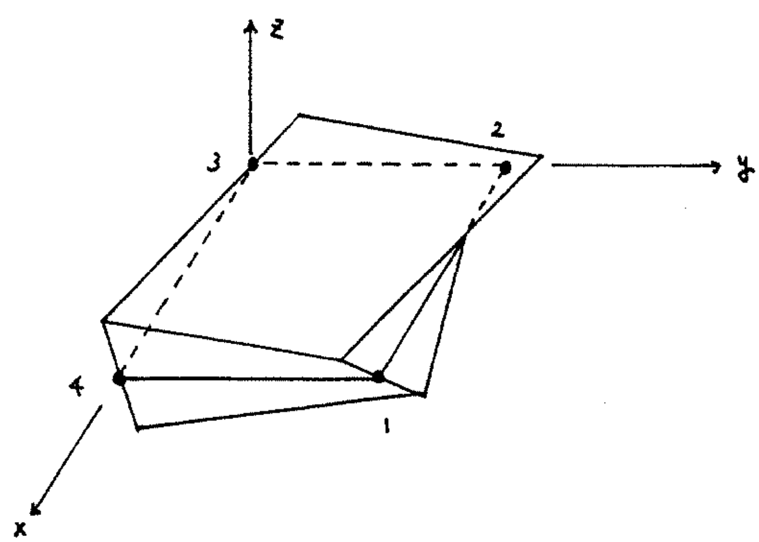
First, assume $w_i = w_0$ ($i = 1, 2, 3$ and 4)

Then,
$$\left.\begin{aligned} \theta_x^1 &= -\theta_x^2 = -\theta_x^3 = \theta_x^4 \\ \theta_y^1 &= \theta_y^2 = -\theta_y^3 = -\theta_y^4 \\ 2\theta_x^1 &= 3\theta_y^1 \end{aligned}\right\}$$

Let $\theta_x^1 = 3\theta_0$, then $\theta_y^1 = 2\theta_0$

$$\therefore \hat{\underline{u}}^T = [\, w_1\ \theta_x^1\ \theta_y^1 \,\vdots\, w_2\ \theta_x^2\ \theta_y^2 \,\vdots\, w_3\ \theta_x^3\ \theta_y^3 \,\vdots\, w_4\ \theta_x^4\ \theta_y^4 \,]$$

$$= [\, w_0\ 3\theta_0\ 2\theta_0 \,\vdots\, w_0\ -3\theta_0\ 2\theta_0 \,\vdots\, w_0\ -3\theta_0\ -2\theta_0 \,\vdots\, w_0\ 3\theta_0\ -2\theta_0 \,]$$



5.62

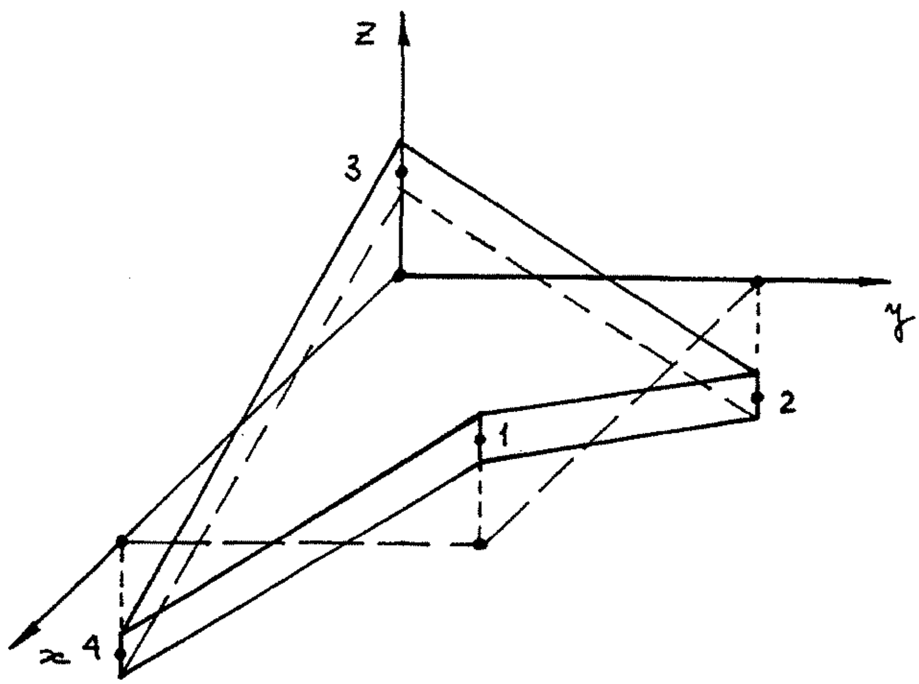
Consider now the case $\theta_x^i = \theta_y^i = 0, \quad i = 1..4.$

Then,

$$\begin{cases} \frac{1}{6} w_1 - \frac{1}{6} w_2 - \frac{1}{6} w_3 + \frac{1}{6} w_4 = 0, \\ \frac{1}{4} w_1 + \frac{1}{4} w_2 - \frac{1}{4} w_3 - \frac{1}{4} w_4 = 0. \end{cases}$$

Let $w_1 = w_0, \quad w_2 = -w_0,$ then $w_3 = w_0, \; w_4 = -w_0,$

and $\hat{\underline{u}}^T = [\, w_0 \;\; 0 \;\; 0 \;\vdots\; -w_0 \;\; 0 \;\; 0 \;\vdots\; w_0 \;\; 0 \;\; 0 \;\vdots\; -w_0 \;\; 0 \;\; 0 \,]$



Therefore, one can see that the element has two spurious zero energy modes, and hence, should not be used in practice.

6.1 (a) $h_1 = (1+{}^0x_1)(1+{}^0x_2)/4, \quad h_2 = (1-{}^0x_1)(1+{}^0x_2)/4$

$h_3 = (1-{}^0x_1)(1-{}^0x_2)/4, \quad h_4 = (1+{}^0x_1)(1-{}^0x_2)/4$

$${}^tu_i = {}^tx_i - {}^0x_i = \sum_{k=1}^{4} h_k \left({}^tx_i^k - {}^0x_i^k\right)$$

Let $${}^0\hat{\underline{x}}^T = [\, 1 \quad 1 \quad -1 \quad 1 \quad -1 \quad -1 \quad 1 \quad -1 \,]$$

$${}^t\hat{\underline{x}}^T = [\, \sqrt{3}+1 \quad 0 \quad \sqrt{3}-1 \quad 0 \quad -1 \quad -1 \quad 1 \quad -1 \,]$$

$$\therefore \; {}^t\underline{u} = \begin{bmatrix} {}^tu \\ {}^tv \end{bmatrix} = \begin{bmatrix} h_1 & 0 & h_2 & 0 & h_3 & 0 & h_4 & 0 \\ 0 & h_1 & 0 & h_2 & 0 & h_3 & 0 & h_4 \end{bmatrix} \left({}^t\hat{\underline{x}} - {}^0\hat{\underline{x}}\right)$$

$$= \begin{bmatrix} \frac{\sqrt{3}}{2}(1+{}^0x_2) \\ -\frac{1}{2}(1+{}^0x_2) \end{bmatrix}$$

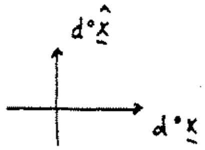
(b) $${}^tx_i = {}^0x_i + {}^tu_i \;, \quad \frac{\partial\, {}^tx_i}{\partial\, {}^0x_j} = \delta_{ij} + \frac{\partial\, {}^tu_i}{\partial\, {}^0x_j}$$

$$\frac{\partial\, {}^tx_1}{\partial\, {}^0x_1} = 1 \;, \quad \frac{\partial\, {}^tx_2}{\partial\, {}^0x_1} = 0 \;, \quad \frac{\partial\, {}^tx_1}{\partial\, {}^0x_2} = \frac{\sqrt{3}}{2} \;, \quad \frac{\partial\, {}^tx_2}{\partial\, {}^0x_2} = \frac{1}{2}$$

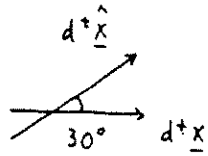
$$\therefore \; {}^t_0\underline{X} = \begin{bmatrix} 1 & \frac{\sqrt{3}}{2} \\ 0 & \frac{1}{2} \end{bmatrix} , \quad {}^t_0\underline{C} = {}^t_0\underline{X}^T\, {}^t_0\underline{X} = \begin{bmatrix} 1 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 1 \end{bmatrix}$$

$${}^t\rho = {}^0\rho / \det {}^t_0\underline{X} = (0.05)/(1/2) = 0.1$$

6.2



time 0



time t

$${}^0\underline{n}^T = [\,1 \quad 0\,]$$

$${}^0\hat{\underline{n}}^T = [\,0 \quad 1\,]$$

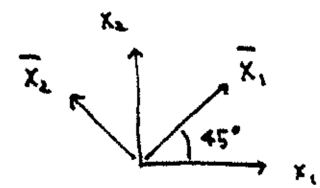
$${}^t\lambda = \left({}^0\underline{n}^T\,{}^t_0\underline{C}\,{}^0\underline{n}\right)^{1/2} = \left([\,1 \quad 0\,]\begin{bmatrix} 1 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)^{1/2} = 1$$

$${}^t\hat{\lambda} = \left({}^0\hat{\underline{n}}^T\,{}^t_0\underline{C}\,{}^0\hat{\underline{n}}\right)^{1/2} = \left([\,0 \quad 1\,]\begin{bmatrix} 1 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)^{1/2} = 1$$

$$\cos{}^t\theta = \frac{{}^0\underline{n}^T\,{}^t_0\underline{C}\,{}^0\hat{\underline{n}}}{{}^t\lambda\,{}^t\hat{\lambda}} = \frac{\sqrt{3}}{2}, \quad {}^t\theta = 30^\circ$$

$$\therefore \text{(ang. distortion)} = 90^\circ - 30^\circ = 60^\circ$$

6.3



$${}^{2\Delta t}_{0}\underline{X} = {}^{2\Delta t}_{\Delta t}\underline{X}\;{}^{\Delta t}_{0}\underline{X}$$

$${}^{\Delta t}_{0}\underline{X} = {}^{\Delta t}_{0}\underline{R} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$${}^{2\Delta t}_{\Delta t}\underline{X} = {}^{\Delta t}_{0}\underline{R}\;{}^{2\Delta t}_{\Delta t}\bar{\underline{X}}\;{}^{\Delta t}_{0}\underline{R}^T \quad \text{where } {}^{2\Delta t}_{\Delta t}\bar{\underline{X}} = \begin{bmatrix} \frac{4}{3} & 0 \\ 0 & \frac{3}{4} \end{bmatrix} = {}^{2\Delta t}_{\Delta t}\bar{\underline{U}}$$

$$\therefore {}^{2\Delta t}_{0}\underline{X} = {}^{\Delta t}_{0}\underline{R}\;{}^{2\Delta t}_{\Delta t}\bar{\underline{U}} = \begin{bmatrix} \frac{2\sqrt{2}}{3} & -\frac{3\sqrt{2}}{8} \\ \frac{2\sqrt{2}}{3} & \frac{3\sqrt{2}}{8} \end{bmatrix}$$

6.4 (a) $${}^{\Delta t}_{0}\underline{U} = \begin{bmatrix} \frac{3}{2} & & \underline{0} \\ & \frac{5}{4} & \\ \underline{0} & & \frac{8}{15} \end{bmatrix}, \quad {}^{2\Delta t}_{\Delta t}\underline{R} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(from incompressible assump.)

$$\therefore \ {}^{2\Delta t}_{0}\underline{X} = {}^{2\Delta t}_{\Delta t}\underline{R}\ {}^{\Delta t}_{0}\underline{U} = \begin{bmatrix} \frac{3\sqrt{3}}{4} & -\frac{5}{8} & 0 \\ \frac{3}{4} & \frac{5\sqrt{3}}{8} & 0 \\ 0 & 0 & \frac{8}{15} \end{bmatrix}$$

(b) Let $${}^{0}\underline{n}_1 = \frac{d\,{}^{0}\underline{s}_1}{d\,{}^{0}s_1}, \quad {}^{0}\underline{n}_2 = \frac{d\,{}^{0}\underline{s}_2}{d\,{}^{0}s_2},$$

$${}^{2\Delta t}\lambda_1 = \left({}^{0}\underline{n}_1^T\ {}^{2\Delta t}_{0}\underline{C}\ {}^{0}\underline{n}_1\right)^{1/2} \qquad {}^{2\Delta t}_{0}\underline{C} = {}^{2\Delta t}_{0}\underline{X}^T\ {}^{2\Delta t}_{0}\underline{X} = \begin{bmatrix} \frac{9}{4} & & \underline{0} \\ & \frac{25}{16} & \\ \underline{0} & & \frac{64}{225} \end{bmatrix}$$

$${}^{2\Delta t}\lambda_2 = \left({}^{0}\underline{n}_2^T\ {}^{2\Delta t}_{0}\underline{C}\ {}^{0}\underline{n}_2\right)^{1/2}$$

$$\therefore \ {}^{2\Delta t}\lambda_1 = \frac{3}{2}, \quad {}^{2\Delta t}\lambda_2 = \sqrt{\frac{61}{32}}$$

6.5 By the chain rule, $${}^{\Delta t}_{0}\underline{\bar{X}} = \frac{\partial\,{}^{\Delta t}\underline{\bar{x}}}{\partial\,{}^{\Delta t}\underline{x}}\ \frac{\partial\,{}^{\Delta t}\underline{x}}{\partial\,{}^{0}\underline{x}}\ \frac{\partial\,{}^{0}\underline{x}}{\partial\,{}^{0}\underline{\bar{x}}}$$

$$\therefore \ {}^{\Delta t}_{0}\underline{\bar{X}} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} {}^{\Delta t}_{0}\underline{X} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{23}{16} & \frac{\sqrt{3}}{16} & 0 \\ \frac{\sqrt{3}}{16} & \frac{21}{16} & 0 \\ 0 & 0 & \frac{8}{15} \end{bmatrix}$$

where ${}^{\Delta t}_{0}\underline{X} = {}^{\Delta t}_{0}\underline{U}$ is given in Exercise 6.4.

Note that ${}^{\Delta t}_{0}\underline{\bar{X}}$ is different from ${}^{2\Delta t}_{0}\underline{X}$ in exercise 6.4.

6.6 (a) Let ${}^{t}_{\circ}\underline{X} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$, then $d^t\underline{x} = {}^{t}_{\circ}\underline{X}\, d^{\circ}\underline{x}$, $d^t\hat{\underline{x}} = {}^{t}_{\circ}\underline{X}\, d^{\circ}\hat{\underline{x}}$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} d^{\circ}s = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} d^{\circ}s \;, \quad \begin{bmatrix} -1 \\ 1 \end{bmatrix} d^{\circ}\hat{s} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} d^{\circ}\hat{s}$$

$\therefore\; a_1 = 2\sqrt{2}+1 ,\; a_2 = -1 ,\; a_3 = \sqrt{2}-1 ,\; a_4 = 1$

$$\longrightarrow \; {}^{t}_{\circ}\underline{X} = \begin{bmatrix} 2\sqrt{2}+1 & -1 \\ \sqrt{2}-1 & 1 \end{bmatrix}$$

(b) By inverting ${}^{t}_{\circ}\underline{X}$:

$${}^{\circ}_{t}\underline{X} = {}^{t}_{\circ}\underline{X}^{-1} = \begin{bmatrix} \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} \\ \frac{-2+\sqrt{2}}{6} & \frac{4+\sqrt{2}}{6} \end{bmatrix}$$

Without inverting ${}^{t}_{\circ}\underline{X}$:

Let ${}^{\circ}_{t}\underline{X} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$, then $d^{\circ}\underline{x} = {}^{\circ}_{t}\underline{X}\, d^t\underline{x}$, $d^{\circ}\hat{\underline{x}} = {}^{\circ}_{t}\underline{X}\, d^t\hat{\underline{x}}$

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} d^{\circ}s = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} d^{\circ}s \;, \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} d^{\circ}\hat{s} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} d^{\circ}\hat{s}$$

$\therefore\; b_1 = b_2 = \frac{1}{3\sqrt{2}} ,\; b_3 = \frac{-2+\sqrt{2}}{6} ,\; b_4 = \frac{4+\sqrt{2}}{6}$

Hence the same ${}^{\circ}_{t}\underline{X}$ is obtained.

(c) $\det {}^{t}_{\circ}\underline{X} = 3\sqrt{2}$, $\therefore\; {}^{t}\rho / {}^{\circ}\rho = 1/3\sqrt{2} = \sqrt{2}/6$

6.7 It is always possible to decompose a deformation gradient $\underline{X}$ into the form:

$$\underline{X} = \underline{R}\,\underline{U} \qquad \text{——— ①}$$

See example 6.8 for a detailed proof.

Now define a symmetric matrix $\underline{V}$ given by

$$\underline{V} = \underline{R}\,\underline{U}\,\underline{R}^T \qquad \text{——— ②}$$

Note that the matrix $\underline{V}$ can always be constructed.

Then $\quad \underline{V}\,\underline{R} = (\underline{R}\,\underline{U}\,\underline{R}^T)\,\underline{R} = \underline{R}\,\underline{U} = \underline{X}$

Hence we see the deformation gradient $\underline{X}$ can always be decomposed into the form of ②.

For the deformation in exercise 6.4,

$$\underline{V} = \underline{R}\,\underline{U}\,\underline{R}^T = \underline{X}\,\underline{R}^T = \begin{bmatrix} \frac{23}{16} & \frac{\sqrt{3}}{16} & 0 \\ \frac{\sqrt{3}}{16} & \frac{21}{16} & 0 \\ 0 & 0 & \frac{8}{15} \end{bmatrix}, \quad \underline{R} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

6.8 (a) $${}^{\Delta t}_{\;0}\underset{\sim}{U} = \begin{bmatrix} \dfrac{\partial\, {}^t x_1}{\partial\, {}^0 x_1} & \dfrac{\partial\, {}^t x_1}{\partial\, {}^0 x_2} \\[2ex] \dfrac{\partial\, {}^t x_2}{\partial\, {}^0 x_1} & \dfrac{\partial\, {}^t x_2}{\partial\, {}^0 x_2} \end{bmatrix} = \begin{bmatrix} 2 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

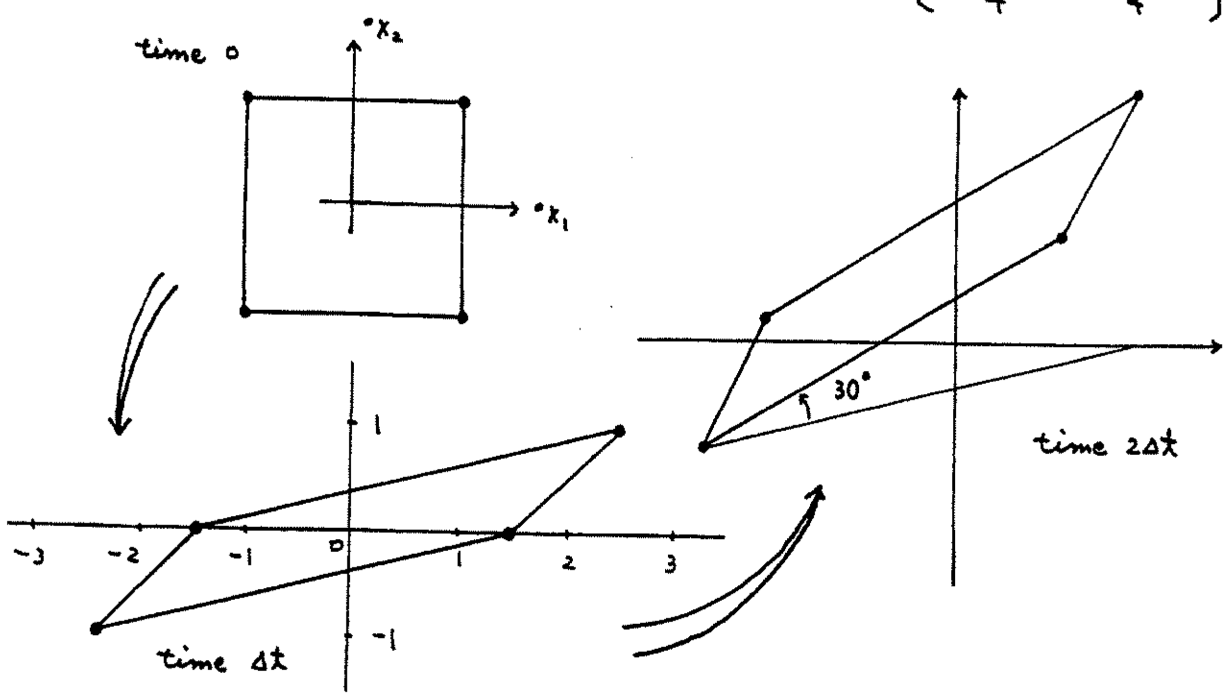
$$\frac{\partial\, {}^t x_1}{\partial\, {}^0 x_1} = 2\,, \quad {}^t x_1 = 2\,{}^0 x_1 + f({}^0 x_2)\,, \quad \frac{\partial\, {}^t x_1}{\partial\, {}^0 x_2} = \frac{\partial f({}^0 x_2)}{\partial\, {}^0 x_2} = 0.5\,, \quad f({}^0 x_2) = 0.5\,{}^0 x_2 + C_1$$

$$\therefore\ {}^t x_1 = 2\,{}^0 x_1 + 0.5\,{}^0 x_2 + C_1$$

$$\frac{\partial\, {}^t x_2}{\partial\, {}^0 x_2} = 0.5\,, \quad {}^t x_2 = 0.5\,{}^0 x_2 + g({}^0 x_1)\,, \quad \frac{\partial\, {}^t x_2}{\partial\, {}^0 x_1} = \frac{\partial g({}^0 x_1)}{\partial\, {}^0 x_1} = 0.5\,, \quad g({}^0 x_1) = 0.5\,{}^0 x_1 + C_2$$

$$\therefore\ {}^t x_2 = 0.5\,{}^0 x_1 + 0.5\,{}^0 x_2 + C_2$$

$${}^{2\Delta t}_{\;0}\underset{\sim}{X} = {}^{2\Delta t}_{\;\Delta t}\underset{\sim}{R}\;{}^{\Delta t}_{\;0}\underset{\sim}{U} = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix} \begin{bmatrix} 2 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} \dfrac{4\sqrt{3}-1}{4} & \dfrac{\sqrt{3}-1}{4} \\[2ex] \dfrac{\sqrt{3}+4}{4} & \dfrac{\sqrt{3}+1}{4} \end{bmatrix}$$



In this sketch the rigid-body motion components C_1 and C_2 are set to zero.

6.8

(b) ${}^{\Delta t}_{0}\underline{U} = \underline{R}_L \underline{\Lambda} \underline{R}_L^T$, $\quad \underline{\Lambda}$: principal stretches

$\underline{R}_L$: stores the directions of these stretches

Consider the eigenproblem $\quad {}^{\Delta t}_{0}\underline{U}\,\underline{p} = \lambda \underline{p}$.

Then
$$\lambda_1 = \frac{5-\sqrt{13}}{4}, \quad \underline{p}_1^T = \left[\frac{3-\sqrt{13}}{\sqrt{2}\sqrt{13-3\sqrt{13}}} \quad \frac{\sqrt{2}}{\sqrt{13-3\sqrt{13}}} \right]$$

$$\lambda_2 = \frac{5+\sqrt{13}}{4}, \quad \underline{p}_2^T = \left[\frac{3+\sqrt{13}}{\sqrt{2}\sqrt{13+3\sqrt{13}}} \quad \frac{\sqrt{2}}{\sqrt{13+3\sqrt{13}}} \right]$$

And $\underline{P} = [\,\underline{p}_1 \;\; \underline{p}_2\,] = \underline{R}_L$

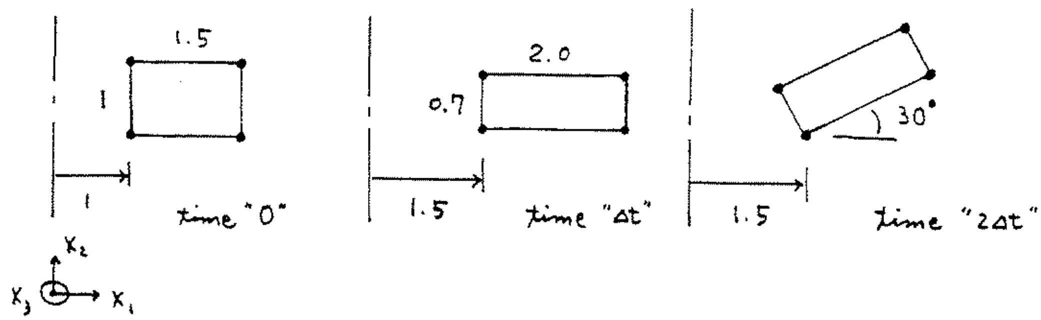
$$\therefore \quad {}^{\Delta t}_{0}\underline{U} = \underbrace{\begin{bmatrix} \frac{3-\sqrt{13}}{\sqrt{2}\sqrt{13-3\sqrt{13}}} & \frac{3+\sqrt{13}}{\sqrt{2}\sqrt{13+3\sqrt{13}}} \\ \frac{\sqrt{2}}{\sqrt{13-3\sqrt{13}}} & \frac{\sqrt{2}}{\sqrt{13+3\sqrt{13}}} \end{bmatrix}}_{\underline{R}_L} \underbrace{\begin{bmatrix} \frac{5-\sqrt{13}}{4} & \\ & \frac{5+\sqrt{13}}{4} \end{bmatrix}}_{\underline{\Lambda}} \begin{bmatrix} \underline{R}_L^T \end{bmatrix}$$

(c) $$\underline{V} = \underline{R}\,\underline{U}\,\underline{R}^T = \begin{bmatrix} \frac{13-2\sqrt{3}}{8} & \frac{2+3\sqrt{3}}{8} \\ \frac{2+3\sqrt{3}}{8} & \frac{7+2\sqrt{3}}{8} \end{bmatrix}, \quad \underline{R} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

$\longrightarrow \quad \underline{X} = \underline{V}\,\underline{R}$

In this representation, conceptually, the element can be thought of as first rotated by 30° and then stretched by the amount of $\underline{V}$.

6.9



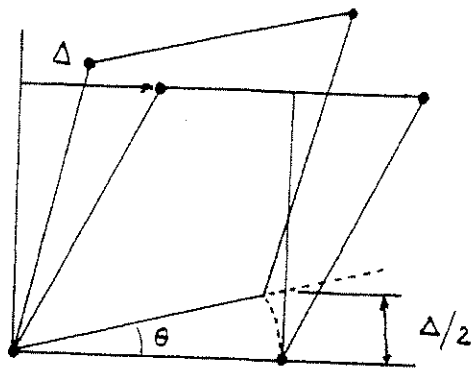
$${}^{2\Delta t}_{0}\underline{X} = {}^{2\Delta t}_{\Delta t}\underline{R}\;{}^{\Delta t}_{0}\underline{U}\;, \qquad \frac{\partial^{t} x_3}{\partial^{0} x_3} = 1 + \frac{{}^{t}u}{R} = 1 + \frac{\frac{1}{6} + \frac{1}{3}\,{}^{0}x_1}{{}^{0}x_1} = \frac{1 + 8\,{}^{0}x_1}{6\,{}^{0}x_1}$$

$$\underline{U} = {}^{\Delta t}_{0}\underline{U} = \begin{bmatrix} \frac{4}{3} & 0 & 0 \\ 0 & \frac{7}{10} & 0 \\ 0 & 0 & \frac{1+8\,{}^{0}x_1}{6\,{}^{0}x_1} \end{bmatrix}, \qquad {}^{2\Delta t}_{\Delta t}\underline{R} = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ & 0 \\ \sin 30^\circ & \cos 30^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{V} = {}^{2\Delta t}_{\Delta t}\underline{R}\;{}^{\Delta t}_{0}\underline{U}\;{}^{2\Delta t}_{\Delta t}\underline{R}^{T} = \begin{bmatrix} \frac{47}{40} & \frac{19}{40\sqrt{3}} & 0 \\ \frac{19}{40\sqrt{3}} & \frac{103}{120} & 0 \\ 0 & 0 & \frac{1+8\,{}^{0}x_1}{6\,{}^{0}x_1} \end{bmatrix}$$

$${}^{2\Delta t}_{0}\underline{X} = {}^{2\Delta t}_{t}\underline{R}\;{}^{t}_{0}\underline{U} = \begin{bmatrix} \frac{2}{3}\sqrt{3} & -\frac{7}{20} & 0 \\ \frac{2}{3} & \frac{7}{20}\sqrt{3} & 0 \\ 0 & 0 & \frac{1+8\,{}^{0}x_1}{6\,{}^{0}x_1} \end{bmatrix}$$

6.10



$\Delta = {}^{t}U = 0.4\ ,\quad \dot{\Delta} = {}^{t}\dot{U} = 0.1$

(a) $\underline{X} = \begin{bmatrix} 1 & \Delta \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.4 \\ 0 & 1 \end{bmatrix}$

By inspection

$$\underline{R} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

where $\cos\theta = 1 \Big/ \sqrt{1+\left(\frac{\Delta}{2}\right)^2} = 5/\sqrt{26}$, $\sin\theta = \frac{\Delta}{2} \Big/ \sqrt{1+\left(\frac{\Delta}{2}\right)^2} = 1/\sqrt{26}$

$$\therefore\ \underline{R} = \frac{1}{\sqrt{1+\left(\frac{\Delta}{2}\right)^2}} \begin{bmatrix} 1 & \frac{\Delta}{2} \\ -\frac{\Delta}{2} & 1 \end{bmatrix} = \frac{1}{\sqrt{26}} \begin{bmatrix} 5 & 1 \\ -1 & 5 \end{bmatrix}$$

$$\underline{U} = \underline{R}^{-1}\underline{X} = \frac{1}{\sqrt{1+\left(\frac{\Delta}{2}\right)^2}} \begin{bmatrix} 1 & \frac{\Delta}{2} \\ \frac{\Delta}{2} & 1+\frac{\Delta^2}{2} \end{bmatrix} = \frac{1}{\sqrt{26}} \begin{bmatrix} 5 & 1 \\ 1 & 27/5 \end{bmatrix}$$

$$\underline{V} = \underline{R}\,\underline{U}\,\underline{R}^T = \underline{X}\,\underline{R}^T = \frac{1}{\sqrt{26}} \begin{bmatrix} 27/5 & 1 \\ 1 & 5 \end{bmatrix}$$

where $\underline{X} = \underline{R}\,\underline{U}$ and $\underline{X} = \underline{V}\,\underline{R}$

(b) Solving $\underline{U}\,\underline{p} = \lambda\,\underline{p}$ we obtain

$$\underline{\Lambda} = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} = \begin{bmatrix} \sqrt{1+\left(\frac{\Delta}{2}\right)^2} + \frac{\Delta}{2} & \\ & \sqrt{1+\left(\frac{\Delta}{2}\right)^2} - \frac{\Delta}{2} \end{bmatrix} = \begin{bmatrix} 1.2198 & 0 \\ 0 & 0.8198 \end{bmatrix}$$

6.10

$$\underline{P} = [\underline{P}_1 \ \underline{P}_2] = \begin{bmatrix} \dfrac{\sqrt{2}}{\left[4+\Delta^2+\Delta\sqrt{4+\Delta^2}\right]^{1/2}} & \dfrac{\sqrt{2}}{\left[4+\Delta^2-\Delta\sqrt{4+\Delta^2}\right]^{1/2}} \\ \dfrac{\Delta+\sqrt{4+\Delta^2}}{\sqrt{2}\left[4+\Delta^2+\Delta\sqrt{4+\Delta^2}\right]^{1/2}} & \dfrac{\Delta-\sqrt{4+\Delta^2}}{\sqrt{2}\left[4+\Delta^2-\Delta\sqrt{4+\Delta^2}\right]^{1/2}} \end{bmatrix}$$

$$= \begin{bmatrix} 0.6340 & 0.7733 \\ 0.7733 & -0.6340 \end{bmatrix} = \underline{R}_L$$

$$\underline{R}_E = \underline{R}\,\underline{R}_L = \begin{bmatrix} \dfrac{\left[4+\Delta^2+\Delta\sqrt{4+\Delta^2}\right]^{1/2}}{\sqrt{2}\sqrt{4+\Delta^2}} & \dfrac{\left[4+\Delta^2-\Delta\sqrt{4+\Delta^2}\right]^{1/2}}{\sqrt{2}\sqrt{4+\Delta^2}} \\ \dfrac{\sqrt{2}}{\left[4+\Delta^2+\Delta\sqrt{4+\Delta^2}\right]^{1/2}} & -\dfrac{\sqrt{2}}{\left[4+\Delta^2-\Delta\sqrt{4+\Delta^2}\right]^{1/2}} \end{bmatrix}$$

$$= \begin{bmatrix} 0.7733 & 0.6340 \\ 0.6340 & -0.7733 \end{bmatrix}$$

$$\therefore \quad \underline{U} = \underline{R}_L \underline{\Lambda} \underline{R}_L^T \ , \quad \underline{V} = \underline{R}_E \underline{\Lambda} \underline{R}_E^T$$

(c) $$\underline{L} = \underline{D} + \underline{W} = \dot{\underline{X}}\,\underline{X}^{-1} = \begin{bmatrix} 0 & \dot{\Delta} \\ 0 & 0 \end{bmatrix}$$

$$\underline{D} = \frac{1}{2}\left(\underline{L} + \underline{L}^T\right) = \begin{bmatrix} 0 & \dot{\Delta}/2 \\ \dot{\Delta}/2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0.05 \\ 0.05 & 0 \end{bmatrix}$$

$$\underline{W} = \frac{1}{2}\left(\underline{L} - \underline{L}^T\right) = \begin{bmatrix} 0 & \dot{\Delta}/2 \\ -\dot{\Delta}/2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0.05 \\ -0.05 & 0 \end{bmatrix}$$

6.11 Relations in (6.46) and (6.47)

$$\mathbf{D} = \frac{1}{2}\mathbf{R}(\dot{\mathbf{U}}\mathbf{U}^{-1} + \mathbf{U}^{-1}\dot{\mathbf{U}})\mathbf{R}^T, \quad \mathbf{W} = \mathbf{\Omega}_R + \frac{1}{2}\mathbf{R}(\dot{\mathbf{U}}\mathbf{U}^{-1} - \mathbf{U}^{-1}\dot{\mathbf{U}})\mathbf{R}^T$$

$$\dot{\mathbf{U}}\mathbf{U}^{-1} = \mathbf{R}_L(\mathbf{\Omega}_L + \dot{\mathbf{\Lambda}}\mathbf{\Lambda}^{-1} - \mathbf{\Lambda}\mathbf{\Omega}_L\mathbf{\Lambda}^{-1})\mathbf{R}_L^T$$

$$\mathbf{U}^{-1}\dot{\mathbf{U}} = \mathbf{R}_L(\mathbf{\Lambda}^{-1}\mathbf{\Omega}_L\mathbf{\Lambda} + \dot{\mathbf{\Lambda}}\mathbf{\Lambda}^{-1} - \mathbf{\Omega}_L)\mathbf{R}_L^T$$

$$\therefore \quad \dot{\mathbf{U}}\mathbf{U}^{-1} + \mathbf{U}^{-1}\dot{\mathbf{U}} = \mathbf{R}_L\left\{2\dot{\mathbf{\Lambda}}\mathbf{\Lambda}^{-1} + \mathbf{\Lambda}^{-1}\mathbf{\Omega}_L\mathbf{\Lambda} - \mathbf{\Lambda}\mathbf{\Omega}_L\mathbf{\Lambda}^{-1}\right\}\mathbf{R}_L^T$$

$$\dot{\mathbf{U}}\mathbf{U}^{-1} - \mathbf{U}^{-1}\dot{\mathbf{U}} = \mathbf{R}_L\left\{2\mathbf{\Omega}_L - (\mathbf{\Lambda}^{-1}\mathbf{\Omega}_L\mathbf{\Lambda} + \mathbf{\Lambda}\mathbf{\Omega}_L\mathbf{\Lambda}^{-1})\right\}\mathbf{R}_L^T$$

$$\therefore \mathbf{D} = \frac{1}{2}\mathbf{R}_E\left\{2\dot{\mathbf{\Lambda}}\mathbf{\Lambda}^{-1} + (\mathbf{\Lambda}^{-1}\mathbf{\Omega}_L\mathbf{\Lambda} - \mathbf{\Lambda}\mathbf{\Omega}_L\mathbf{\Lambda}^{-1})\right\}\mathbf{R}_E^T$$

$$= \mathbf{R}_E\left\{\dot{\mathbf{\Lambda}}\mathbf{\Lambda}^{-1} + \frac{1}{2}(\mathbf{\Lambda}^{-1}\mathbf{\Omega}_L\mathbf{\Lambda} - \mathbf{\Lambda}\mathbf{\Omega}_L\mathbf{\Lambda}^{-1})\right\}\mathbf{R}_E^T$$

$$\longrightarrow \mathbf{D}_E = \dot{\mathbf{\Lambda}}\mathbf{\Lambda}^{-1} + \frac{1}{2}(\mathbf{\Lambda}^{-1}\mathbf{\Omega}_L\mathbf{\Lambda} - \mathbf{\Lambda}\mathbf{\Omega}_L\mathbf{\Lambda}^{-1}) \quad \text{———} \quad (6.46)$$

$$\mathbf{W} = \mathbf{R}_E(\mathbf{\Omega}_E - \cancel{\mathbf{\Omega}_L})\mathbf{R}_E^T + \frac{1}{2}\mathbf{R}_E\left\{2\cancel{\mathbf{\Omega}_L} - (\mathbf{\Lambda}^{-1}\mathbf{\Omega}_L\mathbf{\Lambda} + \mathbf{\Lambda}\mathbf{\Omega}_L\mathbf{\Lambda}^{-1})\right\}\mathbf{R}_E^T$$

$$= \mathbf{R}_E\left\{\mathbf{\Omega}_E - \frac{1}{2}(\mathbf{\Lambda}^{-1}\mathbf{\Omega}_L\mathbf{\Lambda} + \mathbf{\Lambda}\mathbf{\Omega}_L\mathbf{\Lambda}^{-1})\right\}\mathbf{R}_E^T$$

$$\longrightarrow \mathbf{W}_E = \mathbf{\Omega}_E - \frac{1}{2}(\mathbf{\Lambda}^{-1}\mathbf{\Omega}_L\mathbf{\Lambda} + \mathbf{\Lambda}\mathbf{\Omega}_L\mathbf{\Lambda}^{-1}) \quad \text{———} \quad (6.47)$$

Relations in (6.48) to (6.50)

In eq. (6.46) the 2nd term on the r.h.s. is a matrix with zero diagonal elements.

For $\alpha = \beta$: $\quad [\dot{\mathbf{\Lambda}}]_{\alpha\alpha} = \lambda_\alpha [\mathbf{D}_E]_{\alpha\alpha}$ and $\lambda_\alpha \equiv \Lambda_{\alpha\alpha}$,

for $\alpha \neq \beta$: $\quad [\mathbf{D}_E]_{\alpha\beta} = \frac{1}{2}(\Lambda^{-1}_{\alpha\alpha}\Omega_{L\alpha\beta}\Lambda_{\beta\beta} - \Lambda_{\alpha\alpha}\Omega_{L\alpha\beta}\Lambda^{-1}_{\beta\beta})$ — (6.48)

6.11

$$2\,[\underline{D}_E]_{\alpha\beta} = \lambda_\alpha^{-1}[\underline{\Omega}_L]_{\alpha\beta}\lambda_\beta - \lambda_\alpha[\underline{\Omega}_L]_{\alpha\beta}\lambda_\beta^{-1}$$

$$2\lambda_\beta\lambda_\alpha[\underline{D}_E]_{\alpha\beta} = (\lambda_\beta^2 - \lambda_\alpha^2)[\underline{\Omega}_L]_{\alpha\beta}$$

$$\therefore \quad [\underline{\Omega}_L]_{\alpha\beta} = \frac{2\lambda_\beta\lambda_\alpha}{\lambda_\beta^2 - \lambda_\alpha^2}[\underline{D}_E]_{\alpha\beta} \quad (\lambda_\alpha \neq \lambda_\beta) \qquad (6.49)$$

Similarly, $[\underline{\Omega}_E]_{\alpha\beta} = [\underline{W}_E]_{\alpha\beta} + \frac{1}{2}[\underline{\Omega}_L]_{\alpha\beta}\left(\frac{\lambda_\beta}{\lambda_\alpha} + \frac{\lambda_\alpha}{\lambda_\beta}\right)$

$$= [\underline{W}_E]_{\alpha\beta} + \frac{\lambda_\beta^2 + \lambda_\alpha^2}{\lambda_\beta^2 - \lambda_\alpha^2}[\underline{D}_E]_{\alpha\beta} \qquad (6.50)$$

6.12 From (6.51) and (6.52), the Green-Lagrange and Hencky strain tensors are of the general form

$$\underline{E}_g = \underline{R}_L\, g(\underline{\Lambda})\, \underline{R}_L^T \quad \text{where} \quad g(\underline{\Lambda}) = \begin{cases} \frac{1}{2}(\underline{\Lambda}^2 - \underline{I}) & \text{G.-L.} \\ \ln \underline{\Lambda} & \text{Hencky} \end{cases} \qquad \text{— (6.56)}$$

Now the rate of change of $\underline{E}_g$ is

$$\begin{aligned}\dot{\underline{E}}_g &= \dot{\underline{R}}_L\, g(\underline{\Lambda})\underline{R}_L^T + \underline{R}_L\left(\dot{\underline{\Lambda}}\, g'(\underline{\Lambda})\right)\underline{R}_L^T + \underline{R}_L\, g(\underline{\Lambda})\dot{\underline{R}}_L^T \\ &= (\underline{R}_L \underline{\Omega}_L)\, g(\underline{\Lambda})\underline{R}_L^T + \underline{R}_L\left(\dot{\underline{\Lambda}}\, g'(\underline{\Lambda})\right)\underline{R}_L^T + \underline{R}_L\, g(\underline{\Lambda})\,\underline{\Omega}_L^T \underline{R}_L^T \\ &= \underline{R}_L \underbrace{\left[\underline{\Omega}_L\, g(\underline{\Lambda}) + \dot{\underline{\Lambda}}\, g'(\underline{\Lambda}) - g(\underline{\Lambda})\,\underline{\Omega}_L\right]}_{\parallel} \underline{R}_L^T\end{aligned}$$

$$\therefore\ \dot{\underline{E}}_g = \underline{R}_L \dot{\underline{E}}_L \underline{R}_L^T \quad \text{where } \dot{\underline{E}}_L \qquad \text{—— (6.57), (6.58)}$$

For the Green-Lagrange strain tensor,

$$g(\underline{\Lambda}) = \frac{1}{2}(\underline{\Lambda}^2 - \underline{I}), \quad g'(\underline{\Lambda}) = \underline{\Lambda}$$

$$\dot{\underline{E}}_L = \dot{\underline{\Lambda}}\underline{\Lambda} + \frac{1}{2}\underline{\Omega}_L(\underline{\Lambda}^2 - \underline{I}) - \frac{1}{2}(\underline{\Lambda}^2 - \underline{I})\underline{\Omega}_L = \dot{\underline{\Lambda}}\underline{\Lambda} + \frac{1}{2}(\underline{\Omega}_L \underline{\Lambda}^2 - \underline{\Lambda}^2 \underline{\Omega}_L)$$

$$\underline{\Lambda}\,\underline{D}_E\,\underline{\Lambda} = \underline{\Lambda}\dot{\underline{\Lambda}} + \frac{1}{2}(\underline{\Omega}_L \underline{\Lambda}^2 - \underline{\Lambda}^2\underline{\Omega}_L) = \dot{\underline{E}}_L$$

$$\therefore\ [\dot{\underline{E}}_L]_{\alpha\beta} = \lambda_\alpha \lambda_\beta [\underline{D}_E]_{\alpha\beta} = \gamma_{\alpha\beta}[\underline{D}_E]_{\alpha\beta} \quad \text{where } \gamma_{\alpha\beta} = \lambda_\alpha \lambda_\beta$$

For the Hencky strain tensor,

$$g(\underline{\Lambda}) = \ln \underline{\Lambda}, \quad g'(\underline{\Lambda}) = \underline{\Lambda}^{-1}$$

$$\therefore\ \dot{\underline{E}}_L = \dot{\underline{\Lambda}}\underline{\Lambda}^{-1} + \underline{\Omega}_L(\ln \underline{\Lambda}) - (\ln \underline{\Lambda})\underline{\Omega}_L$$

$$\underline{D}_E = \dot{\underline{\Lambda}}\underline{\Lambda}^{-1} + \frac{1}{2}(\underline{\Lambda}^{-1}\underline{\Omega}_L\underline{\Lambda} - \underline{\Lambda}\,\underline{\Omega}_L\underline{\Lambda}^{-1})$$

if $\lambda_\alpha = \lambda_\beta$, $\quad \dot{\underline{E}}_L = \underline{D}_E$

if $\lambda_\alpha \neq \lambda_\beta$, $\quad \dot{E}_{L\alpha\beta} = (\ln \lambda_\beta - \ln \lambda_\alpha)\,\Omega_{L\alpha\beta} = \ln \frac{\lambda_\beta}{\lambda_\alpha}\,\Omega_{L\alpha\beta}$

6.12

$$D_{E\alpha\beta} = \frac{1}{2}\left(\frac{\lambda_\beta}{\lambda_\alpha} - \frac{\lambda_\alpha}{\lambda_\beta}\right)\Omega_{L\alpha\beta} = \frac{\lambda_\beta^2 - \lambda_\alpha^2}{2\lambda_\alpha\lambda_\beta}\,\Omega_{L\alpha\beta}$$

$$\therefore\ \dot{E}_{L\alpha\beta} = \left(\frac{2\lambda_\alpha\lambda_\beta}{\lambda_\beta^2 - \lambda_\alpha^2}\ln\frac{\lambda_\beta}{\lambda_\alpha}\right) D_{E\alpha\beta}$$

Hence $[\dot{\underline{E}}_L]_{\alpha\beta} = \gamma_{\alpha\beta}[\underline{D}_E]_{\alpha\beta}$ —————— (6.59)

where for the Green-Lagrange strain tensor

$$\gamma_{\alpha\beta} = \lambda_\alpha\lambda_\beta \qquad (6.60)$$

and for the Hencky strain tensor

$$\gamma_{\alpha\beta} = \begin{cases} 1 & \text{if } \lambda_\alpha = \lambda_\beta \\ \dfrac{2\lambda_\alpha\lambda_\beta}{\lambda_\beta^2 - \lambda_\alpha^2}\ln\dfrac{\lambda_\beta}{\lambda_\alpha} \end{cases} \qquad (6.61)$$

6.13 From the results in exercise 6.10,

$$\underline{D} = \begin{bmatrix} 0 & 0.05 \\ 0.05 & 0 \end{bmatrix}, \quad \underline{W} = \begin{bmatrix} 0 & 0.05 \\ -0.05 & 0 \end{bmatrix}, \quad \underline{R}_E = \begin{bmatrix} 0.7733 & 0.6340 \\ 0.6340 & -0.7733 \end{bmatrix}$$

$$\rightarrow \underline{D}_E = \underline{R}_E^T \underline{D} \underline{R}_E = \begin{bmatrix} 0.04903 & -0.0098 \\ -0.0098 & -0.04903 \end{bmatrix}, \quad \underline{W}_E = \underline{R}_E^T \underline{W} \underline{R}_E = \begin{bmatrix} 0 & -0.05 \\ 0.05 & 0 \end{bmatrix}$$

$$\underline{\Lambda} = \begin{bmatrix} 1.2198 & \\ 0 & 0.8198 \end{bmatrix}, \quad \underline{\Lambda}^{-1} = \begin{bmatrix} 0.8198 & 0 \\ 0 & 1.2198 \end{bmatrix}$$

Using (6.48), we have $[\dot{\underline{\Lambda}}]_{\alpha\alpha} = \lambda_\alpha [\underline{D}_E]_{\alpha\alpha}$

$$\Rightarrow \quad \dot{\underline{\Lambda}} = \begin{bmatrix} 0.05981 & 0 \\ 0 & -0.04019 \end{bmatrix}$$

Using (6.49), $$[\underline{\Omega}_L]_{\alpha\beta} = \frac{2\lambda_\beta \lambda_\alpha}{\lambda_\beta^2 - \lambda_\alpha^2} [\underline{D}_E]_{\alpha\beta}$$

$$\Rightarrow \quad \underline{\Omega}_L = \begin{bmatrix} 0 & 0.02404 \\ -0.02404 & 0 \end{bmatrix}$$

Using (6.50), $$[\underline{\Omega}_E]_{\alpha\beta} = [\underline{W}_E]_{\alpha\beta} + \frac{\lambda_\beta^2 + \lambda_\alpha^2}{\lambda_\beta^2 - \lambda_\alpha^2} [\underline{D}_E]_{\alpha\beta}$$

$$\Rightarrow \quad \underline{\Omega}_E = \begin{bmatrix} 0 & -0.02404 \\ 0.02404 & 0 \end{bmatrix}$$

Using (6.46) and (6.47) we check:

$$\underline{D}_E = \dot{\underline{\Lambda}} \underline{\Lambda}^{-1} + \frac{1}{2}\left(\underline{\Lambda}^{-1} \underline{\Omega}_L \underline{\Lambda} - \underline{\Lambda} \underline{\Omega}_L \underline{\Lambda}^{-1}\right) = \begin{bmatrix} 0.04903 & -0.0098 \\ -0.0098 & -0.04903 \end{bmatrix}$$

$$\underline{W}_E = \underline{\Omega}_E - \frac{1}{2}\left(\underline{\Lambda}^{-1} \underline{\Omega}_L \underline{\Lambda} + \underline{\Lambda} \underline{\Omega}_L \underline{\Lambda}^{-1}\right) = \begin{bmatrix} 0 & -0.05 \\ 0.05 & 0 \end{bmatrix}$$

6.14 (i) Exercise 6.1

$$ {}_0^t\mathbf{X} = \begin{bmatrix} 1 & \frac{\sqrt{3}}{2} \\ 0 & \frac{1}{2} \end{bmatrix}, \quad {}_0^t\mathbf{C} = \begin{bmatrix} 1 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 1 \end{bmatrix}, \quad {}_0^t\mathbf{U} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} $$

$$ {}^t\mathbf{\Lambda}^2 = \begin{bmatrix} \frac{2-\sqrt{3}}{2} & \\ & \frac{2+\sqrt{3}}{2} \end{bmatrix}, \quad {}_0^t\mathbf{R}_L = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} $$

$$ \therefore {}_0^t\boldsymbol{\varepsilon} = {}_0^t\mathbf{R}_L \left[\frac{1}{2}({}^t\mathbf{\Lambda}^2 - \mathbf{I}) \right] {}_0^t\mathbf{R}_L^T = \begin{bmatrix} 0 & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & 0 \end{bmatrix} = \frac{1}{2}({}_0^t\mathbf{U}^T\,{}_0^t\mathbf{U} - \mathbf{I}) = \frac{1}{2}({}_0^t\mathbf{C} - \mathbf{I}) $$

(← (6.51), (6.53) and (6.54))

Using (6.55),

$$ [{}_0^t u_{i,j}] = {}_0^t\mathbf{X} - \mathbf{I} = \begin{bmatrix} 0 & \frac{\sqrt{3}}{2} \\ 0 & -\frac{1}{2} \end{bmatrix} $$

$$ {}_0^t\varepsilon_{11} = \frac{1}{2}\left({}_0^t u_{1,1} + {}_0^t u_{1,1} + {}_0^t u_{1,1}^2 + {}_0^t u_{2,1}^2\right) = 0 $$

Similarly, ${}_0^t\varepsilon_{22} = 0$, ${}_0^t\varepsilon_{12} = \frac{\sqrt{3}}{4} = {}_0^t\varepsilon_{21}$

$$ \therefore {}_0^t\boldsymbol{\varepsilon} = \begin{bmatrix} 0 & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & 0 \end{bmatrix} $$

(ii) Exercise 6.3 ($t = 2\Delta t$)

$$ {}_0^t\mathbf{X} = \begin{bmatrix} \frac{2\sqrt{2}}{3} & -\frac{3\sqrt{2}}{8} \\ \frac{2\sqrt{2}}{3} & \frac{3\sqrt{2}}{8} \end{bmatrix}, \quad {}_0^t\mathbf{C} = \begin{bmatrix} \frac{16}{9} & 0 \\ 0 & \frac{9}{16} \end{bmatrix} $$

6.14

$$ {}^{t}\underset{\sim}{\Lambda}^2 = \begin{bmatrix} \frac{9}{16} & \\ & \frac{16}{9} \end{bmatrix}, \quad {}^{t}_{0}\underset{\sim}{R}_L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad {}^{t}_{0}\underset{\sim}{U} = \begin{bmatrix} \frac{4}{3} & 0 \\ 0 & \frac{3}{4} \end{bmatrix} $$

$$ \therefore \; {}^{t}_{0}\underset{\sim}{\varepsilon} = {}^{t}_{0}\underset{\sim}{R}_L \left[\frac{1}{2}\left({}^{t}\underset{\sim}{\Lambda}^2 - \underset{\sim}{I}\right)\right] {}^{t}_{0}\underset{\sim}{R}_L^T = \begin{bmatrix} \frac{7}{18} & 0 \\ 0 & -\frac{7}{32} \end{bmatrix} = \frac{1}{2}\left({}^{t}_{0}\underset{\sim}{U}^T\,{}^{t}_{0}\underset{\sim}{U} - \underset{\sim}{I}\right) = \frac{1}{2}\left({}^{t}_{0}C - \underset{\sim}{I}\right) $$

$(\leftarrow$ (6.51), (6.53) and (6.54))

$$ [{}^{t}_{0}u_{i,j}] = {}^{t}_{0}\underset{\sim}{X} - \underset{\sim}{I} = \begin{bmatrix} \frac{2\sqrt{2}}{3} - 1 & -\frac{3\sqrt{2}}{8} \\ \frac{2\sqrt{2}}{3} & \frac{3\sqrt{2}}{8} - 1 \end{bmatrix} $$

$$ {}^{t}_{0}\varepsilon_{11} = \frac{1}{2}\left[2\cdot\left(\frac{2\sqrt{2}}{3} - 1\right) + \left(\frac{2\sqrt{2}}{3} - 1\right)^2 + \left(\frac{2\sqrt{2}}{3}\right)^2 \right] = \frac{7}{18} $$

Similarly, ${}^{t}_{0}\varepsilon_{12} = {}^{t}_{0}\varepsilon_{21} = 0, \quad {}^{t}_{0}\varepsilon_{22} = -\frac{7}{32}$

$$ \therefore \; {}^{t}_{0}\underset{\sim}{\varepsilon} = \begin{bmatrix} \frac{7}{18} & 0 \\ 0 & -\frac{7}{32} \end{bmatrix} \qquad (\leftarrow (6.55)) $$

(iii) Exercise 6.4

$$ {}^{t}_{0}\underset{\sim}{X} = \begin{bmatrix} \frac{3\sqrt{3}}{4} & -\frac{5}{8} & 0 \\ \frac{3}{4} & \frac{5\sqrt{3}}{8} & 0 \\ 0 & 0 & \frac{8}{15} \end{bmatrix}, \quad {}^{t}_{0}\underset{\sim}{U} = \begin{bmatrix} \frac{3}{2} & & \\ & \frac{5}{4} & \\ & & \frac{8}{15} \end{bmatrix} $$

6.14

$$ {}_0^t\underset{\sim}{C} = \begin{bmatrix} \frac{9}{4} & & \\ & \frac{25}{16} & \\ & & \frac{64}{225} \end{bmatrix} = \underset{\sim}{\Lambda}^2 , \quad {}_0^t\underset{\sim}{R}_L = \underset{\sim}{I} $$

$$ \therefore \; {}_0^t\underset{\sim}{\epsilon} = \begin{bmatrix} \frac{5}{8} & & \\ & \frac{9}{32} & \\ & & -\frac{161}{450} \end{bmatrix} \qquad (\leftarrow (6.51), (6.53) \text{ and } (6.54)) $$

$$ [{}_0^t u_{i,j}] = \begin{bmatrix} \frac{3\sqrt{3}}{4}-1 & -\frac{5}{8} & 0 \\ \frac{3}{4} & \frac{5\sqrt{3}}{8}-1 & 0 \\ 0 & 0 & -\frac{7}{15} \end{bmatrix} $$

$$ {}_0^t\epsilon_{11} = \frac{1}{2}\left[2\left(\frac{3\sqrt{3}}{4}-1\right) + \left(\frac{3\sqrt{3}}{4}-1\right)^2 + \left(\frac{3}{4}\right)^2 \right] = \frac{5}{8} $$

Similarly, ${}_0^t\epsilon_{22} = \frac{9}{32}$, ${}_0^t\epsilon_{33} = -\frac{161}{450}$, others are zero.

$$ \therefore \; {}_0^t\underset{\sim}{\epsilon} = \begin{bmatrix} \frac{5}{8} & & \\ & \frac{9}{32} & \\ & & -\frac{161}{450} \end{bmatrix} \qquad (\leftarrow (6.55)) $$

6.15 (i) Exercise 6.1

$$ {}^{t}\underset{\sim}{V} = \begin{bmatrix} \frac{\sqrt{3}-1}{2} & 0 \\ 0 & \frac{\sqrt{3}+1}{2} \end{bmatrix}, \quad {}^{t}_{0}\underset{\sim}{R}_{L} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} $$

$$ \therefore {}^{t}_{0}\underset{\sim}{E}^{H} = {}^{t}_{0}\underset{\sim}{R}_{L} (\ln {}^{t}\underset{\sim}{V})\, {}^{t}_{0}\underset{\sim}{R}_{L}^{T} = \frac{1}{2} \begin{bmatrix} -\ln 2 & \ln(2+\sqrt{3}) \\ \ln(2+\sqrt{3}) & -\ln 2 \end{bmatrix} $$

(ii) Exercise 6.3

$$ {}^{t}\underset{\sim}{V} = \begin{bmatrix} \frac{3}{4} & \\ & \frac{4}{3} \end{bmatrix}, \quad {}^{t}_{0}\underset{\sim}{R}_{L} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} $$

$$ \therefore {}^{t}_{0}\underset{\sim}{E}^{H} = \begin{bmatrix} \ln \frac{4}{3} & 0 \\ 0 & \ln \frac{3}{4} \end{bmatrix} $$

(iii) Exercise 6.4

$$ {}^{t}\underset{\sim}{V} = \begin{bmatrix} \frac{3}{2} & & \\ & \frac{5}{4} & \\ & & \frac{8}{15} \end{bmatrix}, \quad {}^{t}_{0}\underset{\sim}{R}_{L} = \underset{\sim}{I} $$

$$ \therefore {}^{t}_{0}\underset{\sim}{E}^{H} = \ln {}^{t}\underset{\sim}{V} = \begin{bmatrix} \ln \frac{3}{2} & & \\ & \ln \frac{5}{4} & \\ & & \ln \frac{8}{15} \end{bmatrix} $$

6.16 From the results in exercise 6.10 and 6.13,

$$\underline{R}_L = \begin{bmatrix} 0.6340 & 0.7733 \\ 0.7733 & -0.6340 \end{bmatrix}, \quad \dot{\underline{R}}_L = \begin{bmatrix} -0.01859 & 0.01524 \\ 0.01524 & 0.01859 \end{bmatrix}$$

$$\underline{\Lambda} = \begin{bmatrix} 1.2198 & 0 \\ 0 & 0.8198 \end{bmatrix}, \quad \underline{\Lambda}^{-1} = \begin{bmatrix} 0.8198 & 0 \\ 0 & 1.2198 \end{bmatrix}$$

$$\dot{\underline{\Lambda}} = \begin{bmatrix} 0.05981 & 0 \\ 0 & -0.04019 \end{bmatrix}, \quad \underline{\Omega}_L = \begin{bmatrix} 0 & 0.02404 \\ -0.02404 & 0 \end{bmatrix}$$

$$\underline{D}_E = \begin{bmatrix} 0.04903 & -0.0098 \\ -0.0098 & -0.04903 \end{bmatrix}$$

Green-Lagrange strain tensor :

$$g(\underline{\Lambda}) = \frac{1}{2}(\underline{\Lambda}^2 - \underline{I}) = \begin{bmatrix} 0.2439 & 0 \\ 0 & -0.1640 \end{bmatrix}$$

$$\dot{\underline{E}}_L = \dot{\underline{\Lambda}}\, g'(\underline{\Lambda}) + \underline{\Omega}_L\, g(\underline{\Lambda}) - g(\underline{\Lambda})\, \underline{\Omega}_L = \begin{bmatrix} 0.07295 & -0.0098 \\ -0.0098 & -0.03295 \end{bmatrix}$$

$$\therefore \dot{\underline{E}}_g = \underline{R}_L \dot{\underline{E}}_L \underline{R}_L^T = \begin{bmatrix} 0 & 0.05 \\ 0.05 & 0.04 \end{bmatrix}$$

Hencky strain tensor :

$$g(\underline{\Lambda}) = \ln \underline{\Lambda} = \begin{bmatrix} 0.1987 & 0 \\ 0 & -0.1987 \end{bmatrix}, \quad g'(\underline{\Lambda}) = \underline{\Lambda}^{-1} = \begin{bmatrix} 0.8198 & \\ 0 & 1.2198 \end{bmatrix}$$

$$\dot{\underline{E}}_L = \begin{bmatrix} 0.049 & -0.0096 \\ -0.0096 & -0.049 \end{bmatrix}$$

$$\therefore \dot{\underline{E}}_g = \underline{R}_L \dot{\underline{E}}_L \underline{R}_L^T = \begin{bmatrix} -0.019 & 0.0462 \\ 0.0462 & 0.019 \end{bmatrix}$$

6.16

Now use (6.59) to (6.61) for the Green-Lagrange strain tensor.

$$\dot{E}_{L11} = \lambda_1^2 D_{E11} = 0.07295, \quad \dot{E}_{L22} = \lambda_2^2 D_{E22} = -0.03295$$

$$\dot{E}_{L12} = \lambda_1 \lambda_2 D_{E12} = -0.0098 = \dot{E}_{L21}$$

and for the Hencky strain tensor,

$$\dot{E}_{L11} = D_{E11} = 0.049, \qquad \dot{E}_{L22} = D_{E22} = -0.049$$

$$\dot{E}_{L12} = \frac{2\lambda_1\lambda_2}{\lambda_2^2 - \lambda_1^2}\left(\ln \frac{\lambda_2}{\lambda_1}\right) D_{E12} = -0.0096 = \dot{E}_{L21}$$

6.17 (a) $\,{}_0^t\underline{\varepsilon} = \frac{1}{2}\left({}_0^t\underline{X}^T\, {}_0^t\underline{X} - \underline{I} \right)$

$${}_0^t\varepsilon_{ij}\, d^0x_i\, d^0x_j = d^0\underline{x}^T\, {}_0^t\underline{\varepsilon}\, d^0\underline{x} = \frac{1}{2}\left(d^t\underline{x}^T d^t\underline{x} - d^0\underline{x}^T d^0\underline{x} \right)$$

$$(d^ts)^2 = d^tx_i\, d^tx_i = d^t\underline{x}^T d^t\underline{x}$$

$$(d^0s)^2 = d^0x_i\, d^0x_i = d^0\underline{x}^T d^0\underline{x}$$

$$\therefore\quad {}_0^t\varepsilon_{ij}\, d^0x_i\, d^0x_j = \frac{1}{2}\left[(d^ts)^2 - (d^0s)^2 \right]$$

(b) $(d^ts)^2 = (d^0s)^2 + 2\, {}_0^t\varepsilon_{ij}\, d^0x_i\, d^0x_j$

Let $d^0x_i = d^0s\, {}^0n_i$ with ${}^0\underline{n}^T = \left[\frac{\sqrt{3}}{2} \quad \frac{1}{2} \right]$ and $d^0s = \| d^0\underline{x} \|_2$.

Then $$({}^t\lambda)^2 = \frac{(d^ts)^2}{(d^0s)^2} = 1 + 2\, {}_0^t\varepsilon_{ij} \frac{d^0x_i}{d^0s} \frac{d^0x_j}{d^0s} = 1 + 2\, {}_0^t\varepsilon_{ij}\, {}^0n_i\, {}^0n_j$$

$$= 1 + 2\left({}_0^t\varepsilon_{11}\, {}^0n_1\, {}^0n_1 + 2\, {}_0^t\varepsilon_{12}\, {}^0n_1\, {}^0n_2 + {}_0^t\varepsilon_{22}\, {}^0n_2\, {}^0n_2 \right)$$

$$= 1 + 2\left[(0.6)\left(\frac{\sqrt{3}}{2}\right)^2 + 2\,(0.2)\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) + (-0.3)\left(\frac{1}{2}\right)^2 \right]$$

$$= 2.0964$$

$$\therefore\quad {}^t\lambda = 1.4479$$

The Green-Lagrange strain tensor does not contain any rigid body rotational motion. Hence from the given ${}_0^t\underline{\varepsilon}$ it is impossible to calculate the rotation of the line element, while it is still possible to calculate the angular distortion between two fibres.

6.18 Let $h_1 = \frac{1}{4}(1+{}^tx_1)(1+{}^tx_2)$, $h_2 = \frac{1}{4}(1-{}^tx_1)(1+{}^tx_2)$, $h_3 = \frac{1}{4}(1-{}^tx_1)(1-{}^tx_2)$

$h_4 = \frac{1}{4}(1+{}^tx_1)(1-{}^tx_2)$

$${}^tx_i = \sum h_k\, {}^tx_i^k \ , \quad {}^t\dot{u}_i = \sum h_k\, {}^t\dot{u}_i^k \ , \quad L_{ij} = \frac{\partial\, {}^t\dot{u}_i}{\partial\, {}^tx_j}$$

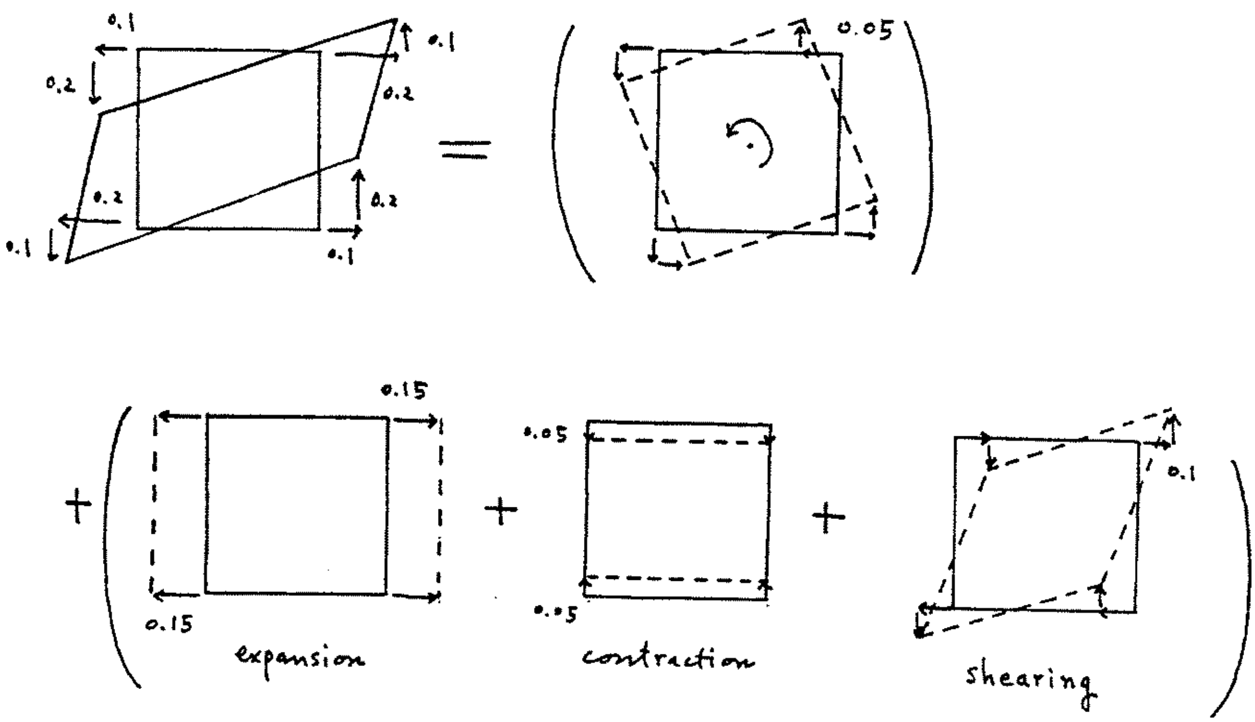
$$L_{11} = \frac{\partial\, {}^t\dot{u}_1}{\partial\, {}^tx_1} = \frac{1}{4}\left[\left({}^t\dot{u}_1^1 - {}^t\dot{u}_1^2 - {}^t\dot{u}_1^3 + {}^t\dot{u}_1^4\right) + {}^tx_2\left({}^t\dot{u}_1^1 - {}^t\dot{u}_1^2 + {}^t\dot{u}_1^3 - {}^t\dot{u}_1^4\right)\right]$$

$$= \frac{1}{4}(0.6) = 0.15$$

Similarly, $L_{12} = 0.05$, $L_{21} = 0.15$, $L_{22} = -0.05$

$$\therefore\ \underline{L} = \begin{bmatrix} 0.15 & 0.05 \\ 0.15 & -0.05 \end{bmatrix},\quad \underline{D} = \begin{bmatrix} 0.15 & 0.10 \\ 0.10 & -0.05 \end{bmatrix},\quad \underline{W} = \begin{bmatrix} 0 & -0.05 \\ 0.05 & 0 \end{bmatrix}$$

Physically, we have



6.19 From the result in exercise 6.10,

$$\underline{X} = \begin{bmatrix} 1 & {}^t u \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.4 \\ 0 & 1 \end{bmatrix}, \quad \underline{\dot{X}} = \begin{bmatrix} 0 & {}^t \dot{u} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0.1 \\ 0 & 0 \end{bmatrix}$$

$$\underline{\dot{\varepsilon}} = \frac{1}{2}\left(\underline{\dot{X}}^T \underline{X} + \underline{X}^T \underline{\dot{X}} \right) = \begin{bmatrix} 0 & 0.05 \\ 0.05 & 0.04 \end{bmatrix}$$

$$\rightarrow \quad \underline{D} = \underline{X}^{-T} \underline{\dot{\varepsilon}}\, \underline{X}^{-1} = \begin{bmatrix} 0 & 0.05 \\ 0.05 & 0 \end{bmatrix}$$

6.20 Let $h_1 = \frac{1}{4}(1+{}^0x_1)(1+{}^0x_2)$, $h_2 = \frac{1}{4}(1-{}^0x_1)(1+{}^0x_2)$

$h_3 = \frac{1}{4}(1-{}^0x_1)(1-{}^0x_2)$, $h_4 = \frac{1}{4}(1+{}^0x_1)(1-{}^0x_2)$

$${}^tx_1 = \sum_{k=1}^{4} h_k \, {}^tx_1^k = \frac{1}{4}(9 + 7\,{}^0x_1 + 3\,{}^0x_2 + {}^0x_1{}^0x_2)$$

$${}^tx_2 = \sum_{k=1}^{4} h_k \, {}^tx_2^k = \frac{1}{4}(7 + 3\,{}^0x_1 + 5\,{}^0x_2 + {}^0x_1{}^0x_2)$$

$$\therefore {}_0^t\underline{X} = \left[\frac{\partial\,{}^tx_i}{\partial\,{}^0x_j}\right] = \frac{1}{4}\begin{bmatrix} 7+{}^0x_2 & 3+{}^0x_1 \\ 3+{}^0x_2 & 5+{}^0x_1 \end{bmatrix}, \qquad {}_t^0\underline{X} = \left({}_0^t\underline{X}\right)^{-1}$$

$${}^tu_1 = {}^tx_1 - {}^0x_1 = \frac{1}{4}(9 + 3\,{}^0x_1 + 3\,{}^0x_2 + {}^0x_1{}^0x_2)$$

$${}^tu_2 = {}^tx_2 - {}^0x_2 = \frac{1}{4}(7 + 3\,{}^0x_1 + {}^0x_2 + {}^0x_1{}^0x_2)$$

$$\delta u_1 = \frac{1}{4}(1+{}^0x_1)(1+{}^0x_2)\,\Delta, \quad \delta u_2 = 0$$

As
$$\delta\,{}_0^t\varepsilon_{ij} = \frac{1}{2}\left[\frac{\partial\,\delta u_i}{\partial\,{}^0x_j} + \frac{\partial\,\delta u_j}{\partial\,{}^0x_i} + \frac{\partial\,\delta u_k}{\partial\,{}^0x_i}\frac{\partial\,{}^tu_k}{\partial\,{}^0x_j} + \frac{\partial\,{}^tu_k}{\partial\,{}^0x_i}\frac{\partial\,\delta u_k}{\partial\,{}^0x_j}\right],$$

$$\delta\,{}_0^t\varepsilon_{11} = \frac{1}{2}\left[\frac{1}{4}(1+{}^0x_2)\Delta\cdot 2 + \left(\frac{3}{4}+\frac{1}{4}{}^0x_2\right)\left(\frac{1}{4}+\frac{1}{4}{}^0x_2\right)\Delta\cdot 2\right]$$

$$= \frac{\Delta}{16}(1+{}^0x_2)(7+{}^0x_2)$$

$$\delta\,{}_0^t\varepsilon_{12} = \frac{\Delta}{16}(5 + 4\,{}^0x_1 + 2\,{}^0x_2 + {}^0x_1{}^0x_2) = \delta\,{}_0^t\varepsilon_{21}$$

$$\delta\,{}_0^t\varepsilon_{22} = \frac{\Delta}{16}(1+{}^0x_1)(3+{}^0x_1)$$

Now from the relation ${}_0^t\underline{X}^T\,\delta_t\underline{e}\,{}_0^t\underline{X} = \delta\,{}_0^t\underline{\varepsilon}$, $\delta_t\underline{e}$ can be calculated as $\delta_t\underline{e} = {}_t^0\underline{X}^T\,\delta\,{}_0^t\underline{\varepsilon}\,{}_t^0\underline{X}$

6.21 (a) $${}^{\Delta t}\underline{\tau} = \underline{R}\,{}^{0}\underline{\tau}\,\underline{R}^T = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 200 & 100 \\ 100 & 300 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} = 25 \begin{bmatrix} 9-2\sqrt{3} & 2-\sqrt{3} \\ 2-\sqrt{3} & 11+2\sqrt{3} \end{bmatrix}$$

(b) The 2nd Piola-Kirchhoff stress components do not change when the body is subjected to a rigid body motion.

$${}^{\Delta t}_{0}\underline{S} = {}^{0}_{0}\underline{S} = {}^{0}\underline{\tau} = \begin{bmatrix} 200 & 100 \\ 100 & 300 \end{bmatrix}$$

(c) $${}^{\Delta t}_{0}\underline{X} = {}^{\Delta t}_{0}\underline{R}\,{}^{\Delta t}_{0}\underline{U} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

(d) From the physical meaning of the Cauchy stresses,

$${}^{0}\bar{\underline{\tau}} = 25 \begin{bmatrix} 9-2\sqrt{3} & 2-\sqrt{3} \\ 2-\sqrt{3} & 11+2\sqrt{3} \end{bmatrix}$$

(e) Using the tensor transformation to get ${}^{0}_{0}\bar{\underline{S}}$,

$${}^{0}_{0}\bar{\underline{S}} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 200 & 100 \\ 100 & 300 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} = 25 \begin{bmatrix} 9-2\sqrt{3} & 2-\sqrt{3} \\ 2-\sqrt{3} & 11+2\sqrt{3} \end{bmatrix}$$

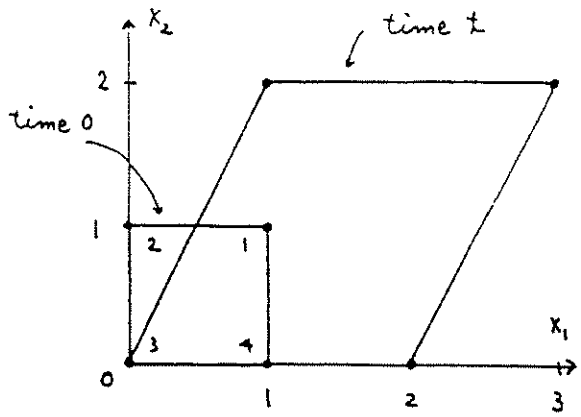
(f) The element undergoes no deformations, hence

$${}^{0}_{0}\bar{\underline{X}} = \underline{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

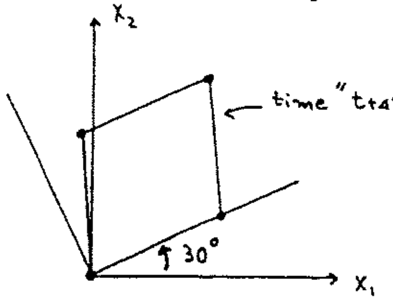
6.22 $\quad {}^{t}_{0}\underset{\sim}{S} = \begin{bmatrix} 100 & 50 & 0 \\ 50 & 200 & 0 \\ 0 & 0 & 100 \end{bmatrix}, \quad {}^{t}_{0}\underset{\sim}{X} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad {}^{t+\Delta t}_{t}\underset{\sim}{R} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(a) From ${}^{t}_{0}\underset{\sim}{X}$ given,

$$ {}^{t}x_1 = 2\,{}^{0}x_1 + {}^{0}x_2 + C_1 $$
$$ {}^{t}x_2 = 2\,{}^{0}x_2 + C_2 $$
$$ {}^{t}x_3 = {}^{0}x_3 + C_3 $$



(b)



(c) (i) $\det {}^{t}_{0}\underset{\sim}{X} = 4$

$$ \therefore\ {}^{t}\underset{\sim}{\tau} = \frac{{}^{t}\rho}{{}^{0}\rho}\, {}^{t}_{0}\underset{\sim}{X}\, {}^{t}_{0}\underset{\sim}{S}\, {}^{t}_{0}\underset{\sim}{X}^T = \frac{1}{\det {}^{t}_{0}\underset{\sim}{X}}\, {}^{t}_{0}\underset{\sim}{X}\, {}^{t}_{0}\underset{\sim}{S}\, {}^{t}_{0}\underset{\sim}{X}^T = \begin{bmatrix} 200 & 150 & 0 \\ 150 & 200 & 0 \\ 0 & 0 & 25 \end{bmatrix} $$

(ii)
$$ {}^{t+\Delta t}\underset{\sim}{\tau} = {}^{t+\Delta t}_{t}\underset{\sim}{R}\, {}^{t}\underset{\sim}{\tau}\, {}^{t+\Delta t}_{t}\underset{\sim}{R}^T = \begin{bmatrix} 200-75\sqrt{3} & 75 & 0 \\ 75 & 200+75\sqrt{3} & 0 \\ 0 & 0 & 25 \end{bmatrix} $$

(iii) As the body is undergoing a rigid body motion from time t to time t+Δt,

$$ {}^{t+\Delta t}\underset{\sim}{S} = {}^{t}\underset{\sim}{S} = \begin{bmatrix} 100 & 50 & 0 \\ 50 & 200 & 0 \\ 0 & 0 & 100 \end{bmatrix} $$

6.23 (a) $\,{}^{t}_{0}\underset{\sim}{S} = \begin{bmatrix} 40 & 0 & 0 \\ 0 & -60 & 0 \\ 0 & 0 & -15 \end{bmatrix}$, $\,{}^{t}_{0}\underset{\sim}{X} = \begin{bmatrix} \frac{3}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\det {}^{t}_{0}\underset{\sim}{X} = \frac{3}{4}$

$$\therefore \; {}^{t}\underset{\sim}{\tau} = \frac{1}{(3/4)} \begin{bmatrix} \frac{3}{2} & & \\ & \frac{1}{2} & \\ & & 1 \end{bmatrix} \begin{bmatrix} 40 & & \\ & -60 & \\ & & -15 \end{bmatrix} \begin{bmatrix} \frac{3}{2} & & \\ & \frac{1}{2} & \\ & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 120 & & \\ & -20 & \\ & & -20 \end{bmatrix}$$

(b) $\,{}^{t+\Delta t}_{0}\underset{\sim}{S} = {}^{t}_{0}\underset{\sim}{S} = \begin{bmatrix} 40 & & \\ & -60 & \\ & & -15 \end{bmatrix}$, $\,{}^{t+\Delta t}_{t}\underset{\sim}{R} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \\ & & 1 \end{bmatrix}$

$$\therefore \; {}^{t+\Delta t}\underset{\sim}{\tau} = {}^{t+\Delta t}_{t}\underset{\sim}{R} \; {}^{t}\underset{\sim}{\tau} \; {}^{t+\Delta t}_{t}\underset{\sim}{R}^{T} = \begin{bmatrix} 50 & 70 & 0 \\ 70 & 50 & 0 \\ 0 & 0 & -20 \end{bmatrix}$$

6.24 (a) $${}^{0}_{t}\underline{X} = {}^{0}_{t}\underline{R}\,{}^{0}_{t}\underline{U} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{4} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{2} & -\frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

$$\therefore\ {}^{t}_{0}\underline{X} = \left({}^{0}_{t}\underline{X}\right)^{-1} = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ -\frac{2\sqrt{2}}{3} & \frac{2\sqrt{2}}{3} \end{bmatrix}$$

(b) $${}^{t}_{0}\underline{\epsilon} = \frac{1}{2}\left({}^{t}_{0}\underline{X}^{T}\,{}^{t}_{0}\underline{X} - \underline{I}\right) = \begin{bmatrix} \frac{17}{18} & \frac{5}{9} \\ \frac{5}{9} & \frac{17}{18} \end{bmatrix}$$

$${}^{t}_{0}\underline{S} = \begin{bmatrix} 11 & 7 & 0 \\ 7 & 11 & 0 \\ 0 & 0 & 9 \end{bmatrix}\begin{bmatrix} 17/18 \\ 17/18 \\ 5/9 \end{bmatrix} = \begin{bmatrix} 17 \\ 17 \\ 5 \end{bmatrix}$$

From the plane strain assumption, $\frac{{}^{t}\rho}{{}^{0}\rho} = \frac{(1.5)(1)(1)}{(2)(2)(1)} = \frac{3}{8}$

$$\therefore\ {}^{t}\underline{\tau} = \frac{{}^{t}\rho}{{}^{0}\rho}\,{}^{t}_{0}\underline{X}\,{}^{t}_{0}\underline{S}\,{}^{t}_{0}\underline{X}^{T} = \begin{bmatrix} 33 & 0 \\ 0 & 8 \end{bmatrix}$$

i.e., ${}^{t}\tau_{11} = 33$, ${}^{t}\tau_{22} = 8$, ${}^{t}\tau_{12} = 0$

Hence the Cauchy stress ${}^{t}\underline{\tau}$ given by the program is not correct. We can identify the program error by noting that $\frac{33+8}{2} = 20.5$ and $20.5 + 12.5 = 33$, $20.5 - 12.5 = 8$. Hence a rotation of 45° was wrongly applied. Therefore,

$${}^{t}\underline{\tau}\Big|_{book} = \underline{R}\,{}^{t}\underline{\tau}\Big|_{above}\,\underline{R}^{T}, \quad \text{where } \underline{R} = \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{bmatrix}$$

6.25 P.V.D. $\int_{^tV} {}^t\tau_{ij}\,\delta_t e_{ij}\,d^tV = \int_{^tV} {}^tf_i^B\,\delta u_i\,d^tV + \int_{^tS} f_i^S\,\delta u_i^S\,d^tS$

Six independent virtual displacement patterns are considered here.

Case 1 shear

$$\left.\begin{array}{l}\delta u_1 = {}^tx_2\\ \delta u_2 = {}^tx_1\end{array}\right\}\quad \delta_t e_{11} = \delta_t e_{22} = 0,\quad \delta_t e_{12} = 1$$

$$(\text{l.h.s.}) = 0$$

$$(\text{r.h.s.}) = \int_1^4 (-10)\,{}^tx_2\,d^tx_2 + \int_1^4 10\,{}^tx_2\,d^tx_2 + \int_{2.5}^{3.5} 20\,{}^tx_1\,d^tx_1 + \int_{2.5}^{3.5}(-20)\,{}^tx_1\,d^tx_1 = 0$$

Case 2 expansion

$$\left.\begin{array}{l}\delta u_1 = {}^tx_1\\ \delta u_2 = {}^tx_2\end{array}\right\}\quad \delta_t e_{11} = 1,\quad \delta_t e_{22} = 1,\quad \delta_t e_{12} = 0$$

$$(\text{l.h.s.}) = \int_{^tV}\left[(-10)(1) + (20)(1)\right] d^tV = 30$$

$$(\text{r.h.s.}) = \int_{2.5}^{3.5}\left[(20)(4)\right] d^tx_1 + \int_{2.5}^{3.5}\left[(-20)(1)\right] d^tx_1 + \int_1^4\left[(-10)(3.5)\right] d^tx_2 + \int_1^4\left[(10)(2.5)\right] d^tx_2 = 30$$

Case 3 translation

$$\left.\begin{array}{l}\delta u_1 = c_1\\ \delta u_2 = c_2\end{array}\right\}\quad \delta_t e_{11} = \delta_t e_{22} = \delta_t e_{12} = 0$$

$$(\text{l.h.s.}) = 0$$

$$(\text{r.h.s.}) = \int_{2.5}^{3.5}(20)\,c_2\,d^tx_1 + \int_{2.5}^{3.5}(-20)\,c_2\,d^tx_2 + \int_1^4 (10)\,c_1\,d^tx_2 + \int_1^4(-10)\,c_1\,d^tx_2 = 0$$

6.25

Case 4 $\left.\begin{array}{l}\delta u_1 = c_1 \\ \delta u_2 = {}^t x_1\end{array}\right\}$ $\delta_t e_{11} = \delta_t e_{22} = 0$, $\delta_t e_{12} = \frac{1}{2}$

$$(\text{l.h.s.}) = 0$$

$$(\text{r.h.s.}) = \int_{2.5}^{3.5} (20)\,{}^t x_1\, d\,{}^t x_1 + \int_{2.5}^{3.5} (-20)\,{}^t x_1\, d\,{}^t x_1 + \int_1^4 (10)\, c_1\, d\,{}^t x_2 + \int_1^4 (-10)\, c_1\, d\,{}^t x_2 = 0$$

Case 5 $\left.\begin{array}{l}\delta u_1 = {}^t x_2 \\ \delta u_2 = c_2\end{array}\right\}$ $\delta_t e_{11} = \delta_t e_{22} = 0$, $\delta_t e_{12} = \frac{1}{2}$

$$(\text{l.h.s.}) = 0$$

$$(\text{r.h.s.}) = \int_{2.5}^{3.5} (20)\, c_2\, d\,{}^t x_1 + \int_{2.5}^{3.5} (-20)\, c_2\, d\,{}^t x_1 + \int_1^4 (10)\,{}^t x_2\, d\,{}^t x_2 + \int_1^4 (-10)\,{}^t x_2\, d\,{}^t x_2 = 0$$

Case 6 rotation

$\left.\begin{array}{l}\delta u_1 = {}^t x_2 \\ \delta u_2 = -{}^t x_1\end{array}\right\}$ $\delta_t e_{11} = \delta_t e_{22} = \delta_t e_{12} = 0$

$$(\text{l.h.s.}) = 0$$

$$(\text{r.h.s.}) = \int_{2.5}^{3.5} (20)(-{}^t x_1)\, d\,{}^t x_1 + \int_{2.5}^{3.5} (-20)(-{}^t x_1)\, d\,{}^t x_1 + \int_1^4 (10)\,{}^t x_2\, d\,{}^t x_2 + \int_1^4 (-10)\,{}^t x_2\, d\,{}^t x_2 = 0$$

6.26 Considering the bar,

$$ {}^t_0\underline{X} = \begin{bmatrix} \frac{\partial\,{}^t x_1}{\partial\,{}^0 x_1} & 0 & 0 \\ 0 & \left[\frac{\partial\,{}^t x_1}{\partial\,{}^0 x_1}\right]^{-\frac{1}{2}} & 0 \\ 0 & 0 & \left[\frac{\partial\,{}^t x_1}{\partial\,{}^0 x_1}\right]^{-\frac{1}{2}} \end{bmatrix} $$

(a) $$ {}^t_0 S_{11} = \frac{{}^0\rho}{{}^t\rho}\,\frac{1}{\left(\frac{\partial\,{}^t x_1}{\partial\,{}^0 x_1}\right)^2}\,{}^t\tau_{11} = \left[1 \Big/ \left(\frac{\partial\,{}^t x_1}{\partial\,{}^0 x_1}\right)^2 \right] {}^t\tau_{11} $$

where ${}^t\tau_{11}$ is the Cauchy stress.

(b) $$ \int_0^{{}^0L} {}^t_0 S_{11}\,\delta\,{}^t_0\epsilon_{11}\,{}^0A\,d^0x = \int_0^{{}^0L} f^B \delta u_1\,{}^0A\,d^0x \qquad (f^B\text{: force per unit original volume}) $$

$$ \delta\,{}^t_0\epsilon_{11} = \frac{\partial \delta u_1}{\partial\,{}^0 x_1} + \frac{\partial\,{}^t u_1}{\partial\,{}^0 x_1}\frac{\partial \delta u_1}{\partial\,{}^0 x_1} = \frac{\partial\,{}^t x_1}{\partial\,{}^0 x_1}\frac{\partial \delta u_1}{\partial\,{}^0 x_1} $$

$$ \left({}^t x_1 = {}^0 x_1 + {}^t u_1\ ,\quad \frac{\partial\,{}^t x_1}{\partial\,{}^0 x_1} = 1 + \frac{\partial\,{}^t u_1}{\partial\,{}^0 x_1} \right) $$

$$ (\text{L.H.S.}) = \int_0^{{}^0L} {}^t_0 S_{11}\,\frac{\partial\,{}^t x_1}{\partial\,{}^0 x_1}\,\frac{\partial \delta u_1}{\partial\,{}^0 x_1}\,{}^0A\,d^0x $$

$$ = {}^t_0 S_{11}\,\frac{\partial\,{}^t x_1}{\partial\,{}^0 x_1}\,{}^0A\,\delta u_1 \Big|_0^{{}^0L} - \int_0^{{}^0L} \frac{\partial}{\partial\,{}^0 x_1}\left({}^t_0 S_{11}\,{}^0A\,\frac{\partial\,{}^t x_1}{\partial\,{}^0 x_1} \right) \delta u_1\,d^0x $$

Hence we have

$$ \text{(i)} \text{ ——— } \begin{cases} \frac{\partial}{\partial\,{}^0 x_1}\left({}^t_0 S_{11}\,{}^0A\,\frac{\partial\,{}^t x_1}{\partial\,{}^0 x_1} \right) + f^B\,{}^0A = 0\ , & 0 < {}^0x_1 < {}^0L \\ {}^t_0 S_{11}\,\frac{\partial\,{}^t x_1}{\partial\,{}^0 x_1}\,{}^0A = 0 \quad \text{at}\quad {}^0x_1 = {}^0L; & {}^t u_1 = 0 \ \text{at}\ {}^0x_1 = \end{cases} $$

(c) $$ \int_0^{{}^tL} {}^t\tau_{11}\,\delta_t e_{11}\,{}^tA\,d^tx = \int_0^{{}^tL} f^B \delta u_1\,{}^tA\,d^tx \qquad ({}^tA\,d^tx = {}^0A\,d^0x) $$

$$ (\text{L.H.S.}) = \int_0^{{}^tL} {}^t\tau_{11}\,\frac{\partial \delta u_1}{\partial\,{}^t x_1}\,{}^tA\,d^tx = {}^t\tau_{11}\,{}^tA\,\delta u_1 \Big|_0^{{}^tL} - \int_0^{{}^tL} \frac{\partial}{\partial\,{}^t x_1}({}^t\tau_{11}\,{}^tA)\,\delta u_1\,d^tx $$

6.26

Hence we have

(ii) —— $$\begin{cases} \frac{\partial}{\partial\, {}^{t}x_1}\left({}^{t}\tau_{11}\,{}^{t}A\right) + f^{B}\,{}^{t}A = 0 \quad , \quad 0 < {}^{t}x < {}^{t}L \\ {}^{t}\tau_{11}\,{}^{t}A = 0 \quad \text{at } {}^{t}x_1 = {}^{t}L \,; \quad {}^{t}u_1 = 0 \quad \text{at } {}^{t}x_1 = 0. \end{cases}$$

For infinitesimally small strain analysis,

$$ {}^{t}A = {}^{0}A \,, \quad {}^{t}L = {}^{0}L \,, \quad {}^{t}_{0}S_{11} = {}^{t}\tau_{11} \,, \quad \frac{\partial\, {}^{t}x_1}{\partial\, {}^{0}x_1} = 1 $$

and the relations (i) and (ii) are the usual small strain analysis equations.

6.27 Since $H, h << b$, we only consider the displacement u_r in the x_1-direction with the plane stress assumption.

T.L. formulation

$${}^{t}f^{B} = {}^{t}\rho\,\omega^2\,{}^{t}r\,, \quad {}^{0}t = H\left(\frac{r-b}{a-b}\right) + h\left(\frac{r-a}{b-a}\right), \quad {}^{t}t = {}^{t}H\left(\frac{r-{}^{t}b}{{}^{t}a-{}^{t}b}\right) + {}^{t}h\left(\frac{r-{}^{t}a}{{}^{t}b-{}^{t}a}\right)$$

$${}_{0}\epsilon_{rr} = {}_{0}e_{rr} + {}_{0}\eta_{rr}\,, \quad {}_{0}\epsilon_{\theta\theta} = {}_{0}e_{\theta\theta} + {}_{0}\eta_{\theta\theta}$$

$${}_{0}e_{rr} = \frac{\partial u_r}{\partial\,{}^{0}r} + \frac{\partial\,{}^{t}u_r}{\partial\,{}^{0}r}\,\frac{\partial u_r}{\partial\,{}^{0}r}\,, \quad {}_{0}\eta_{rr} = \frac{1}{2}\left(\frac{\partial u_r}{\partial\,{}^{0}r}\right)^2$$

$${}_{0}e_{\theta\theta} = \frac{u_r}{{}^{0}r} + \frac{{}^{t}u_r\,u_r}{{}^{0}r^2}\,, \quad {}_{0}\eta_{\theta\theta} = \frac{1}{2}\left(\frac{u_r}{{}^{0}r}\right)^2$$

$$\therefore \int_a^b {}_{0}C_{ijrs}\,{}_{0}e_{rs}\,\delta\,{}_{0}e_{ij}\,{}^{0}r\,{}^{0}t\,dr + \int_a^b {}^{t}_{0}S_{ij}\,\delta\,{}_{0}\eta_{ij}\,{}^{0}r\,{}^{0}t\,dr$$

$$= \int_{{}^{t+\Delta t}a}^{{}^{t+\Delta t}b} {}^{t+\Delta t}\rho\,\omega^2\,{}^{t+\Delta t}r^2\,{}^{t+\Delta t}t\,\delta u_r\,dr - \int_a^b {}^{t}_{0}S_{ij}\,\delta\,{}_{0}e_{ij}\,{}^{0}r\,{}^{0}t\,dr$$

U.L. formulation

$${}_{t}\epsilon_{rr} = {}_{t}e_{rr} + {}_{t}\eta_{rr}\,, \quad {}_{t}\epsilon_{\theta\theta} = {}_{t}e_{\theta\theta} + {}_{t}\eta_{\theta\theta}$$

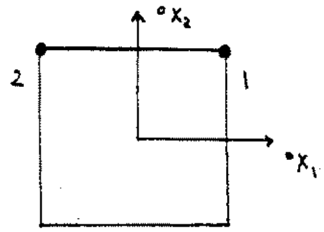
$${}_{t}e_{rr} = \frac{\partial u_r}{\partial\,{}^{t}r}\,, \quad {}_{t}\eta_{rr} = \frac{1}{2}\left(\frac{\partial u_r}{\partial\,{}^{t}r}\right)^2$$

$${}_{t}e_{\theta\theta} = \frac{u_r}{{}^{t}r}\,, \quad {}_{t}\eta_{\theta\theta} = \frac{1}{2}\left(\frac{u_r}{{}^{t}r}\right)^2$$

$$\therefore \int_{{}^{t}a}^{{}^{t}b} {}_{t}C_{ijrs}\,{}_{t}e_{rs}\,\delta\,{}_{t}e_{ij}\,{}^{t}r\,{}^{t}t\,dr + \int_{{}^{t}a}^{{}^{t}b} {}^{t}\tau_{ij}\,\delta\,{}_{t}\eta_{ij}\,{}^{t}r\,{}^{t}t\,dr$$

$$= \int_{{}^{t+\Delta t}a}^{{}^{t+\Delta t}b} {}^{t+\Delta t}\rho\,\omega^2\,{}^{t+\Delta t}r^2\,{}^{t+\Delta t}t\,\delta u_r\,dr - \int_{{}^{t}a}^{{}^{t}b} {}^{t}\tau_{ij}\,\delta\,{}_{t}e_{ij}\,{}^{t}r\,{}^{t}t\,dr$$

6.28



$$h_1 = \frac{1}{16}(2 + {}^0x_1)(2 + {}^0x_2)$$

$$h_2 = \frac{1}{16}(2 - {}^0x_1)(2 + {}^0x_2)$$

$$ {}^tu_1 = \frac{1}{16}(2+{}^0x_1)(2+{}^0x_2)(1) + \frac{1}{16}(2-{}^0x_1)(2+{}^0x_2)(1) = \frac{1}{4}(2+{}^0x_2)$$

$$ {}^tu_2 = \frac{1}{16}(2+{}^0x_1)(2+{}^0x_2)(2) + \frac{1}{16}(2-{}^0x_1)(2+{}^0x_2)(2) = \frac{1}{2}(2+{}^0x_2)$$

$$u_1 = 0, \quad u_2 = \frac{1}{4}(2+{}^0x_2)$$

$${}_0\epsilon_{ij} = \frac{1}{2}\left({}_0u_{i,j} + {}_0u_{j,i} + {}^t_0u_{k,i}\,{}_0u_{k,j} + {}_0u_{k,i}\,{}^t_0u_{k,j}\right) + \frac{1}{2}\,{}_0u_{k,i}\;{}_0u_{k,j}$$

$${}_0\epsilon_{11} = \frac{1}{2}\left[2\,{}_0u_{1,1} + 2\left({}^t_0u_{1,1}\,{}_0u_{1,1} + {}^t_0u_{2,1}\,{}_0u_{2,1}\right)\right] + \frac{1}{2}\left[{}_0u_{1,1}\,{}_0u_{1,1} + {}_0u_{2,1}\,{}_0u_{2,1}\right] = 0$$

Similarly, ${}_0\epsilon_{22} = \frac{13}{32}$, ${}_0\epsilon_{12} = 0 = {}_0\epsilon_{21}$

$$\therefore\; {}_0\underline{\epsilon} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{13}{32} \end{bmatrix}$$

6.29 $$\int_{{}^0V} {}^{t+\Delta t}_{\ 0}S_{ij}\, \delta\, {}^{t+\Delta t}_{\ 0}\epsilon_{ij}\, d^0V = \int_{{}^0V} {}^{t+\Delta t}_{\ 0}S_{ij}\, \delta\, {}_0\epsilon_{ij}\, d^0V \quad \text{where } \delta\, {}^t_0\epsilon_{ij} = 0 \qquad \text{— ①}$$

$$\int_{{}^0V} {}^t_0S_{ij}\, \delta\, {}_0e_{ij}\, d^0V = \int_{{}^0V} {}^t_0S_{ij}\, \delta\, {}^t_0\epsilon_{ij}\, d^0V \quad \text{where } \delta\, {}_0e_{ij} = \delta\, {}^t_0\epsilon_{ij} \neq 0 \qquad \text{— ②}$$

In eq. ①, we have already calculated the configuration at time t and we now consider the variation at time $t+\Delta t$.

$$\delta\, {}^{t+\Delta t}_{\ 0}\epsilon_{ij} = \delta\left({}^t_0\epsilon_{ij} + {}_0\epsilon_{ij}\right) = \delta\, {}^t_0\epsilon_{ij} + \delta\, {}_0\epsilon_{ij}$$

$$\delta\, {}^t_0\epsilon_{ij} = \delta\,(\text{const.}) = 0$$

$$\therefore\ \delta\, {}^{t+\Delta t}_{\ 0}\epsilon_{ij} = \delta\, {}_0\epsilon_{ij}$$

In eq. ②, $\delta\, {}^t_0\epsilon_{ij}$ is due to variation in ${}^tu_i$,

$${}^t_0\epsilon_{ij} = \frac{1}{2}\left({}^t_0u_{i,j} + {}^t_0u_{j,i} + {}^t_0u_{k,i}\, {}^t_0u_{k,j}\right)$$

and $$\delta\, {}^t_0\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial \delta\, {}^tu_i}{\partial\, {}^0x_j} + \frac{\partial \delta\, {}^tu_j}{\partial\, {}^0x_i} + \frac{\partial \delta\, {}^tu_k}{\partial\, {}^0x_i}\, {}^t_0u_{k,j} + {}^t_0u_{k,i}\, \frac{\partial \delta\, {}^tu_k}{\partial\, {}^0x_j}\right)$$

Let $\delta\, {}^tu_i = \delta u_i$,

$$\delta\, {}^t_0\epsilon_{ij} = \frac{1}{2}\left(\delta\, {}_0u_{i,j} + \delta\, {}_0u_{j,i} + {}^t_0u_{k,i}\, \delta\, {}_0u_{k,j} + {}^t_0u_{k,j}\, \delta\, {}_0u_{k,i}\right)$$

and hence $\delta\, {}^t_0\epsilon_{ij} = \delta\, {}_0e_{ij}$.

$\underline{6.30}$ $\quad \delta_t e_{rr} = \dfrac{\partial \delta u_r}{\partial\, {}^t r}, \quad \delta_t e_{\theta\theta} = \dfrac{\delta u_r}{{}^t r}, \quad \delta\, {}^t_0\epsilon_{\theta\theta} = \dfrac{\delta u_r}{{}^0 r} + \dfrac{{}^t u_r\, \delta u_r}{{}^0 r^2} = \dfrac{{}^t r}{{}^0 r^2}\, \delta u_r$

$$\delta\, {}^t_0\epsilon_{rr} = \frac{\partial \delta u_r}{\partial\, {}^0 r} + \frac{\partial\, {}^t u_r}{\partial\, {}^0 r}\,\frac{\partial \delta u_r}{\partial\, {}^0 r} = \frac{\partial\, {}^t r}{\partial\, {}^0 r}\,\frac{\partial \delta u_r}{\partial\, {}^0 r}$$

$${}^t_0 S_{rr} = \frac{{}^0\rho}{{}^t\rho}\left(\frac{\partial\, {}^0 r}{\partial\, {}^t r}\right)^2 {}^t\tau_{rr}, \quad {}^t_0 S_{\theta\theta} = \frac{{}^0\rho}{{}^t\rho}\left(\frac{{}^0 r}{{}^t r}\right)^2 {}^t\tau_{\theta\theta}$$

$$(\text{L.H.S.}) = \int_{{}^t V}\left({}^t\tau_{rr}\,\frac{\partial \delta u_r}{\partial\, {}^t r} + {}^t\tau_{\theta\theta}\,\frac{\delta u_r}{{}^t r}\right) d\,{}^t V$$

$$= \int_{{}^0 V}\frac{{}^0\rho}{{}^t\rho}\left({}^t\tau_{rr}\,\frac{\partial \delta u_r}{\partial\, {}^t r} + {}^t\tau_{\theta\theta}\,\frac{\delta u_r}{{}^t r}\right) d\,{}^0 V$$

$$(\text{R.H.S.}) = \int_{{}^0 V}\frac{{}^0\rho}{{}^t\rho}\left(\frac{\partial\, {}^0 r}{\partial\, {}^t r}\right)^2 {}^t\tau_{rr}\,\frac{\partial\, {}^t r}{\partial\, {}^0 r}\,\underline{\frac{\partial \delta u_r}{\partial\, {}^0 r}}\, d\,{}^0 V$$

$$+ \int_{{}^0 V}\frac{{}^0\rho}{{}^t\rho}\left(\frac{{}^0 r}{{}^t r}\right)^2 {}^t\tau_{\theta\theta}\,\frac{{}^t r}{{}^0 r^2}\,\delta u_r\, d\,{}^0 V = (\text{L.H.S.})$$

$$\left(\text{here} \quad \frac{\partial \delta u_r}{\partial\, {}^0 r} = \frac{\partial \delta u_r}{\partial\, {}^t r}\,\frac{\partial\, {}^t r}{\partial\, {}^0 r}\right)$$

6.31 Node 2 is restrained and node 1 has displacements $^{t}u_1^1$, u_1^1 and coordinates $^{t}x_1^1$, $^{0}x_1^1$ with $u_1^1 = u$.

$$^{t}u_1 = \frac{^{0}x_1}{^{0}L}\,{}^{t}u_1^1 \ , \quad u_1 = \frac{^{0}x_1}{^{0}L}\,u \ , \quad {}^{t}x_1 = \frac{^{0}x_1}{^{0}L}\,{}^{t}x_1^1$$

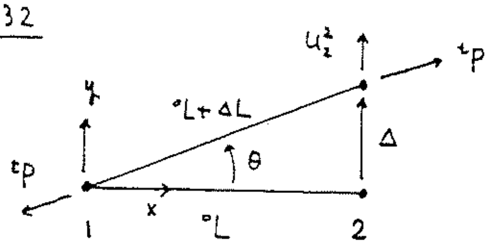
$$_{0}e_{11} = \frac{1}{2}\left(\frac{\partial u_1}{\partial {}^{0}x_1} + \frac{\partial u_1}{\partial {}^{0}x_1} + \frac{\partial {}^{t}u_1}{\partial {}^{0}x_1}\frac{\partial u_1}{\partial {}^{0}x_1} + \frac{\partial u_1}{\partial {}^{0}x_1}\frac{\partial {}^{t}u_1}{\partial {}^{0}x_1}\right) = 0.7\,u$$

$$_{0}\eta_{11} = \frac{1}{2}\frac{\partial u_1}{\partial {}^{0}x_1}\frac{\partial u_1}{\partial {}^{0}x_1} = \frac{u^2}{8}$$

$$^{t}_{0}u_{1,1} = \frac{\partial {}^{t}u_1}{\partial {}^{0}x_1} = \frac{^{t}u_1^1}{^{0}L} = 0.4 \ , \quad {}_{0}u_{1,1} = \frac{\partial u_1}{\partial {}^{0}x_1} = \frac{u}{^{0}L} = \frac{u}{2}$$

$$^{t}_{0}x_{1,1} = \frac{\partial {}^{t}x_1}{\partial {}^{0}x_1} = \frac{^{t}x_1^1}{^{0}L} = 1.4$$

6.32



(a) Using the derivations given in Example 6.16 with ${}^tu_1^1 = {}^tu_2^1 = {}^tu_1^2 = 0$ we have

$${}^t_0K\, u_2^2 = {}^{t+\Delta t}R - {}^t_0F$$

where ${}^t_0K = {}^t_0K_L + {}^t_0K_{NL}$

$${}^t_0K_L = E\,{}^0A\,\frac{({}^0L+\Delta L)^2}{({}^0L)^3}\sin^2\theta\ , \quad {}^t_0K_{NL} = \frac{{}^tP}{{}^0L+\Delta L}$$

$${}^t_0F = {}^tP\sin\theta .$$

Since $\sin\theta = \dfrac{\Delta}{{}^0L+\Delta L}$, ${}^t_0K_L = E\,{}^0A\,\dfrac{({}^0L+\Delta L)^2}{({}^0L)^3}\left(\dfrac{\Delta}{{}^0L+\Delta L}\right)^2 = \dfrac{E\,{}^0A}{{}^0L}\left(\dfrac{\Delta}{{}^0L}\right)^2$

And $${}^tP = {}^t_0S_{11}\,{}^0A\,\frac{{}^0L+\Delta L}{{}^0L} = E\,{}^0A\left[\frac{\Delta L}{{}^0L} + \frac{1}{2}\left(\frac{\Delta L}{{}^0L}\right)^2\right]\frac{{}^0L+\Delta L}{{}^0L}$$

$$\Delta L = \sqrt{{}^0L^2+\Delta^2} - {}^0L\ , \quad \frac{\Delta L}{{}^0L} = \sqrt{1+\left(\frac{\Delta}{{}^0L}\right)^2} - 1$$

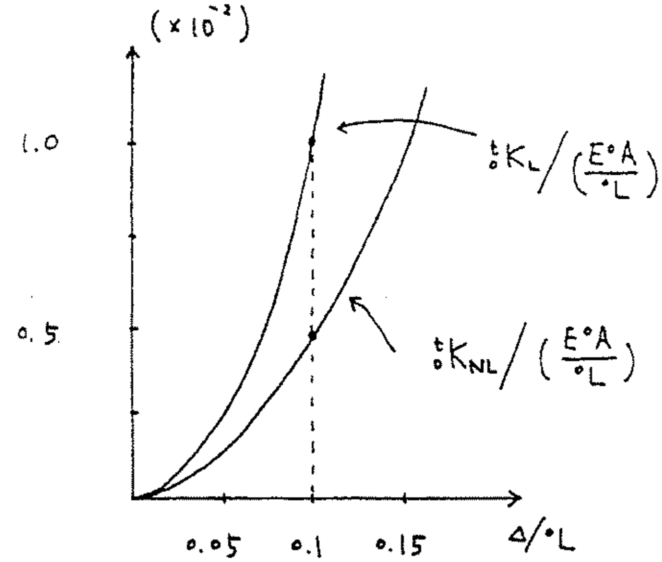
$$\frac{\Delta L}{{}^0L} + \frac{1}{2}\left(\frac{\Delta L}{{}^0L}\right)^2 = \frac{1}{2}\left(\frac{\Delta}{{}^0L}\right)^2\ , \quad {}^tP = E\,{}^0A\left[\frac{1}{2}\left(\frac{\Delta}{{}^0L}\right)^2\right]\frac{{}^0L+\Delta L}{{}^0L}$$

$$\therefore\ {}^t_0K_{NL} = \frac{{}^tP}{{}^0L+\Delta L} = \frac{E\,{}^0A}{{}^0L}\left[\frac{1}{2}\left(\frac{\Delta}{{}^0L}\right)^2\right]$$

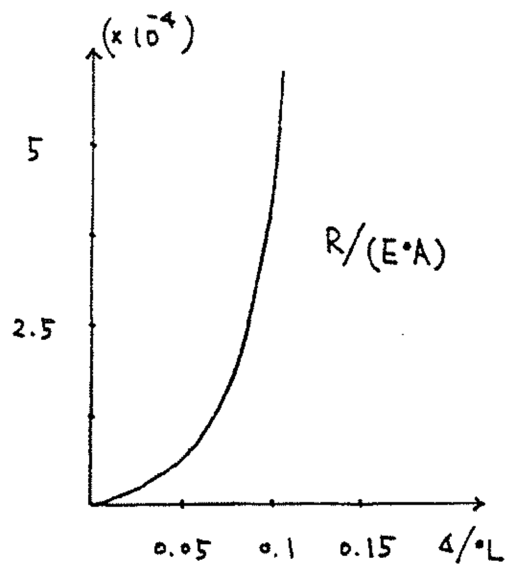
(b) $${}^tR = {}^tP\sin\theta = E\,{}^0A\left[\frac{1}{2}\left(\frac{\Delta}{{}^0L}\right)^2\right]\frac{{}^0L+\Delta L}{{}^0L}\cdot\frac{\Delta}{{}^0L+\Delta L} = E\,{}^0A\left[\frac{1}{2}\left(\frac{\Delta}{{}^0L}\right)^3\right]$$

6.32

a)

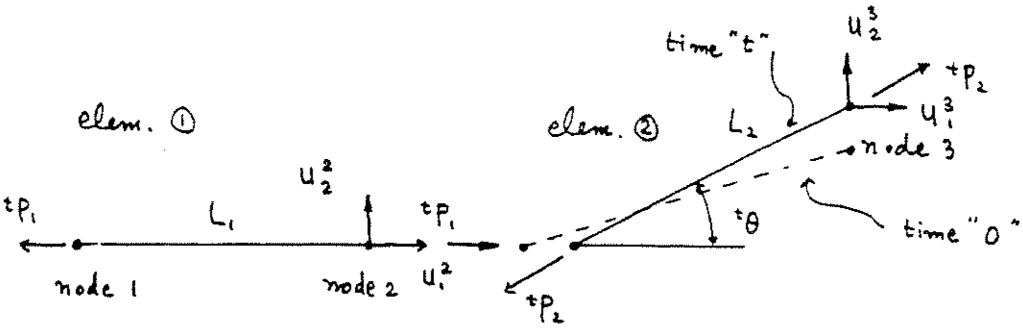


b)



We note that for the $\frac{\Delta}{^\circ L}$ values given, small strains are encountered and we can use $^\circ A = A$ (i.e., the given constant value).

6.33



(a) element ① :

$$^t\underline{K}_L = \frac{EA}{L_1}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad {}^t\underline{K}_{NL} = \frac{^tP_1}{L_1}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\qquad u_1^1 \quad u_2^1 \quad u_1^2 \quad u_2^2$$

$$^t\underline{F}^T = {}^tP_1 \begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}$$

element ② :

$$^t\underline{K}_L = \underline{T}^T \underline{\tilde{K}}_L \underline{T}, \quad {}^t\underline{K}_{NL} = \underline{T}^T \underline{\tilde{K}}_{NL} \underline{T}, \quad {}^t\underline{F} = \underline{T}^T\, {}^t\underline{\tilde{F}}$$

$$\text{where} \quad \underline{T} = \begin{bmatrix} \cos{}^t\theta & \sin{}^t\theta & 0 & 0 \\ -\sin{}^t\theta & \cos{}^t\theta & 0 & 0 \\ 0 & 0 & \cos{}^t\theta & \sin{}^t\theta \\ 0 & 0 & -\sin{}^t\theta & \cos{}^t\theta \end{bmatrix}$$

$$\underline{\tilde{K}}_L = \frac{EA}{L_2}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \underline{\tilde{K}}_{NL} = \frac{^tP_2}{L_2}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{matrix} u_1^2 \\ u_2^2 \\ u_1^3 \\ u_2^3 \end{matrix}$$

$$^t\underline{\tilde{F}}^T = {}^tP_2 \begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}$$

6.33

(b) Assembling the element matrices and applying the boundary conditions, we have

$$
{}^t\underline{K}_L = \begin{bmatrix} \frac{EA}{L_1} + \frac{EA}{L_2}\cos^2 {}^t\theta & -\sin^t\theta\cos^t\theta\cdot\frac{EA}{L_2} \\ -\sin^t\theta\cos^t\theta\cdot\frac{EA}{L_2} & \sin^2{}^t\theta\cdot\frac{EA}{L_2} \end{bmatrix} \begin{matrix} u_1^2 \\ u_2^3 \end{matrix}
$$

$$
{}^t\underline{K}_{NL} = \begin{bmatrix} \frac{{}^tP_1}{L_1} + \frac{{}^tP_2}{L_2} & 0 \\ 0 & \frac{{}^tP_2}{L_2} \end{bmatrix}, \quad {}^t\underline{F} = \begin{bmatrix} {}^tP_1 - {}^tP_2\cos^t\theta \\ {}^tP_2\sin^t\theta \end{bmatrix}
$$

Considering the force and moment equilibrium,

$$
{}^tP_2 = -P/\sin^t\theta, \quad {}^tP_1 = -P\cos^t\theta/\sin^t\theta
$$

Note that ${}^t\underline{R} = \begin{bmatrix} 0 \\ -P \end{bmatrix}$ and ${}^t\underline{R} = {}^t\underline{F}$

(c) ${}^tP_1 - {}^tP_2\cos^t\theta = 0, \quad {}^tP_2\sin^t\theta = -P$ —— ①

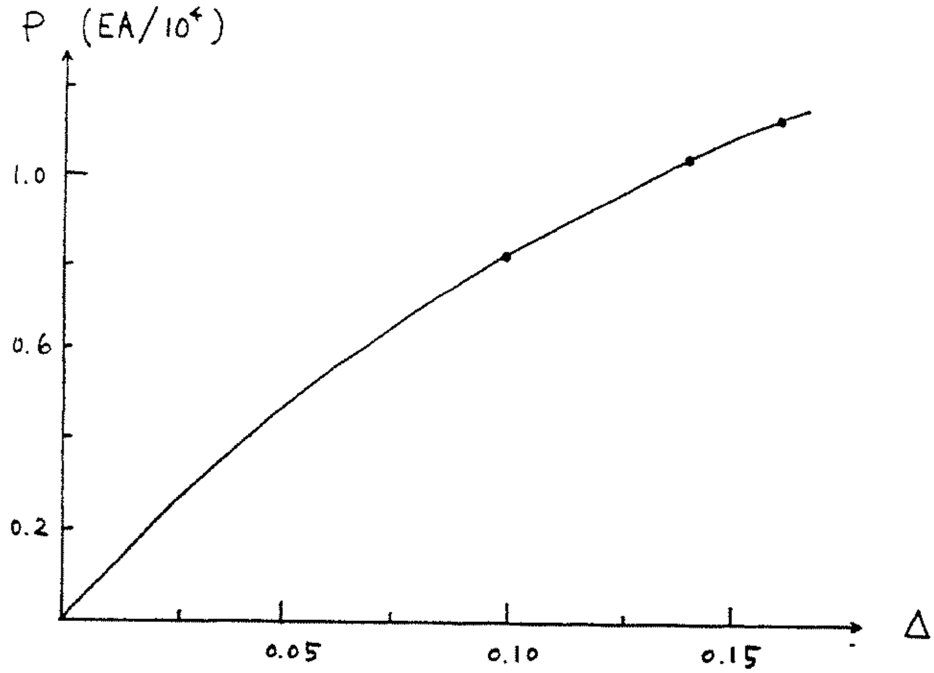
${}^tP_1 = \frac{EA}{L_1}\,{}^tu_1^2, \quad {}^tP_2 = \frac{EA}{L_2}\,\delta L_2$ —— ②

where $\delta L_2 = -{}^tu_1^2\cos^t\theta + {}^tu_2^3\sin^t\theta$ —— ③

From the geometry $\tan^t\theta = \frac{0.5 + {}^tu_2^3}{5 - {}^tu_1^2}, \quad {}^tu_2^3 = -\Delta$ —— ④

Eq.① are two equations with two unknowns ${}^tu_1^2$ and Δ. We use them by assuming a ${}^tu_1^2$, solving from the 1st equation for Δ, then substituting ${}^tu_1^2$ and Δ into the 2nd equation to obtain the corresponding P.

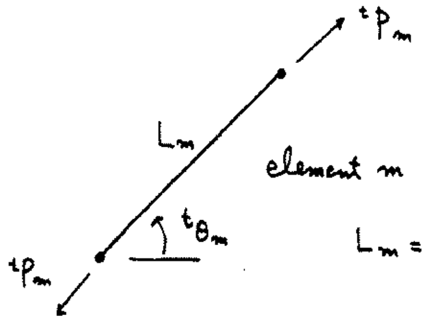
6.33



6.34 Using the result of Exercise 6.33 for ${}^t\tilde{\underline{K}}_L$, ${}^t\tilde{\underline{K}}_{NL}$ and ${}^t\tilde{\underline{F}}$,

$$
{}^t\tilde{\underline{K}}_L^{(m)} = C\,\frac{({}^0L_m + \Delta L_m)^2}{({}^0L_m)^3}A_0 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad {}^t\tilde{\underline{K}}_{NL}^{(m)} = \frac{{}^tP_m}{L_m}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}
$$

$$
{}^t\tilde{\underline{F}}^{(m)} = {}^tP_m \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}
$$



$$L_m = {}^0L_m + \Delta L_m$$

with

$$
\underline{T}^{(m)} = \begin{bmatrix} \cos{}^t\theta_m & \sin{}^t\theta_m & & \\ -\sin{}^t\theta_m & \cos{}^t\theta_m & \multicolumn{2}{c}{\underline{0}} \\ & & \cos{}^t\theta_m & \sin{}^t\theta_m \\ \multicolumn{2}{c}{\underline{0}} & -\sin{}^t\theta_m & \cos{}^t\theta_m \end{bmatrix}
$$

$$
\therefore \quad {}^t\underline{K}_L^{(m)} = \underline{T}^{(m)T}\,\tilde{\underline{K}}_L^{(m)}\,\underline{T}^{(m)} = C\,\frac{({}^0L_m + \Delta L_m)^2}{({}^0L_m)^3}A_0 \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cos^2{}^t\theta_m & \sin{}^t\theta_m\cos{}^t\theta_m \\ \cdot & \cdot & \sin{}^t\theta_m\cos{}^t\theta_m & \sin^2{}^t\theta_m \end{bmatrix}
$$

$$
{}^t\underline{K}_{NL}^{(m)} = \underline{T}^{(m)T}\,{}^t\tilde{\underline{K}}_{NL}^{(m)}\,\underline{T}^{(m)} = {}^t\tilde{\underline{K}}_{NL}^{(m)}
$$

$$
{}^t\underline{F}^{(m)} = {}^tP_m \begin{bmatrix} \vdots \\ \cos{}^t\theta_m \\ \sin{}^t\theta_m \end{bmatrix}
$$

6.34

$$\therefore \quad {}^{t}\underline{K} = {}^{t}\underline{K}_{L} + {}^{t}\underline{K}_{NL}$$

where
$$ {}^{t}\underline{K}_{L} = \sum_{m} {}^{t}\underline{K}_{L}^{(m)} = c\,\frac{({}^{0}L_m + \Delta L_m)^2}{({}^{0}L_m)^3} A_0 \begin{bmatrix} \cos^2{}^{t}\theta_m & \sin{}^{t}\theta_m \cos{}^{t}\theta_m \\ \sin{}^{t}\theta_m \cos{}^{t}\theta_m & \sin^2{}^{t}\theta_m \end{bmatrix}$$

$$ {}^{t}\underline{K}_{NL} = \sum_{m} {}^{t}\underline{K}_{NL}^{(m)} = \sum_{m=1}^{3} \frac{{}^{t}P_m}{L_m} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$ {}^{t}\underline{F} = \sum_{m} {}^{t}\underline{F}^{(m)} = \sum_{m=1}^{3} {}^{t}P_m \begin{bmatrix} \cos{}^{t}\theta_m \\ \sin{}^{t}\theta_m \end{bmatrix}$$

6.35 We consider the following

$$\delta\, {}^{t}W_{ext} = \int_{{}^{t}A} {}^{t}p_i\, \delta u_i\, d\,{}^{t}A \qquad \text{with} \quad {}^{t}p_i = {}^{t}f\, {}^{t}n_i \qquad \text{—— (1)}$$

where

${}^{t}f = f({}^{t}x_j)$: the load distribution depending on the coordinates ${}^{t}x_j$ of configuration at time t

${}^{t}n_i$: the component of the surface normal in configuration at time t.

The coordinates of the surface are

$${}^{t}x_i = {}^{t}x_i(r,s)\ , \quad {}^{t+\Delta t}x_i(r,s) = {}^{t}x_i(r,s) + u_i(r,s)$$

From eq. (1)

$$\delta\, {}^{t+\Delta t}W_{ext} = \int_{{}^{t+\Delta t}A} {}^{t+\Delta t}f\ {}^{t+\Delta t}n_i\ \delta u_i\ d\,{}^{t+\Delta t}A \qquad \text{—— (2)}$$

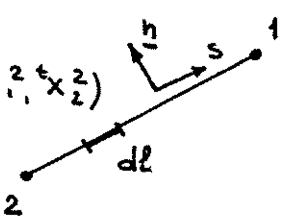
In general, according to vector analysis the product of the normal and surface element is

$${}^{t+\Delta t}n_i\, d\,{}^{t+\Delta t}A = e_{ijk}\, \frac{\partial\, {}^{t+\Delta t}x_j}{\partial r}\, \frac{\partial\, {}^{t+\Delta t}x_k}{\partial s}\, dr\, ds\ ,$$

where e_{ijk} is the permutation symbol.

From Taylor's expansion we can actually derive the tangent stiffness matrix from (2).

$({}^{t}x_1^2, {}^{t}x_2^2)$



$({}^{t}x_1^1, {}^{t}x_2^1)$ Let $d\,{}^{t}S_f = t\, dl$,

6.35

where t is the constant thickness for plane stress and plane strain cases. We have:

$$\delta\, {}^tW_{ext} = \int_{\ell} (-\,{}^tp)\; {}^t\underline{n} \cdot \delta\underline{u} \;\; t \; d\ell \qquad \text{—— (3)}.$$

Since $\quad {}^t\underline{n}\, d\ell = \left(-\dfrac{\partial\, {}^tx_2}{\partial s}\, ds,\; \dfrac{\partial\, {}^tx_1}{\partial s}\, ds\right)$, and let

${}^tp = p_0 + \dfrac{p}{h}\, {}^tx_1$, by substituting into (3)

$$\begin{cases} {}^tx_i = \dfrac{(1-s)}{2}\, {}^tx_i^2 + \dfrac{(1+s)}{2}\, {}^tx_i^1, \\ \delta u_i = \dfrac{(1-s)}{2}\, \delta u_i^2 + \dfrac{(1+s)}{2}\, \delta u_i^1, \end{cases}$$

we have: $\quad \delta\, {}^tW_{ext} = \displaystyle\int_{-1}^{1} \left(p_0 + \frac{p}{h}\, {}^tx_1\right) \left\{ \frac{({}^tx_2^1 - {}^tx_2^2)}{2} \times \right.$

$\times \left[\left(\dfrac{1-s}{2}\right) \delta u_1^2 + \left(\dfrac{1+s}{2}\right) \delta u_1^1 \right] - \dfrac{({}^tx_1^1 - {}^tx_1^2)}{2} \left[\left(\dfrac{1-s}{2}\right) \delta u_2^2 + \right.$

$\left. \left. + \dfrac{(1+s)}{2}\, \delta u_2^1 \right] \right\} ds$, from which we can explicitely get the incremental stiffness matrix. Note that when we move $\delta\, {}^tW_{ext}$ to the l.h.s., there is a sign change.

6.35

Using $\begin{cases} {}^tp^1 = p_0 + \dfrac{p}{h}\,{}^tx_1^1 \\ {}^tp^2 = p_0 + \dfrac{p}{h}\,{}^tx_1^2 \end{cases}$, we have

the incremental stiffness matrix $\underline{K}_L = \underline{K}_L^{I} + \underline{K}_L^{II} + \underline{K}_L^{III}$,

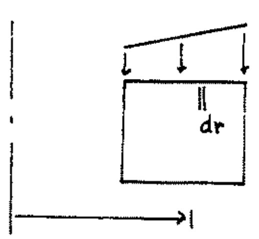
where

$$\underline{K}_L^{III} = -\begin{bmatrix} 0 & 0 & {}^tp^1/2 & 0 \\ 0 & 0 & 0 & -{}^tp^2/2 \\ -{}^tp^1/2 & 0 & 0 & 0 \\ 0 & {}^tp^2/2 & 0 & 0 \end{bmatrix}$$

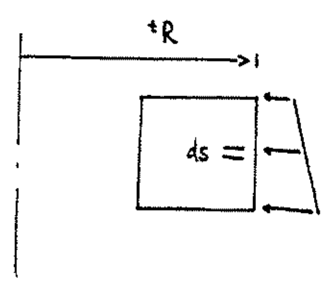
$$\underline{K}_L^{I} = -\begin{bmatrix} 0 & 0 & 0 & -1/2 \\ 0 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 0 \\ -1/2 & 0 & 0 & 0 \end{bmatrix} \left\{ p_0 + \frac{p}{2h}\left({}^tx_1^1 + {}^tx_1^2\right) \right\}$$

$$\underline{K}_L^{II} = -\frac{p}{3h}\begin{bmatrix} -({}^tx_2^2 - {}^tx_2^1) & -\dfrac{{}^tx_2^2 - {}^tx_2^1}{2} & \dfrac{{}^tx_1^2 - {}^tx_1^1}{2} & \dfrac{{}^tx_1^2 - {}^tx_1^1}{4} \\ -\dfrac{x_2^2 - {}^tx_2^1}{2} & -({}^tx_2^2 - {}^tx_2^1) & \dfrac{{}^tx_1^2 - {}^tx_1^1}{4} & \dfrac{{}^tx_1^2 - {}^tx_1^1}{2} \\ \dfrac{{}^tx_1^2 - {}^tx_1^1}{2} & \dfrac{{}^tx_1^2 - {}^tx_1^1}{4} & 0 & 0 \\ \dfrac{{}^tx_1^2 - {}^tx_1^1}{4} & \dfrac{{}^tx_1^2 - {}^tx_1^1}{2} & 0 & 0 \end{bmatrix}$$

For axisymmetric analysis the same procedure can be applied but small modifications are needed as shown below:



$d\,{}^tA = {}^tr\,dr$



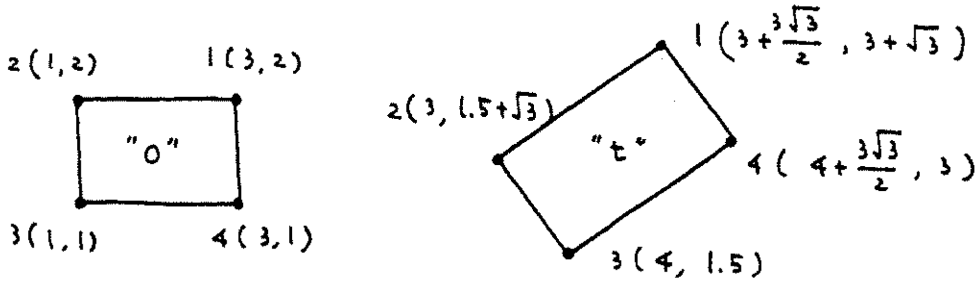
$d\,{}^tA = {}^tR\,ds$

6.36 (a) $\;{}^t_0\underline{X} = {}^t_0\underline{R}\,{}^t_0\underline{U} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{3\sqrt{3}}{4} & -1 \\ \frac{3}{4} & \sqrt{3} \end{bmatrix}, \quad \det {}^t_0\underline{X} = 3$

$${}^t_0\underline{\epsilon} = \frac{1}{2}\left({}^t_0\underline{X}^T\,{}^t_0\underline{X} - \underline{I} \right) = \begin{bmatrix} \frac{5}{8} & 0 \\ 0 & \frac{3}{2} \end{bmatrix}, \quad {}^t_0\underline{S} = {}^t_0\underline{C}\,{}^t_0\underline{\epsilon} = \begin{bmatrix} 34134.6 & 0 \\ 0 & 47596.2 \end{bmatrix}$$

$${}^t_0\hat{\underline{S}}^T = \left[{}^t_0S_{11} \quad {}^t_0S_{22} \quad {}^t_0S_{12} \right] = \left[34134.6 \quad 47596.2 \quad 0 \right]$$

We are next using the finite element formulation. For a physical procedure see the note below.



$${}^tu_i^k = {}^tx_i^k - {}^0x_i^k$$

$$\begin{cases} {}^tu_1^1 = \left(3 + \frac{3\sqrt{3}}{2}\right) - 3 = \frac{3\sqrt{3}}{2}, & \text{Similarly } {}^tu_2^1 = 1 + \sqrt{3} \\ {}^tu_1^2 = 2, \quad {}^tu_2^2 = -0.5 + \sqrt{3}, & {}^tu_1^3 = 3, \quad {}^tu_2^3 = 0.5 \\ {}^tu_1^4 = 1 + \frac{3\sqrt{3}}{2}, & {}^tu_2^4 = 2 \end{cases}$$

$${}^0\underline{J} = \begin{bmatrix} 1 & \\ & \frac{1}{2} \end{bmatrix}$$

$$\begin{cases} l_{11} = \sum {}_0h_{k,1}\,{}^tu_1^k = -1 + \frac{3\sqrt{3}}{4}, & l_{22} = \sum {}_0h_{k,2}\,{}^tu_2^k = -1 + \sqrt{3} \\ l_{21} = \sum {}_0h_{k,1}\,{}^tu_2^k = \frac{3}{4}, & l_{12} = \sum {}_0h_{k,2}\,{}^tu_1^k = -1 \end{cases}$$

6.36

$$
{}^{t}_{0}\underset{\sim}{B}_{L0} = \left[\begin{array}{cc:cc:cc:cc}
\frac{1+s}{4} & 0 & -\frac{1+s}{4} & 0 & -\frac{1-s}{4} & 0 & \frac{1-s}{4} & 0 \\
0 & \frac{1+r}{2} & 0 & \frac{1-r}{2} & 0 & -\frac{1-r}{2} & 0 & -\frac{1+r}{2} \\
\frac{1+r}{2} & \frac{1+s}{4} & \frac{1-r}{2} & -\frac{1+s}{4} & -\frac{1-r}{2} & -\frac{1-s}{4} & -\frac{1+r}{2} & \frac{1-s}{4}
\end{array}\right]
$$

$$
{}^{t}_{0}\underset{\sim}{B}_{L1} = \left[\begin{array}{cc:}
\left(-1+\frac{3\sqrt{3}}{4}\right)\frac{1+s}{4} & \frac{3}{4}\,\frac{1+s}{4} \\
-\frac{1+r}{2} & \left(-1+\sqrt{3}\right)\frac{1+r}{2} \\
\left(-1+\frac{3\sqrt{3}}{4}\right)\frac{1+r}{2}-\frac{1+s}{4} & \frac{3}{4}\,\frac{1+r}{2}+\left(-1+\sqrt{3}\right)\frac{1+s}{4}
\end{array}\right.
$$

$$
\begin{array}{cc:}
\left(-1+\frac{3\sqrt{3}}{4}\right)\left(-\frac{1+s}{4}\right) & \frac{3}{4}\left(-\frac{1+s}{4}\right) \\
-\frac{1-r}{2} & \left(-1+\sqrt{3}\right)\frac{1-r}{2} \\
\left(-1+\frac{3\sqrt{3}}{4}\right)\frac{1-r}{2}-\left(-\frac{1+s}{4}\right) & \frac{3}{4}\left(\frac{1-r}{2}\right)+\left(-1+\sqrt{3}\right)\left(-\frac{1+s}{4}\right)
\end{array}
$$

$$
\begin{array}{cc:}
\left(-1+\frac{3\sqrt{3}}{4}\right)\left(-\frac{1-s}{4}\right) & \frac{3}{4}\left(-\frac{1-s}{4}\right) \\
-\left(-\frac{1-r}{2}\right) & \left(-1+\sqrt{3}\right)\left(-\frac{1-r}{2}\right) \\
\left(-1+\frac{3\sqrt{3}}{4}\right)\left(-\frac{1-r}{2}\right)-\left(-\frac{1-s}{4}\right) & \frac{3}{4}\left(-\frac{1-r}{2}\right)+\left(-1+\sqrt{3}\right)\left(-\frac{1-s}{4}\right)
\end{array}
$$

$$
\left.\begin{array}{cc}
\left(-1+\frac{3\sqrt{3}}{4}\right)\left(\frac{1-s}{4}\right) & \frac{3}{4}\left(\frac{1-s}{4}\right) \\
-\left(-\frac{1+r}{2}\right) & \left(-1+\sqrt{3}\right)\left(-\frac{1+r}{2}\right) \\
\left(-1+\frac{3\sqrt{3}}{4}\right)\left(-\frac{1+r}{2}\right)-\frac{1-s}{4} & \frac{3}{4}\left(-\frac{1+r}{2}\right)+\left(-1+\sqrt{3}\right)\left(\frac{1-s}{4}\right)
\end{array}\right]
$$

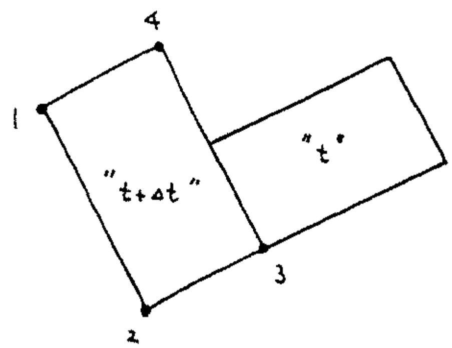
$$
{}^{t}_{0}\underset{\sim}{B}_{L} = {}^{t}_{0}\underset{\sim}{B}_{L0} + {}^{t}_{0}\underset{\sim}{B}_{L1}
$$

$$
{}^{t}_{0}\underset{\sim}{F} = \int_{-1}^{1}\int_{-1}^{1} {}^{t}_{0}\underset{\sim}{B}_{L}^{T}\, {}^{t}_{0}\hat{\underset{\sim}{S}} \left(\frac{1}{2}\right) dr\, ds
$$

$$
\therefore\ {}^{t}_{0}\underset{\sim}{F}^{T} = \left[\begin{array}{cc:cc:cc:cc}
-25425.1 & 95239.5 & -69767.3 & 69638.6 & 25425.1 & -95239.5 & 69767.3 & -69638.6
\end{array}\right]
$$

6.36

(b) Since we are dealing with a rigid body motion, it does not matter about what point we rotate the element



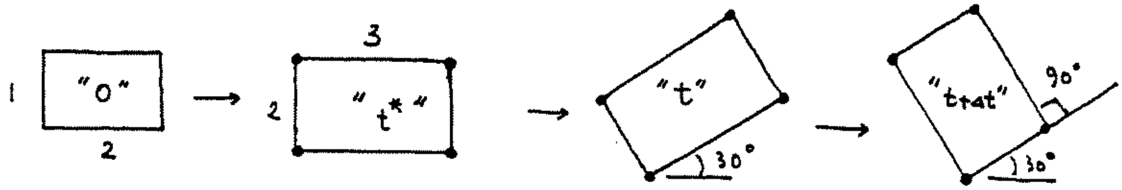
$1\left(2.5-\sqrt{3},\ 0.5+\frac{3\sqrt{3}}{2}\right)$ $\quad 2\left(4-\sqrt{3},\ 0.5\right)$

$3\left(4,\ 1.5\right)$ $\quad 4\left(2.5,\ 1.5+\frac{3\sqrt{3}}{2}\right)$

Similarly as in part (a) with ${}^{t+\Delta t}_{0}\hat{\underline{S}} = {}^{t}_{0}\hat{\underline{S}}$, we obtain

$$ {}^{t+\Delta t}\underline{F}^T = \left[\begin{array}{cc:cc:} -95239.5 & -25425.1 & -69638.6 & -69767.3 \end{array} \right. $$
$$ \left. \begin{array}{cc:cc} 95239.5 & 25425.1 & 69638.6 & 69767.3 \end{array} \right] $$

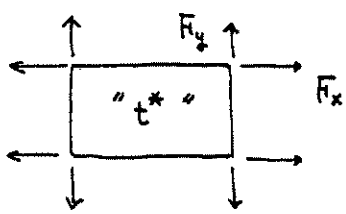
Note



At time t^*, ${}^{t^*}_{0}\underline{S} = {}^{t}_{0}\underline{S}$, ${}^{t^*}_{0}\underline{X} = {}^{t}_{0}\underline{U}$, $\det {}^{t^*}_{0}\underline{X} = 3$

$$ {}^{t^*}\underline{\tau} = \frac{{}^{t^*}\rho}{{}^{0}\rho}\, {}^{t^*}_{0}\underline{X}\, {}^{t^*}_{0}\underline{S}\, {}^{t^*}_{0}\underline{X}^T = \frac{1}{\det {}^{t^*}_{0}X}\, {}^{t}_{0}\underline{U}\, {}^{t}_{0}\underline{S}\, {}^{t}_{0}\underline{U}^T = \begin{bmatrix} 25601.0 & 0 \\ 0 & 63461.6 \end{bmatrix} $$

Hence the nodal point forces are



$$ F_x = \frac{(25601.0)(2)}{(2)} = 25601.0 $$

$$ F_y = \frac{(63461.6)(3)}{(2)} = 95192.4 $$

6.36

Now at time t,

$$\begin{bmatrix} {}^{t}F_x \\ {}^{t}F_y \end{bmatrix}_{node\ 1} = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix} \begin{bmatrix} {}^{t*}F_x \\ {}^{t*}F_y \end{bmatrix}_{node\ 1} = \begin{bmatrix} -25425.1 \\ 95239.5 \end{bmatrix}$$

Similarly, $\begin{bmatrix} {}^{t}F_x \\ {}^{t}F_y \end{bmatrix}_{node\ 2} = \begin{bmatrix} -69767.3 \\ 69638.6 \end{bmatrix}$

$$\begin{bmatrix} {}^{t}F_x \\ {}^{t}F_y \end{bmatrix}_{node\ 3} = \begin{bmatrix} 25425.1 \\ -95239.5 \end{bmatrix}, \quad \begin{bmatrix} {}^{t}F_x \\ {}^{t}F_y \end{bmatrix}_{node\ 4} = \begin{bmatrix} 69767.3 \\ -69638.6 \end{bmatrix}$$

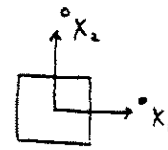
At time $t+\Delta t$, applying the rotation of $\theta = 90^\circ$, we have

$$\begin{bmatrix} {}^{t+\Delta t}F_x \\ {}^{t+\Delta t}F_y \end{bmatrix}_{node\ 1} = \begin{bmatrix} -95239.5 \\ -25425.1 \end{bmatrix}, \quad \begin{bmatrix} \quad \end{bmatrix}_{node\ 2} = \begin{bmatrix} -69638.6 \\ -69767.3 \end{bmatrix},$$

$$\begin{bmatrix} \quad \end{bmatrix}_{node\ 3} = \begin{bmatrix} 95239.5 \\ 25425.1 \end{bmatrix}, \quad \begin{bmatrix} \quad \end{bmatrix}_{node\ 4} = \begin{bmatrix} 69638.6 \\ 69767.3 \end{bmatrix}.$$

We find that the same results are obtained.

6.37



$$\begin{cases} {}^tx_1 = {}^0x_1 + (0.1)\,\dfrac{1}{0.2}(0.1 + {}^0x_2) = {}^0x_1 + \dfrac{1}{2}\,{}^0x_2 + \dfrac{1}{20} \\ {}^tx_2 = {}^0x_2 + (0.1)\,\dfrac{1}{0.2}(0.1 + {}^0x_2) = \dfrac{3}{2}\,{}^0x_2 + \dfrac{1}{20} \end{cases}$$

$$\therefore\ {}^t_0\underline{X} = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & \frac{3}{2} \end{bmatrix}, \quad \det {}^t_0\underline{X} = \frac{3}{2}, \quad {}^0_t\underline{X} = \begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & \frac{2}{3} \end{bmatrix}$$

$${}^t_0\underline{S} = \frac{{}^0\rho}{{}^t\rho}\,{}^0_t\underline{X}\,{}^t\underline{\tau}\,{}^0_t\underline{X}^T = \begin{bmatrix} 4.326\times 10^7 & 1.924\times 10^7 \\ 1.924\times 10^7 & 1.009\times 10^8 \end{bmatrix}$$

$${}^tu_1^1 = 0.1,\ {}^tu_2^1 = 0.1,\ {}^tu_1^2 = 0.1,\ {}^tu_2^2 = 0.1,\ (\text{others}) = 0$$

$$\ell_{11} = 10\left[\frac{1+s}{4}(0.1) - \frac{1+s}{4}(0.1)\right] = 0$$

$$\ell_{12} = 10\left[\frac{1+r}{4}(0.1) + \frac{1-r}{4}(0.1)\right] = \frac{1}{2}$$

$${}^t_0\underline{B}_{L0} = \frac{5}{2}\begin{bmatrix} 1+s & \\ 0 & \cdots \\ 1+r & \end{bmatrix}, \quad {}^t_0\underline{B}_{L1} = \frac{5}{4}\begin{bmatrix} 0 & \\ 1+r & \cdots \\ 1+s & \end{bmatrix}$$

$$\therefore\ {}^t_0\underline{B}_L = \begin{bmatrix} \frac{5}{2}(1+s) & \\ \frac{5}{4}(1+r) & \cdots \\ \frac{5}{4}\{2(1+r) + (1+s)\} & \end{bmatrix}$$

$${}^t_0\underline{B}_{NL} = \begin{bmatrix} \frac{5}{2}(1+s) & \\ \frac{5}{2}(1+r) & \cdots \\ 0 & \\ 0 & \end{bmatrix}$$

6.37

$$ {}^t_0\tilde{S} = \begin{bmatrix} {}^t_0S_{11} & {}^t_0S_{12} & \multicolumn{2}{c}{\underline{0}} \\ {}^t_0S_{12} & {}^t_0S_{22} & & \\ \multicolumn{2}{c}{\underline{0}} & {}^t_0S_{11} & {}^t_0S_{12} \\ & & {}^t_0S_{12} & {}^t_0S_{22} \end{bmatrix}, \quad {}_0\underline{C} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix} $$

$$ \Rightarrow \quad {}^t_0K_{L11} = \int_{-1}^{1}\int_{-1}^{1} \left[\tfrac{5}{2}(1+s) \quad \tfrac{5}{4}(1+r) \quad \tfrac{5}{4}(3+2r+s) \right] {}_0\underline{C} \begin{bmatrix} \frac{5}{2}(1+s) \\ \frac{5}{4}(1+r) \\ \frac{5}{4}(3+2r+s) \end{bmatrix} \left(\frac{1}{10}\right)^2 dr\,ds $$

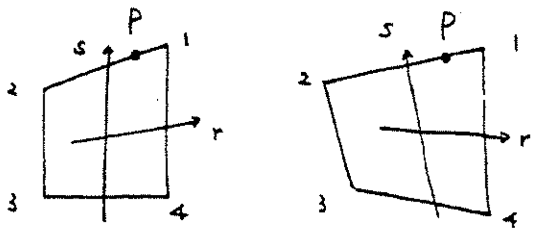
$$ = 0.9615\,E $$

$$ {}^t_0K_{NL11} = \int_{-1}^{1}\int_{-1}^{1} \left[\tfrac{5}{2}(1+s) \quad \tfrac{5}{2}(1+r) \quad 0 \quad 0 \right] {}^t_0\tilde{S} \begin{bmatrix} \frac{5}{2}(1+s) \\ \frac{5}{2}(1+r) \\ 0 \\ 0 \end{bmatrix} \left(\frac{1}{10}\right)^2 dr\,ds $$

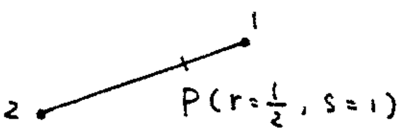
$$ = 5.7686 \times 10^7 $$

$$ \therefore \quad {}^t_0K_{11} = 0.9615\,E + 5.7686 \times 10^7 $$

6.38



time "0" time "t"



$${}^{0}x_1 = \frac{1}{2}(17 + 7r + s + rs)$$

$${}^{t}x_1 = \frac{1}{4}(53 + 23r - 5s + rs)$$

$${}^{t}u_1^1 = 5 \quad {}^{t}u_1^2 = 1 \quad {}^{t}u_1^3 = 4 \quad {}^{t}u_1^4 = 9$$

$${}^{t}u_1\big|_p = \Sigma h_k\, {}^{t}u_1^k\big|_p = 4$$

$$l_{33}\big|_p = \left(\frac{{}^{t}u_1}{{}^{0}x_1}\right)\Big|_p = \frac{4}{11}$$

$${}^{t}x_1\big|_p = 15 \quad , \quad {}^{0}x_1\big|_p = 11$$

$$h_1\big|_p = \frac{3}{4} \ , \ h_2\big|_p = \frac{1}{4} \ , \ h_3\big|_p = h_4\big|_p = 0$$

T.L. $\quad {}^{t}_{0}\underline{\tilde{B}}_L\big|_p = \left({}^{t}_{0}\underline{\tilde{B}}_{L0} + {}^{t}_{0}\underline{\tilde{B}}_{L1}\right)\Big|_p$

$${}^{t}_{0}\underline{\tilde{B}}_{L0}\big|_p = \left[\frac{h_1}{{}^{0}x_1} \quad 0 \quad \frac{h_2}{{}^{0}x_1} \quad 0 \quad \frac{h_3}{{}^{0}x_1} \quad 0 \quad \frac{h_4}{{}^{0}x_1} \quad 0\right]\Big|_p$$

$$= \left[\frac{3}{44} \quad 0 \quad \frac{1}{44} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\right]$$

$${}^{t}_{0}\underline{\tilde{B}}_{L1}\big|_p = \left[l_{33}\frac{h_1}{{}^{0}x_1} \quad 0 \quad l_{33}\frac{h_2}{{}^{0}x_1} \quad 0 \quad \cdots \quad l_{33}\frac{h_4}{{}^{0}x_1} \quad 0\right]\Big|_p$$

$$= \left[\frac{3}{121} \quad 0 \quad \frac{1}{121} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\right]$$

$$\therefore\ {}^{t}_{0}\underline{\tilde{B}}_L\big|_p = \left[\frac{45}{484} \quad 0 \quad \frac{15}{484} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\right]$$

$${}^{t}_{0}\underline{\tilde{B}}_{NL}\big|_p = \left[\frac{h_1}{{}^{0}x_1} \quad 0 \quad \frac{h_2}{{}^{0}x_1} \quad 0 \quad \frac{h_3}{{}^{0}x_1} \quad 0 \quad \frac{h_4}{{}^{0}x_1} \quad 0\right]\Big|_p$$

$$= \left[\frac{3}{44} \quad 0 \quad \frac{1}{44} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\right]$$

Here the curl denotes the last row in ${}^{t}_{0}\underline{B}_L$ and ${}^{t}_{0}\underline{B}_{NL}$.

6.38

U.L. $\quad {}^t_t\tilde{\underline{B}}_L\Big|_P = \left[\frac{h_1}{{}^tX_1} \quad 0 \quad \frac{h_2}{{}^tX_1} \quad 0 \quad \frac{h_3}{{}^tX_1} \quad 0 \quad \frac{h_4}{{}^tX_1} \quad 0 \right]\Big|_P$

$$= \left[\frac{1}{20} \quad 0 \quad \frac{1}{60} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \right]$$

$$= {}^t_t\tilde{\underline{B}}_{NL}\Big|_P$$

6.39 (a) $u_1 = \frac{1}{4}(1+\frac{{}^0x_1}{3})(1+\frac{{}^0x_2}{2})u_1^1 , \quad u_2 = 0$

$${}^tu_1 = \frac{1}{2}(1+\frac{{}^0x_1}{3})(1.5) , \quad {}^tu_2 = \frac{1}{2}(1+\frac{{}^0x_2}{2})(0.5)$$

$$\left(\begin{array}{l} {}_0e_{11} = {}_0u_{1,1} + {}^t_0u_{k,1}\,{}_0u_{k,1} = \frac{5}{48}(1+\frac{{}^0x_2}{2})u_1^1 \\ {}_0e_{12} = \frac{1}{2}({}_0u_{1,2} + {}_0u_{2,1} + {}^t_0u_{k,1}\,{}_0u_{k,2} + {}_0u_{k,1}\,{}^t_0u_{k,2}) = \frac{5}{64}(1+\frac{{}^0x_1}{3})u_1^1 \\ {}_0e_{22} = {}_0u_{2,2} + {}^t_0u_{k,2}\,{}_0u_{k,2} = 0 \end{array}\right.$$

$$\left(\begin{array}{l} {}_0\eta_{11} = \frac{1}{2}\,{}_0u_{k,1}\,{}_0u_{k,1} = \frac{1}{2}\left\{\frac{1}{12}(1+\frac{{}^0x_2}{2})u_1^1\right\}^2 \\ {}_0\eta_{12} = \frac{1}{2}\,{}_0u_{k,1}\,{}_0u_{k,2} = \frac{1}{2}\left[\frac{1}{12}(1+\frac{{}^0x_2}{2})u_1^1\right]\left[\frac{1}{8}(1+\frac{{}^0x_1}{3})u_1^1\right] \\ {}_0\eta_{22} = \frac{1}{2}\,{}_0u_{k,2}\,{}_0u_{k,2} = \frac{1}{2}\left\{\frac{1}{8}(1+\frac{{}^0x_1}{3})u_1^1\right\}^2 \end{array}\right.$$

$${}^t_0\underline{S} = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 60 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \quad {}_0\underline{C} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

$$({}^t_0K_L)_{11} = \int_{-3}^{3}\int_{-2}^{2}\left[\frac{5}{48}(1+\frac{{}^0x_2}{2}) \quad 0 \quad 2\cdot\frac{5}{64}(1+\frac{{}^0x_1}{3})\right]{}_0\underline{C}\begin{bmatrix} \frac{5}{48}(1+\frac{{}^0x_2}{2}) \\ 0 \\ 2\cdot\frac{5}{64}(1+\frac{{}^0x_1}{3}) \end{bmatrix} h\,d{}^0x_1 d{}^0x_2$$

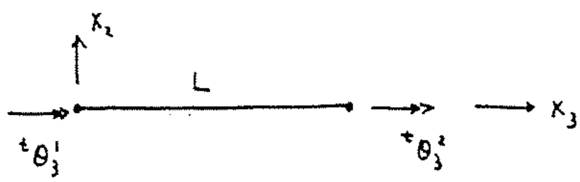
$$= 0.682\,Eh$$

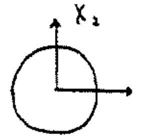
$$({}^t_0K_{NL})_{11} = \int_{-3}^{3}\int_{-2}^{2}\left[\left\{\frac{1}{12}(1+\frac{{}^0x_2}{2})\right\}^2 {}^t_0S_{11} + \left\{\frac{1}{8}(1+\frac{{}^0x_1}{3})\right\}^2 {}^t_0S_{22}\right] h\,d{}^0x_1 d{}^0x_2$$

$$= \frac{470}{9}h = 52.2\,h$$

(b) $$({}^t_0\hat{F})_1 = \int_{-3}^{3}\int_{-2}^{2}\left[\frac{5}{48}(1+\frac{{}^0x_2}{2}) \quad 0 \quad 2\cdot\frac{5}{64}(1+\frac{{}^0x_1}{3})\right]\begin{bmatrix} 100 \\ 60 \\ 0 \end{bmatrix} h\,d{}^0x_1 d{}^0x_2 = 250h$$

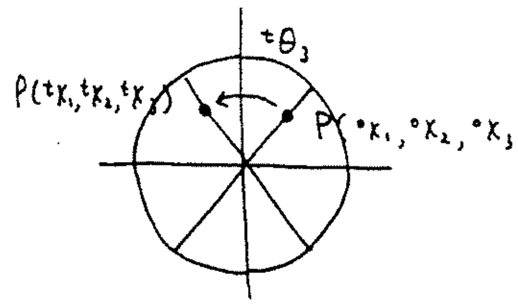
6.40





(a) $${}^t\theta_3 = \left(1 - \frac{{}^0x_3}{L}\right){}^t\theta_3^1 + \left(\frac{{}^0x_3}{L}\right){}^t\theta_3^2$$

$$\theta_3 = \left(1 - \frac{{}^0x_3}{L}\right)\theta_3^1 + \left(\frac{{}^0x_3}{L}\right)\theta_3^2$$



$$\begin{bmatrix} {}^tx_1 \\ {}^tx_2 \\ {}^tx_3 \end{bmatrix} = \begin{bmatrix} \cos {}^t\theta_3 & -\sin {}^t\theta_3 & 0 \\ \sin {}^t\theta_3 & \cos {}^t\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^0x_1 \\ {}^0x_2 \\ {}^0x_3 \end{bmatrix}$$

$$\therefore\ {}_0^t\underline{X} = \left[\frac{\partial\, {}^tx_i}{\partial\, {}^0x_j}\right] = \begin{bmatrix} \cos {}^t\theta_3 & -\sin {}^t\theta_3 & -\frac{{}^t\theta_3^2 - {}^t\theta_3^1}{L}\left({}^0x_1 \sin {}^t\theta_3 + {}^0x_2 \cos {}^t\theta_3\right) \\ \sin {}^t\theta_3 & \cos {}^t\theta_3 & \frac{{}^t\theta_3^2 - {}^t\theta_3^1}{L}\left({}^0x_1 \cos {}^t\theta_3 - {}^0x_2 \sin {}^t\theta_3\right) \\ 0 & 0 & 1 \end{bmatrix}$$

(b) $${}_0^t\underline{\epsilon} = \frac{1}{2}\left({}_0^t\underline{X}^T\, {}_0^t\underline{X} - \underline{I}\right) = \frac{1}{2}\begin{bmatrix} 0 & 0 & -\Delta\, {}^0x_2 \\ 0 & 0 & \Delta\, {}^0x_1 \\ -\Delta\, {}^0x_2 & \Delta\, {}^0x_1 & \Delta^2\left({}^0x_1^2 + {}^0x_2^2\right) \end{bmatrix}$$

where $\Delta = \left({}^t\theta_3^2 - {}^t\theta_3^1\right)/L$

Hence the term which is associated with large strain effects is $${}_0^t\epsilon_{33} = \frac{1}{2}\left({}^0x_1^2 + {}^0x_2^2\right)\left(\frac{{}^t\theta_3^2 - {}^t\theta_3^1}{L}\right)^2 .$$

(c) $$\frac{{}^t\rho}{{}^0\rho} = \frac{1}{\det {}_0^t\underline{X}} = \frac{1}{1} = 1$$

6.40

d) Let ${}^t_0\hat{\underline{\varepsilon}}^T = [\ {}^t_0\varepsilon_{33} \quad 2\,{}^t_0\varepsilon_{31} \quad 2\,{}^t_0\varepsilon_{32}\ ]$, and

$\hat{\underline{u}}^T = [\ {}^t\theta_3^1 \quad {}^t\theta_3^2\ ]$.

Then, from part (b) we have

$${}^t_0\hat{\varepsilon}_{33} = \frac{1}{2}\left({}^0x_1^2 + {}^0x_2^2\right)\left(\frac{{}^t\theta_3^2 - {}^t\theta_3^1}{L}\right)^2,$$

$${}^t_0\hat{\varepsilon}_{31} = -{}^0x_2\left(\frac{{}^t\theta_3^2 - {}^t\theta_3^1}{L}\right),$$

$${}^t_0\hat{\varepsilon}_{32} = {}^0x_1\left(\frac{{}^t\theta_3^2 - {}^t\theta_3^1}{L}\right),$$ and using the linearization process stated in Section 6.3.1 (eqn's 6.90-6.92), we obtain

$$\delta\,{}^t_0\hat{\varepsilon}_{33} = \frac{\left({}^0x_1^2 + {}^0x_2^2\right)}{L^2}\left({}^t\theta_3^2 - {}^t\theta_3^1\right)\left(\delta\,{}^t\theta_3^2 - \delta\theta_3^1\right),$$

$$\delta\,{}^t_0\hat{\varepsilon}_{31} = -\frac{{}^0x_2}{L}\left(\delta\,{}^t\theta_3^2 - \delta\,{}^t\theta_3^1\right),$$

$$\delta\,{}^t_0\hat{\varepsilon}_{32} = \frac{{}^0x_1}{L}\left(\delta\,{}^t\theta_3^2 - \delta\,{}^t\theta_3^1\right).$$

Using these relations, we obtain the following matrices:

$${}^t_0\underline{B}_L = \begin{bmatrix} -\dfrac{{}^0x_1^2 + {}^0x_2^2}{L^2}\left({}^t_0\theta_3^2 - {}^t_0\theta_3^1\right) & \dfrac{{}^0x_1^2 + {}^0x_2^2}{L^2}\left({}^t_0\theta_3^2 - {}^t_0\theta_3^1\right) \\ {}^0x_2/L & -{}^0x_2/L \\ -{}^0x_1/L & {}^0x_1/L \end{bmatrix},$$

$${}^t_0\underline{B}_{NL} = \begin{bmatrix} -\sqrt{{}^0x_1^2 + {}^0x_2^2}/L & \sqrt{{}^0x_1^2 + {}^0x_2^2}/L \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

6.41 $\quad {}_0^tW = {}_0^t\bar{W} - \frac{1}{2\kappa}\left({}^t\bar{p} - {}^t\tilde{p}\right)^2 \qquad \leftarrow (6.136)$

$${}^t\bar{\tau}_{ij} = \frac{1}{\det {}_0^t\underline{X}}\ {}_0^tX_{i,m}\ {}_0^tX_{j,n}\ {}_0^t\bar{S}_{mn}$$

$${}^t\bar{p} = -\frac{1}{3}\,{}^t\bar{\tau}_{ii} = -\frac{1}{3}\frac{1}{\det {}_0^t\underline{X}}\ {}_0^tC_{mn}\ {}_0^t\bar{S}_{mn} = -\frac{1}{3\det {}_0^t\underline{X}}\ {}_0^tC_{mn}\frac{\partial\, {}_0^t\bar{W}}{\partial\, {}_0^t\varepsilon_{mn}}$$

where ${}_0^tC_{mn} = {}_0^tX_{i,m}\,{}_0^tX_{i,n}$: right Cauchy-Green deformation tensor.

Let the operator ${}_0P = -\frac{1}{3\det {}_0^t\underline{X}}\ {}_0^tC_{mn}\frac{\partial}{\partial\, {}_0^t\varepsilon_{mn}}$ and we now show

that ${}_0P\left({}_0^t\bar{W} + {}_0^tQ\right) = {}^t\tilde{p}$.

$${}_0P({}_0^t\bar{W} + {}_0^tQ) = {}_0P({}_0^t\bar{W}) + {}_0P({}_0^tQ) \qquad ({}_0P : \text{linear operator})$$

$$= {}^t\bar{p} + {}_0P\left(-\frac{1}{2\kappa}\left({}^t\bar{p} - {}^t\tilde{p}\right)^2\right)$$

$$= {}^t\bar{p} - \frac{1}{\kappa}\left({}^t\bar{p} - {}^t\tilde{p}\right){}_0P({}^t\bar{p}) \qquad \text{——— (1)}$$

Now the operator ${}_0P$ can be written as

$${}^t\bar{p} = {}_0P({}_0^t\bar{W}) = -\frac{d({}_0^t\bar{W})}{d(\det {}_0^t\underline{X})} \qquad \text{——— (2)}$$

in which $d(\det {}_0^t\underline{X})$ is an independent infinitesimal quantity. The reason for (2) is as follows. Considering the increment in the Cauchy-Green deformation tensor we have

$$d\,{}_0^tC_{ij} = ({}_0^tC_{ij})(d\alpha) \quad \text{where} \quad d\alpha = \frac{2}{3\det {}_0^t\underline{X}}\, d(\det {}_0^t\underline{X})$$

6.41

This change in the Cauchy-Green deformation tensor causes a corresponding change in the potential

$$d\,{}_0^t\bar{W} = \frac{\partial\,{}_0^t\bar{W}}{\partial\,{}_0^tC_{ij}}\,d\,{}_0^tC_{ij} = \frac{1}{2}\,\frac{\partial\,{}_0^t\bar{W}}{\partial\,{}_0^t\varepsilon_{ij}}\,d\,{}_0^tC_{ij} \qquad \leftarrow \text{since } {}_0^t\varepsilon_{ij} = \frac{1}{2}({}_0^tC_{ij} - \delta_{ij})$$

$$= \frac{1}{2}\,\frac{\partial\,{}_0^t\bar{W}}{\partial\,{}_0^t\varepsilon_{ij}}\,{}_0^tC_{ij}\,d\alpha$$

$$= \frac{1}{3\det{}_0^t\underline{X}}\,{}_0^tC_{ij}\,\frac{\partial\,{}_0^t\bar{W}}{\partial\,{}_0^t\varepsilon_{ij}}\,d(\det{}_0^t\underline{X})$$

that is

$${}_0P({}_0^t\bar{W}) = -\frac{d({}_0^t\bar{W})}{d(\det{}_0^t\underline{X})}$$

In an isotropic material with a constant bulk modulus κ, we note that the bulk modulus is insensitive to changes in the volume ratio $\det{}_0^t\underline{X}$, that is,

$${}^t\bar{p} = -\kappa\,(\det{}_0^t\underline{X} - 1)$$

Hence $\quad {}_0P({}^t\bar{p}) = \kappa \qquad \text{——— ③}$

From ① and ③

$${}_0P({}_0^t\bar{W} + {}_0^tQ) = {}^t\bar{p} - ({}^t\bar{p} - {}^t\tilde{p}) = {}^t\tilde{p}\,.$$

6.42

$$\frac{\partial\, {}_0^t\varepsilon_{kl}}{\partial\, {}^tU_n^L} = \frac{1}{2}\left(\frac{\partial\, {}_0^tu_{k,l}}{\partial\, {}^tU_n^L} + \frac{\partial\, {}_0^tu_{l,k}}{\partial\, {}^tU_n^L} + {}_0^tu_{m,k}\frac{\partial\, {}_0^tu_{m,l}}{\partial\, {}^tU_n^L} + {}_0^tu_{m,l}\frac{\partial\, {}_0^tu_{m,k}}{\partial\, {}^tU_n^L} \right)$$

$$\frac{\partial\, {}_0^tu_{k,l}}{\partial\, {}^tU_n^L} = \frac{\partial\left({}_0h_{M,l}\, {}^tU_k^M\right)}{\partial\, {}^tU_n^L} = {}_0h_{M,l}\,\delta_{kn}\,\delta_{ML} = \delta_{kn}\, {}_0h_{L,l}$$

$$\left({}^tu_k = h_M\, {}^tU_k^M \;,\; {}_0^tu_{k,l} = {}_0h_{M,l}\, {}^tU_k^M \right)$$

$$\frac{\partial\, {}_0^tu_{l,k}}{\partial\, {}^tU_n^L} = \delta_{ln}\, {}_0h_{L,k}$$

$${}_0^tu_{m,k}\frac{\partial\, {}_0^tu_{m,l}}{\partial\, {}^tU_n^L} = \left({}_0h_{M,k}\, {}^tU_m^M\right)\left(\delta_{mn}\, {}_0h_{L,l}\right) = {}_0h_{M,k}\, {}_0h_{L,l}\, {}^tU_n^M$$

$${}_0^tu_{m,l}\frac{\partial\, {}_0^tu_{m,k}}{\partial\, {}^tU_n^L} = {}_0h_{M,l}\, {}_0h_{L,k}\, {}^tU_n^M$$

$$\therefore\quad \frac{\partial\, {}_0^t\varepsilon_{kl}}{\partial\, {}^tU_n^L} = \frac{1}{2}\left[{}_0h_{L,l}\left(\delta_{nk} + {}_0h_{M,k}\, {}^tU_n^M\right) + {}_0h_{L,k}\left(\delta_{nl} + {}_0h_{M,l}\, {}^tU_n^M\right) \right]$$

$$= \frac{1}{2}\left[{}_0h_{L,l}\left(\delta_{nk} + {}_0^tu_{n,k}\right) + {}_0h_{L,k}\left(\delta_{nl} + {}_0^tu_{n,l}\right) \right]$$

$$= \frac{1}{2}\left({}_0h_{L,l}\, {}_0^tX_{n,k} + {}_0h_{L,k}\, {}_0^tX_{n,l} \right) \qquad \longleftarrow \quad (6.144)$$

$$\frac{\partial^2\, {}_0^t\varepsilon_{kl}}{\partial\, {}^tU_n^L\, \partial\, {}^tU_m^M} = \frac{1}{2}\frac{\partial}{\partial\, {}^tU_m^M}\left\{ {}_0h_{L,l}\left(\delta_{nk} + {}_0^tu_{n,k}\right) + {}_0h_{L,k}\left(\delta_{nl} + {}_0^tu_{n,l}\right) \right\}$$

$$= \frac{1}{2}\left({}_0h_{L,l}\frac{\partial\, {}_0^tu_{n,k}}{\partial\, {}^tU_m^M} + {}_0h_{L,k}\frac{\partial\, {}_0^tu_{n,l}}{\partial\, {}^tU_m^M} \right)$$

$$= \frac{1}{2}\left\{ {}_0h_{L,l}\left(\delta_{nm}\, {}_0h_{M,k}\right) + {}_0h_{L,k}\left(\delta_{nm}\, {}_0h_{M,l}\right) \right\}$$

$$= \frac{1}{2}\left({}_0h_{L,k}\, {}_0h_{M,l} + {}_0h_{L,l}\, {}_0h_{M,k} \right)\delta_{nm} \qquad \longleftarrow \quad (6.145)$$

6.43

Consider equations (6.137) to (6.142). We now do not include the pressure interpolation, and hence not the equations for $\hat{\underline{P}}$. We notice that

$$ {}^{t}\underline{KUU} = {}^{t}_{0}\underline{K} \quad \text{of Eq. (6.100)} $$

$$ {}^{t}\underline{FU} = {}^{t}_{0}\underline{F} \quad \text{of Eq. (6.100)} $$

6.44

$$ {}^tFU_i = \frac{\partial}{\partial\, {}^t\hat{u}_i}\left[\int_{{}^0V} {}^t_0W \, d\,{}^0V\right] $$

$$ \frac{\partial\, {}^t_0W}{\partial\, {}^t\hat{u}_i} = \frac{\partial}{\partial\, {}^t\hat{u}_i}\left[{}^t_0\bar{W} - \frac{1}{2\kappa}\left({}^t\bar{p} - {}^t\tilde{p}\right)^2\right] $$

$$ = \frac{\partial\, {}^t_0\bar{W}}{\partial\, {}^t_0\varepsilon_{kl}} \frac{\partial\, {}^t_0\varepsilon_{kl}}{\partial\, {}^t\hat{u}_i} - \frac{1}{\kappa}\left({}^t\bar{p} - {}^t\tilde{p}\right) \frac{\partial\, {}^t\bar{p}}{\partial\, {}^t_0\varepsilon_{kl}} \frac{\partial\, {}^t_0\varepsilon_{kl}}{\partial\, {}^t\hat{u}_i} $$

$$ = \left[{}^t_0\bar{S}_{kl} - \frac{1}{\kappa}\left({}^t\bar{p} - {}^t\tilde{p}\right)\frac{\partial\, {}^t\bar{p}}{\partial\, {}^t_0\varepsilon_{kl}}\right] \frac{\partial\, {}^t_0\varepsilon_{kl}}{\partial\, {}^t\hat{u}_i} $$

$$ \therefore\ {}^tFU_i = \int_{{}^0V} {}^t_0S_{kl}\, \frac{\partial\, {}^t_0\varepsilon_{kl}}{\partial\, {}^t\hat{u}_i}\, d\,{}^0V \quad \text{where} \quad {}^t_0S_{kl} = {}^t_0\bar{S}_{kl} - \frac{1}{\kappa}\left({}^t\bar{p} - {}^t\tilde{p}\right)\frac{\partial\, {}^t\bar{p}}{\partial\, {}^t_0\varepsilon_{kl}} $$

$$ \frac{\partial\, {}^t_0W}{\partial\, {}^t\hat{p}_i} = 0 + \frac{1}{\kappa}\left({}^t\bar{p} - {}^t\tilde{p}\right)\frac{\partial\, {}^t\tilde{p}}{\partial\, {}^t\hat{p}_i} $$

$$ \therefore\ {}^tFP_i = \int_{{}^0V} \frac{1}{\kappa}\left({}^t\bar{p} - {}^t\tilde{p}\right)\frac{\partial\, {}^t\tilde{p}}{\partial\, {}^t\hat{p}_i}\, d\,{}^0V $$

$$ {}^tKUU_{ij} = \frac{\partial\, {}^tFU_i}{\partial\, {}^t\hat{u}_j} = \frac{\partial}{\partial\, {}^t\hat{u}_j}\left[\int_{{}^0V} \overbrace{\left\{ {}^t_0\bar{S}_{kl} - \frac{1}{\kappa}\left({}^t\bar{p} - {}^t\tilde{p}\right)\frac{\partial\, {}^t\bar{p}}{\partial\, {}^t_0\varepsilon_{kl}} \right\}}^{A} \frac{\partial\, {}^t_0\varepsilon_{kl}}{\partial\, {}^t\hat{u}_i}\, d\,{}^0V\right] $$

$$ \frac{\partial}{\partial\, {}^t\hat{u}_j}\{A\} = \frac{\partial\, {}^t_0\bar{S}_{kl}}{\partial\, {}^t_0\varepsilon_{rs}} \frac{\partial\, {}^t_0\varepsilon_{rs}}{\partial\, {}^t\hat{u}_j} - \frac{1}{\kappa}\left\{\frac{\partial\, {}^t\bar{p}}{\partial\, {}^t_0\varepsilon_{kl}} \frac{\partial\, {}^t\bar{p}}{\partial\, {}^t_0\varepsilon_{rs}} + \left({}^t\bar{p} - {}^t\tilde{p}\right) \frac{\partial^2\, {}^t\bar{p}}{\partial\, {}^t_0\varepsilon_{kl}\, \partial\, {}^t_0\varepsilon_{rs}}\right\} \frac{\partial\, {}^t_0\varepsilon_{rs}}{\partial\, {}^t\hat{u}_j} $$

$$ \therefore\ {}^tKUU_{ij} = \int_{{}^0V}\left({}_0CUU_{klrs} \frac{\partial\, {}^t_0\varepsilon_{kl}}{\partial\, {}^t\hat{u}_i} \frac{\partial\, {}^t_0\varepsilon_{rs}}{\partial\, {}^t\hat{u}_j} + {}^t_0S_{kl} \frac{\partial^2\, {}^t_0\varepsilon_{kl}}{\partial\, {}^t\hat{u}_i\, \partial\, {}^t\hat{u}_j}\right) d\,{}^0V $$

$$ \text{where} \quad {}_0CUU_{klrs} = {}_0\bar{C}_{klrs} - \frac{1}{\kappa} \frac{\partial\, {}^t\bar{p}}{\partial\, {}^t_0\varepsilon_{kl}} \frac{\partial\, {}^t\bar{p}}{\partial\, {}^t_0\varepsilon_{rs}} - \frac{1}{\kappa}\left({}^t\bar{p} - {}^t\tilde{p}\right) \frac{\partial^2\, {}^t\bar{p}}{\partial\, {}^t_0\varepsilon_{kl}\, \partial\, {}^t_0\varepsilon_{rs}} $$

6.44

$$ {}^{t}KUP_{ij} = \frac{\partial\, {}^{t}FU_i}{\partial\, {}^{t}\hat{P}_j} = \frac{\partial}{\partial\, {}^{t}\hat{P}_j}\left[\int_{{}^{0}V} \left\{ {}^{t}_{0}\bar{S}_{kl} - \frac{1}{K}({}^{t}\bar{p} - {}^{t}\tilde{p}) \frac{\partial\, {}^{t}\bar{p}}{\partial\, {}^{t}_{0}\varepsilon_{kl}} \right\} \frac{\partial\, {}^{t}_{0}\varepsilon_{kl}}{\partial\, {}^{t}\hat{U}_i}\, d\,{}^{0}V \right] $$

$$ = \int_{{}^{0}V} \left[-\frac{1}{K}\left\{ -\frac{\partial\, {}^{t}\tilde{p}}{\partial\, {}^{t}\hat{P}_j} \frac{\partial\, {}^{t}\bar{p}}{\partial\, {}^{t}_{0}\varepsilon_{kl}} \right\} \frac{\partial\, {}^{t}_{0}\varepsilon_{kl}}{\partial\, {}^{t}\hat{U}_i} \right] d\,{}^{0}V $$

$$ \therefore\ {}^{t}KUP_{ij} = \int_{{}^{0}V} {}_{0}CUP_{kl} \frac{\partial\, {}^{t}_{0}\varepsilon_{kl}}{\partial\, {}^{t}\hat{U}_i} \frac{\partial\, {}^{t}\tilde{p}}{\partial\, {}^{t}\hat{P}_j}\, d\,{}^{0}V \quad \text{where } {}_{0}CUP_{kl} = \frac{1}{K}\frac{\partial\, {}^{t}\bar{p}}{\partial\, {}^{t}_{0}\varepsilon_{kl}} $$

$$ {}^{t}KPP_{ij} = \frac{\partial\, {}^{t}FP_i}{\partial\, {}^{t}\hat{P}_j} = \int_{{}^{0}V} \frac{\partial}{\partial\, {}^{t}\hat{P}_j}\left[\frac{1}{K}({}^{t}\bar{p} - {}^{t}\tilde{p}) \frac{\partial\, {}^{t}\tilde{p}}{\partial\, {}^{t}\hat{P}_i} \right] d\,{}^{0}V $$

$$ = \int_{{}^{0}V} \left[-\frac{1}{K} \frac{\partial\, {}^{t}\tilde{p}}{\partial\, {}^{t}\hat{P}_i} \frac{\partial\, {}^{t}\tilde{p}}{\partial\, {}^{t}\hat{P}_j} \right] d\,{}^{0}V $$

6.45

$$\begin{bmatrix} {}^t\underline{KUU} & {}^t\underline{KUP} \\ {}^t\underline{KPU} & {}^t\underline{KPP} \end{bmatrix} \begin{bmatrix} \hat{\underline{u}} \\ \hat{\underline{P}} \end{bmatrix} = \begin{bmatrix} {}^{t+\Delta t}\underline{R} \\ \underline{0} \end{bmatrix} - \begin{bmatrix} {}^t\underline{FU} \\ {}^t\underline{FP} \end{bmatrix}$$

$${}^tFU_i = \int_{{}^0V} {}^t_0S_{kl} \frac{\partial\, {}^t_0\varepsilon_{kl}}{\partial \hat{u}_i} d^0V, \quad {}^t_0S_{kl} = {}^t_0\bar{S}_{kl} - \frac{1}{\kappa}({}^t\bar{p} - {}^t\tilde{p}) \frac{\partial\, {}^t\bar{p}}{\partial\, {}^t_0\varepsilon_{kl}}$$

$${}^t\underline{FU} = \int_{{}^0V} {}^t_0\underline{B}_L^T \, {}^t_0\hat{\underline{S}} \, d^0V$$ where ${}^t_0B_L$, ${}^t_0\hat{\underline{S}}$ are given in Table 6.5.

$${}^tFP_i = \int_{{}^0V} \frac{1}{\kappa}({}^t\bar{p} - {}^t\tilde{p}) \frac{\partial\, {}^t\tilde{p}}{\partial \hat{P}_i} d^0V = \frac{1}{\kappa}\int_{{}^0V} ({}^t\bar{p} - {}^tP_0)\, d^0V \quad (\text{here } {}^t\tilde{p} = {}^tP_0)$$

$${}^tKUU_{ij} = \int_{{}^0V} {}_0CUU_{klrs} \frac{\partial\, {}^t_0\varepsilon_{kl}}{\partial \hat{u}_i} \frac{\partial\, {}^t_0\varepsilon_{rs}}{\partial \hat{u}_j} d^0V + \int_{{}^0V} {}^t_0S_{kl} \frac{\partial^2\, {}^t_0\varepsilon_{kl}}{\partial \hat{u}_i \partial \hat{u}_j} d^0V$$

$${}^t\underline{KUU} = \int_{{}^0V} {}^t_0\underline{B}_L^T \, {}_0\underline{CUU}\, {}^t_0\underline{B}_L \, d^0V + \int_{{}^0V} {}^t_0\underline{B}_{NL}^T \, {}^t_0\underline{S} \, {}^t_0\underline{B}_{NL} \, d^0V$$

where ${}^t_0\underline{S}$, ${}^t_0\underline{B}_{NL}$ are given in Table 6.5.

$${}^tKUP_{ij} = \int_{{}^0V} {}_0CUP_{kl} \frac{\partial\, {}^t_0\varepsilon_{kl}}{\partial \hat{u}_i} \frac{\partial\, {}^t\tilde{p}}{\partial \hat{P}_j} d^0V$$

$${}^t\underline{KUP} = \int_{{}^0V} \underline{B}_L^T \, {}_0\underline{CUP} \, d^0V$$

$${}^tKPP_{ij} = \int_{{}^0V} {}_0CPP \frac{\partial\, {}^t\tilde{p}}{\partial \hat{P}_i} \frac{\partial\, {}^t\tilde{p}}{\partial \hat{P}_j} d^0V = -\frac{1}{\kappa}\int_{{}^0V} d^0V$$

Therefore,

$${}_0CUU_{klrs} = {}_0\bar{C}_{klrs} - \frac{1}{\kappa} \frac{\partial \bar{p}}{\partial\, {}^t_0\varepsilon_{kl}} \frac{\partial \bar{p}}{\partial\, {}^t_0\varepsilon_{rs}} - \frac{1}{\kappa}({}^t\bar{p} - {}^tP_0) \frac{\partial^2 \bar{p}}{\partial\, {}^t_0\varepsilon_{kl} \partial\, {}^t_0\varepsilon_{rs}},$$

$${}_0CUP_{kl} = \frac{1}{\kappa} \frac{\partial\, {}^t\bar{p}}{\partial\, {}^t_0\varepsilon_{kl}}, \quad \frac{\partial\, {}^t\bar{p}}{\partial\, {}^t_0\varepsilon_{kl}} = -\kappa \left(\frac{\partial (\det {}^t_0\underline{X})}{\partial\, {}^t_0\varepsilon_{kl}} \right),$$

where ${}_0\bar{C}_{klrs}$ corresponds to that of plane strain condition.

6.46

$$\begin{bmatrix} {}^t\underline{KUU} & {}^t\underline{KUP} \\ {}^t\underline{KUP}^T & {}^t\underline{KPP} \end{bmatrix} \begin{bmatrix} \underline{\hat{u}} \\ \underline{\hat{p}} \end{bmatrix} = \begin{bmatrix} {}^{t+\Delta t}\underline{R} \\ \underline{0} \end{bmatrix} - \begin{bmatrix} {}^t\underline{FU} \\ {}^t\underline{FP} \end{bmatrix}$$

$${}^tFU_i = \int_{{}^tV} {}^t\tau_{kl} \frac{\partial_t e_{kl}}{\partial \hat{u}_i} d^tV \quad \text{with} \quad {}^t\tau_{kl} = {}^t\bar{\tau}_{kl} - \frac{1}{\kappa}({}^t\bar{p} - {}^t\tilde{p}) \frac{\partial {}^t\bar{p}}{\partial_t e_{kl}}$$

$$\frac{\partial {}^t\bar{p}}{\partial_t e_{kl}} = -\kappa \delta_{kl} \quad , \quad {}^t\tau_{kl} = {}^t\bar{\tau}_{kl} + ({}^t\bar{p} - {}^t\tilde{p})\delta_{kl}$$

$\therefore {}^t\underline{FU} = \int_{{}^tV} {}^t_t\underline{B}_L^T \, {}^t\underline{\hat{\tau}} \, d^tV$ where ${}^t_t\underline{B}_L$, ${}^t\underline{\hat{\tau}}$ are given in Table 6.5.

$${}^tFP_i = \int_{{}^tV} \frac{1}{\kappa}({}^t\bar{p} - {}^t\tilde{p}) \frac{\partial {}^t\tilde{p}}{\partial \hat{p}_i} d^tV = \frac{1}{\kappa}\int_{{}^tV} ({}^t\bar{p} - {}^tp_0) d^tV \quad \left(\text{here } {}^t\tilde{p} = {}^tp_0\right)$$

$${}^tKUU_{ij} = \int_{{}^tV} {}_\tau CUU_{klrs} \frac{\partial_t e_{kl}}{\partial \hat{u}_i} \frac{\partial_t e_{rs}}{\partial \hat{u}_j} d^tV + \int_{{}^tV} {}^t\tau_{kl} \frac{\partial^2 {}_t e_{kl}}{\partial \hat{u}_i \partial \hat{u}_j} d^tV$$

$${}_\tau CUU_{klrs} = {}_\tau \bar{C}_{klrs} - \frac{1}{\kappa} \frac{\partial {}^t\bar{p}}{\partial_t e_{kl}} \frac{\partial {}^t\bar{p}}{\partial_t e_{rs}} = {}_\tau \bar{C}_{klrs} - \kappa \delta_{kl} \delta_{rs}$$

where ${}_\tau \bar{C}_{klrs}$ corresponds to the constitutive matrix in plane strain condition.

$$\therefore {}^t\underline{KUU} = \int_{{}^tV} {}^t_t\underline{B}_L^T \, {}_\tau\underline{CUU} \, {}^t_t\underline{B}_L \, d^tV + \int_{{}^tV} {}^t_t\underline{B}_{NL}^T \, {}^t\underline{\tau} \, {}^t_t\underline{B}_{NL} \, d^tV$$

where ${}^t_t\underline{B}_{NL}$, ${}^t\underline{\tau}$ are given in Table 6.5.

$${}^tKUP_{ij} = \int_{{}^tV} {}_\tau CUP_{kl} \frac{\partial_t e_{kl}}{\partial \hat{u}_i} \frac{\partial {}^t\tilde{p}}{\partial \hat{p}_j} d^tV \qquad \therefore {}^t\underline{KUP} = \int_{{}^tV} {}^t_t\underline{B}_L^T d^tV$$

$${}_\tau CUP_{kl} = \frac{1}{\kappa} \frac{\partial {}^t\bar{p}}{\partial_t e_{kl}} = \delta_{kl}$$

$${}^tKPP_{ij} = \int_{{}^tV} {}_\tau C_{pp} \frac{\partial {}^t\tilde{p}}{\partial \hat{p}_i} \frac{\partial {}^t\tilde{p}}{\partial \hat{p}_j} d^tV \qquad \therefore {}^t\underline{KPP} = -\frac{1}{\kappa}\int_{{}^tV} d^tV \,,$$

6.46

where $$ {}_{\tau}CPP = -\frac{1}{K(\det {}^{\tau}_{0}\underline{X})}, $$

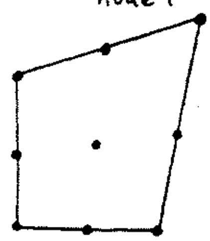
$$ {}_{\tau}CUP_{kl} = \frac{1}{K(\det {}^{\tau}_{0}\underline{X})} \frac{\partial\, {}^{t}\bar{p}}{\partial_{t} e_{kl}}, $$

$$ {}_{\tau}CUU_{klrs} = {}_{\tau}\bar{C}_{klrs} + {}_{\tau}CPP \frac{\partial\, {}^{t}\bar{p}}{\partial_{t} e_{kl}} \frac{\partial\, {}^{t}\bar{p}}{\partial_{t} e_{rs}} + $$

$$ + {}_{\tau}CPP \left({}^{t}\bar{p} - {}^{t}p_{0} \right) \frac{\partial^{2}\, {}^{t}\bar{p}}{\partial_{t} e_{kl}\, \partial_{t} e_{rs}} $$

$$ \frac{\partial({}^{t}\bar{p})}{\partial_{t} e_{kl}} = -K \frac{\partial(\det {}^{t}_{0}\underline{X})}{\partial_{t} e_{kl}} $$

6.47



$\rightarrow {}^{t}U_1^1$ Since the ${}^{t}F$ calculation is correct we can evaluate ${}^{t}_{0}K_{11}$ as follows:

Step 1. Consider the configuration at time t. Impose using the program an additional small displacement ε at d.o.f. ${}^{t}U_1^1$ with all other d.o.f. fixed. The program will give the two forces ${}^{t}F_1^1$ and ${}^{t}F_{1+\varepsilon}^1$ (= nodal force corresponding to $1+\varepsilon$.) Then we have

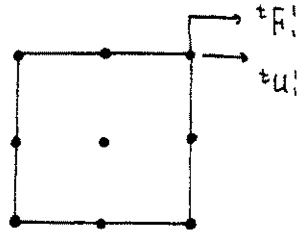
$$ {}^{t}_{0}K_{11}\Big|_{exact} \doteq \frac{{}^{t}F_{1+\varepsilon}^1 - {}^{t}F_1^1}{\varepsilon} \qquad \text{——} \quad ① $$

Step 2. Next put the force ${}^{t}F_1^1$ + (unit force) at the d.o.f. ${}^{t}U_1^1$ with all other d.o.f. fixed, and have the program calculate the corresponding incremental displacement ${}^{t}\tilde{U}_1^1$ from time t without any equilibrium iteration. Then

$$ {}^{t}_{0}K_{11}\Big|_{actually\ used} = \frac{1}{{}^{t}\tilde{U}_1^1} \qquad \text{——} \quad ② $$

Step 3. The above value for the stiffness coefficient in ② should be close enough to that in ①. If so, the stiffness calculation for node 1 is correct. Note that the value obtained in ① is actually carrying a small error because of a finite difference scheme used.

6.48



Step. 1. Consider the configuration at time t. Impose using the program an additional displacement ε at d.o.f. ${}^tU_1^1$ with all other d.o.f. fixed to obtain ${}^tF_{1+\varepsilon}^1$.

Then $${}^t_0K_{11}\Big|_{exact} \doteq \frac{{}^tF_{1+\varepsilon}^1 - {}^tF_1^1}{\varepsilon} \quad \text{——— } ①$$

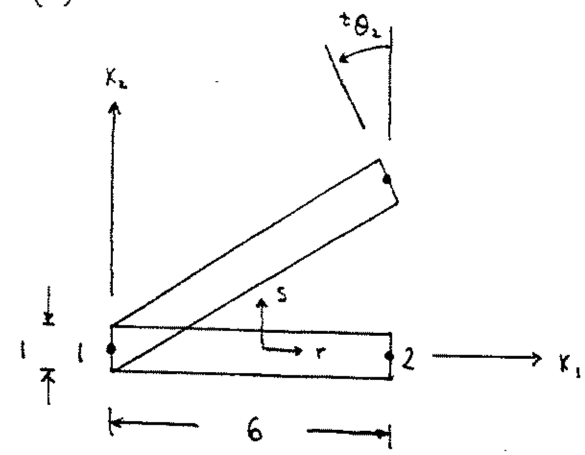
Step 2. Next put the force ${}^tF_1^1$ + (unit force) at the d.o.f. ${}^tU_1^1$ with all other d.o.f. fixed, and have the program calculate the corresponding incremental displacement ${}^t\tilde{U}_1^1$ without any equilibrium iteration. Then

$${}^t_0K_{11}\Big|_{actually\ used} = \frac{1}{{}^t\tilde{U}_1^1} \quad \text{——— } ②$$

Step 3. Check if ${}^t_0K_{11}\big|_{exact} \doteq {}^t_0K_{11}\big|_{actually\ used}$. Note the value obtained in ① is actually carrying a small error because of a finite difference scheme used. If so, the stiffness calculation for node 1 is correct.

You may want to perform this test using the computer program ADINA.

6.51 (a)



$$ {}^0x_1 = 3(1+r), \quad {}^0x_2 = \frac{s}{2} $$

$$ {}^tx_1 = 3(1+r) - \frac{\sin {}^t\theta_2}{4}(1+r)\,s $$

$$ {}^tx_2 = 1+r+\frac{s}{4}\left[(1-r)+(1+r)\cos {}^t\theta_2\right] $$

$$ \therefore\ {}^tu_1 = {}^tx_1 - {}^0x_1 = -\frac{(1+r)s}{4}\sin {}^t\theta_2 $$

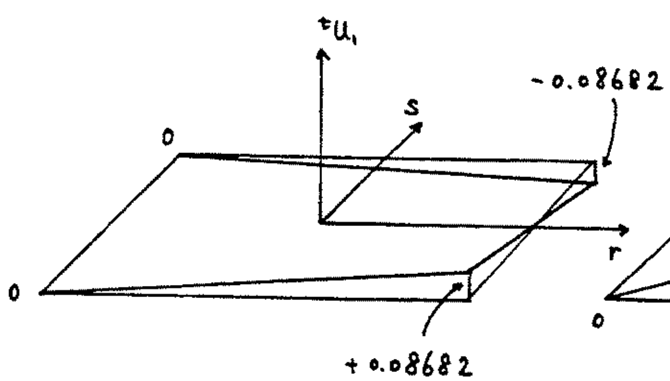
$$ {}^tu_2 = {}^tx_2 - {}^0x_2 $$

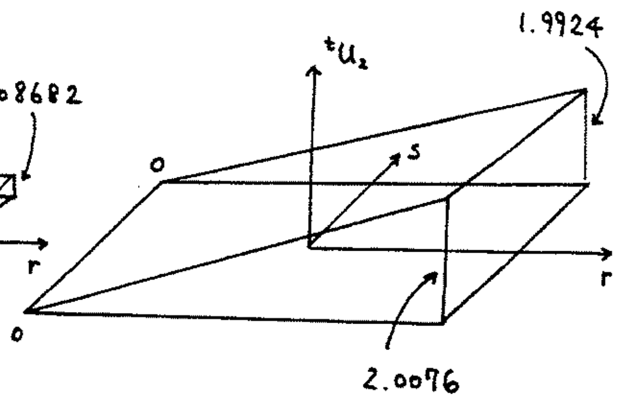
$$ = 1+r+\frac{s}{4}\left[(1-r)+(1+r)\cos {}^t\theta_2\right]-\frac{s}{2} $$

where ${}^t\theta_2 = 10°$

$$ {}^0\underline{V}_s^1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = {}^0\underline{V}_s^2 = {}^t\underline{V}_s^1 $$

$$ {}^t\underline{V}_s^2 = \begin{bmatrix} -\sin {}^t\theta_2 \\ \cos {}^t\theta_2 \end{bmatrix} $$





Now using $\frac{\partial}{\partial\,{}^0x_1} = \frac{1}{3}\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial\,{}^0x_2} = 2\frac{\partial}{\partial s}$ we calculate ${}^t_0X_{ij}$.

$$ \frac{\partial\,{}^tx_1}{\partial\,{}^0x_1} = 1 - \frac{s}{12}\sin {}^t\theta_2, \quad \frac{\partial\,{}^tx_1}{\partial\,{}^0x_2} = -\frac{1+r}{2}\sin {}^t\theta_2 $$

$$ \frac{\partial\,{}^tx_2}{\partial\,{}^0x_1} = \frac{1}{3}\left\{1-\frac{s}{4}(1-\cos {}^t\theta_2)\right\}, \quad \frac{\partial\,{}^tx_2}{\partial\,{}^0x_2} = \frac{1}{2}\left\{(1-r)+(1+r)\cos {}^t\theta_2\right\} $$

6.51

Hence $\ {}^t_0\underline{\varepsilon}\Big|_{(0,0)} = \frac{1}{2}\left({}^t_0\underline{X}^T\,{}^t_0\underline{X} - \underline{I}\right)\Big|_{(0,0)} = \begin{bmatrix} 0.05556 & 0.1220 \\ 0.1220 & -0.003798 \end{bmatrix}$

(b) $$u_1 = \frac{1+r}{2}u_1^2 + \frac{(1+r)s}{4}\left[-\theta_2 \cos{}^t\theta_2 + \frac{1}{2}(\theta_2)^2 \sin{}^t\theta_2\right]$$

$$u_2 = \frac{1+r}{2}u_2^2 - \frac{(1+r)s}{4}\left[\theta_2 \sin{}^t\theta_2 + \frac{1}{2}(\theta_2)^2 \cos{}^t\theta_2\right]$$

Using $\frac{\partial}{\partial\,{}^0x_1} = \frac{1}{3}\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial\,{}^0x_2} = 2\frac{\partial}{\partial s}$,

$${}_0u_{1,1} = \frac{1}{3}\left[\frac{1}{2}u_1^2 + \frac{s}{4}\left\{-\theta_2 \cos{}^t\theta_2 + \frac{1}{2}(\theta_2)^2 \sin{}^t\theta_2\right\}\right]$$

$${}_0u_{1,2} = \frac{1+r}{2}\left[-\theta_2 \cos{}^t\theta_2 + \frac{1}{2}(\theta_2)^2 \sin{}^t\theta_2\right]$$

$${}_0u_{2,1} = \frac{1}{3}\left[\frac{1}{2}u_2^2 - \frac{s}{4}\left\{\theta_2 \sin{}^t\theta_2 + \frac{1}{2}(\theta_2)^2 \cos{}^t\theta_2\right\}\right]$$

$${}_0u_{2,2} = -\frac{1+r}{2}\left[\theta_2 \sin{}^t\theta_2 + \frac{1}{2}(\theta_2)^2 \cos{}^t\theta_2\right]$$

with ${}^t\theta_2 = 10^\circ$

6.52 Using eq. (6.154) and (6.162) we obtain

$$u_1 = \frac{1+r}{2} u_1^2 + \frac{1+r}{2}\frac{s}{2}(-\theta_2) \qquad u_2 = \frac{1+r}{2} u_2^2 + \frac{1+r}{2}\frac{s}{2}\left\{-\frac{(\theta_2)^2}{2}\right\}$$

$$^t u_1 = \frac{1+r}{2}\, {}^t u_1^1 = \frac{1+r}{2}(0.1) \qquad {}^t u_2 = 0$$

$$_0 u_{1,1} = \frac{2}{L}\left(\frac{1}{2}u_1^2 - \frac{s}{4}\theta_2\right) \qquad {}_0 u_{1,2} = \frac{2}{h}\left(-\frac{1+r}{4}\theta_2\right)$$

$$_0 u_{2,1} = \frac{2}{L}\left[\frac{1}{2}u_2^2 - \frac{s}{8}(\theta_2)^2\right] \qquad {}_0 u_{2,2} = \frac{2}{h}\left[-\frac{1+r}{8}(\theta_2)^2\right]$$

$$^t_0 u_{1,1} = \frac{2}{L}\left(\frac{0.1}{2}\right) = \frac{0.1}{L} \qquad {}^t_0 u_{1,2} = {}^t_0 u_{2,1} = {}^t_0 u_{2,2} = 0$$

Hence we have

$$\left.\begin{aligned} {}_0 e_{11} &= \left(1+\frac{0.1}{L}\right)\left(\frac{1}{L}u_1^2 - \frac{s}{2L}\theta_2\right) \\ 2\,{}_0 e_{12} &= \frac{1}{L}u_2^2 - \left(1+\frac{0.1}{L}\right)\frac{1+r}{2h}\theta_2 - \frac{s}{4L}(\theta_2)^2 \end{aligned}\right\} \quad \text{—— ①}$$

$$\left.\begin{aligned} {}_0 \eta_{11} &= \frac{1}{2}\left({}_0 u_{1,1}^2 + {}_0 u_{2,1}^2\right) \\ 2\,{}_0 \eta_{12} &= {}_0 u_{1,1}\,{}_0 u_{1,2} + {}_0 u_{2,1}\,{}_0 u_{2,2} \end{aligned}\right\} \quad \text{—— ②}$$

(a) P.V.D. states

$$\underbrace{\int_{^0V} {}_0 C_{ijrs}\, {}_0 e_{rs}\, \delta\, {}_0 e_{ij}\, d\,{}^0V}_{\text{Ⓘ}} + \underbrace{\int_{^0V} {}^t_0 S_{ij}\, \delta\, {}_0 \eta_{ij}\, d\,{}^0V}_{\text{Ⓘ}} = {}^{t+\Delta t}\mathcal{R} - \underbrace{\int_{^0V} {}^t_0 S_{ij}\, \delta\, {}_0 e_{ij}\, d\,{}^0V}_{\text{Ⓘ Ⓘ}}$$

6.52

As can be seen in eq. ①, ${}_0e_{ij}$ contains quadratic terms due to θ_2. For the terms of Ⓘ and Ⓘⅰ the effect of $(\theta_2)^2$ does not enter in the linearization.

However, for Ⓘⅰⅰ we obtain $\int_{{}^0V} {}^t_0S_{ij}\, \delta_0 e_{ij}^L \, d^0V + \int_{{}^0V} {}^t_0S_{ij}\, \delta_0 e_{ij}^Q \, dV$, where L and Q denote the linear and quadratic terms of ${}_0e_{ij}$ respectively.

$$\text{Ⓘ} = \int_{{}^0V} [\, \delta_0 e_{11} (C_{1111})\, e_{11} + k\, \delta(2\,{}_0e_{12})(C_{1212})(2\,{}_0e_{12})\,]\, d^0V$$

$C_{1111} = E$, $C_{1212} = G$, k = shear correction factor

$$\therefore\ {}^t\underline{K}_L = \begin{bmatrix} 0.05050\,E & 0 & 0 \\ 0 & 0.04167\,G & -0.4188\,G \\ 0 & -0.4188\,G & 0.004208\,E + 5.611\,G \end{bmatrix}$$

$$\text{Ⓘⅰ} = \int_{{}^0V} [\, {}^t_0S_{11}\, \delta_0\eta_{11}\,]\, d^0V$$

$$\therefore\ {}^t\underline{K}_{NL} = \begin{bmatrix} 0.05\ {}^t_0S_{11} & 0 & 0 \\ 0 & 0.05\ {}^t_0S_{11} & 0 \\ 0 & 0 & 0.004167\ {}^t_0S_{11} \end{bmatrix}$$

$$\text{Ⓘⅰⅰ} = \int_{{}^0V} [\ {}^t_0S_{11}\, \delta_0 e_{11}]\, d^0V$$

6.52

Since ${}^t_0S_{12} = 0$ and ${}_0e_{11}$ does not contain quadratic terms due to θ_2, the contribution from $\int_{{}^0V} {}^t_0S_{ij}\,\delta\,{}_0e_{ij}\,d\,{}^0V$ to the tangent stiffness matrix is zero.

$$\int_{{}^0V} {}^t_0S_{ij}\,\delta\,{}_0e^L_{ij}\,d\,{}^0V = [\,1.005\,{}^t_0S_{11} \quad 0 \quad 0\,]^T, \text{ and } \int_{{}^0V} {}^t_0S_{ij}\,\delta\,{}_0e^Q_{ij}\,d\,{}^0V = \underline{0}$$

$${}^t\underline{F}^T = [\,1.005\,{}^t_0S_{11} \quad 0 \quad 0\,]$$

(b) One-point Gauss integration for the r-direction

$$\therefore \quad {}^t\underline{K}_L = \begin{bmatrix} 0.05050\,E & 0 & 0 \\ 0 & 0.04167\,G & -0.4188\,G \\ 0 & -0.4188\,G & \underbrace{4.208\,G}_{\text{different}} \end{bmatrix}$$

$${}^t\underline{K}_{NL} = \begin{bmatrix} 0.05\,{}^t_0S_{11} & 0 & 0 \\ 0 & 0.05\,{}^t_0S_{11} & 0 \\ 0 & 0 & \underbrace{0}_{\text{different}} \end{bmatrix}$$

${}^t\underline{F}^T = [\,1.005\,{}^t_0S_{11} \quad 0 \quad 0\,]$ ← the same

6.53 $\quad {}^{t}u_1 = \frac{1+r}{2}(0.1) \qquad {}^{0}\bar{x}_1 = \frac{1-r}{2}(5) + \frac{1+r}{2}(25) = 15 + 10r$

$${}_{0}e_{33} = {}_{0}u_{3,3} + {}^{t}_{0}u_{3,3}\,{}_{0}u_{3,3} = \frac{u_1}{{}^{0}\bar{x}_1} + \frac{{}^{t}u_1 u_1}{({}^{0}\bar{x}_1)^2} = \left(1 + \frac{{}^{t}u_1}{{}^{0}\bar{x}_1}\right)\frac{u_1}{{}^{0}\bar{x}_1}$$

$${}_{0}\eta_{33} = \frac{1}{2}\,{}_{0}u_{3,3}^2 = \frac{1}{2}\left(\frac{u_1}{{}^{0}\bar{x}_1}\right)^2$$

(a) Using the equations above and the results in exercise 6.52,

$${}^{t}\underline{K}_L \longrightarrow \int_{{}^{0}V} [\delta_0 e_{11} \;\; \delta_0 e_{33} \;\; \delta_0 \gamma_{12}] \begin{bmatrix} C_{1111} & C_{1133} & 0 \\ C_{1133} & C_{3333} & 0 \\ 0 & 0 & C_{1212} \end{bmatrix} \begin{bmatrix} {}_{0}e \\ {}_{0}e_{33} \\ {}_{0}\gamma_{12} \end{bmatrix} {}^{0}\bar{x}_1\, d^{0}V$$

$${}^{t}\underline{K}_{NL} \longrightarrow \int_{{}^{0}V} \left({}^{t}_{0}S_{11}\, \delta_0 \eta_{11} + {}^{t}_{0}S_{33}\, \delta_0 \eta_{33} \right) {}^{0}\bar{x}_1\, d^{0}V$$

$${}^{t}\underline{F} \longrightarrow \int_{{}^{0}V} \left({}^{t}_{0}S_{11}\, \delta_0 e_{11} + {}^{t}_{0}S_{33}\, \delta_0 e_{33} \right) {}^{0}\bar{x}_1\, d^{0}V$$

As in exercise 6.52, ${}^{t}_{0}S_{12} = 0$, and ${}_{0}e_{11}$, ${}_{0}e_{33}$ do not contain the quadratic terms due to θ_2; therefore, the contribution to the tangent stiffness matrix from $\int_{{}^{0}V} {}^{t}_{0}S_{ij}\, \delta_0 e_{ij}\, d^{0}V$ is zero.

$${}^{t}\underline{K}_L = \begin{bmatrix} 1.111\,C_{1111} + 1.009\,C_{1133} & 0 & 0 \\ 0 & 0.625\,C_{1212} & -7.678\,C_{1212} \\ 0 & -7.678\,C_{1212} & 0.09255\,C_{1111} + 0.08404\,C_{1133} + 112.2\,C_{1212} \end{bmatrix}$$

where $C_{1111} = C_{3333} = \frac{E}{1-\nu^2}$, $\quad C_{1133} = \frac{E\nu}{1-\nu^2}$, $\quad C_{1212} = Gk$

6.53

$$
{}^{t}\underline{K}_{NL} = \begin{bmatrix} 0.75\,{}^{t}_{0}S_{11} + 0.3506\,{}^{t}_{0}S_{33} & 0 & 0 \\ 0 & 0.75\,{}^{t}_{0}S_{11} & 0 \\ 0 & 0 & 0.0625\,{}^{t}_{0}S_{11} + 0.02921\,{}^{t}_{0}S_{33} \end{bmatrix}
$$

$$
{}^{t}\underline{F}^{T} = \begin{bmatrix} 15.08\,{}^{t}_{0}S_{11} + 10.04\,{}^{t}_{0}S_{33} & 0 & 0 \end{bmatrix}
$$

(b)

$$
{}^{t}\underline{K}_{L} = \begin{bmatrix} 1.093\,C_{1111} + 1.008\,C_{1133} & 0 & 0 \\ 0 & 0.625\,C_{1212} & -6.281\,C_{1212} \\ 0 & -6.281\,C_{1212} & 63.13\,C_{1212} \end{bmatrix}
$$

$$
{}^{t}\underline{K}_{NL} = \begin{bmatrix} 0.75\,{}^{t}_{0}S_{11} + 0.3333\,{}^{t}_{0}S_{33} & 0 & 0 \\ 0 & 0.75\,{}^{t}_{0}S_{11} & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

$$
{}^{t}\underline{F}^{T} = \begin{bmatrix} 15.08\,{}^{t}_{0}S_{11} + 10.0\,{}^{t}_{0}S_{33} & 0 & 0 \end{bmatrix}
$$

6.54

Based on the discussion in Sec. 5.4, we calculate the shear strain at the Gauss point $r=0$ (for the 2-node beam element) in the mixed interpolation.

In fact, if we assume ${}_0e_{12}^*$ is the constant shear strain unknown, we have

$$\int_{^0V} 1\cdot ({}_0e_{12}^* - {}_0e_{12})\, d^0V = 0 \qquad \text{(1)}$$

From Excercise 6.52 we find that ${}_0e_{12}$ is a linear function of r, therefore, Eq. (1) is equivalent to

$${}_0e_{12}^* = {}_0e_{12}\Big|_{r=0}.$$

Similarly, we have

$${}_0\eta_{12}^* = {}_0\eta_{12}\Big|_{r=0}$$

From the calculation results in Excercise 6.52 we find that ${}_0^tS_{12} = {}_0^tS_{22} = 0$, and ${}_0\eta_{11}$, ${}_0e_{11}$ are not functions of r. Therefore, in this case, using the mixed interpolation of linear displacements and constant transverse shear strain is equivalent to using the reduced integration technique.

6.55 $\quad {}^{t+\Delta t}\underline{V}_m^k = \underline{Q}\,{}^t\underline{V}_m^k \quad (m = t, s)$ —— ①

$$\underline{Q} = \underline{I} + \frac{\sin\gamma_k}{\gamma_k}\underline{S}_k + \frac{1}{2}\left(\frac{\sin\frac{\gamma_k}{2}}{\left(\frac{\gamma_k}{2}\right)}\right)^2 \underline{S}_k^2 \quad \text{—— ②}$$

where $\gamma_k = (\theta_{k_1}^2 + \theta_{k_2}^2 + \theta_{k_3}^2)^{1/2}$, $\quad \underline{S}_k = \begin{bmatrix} 0 & -\theta_{k_3} & \theta_{k_2} \\ \theta_{k_3} & 0 & -\theta_{k_1} \\ -\theta_{k_2} & \theta_{k_1} & 0 \end{bmatrix}$

Using $\sin\gamma_k = \gamma_k - \frac{\gamma_k^3}{3!} + \cdots$ we can write eq. ② as

$$\underline{Q} = \underline{I} + \underline{S}_k + \frac{1}{2}(\underline{S}_k)^2 + \cdots\cdots \quad \text{—— ③}$$

Note that if the incremental rotations are infinitesimal,

$$\underline{Q} = \underline{I} + \underline{S}_k \quad \text{and} \quad {}^{t+\Delta t}\underline{V}_m^k - {}^t\underline{V}_m^k = \underline{S}_k\,{}^t\underline{V}_m^k$$

Because the incremental rotations are finite we keep the linear and quadratic terms in eq. ③.

Hence, $\quad {}^{t+\Delta t}\underline{V}_m^k - {}^t\underline{V}_m^k = \underline{S}_k\,{}^t\underline{V}_m^k + \frac{1}{2}(\underline{S}_k)^2\,\underline{V}_m^k$

or $\quad {}^{t+\Delta t}\underline{V}_m^k - {}^t\underline{V}_m^k = \underline{\theta}_k \times {}^t\underline{V}_m^k + \frac{1}{2}\underline{\theta}_k \times (\underline{\theta}_k \times {}^t\underline{V}_m^k)$

where $\underline{\theta}_k^T = [\,\theta_1^k \;\; \theta_2^k \;\; \theta_3^k\,]$

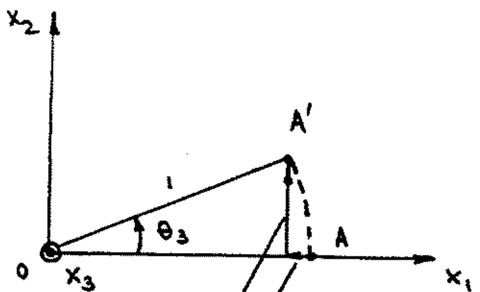
That is, in each direction

$$\left.\begin{aligned} \underline{V}_t^k &= \underline{\theta}_k \times {}^t\underline{V}_t^k + \frac{1}{2}\underline{\theta}_k \times (\underline{\theta}_k \times {}^t\underline{V}_t^k) \\ \underline{V}_s^k &= \underline{\theta}_k \times {}^t\underline{V}_s^k + \frac{1}{2}\underline{\theta}_k \times (\underline{\theta}_k \times {}^t\underline{V}_s^k) \end{aligned}\right\} \quad \text{—— (6.161 and 162)}$$

These director vector increments contain all second-order terms in $\underline{\theta}_k$.

6.55

For a geometric interpretation, let us consider the case $\underline{\theta}_k = [\,0 \quad 0 \quad \theta_3\,]$



$$\underline{Q} = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\cos\theta_3 - 1 \doteq -\frac{1}{2}\theta_3^2 + 0(\theta_3^3)$$

$$\sin\theta_3 \doteq \theta_3 + 0(\theta_3^3)$$

$$\underline{Q} - \underline{I} = \begin{bmatrix} \cos\theta_3 - 1 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \doteq \begin{bmatrix} -\dfrac{\theta_3^2}{2} & -\theta_3 & 0 \\ \theta_3 & -\dfrac{\theta_3^2}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\theta_3 & 0 \\ \theta_3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & -\theta_3 & 0 \\ \theta_3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^2$$

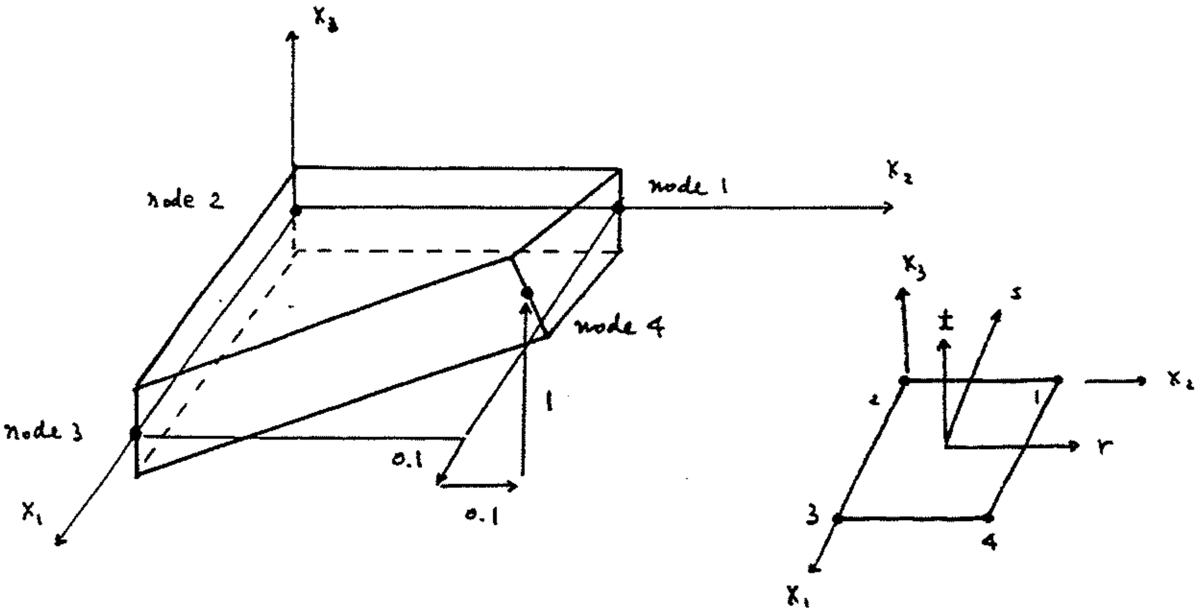
As shown in the above figure, the x_1 increment due to the vector $\overline{OA}$ subjected to the rotation $\underline{\theta}_k$ is $\cos\theta_3 - 1$, and the corresponding x_2 increment is $\sin\theta_3$.

6.56 $\;{}^{t}u_i = \sum h_k\, {}^{t}u_i^k + \frac{t}{2} \sum a_k h_k \left({}^{t}V_{ni}^k - {}^{0}V_{ni}^k \right)$

$$\rightarrow \; {}^{t}u_1 = h_4\, {}^{t}u_1^4 = \frac{(1+r)(1-s)}{4} (0.1)$$

$${}^{t}u_2 = h_4\, {}^{t}u_2^4 + \frac{t}{2} h_4 \left(-\frac{1}{2}\right) = \frac{(1+r)(1-s)}{4} \left(0.1 - \frac{t}{4} \right)$$

$${}^{t}u_3 = h_4\, {}^{t}u_3^4 + \frac{t}{2} h_4 \left(\frac{\sqrt{3}}{2} - 1 \right) = \frac{(1+r)(1-s)}{4} \left[1 + \frac{t}{2} \left(\frac{\sqrt{3}}{2} - 1 \right) \right]$$



6.57 $\quad {}^{t+\Delta t}\underline{V}_n^k = \underline{Q}\,{}^t\underline{V}_n^k \qquad$ —— ①

where $\quad \underline{Q} = \underline{I} + \dfrac{\sin\gamma_k}{\gamma_k}\underline{S}_k + \dfrac{1}{2}\left(\dfrac{\sin\frac{\gamma_k}{2}}{\frac{\gamma_k}{2}}\right)^2(\underline{S}_k)^2 \qquad$ —— ②

$$\gamma_k = (\alpha_k^2 + \beta_k^2)^{1/2}, \quad \underline{S}_k = \begin{bmatrix} 0 & 0 & \beta_k \\ 0 & 0 & -\alpha_k \\ -\beta_k & \alpha_k & 0 \end{bmatrix}$$

Using $\sin\gamma_k = \gamma_k - \dfrac{\gamma_k^3}{3!} + \cdots$,

$$\underline{Q} = \underline{I} + \underline{S}_k + \frac{1}{2}(\underline{S}_k)^2 + \cdots$$

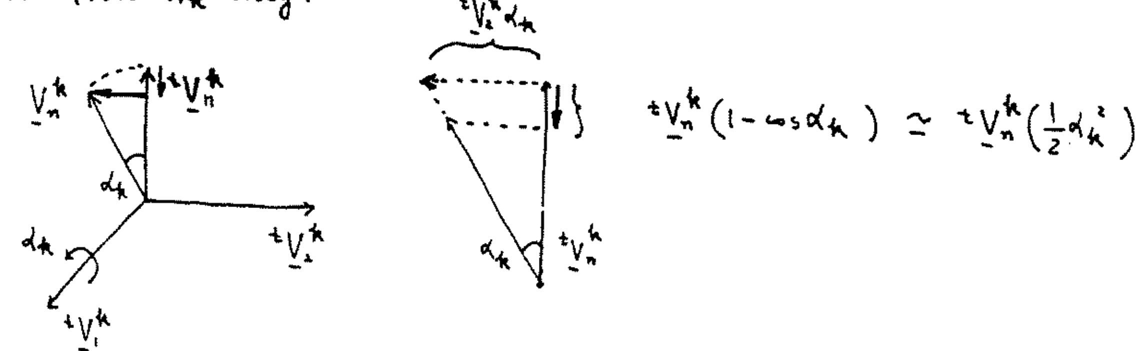
Now we keep the linear and quadratic terms in $\underline{S}_k$; hence eq. ① gives

$${}^{t+\Delta t}\underline{V}_n^k - {}^t\underline{V}_n^k = \underline{V}_n^k = \underline{S}_k\,{}^t\underline{V}_n^k + \frac{1}{2}(\underline{S}_k)^2\,{}^t\underline{V}_n^k \qquad \text{—— ③}$$

Note here that eq. ③ includes all the 2^{nd}-order terms in α_k and β_k. The equation can be simplified as follows.

$$\underline{V}_n^k = -{}^t\underline{V}_2^k\alpha_k + {}^t\underline{V}_1^k\beta_k - \frac{1}{2}(\alpha_k^2 + \beta_k^2)\,{}^t\underline{V}_n^k$$

For a geometric interpretation, consider first the effect of rotation α_k only.



6.57

As can be seen in the figure,

$$^{t+\Delta t}\underline{V}_n^k - {}^t\underline{V}_n^k = -{}^t\underline{V}_2^k \alpha_k - \frac{1}{2}(\alpha_k^2)\,{}^t\underline{V}_n^k$$

Similarly including the effect of β_k we have

$$^{t+\Delta t}\underline{V}_n^k - {}^t\underline{V}_n^k = -{}^t\underline{V}_2^k \alpha_k + {}^t\underline{V}_1^k \beta_k - \frac{1}{2}(\alpha_k^2 + \beta_k^2)\,{}^t\underline{V}_n^k$$

6.58 $\quad {}^t_0\tilde{\varepsilon}^{DI}_{ij} = \frac{1}{2}\left({}^t\underline{g}_i \cdot {}^t\underline{g}_j - {}^0\underline{g}_i \cdot {}^0\underline{g}_j \right)$ where ${}^t\underline{g}_i = \frac{\partial\, {}^t\underline{x}}{\partial r_i}$, ${}^0\underline{g}_i = \frac{\partial\, {}^0\underline{x}}{\partial r_i}$

$${}^0x_i = \sum h_k\, {}^0x_i^k + \frac{t}{2} \sum a_k h_k\, {}^0V_{ni}^k$$

$${}^0x_1 = 5(1-s) \quad , \quad {}^0x_2 = 10(1+r) , \quad {}^0x_3 = \frac{t}{2}$$

$${}^tu_i = \sum h_k\, {}^tu_i^k + \frac{t}{2} \sum a_k h_k \left({}^tV_{ni}^k - {}^0V_{ni}^k \right)$$

$${}^tu_1 = \frac{(1+r)(1-s)}{4}(0.1) \quad , \quad {}^tu_2 = \frac{(1+r)(1-s)}{4}\left(0.1 - \frac{t}{4}\right) .$$

$${}^tu_3 = \frac{(1+r)(1-s)}{4}\left[1 + \frac{t}{2}\left(\frac{\sqrt{3}}{2} - 1\right) \right]$$

$${}^t_0\tilde{\varepsilon}^{DI}_{ij} = \frac{1}{2}\left(\frac{\partial\, {}^t\underline{x}}{\partial r_i} \cdot \frac{\partial\, {}^t\underline{x}}{\partial r_j} - \frac{\partial\, {}^0\underline{x}}{\partial r_i} \cdot \frac{\partial\, {}^0\underline{x}}{\partial r_j} \right) \qquad (\leftarrow \text{use } \; {}^t\underline{x} = {}^0\underline{x} + {}^t\underline{u} \;)$$

$$= \frac{1}{2}\left(\frac{\partial\, {}^0\underline{x}}{\partial r_i} \cdot \frac{\partial\, {}^t\underline{u}}{\partial r_j} + \frac{\partial\, {}^t\underline{u}}{\partial r_i} \cdot \frac{\partial\, {}^0\underline{x}}{\partial r_j} + \frac{\partial\, {}^t\underline{u}}{\partial r_i} \cdot \frac{\partial\, {}^t\underline{u}}{\partial r_j} \right)$$

$$\therefore \; {}^t_0\tilde{\varepsilon}^{DI}_{11} = \frac{1}{2}\left(2\, \frac{\partial\, {}^0\underline{x}}{\partial r} \cdot \frac{\partial\, {}^t\underline{u}}{\partial r} + \frac{\partial\, {}^t\underline{u}}{\partial r} \cdot \frac{\partial\, {}^t\underline{u}}{\partial r} \right)$$

$$= \frac{1}{2}\left[2 \cdot 10 \cdot \frac{1-s}{4}\left(0.1 - \frac{t}{4}\right) + \left((0.1)^2 + \left(0.1 - \frac{t}{4}\right)^2 + \left\{ 1 + \frac{t}{2}\left(\frac{\sqrt{3}}{2} - 1\right) \right\}^2 \right) \frac{(1-s)^2}{16} \right]$$

Similarly we have

$${}^t_0\tilde{\varepsilon}^{DI}_{22} = \frac{1}{2}\left[2(-5)\left(-\frac{1+r}{4}\right)(0.1) + \left((0.1)^2 + \left(0.1 - \frac{t}{4}\right)^2 + \left\{ 1 + \frac{t}{2}\left(\frac{\sqrt{3}}{2} - 1\right) \right\}^2 \right) \frac{(1+r)^2}{16} \right]$$

$${}^t_0\tilde{\varepsilon}^{DI}_{33} = \frac{1}{2}\left[2 \cdot \frac{1}{2} \cdot \frac{1}{2}\left(\frac{\sqrt{3}}{2} - 1\right) \frac{(1+r)(1-s)}{4} + \left(\left(\frac{1}{4}\right)^2 + \left\{ \frac{1}{2}\left(\frac{\sqrt{3}}{2} - 1\right) \right\}^2 \right) \frac{(1+r)^2(1-s)^2}{16} \right]$$

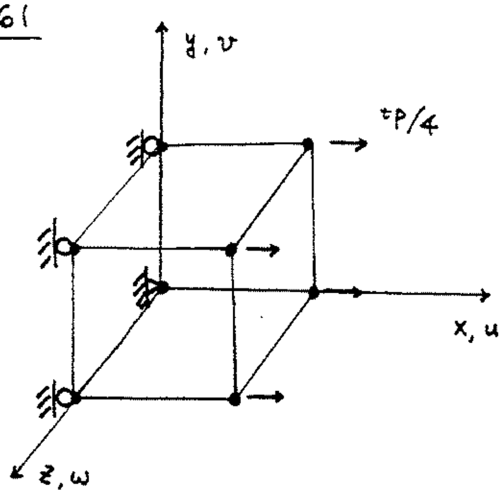
$${}^t_0\tilde{\varepsilon}^{DI}_{12} = \frac{1}{2}\left[10\left(-\frac{1+r}{4}\right)\left(0.1 - \frac{t}{4}\right) + (-5)\, \frac{1-s}{4}(0.1) - \left((0.1)^2 + \left(0.1 - \frac{t}{4}\right)^2 + \left\{ 1 + \frac{t}{2}\left(\frac{\sqrt{3}}{2} - 1\right) \right\}^2 \right) \frac{(1+r)(1-s)}{16} \right]$$

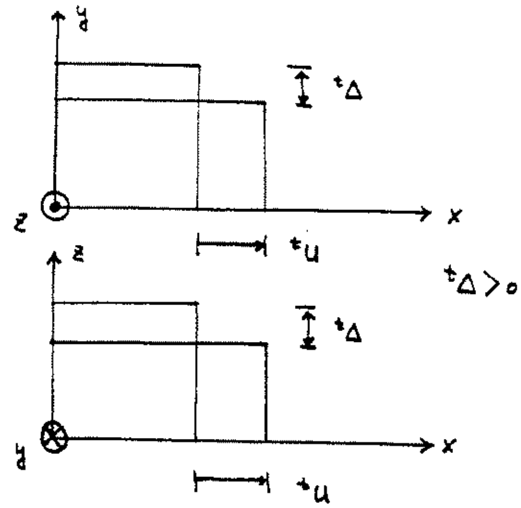
6.58

$$ {}^{t}_{0}\tilde{\varepsilon}^{DI}_{23} = \frac{1}{2}\Big[\left(\frac{1}{2}\right)\left(-\frac{1+r}{4}\right)\left\{1+\frac{t}{2}\left(\frac{\sqrt{3}}{2}-1\right)\right\} - \left(\left(0.1-\frac{t}{4}\right)\left(-\frac{1}{4}\right)+\left\{1+\frac{t}{2}\left(\frac{\sqrt{3}}{2}-1\right)\right\}\left\{\frac{1}{2}\left(\frac{\sqrt{3}}{2}-1\right)\right\}\right)\frac{(1+r)^2(1-s)}{16}\Big] $$

$$ {}^{t}_{0}\tilde{\varepsilon}^{DI}_{31} = \frac{1}{2}\Big[\left(\frac{1}{2}\right)\frac{1-s}{4}\left\{1+\frac{t}{2}\left(\frac{\sqrt{3}}{2}-1\right)\right\} + 10\cdot\frac{(1+r)(1-s)}{4}\left(-\frac{1}{4}\right) + \left(\left(-\frac{1}{4}\right)\left(0.1-\frac{t}{4}\right)+\left\{\frac{1}{2}\left(\frac{\sqrt{3}}{2}-1\right)\right\}\left\{1+\frac{t}{2}\left(\frac{\sqrt{3}}{2}-1\right)\right\}\right)\frac{(1+r)(1-s)^2}{16}\Big] $$

6.61





$^0L = 2, \quad {}^tL = {}^0L + {}^tu$

$$ {}^t_0\underline{X} = \begin{bmatrix} {}^t_0X_{11} & & \\ & {}^t_0X_{22} & \\ & & {}^t_0X_{22} \end{bmatrix} = \begin{bmatrix} 1+\frac{{}^tu}{2} & & \\ & 1-\frac{{}^t\Delta}{2} & \\ & & 1-\frac{{}^t\Delta}{2} \end{bmatrix} $$

$$ {}^t_0\underline{\epsilon} = \frac{1}{2}\left({}^t_0\underline{X}^T\,{}^t_0\underline{X} - \underline{I}\right) = \frac{1}{2}\begin{bmatrix} {}^t_0X_{11}^2-1 & & \\ & {}^t_0X_{22}^2-1 & \\ & & {}^t_0X_{33}^2-1 \end{bmatrix} $$

(i) T.L. formulation

$$ {}^t_0S_{ij} = {}^t_0C_{ijrs}\,{}^t_0\epsilon_{rs}\ , \quad {}^t_0C_{ijrs} = \lambda\delta_{ij}\delta_{rs} + \mu(\delta_{ir}\delta_{js} + \delta_{is}\delta_{jr}) $$

$$ {}^t_0C_{1111} = {}^t_0C_{2222} = {}^t_0C_{3333} = \tilde{E} \quad \text{where} \quad \tilde{E} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} $$

$$ {}^t_0C_{1122} = {}^t_0C_{2211} = {}^t_0C_{2233} = {}^t_0C_{3322} = {}^t_0C_{3311} = {}^t_0C_{1133} = \tilde{E}\,\frac{\nu}{1-\nu} $$

$$ {}^t_0S_{11} = \tilde{E}\left({}^t_0\epsilon_{11} + \frac{\nu}{1-\nu}\,{}^t_0\epsilon_{22} + \frac{\nu}{1-\nu}\,{}^t_0\epsilon_{33}\right) $$

$$ {}^t_0S_{22} = \tilde{E}\left(\frac{\nu}{1-\nu}\,{}^t_0\epsilon_{11} + {}^t_0\epsilon_{22} + \frac{\nu}{1-\nu}\,{}^t_0\epsilon_{33}\right) = {}^t_0S_{33} $$

$$ {}^t_0S_{12} = {}^t_0S_{23} = {}^t_0S_{31} = 0\,. $$

6.61

Using ${}^t\underline{\tau} = \frac{{}^t\rho}{{}^0\rho}\, {}^t_0\underline{X}\, {}^t_0\underline{S}\, {}^t_0\underline{X}^T$,

$${}^t\tau_{11} = \frac{{}^t\rho}{{}^0\rho}\, {}^t_0X_{11}^2\, {}^t_0S_{11}, \quad {}^t\tau_{22} = \frac{{}^t\rho}{{}^0\rho}\, {}^t_0X_{22}^2\, {}^t_0S_{22} = {}^t\tau_{33}$$

As ${}^t\tau_{22} = {}^t\tau_{33} = 0$, ${}^t_0S_{22} = {}^t_0S_{33} = 0 \longrightarrow {}^t_0\epsilon_{22} = -\nu\, {}^t_0\epsilon_{11} = {}^t_0\epsilon_{33}$

$$\therefore\ {}^t_0S_{11} = \widetilde{E}\, {}^t_0\epsilon_{11}$$

$${}^t\tau_{11} = \left(\frac{{}^0\bar{A}}{{}^t\bar{A}}\, \frac{{}^0L}{{}^tL}\right) {}^t_0X_{11}^2\, {}^t_0S_{11} = \frac{\widetilde{E}\, {}^0\bar{A}}{2\, {}^t\bar{A}}\, {}^t_0X_{11}\left({}^t_0X_{11}^2 - 1\right) = \frac{{}^tP}{{}^t\bar{A}}$$

$$\therefore\ \frac{{}^tP}{{}^0\bar{A}} = \frac{\widetilde{E}}{2}\, {}^t_0X_{11}\left({}^t_0X_{11}^2 - 1\right) \quad \text{where } {}^t_0X_{11} = 1 + \frac{{}^tu}{2}$$

(ii) U.L. formulation

$${}^t\tau_{ij} = {}^t_tC_{ijrs}\, {}^t_t\epsilon^A_{rs} \quad \text{where } {}^t_tC_{ijrs} = \lambda\delta_{ij}\delta_{rs} + \mu(\delta_{ir}\delta_{js} + \delta_{is}\delta_{jr})$$

From ${}^t_t\epsilon^A_{rs} = {}^0_tX_{i,m}\, {}^0_tX_{j,n}\, {}^t_0\epsilon_{ij}$ we have

$${}^t_t\underline{\epsilon}^A = \begin{bmatrix} {}^t_t\epsilon^A_{11} & & \\ & {}^t_t\epsilon^A_{22} & \\ & & {}^t_t\epsilon^A_{33} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \frac{{}^t_0X_{11}^2 - 1}{{}^t_0X_{11}^2} & & \\ & \frac{{}^t_0X_{22}^2 - 1}{{}^t_0X_{22}^2} & \\ & & \frac{{}^t_0X_{33}^2 - 1}{{}^t_0X_{33}^2} \end{bmatrix}$$

Similarly as in part (i),

$${}^t\tau_{11} = \widetilde{E}\left({}^t_t\epsilon^A_{11} + \frac{2\nu}{1-\nu}\, {}^t_t\epsilon^A_{22}\right)$$

$${}^t\tau_{22} = \widetilde{E}\left[\frac{\nu}{1-\nu}\, {}^t_t\epsilon^A_{11} + \left(1 + \frac{\nu}{1-\nu}\right) {}^t_t\epsilon^A_{22}\right] = {}^t\tau_{33}$$

With ${}^t\tau_{22} = {}^t\tau_{33} = 0$, ${}^t_t\epsilon^A_{22} = -\nu\, {}^t_t\epsilon^A_{11} = {}^t_t\epsilon^A_{33}$

$$\frac{1}{2}\, \frac{{}^t_0X_{22}^2 - 1}{{}^t_0X_{22}^2} = -\nu \cdot \frac{1}{2}\, \frac{{}^t_0X_{11}^2 - 1}{{}^t_0X_{11}^2}, \quad {}^t_0X_{22}^2 = \frac{{}^t_0X_{11}^2}{(1+\nu)\, {}^t_0X_{11}^2 - \nu}$$

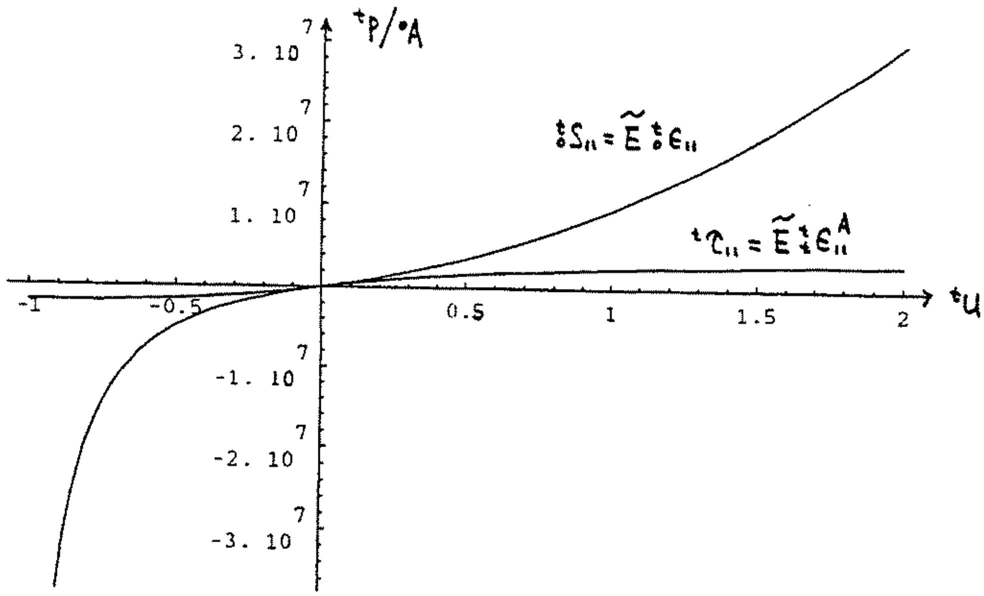
6.61

Note $\,{}^t_0X_{22}^2 = \left(1 - \frac{{}^t\Delta}{2}\right)^2 = (2 - {}^t\Delta)^2/4 = {}^t\bar{A}/{}^0\bar{A}$

$$\therefore \quad {}^t\bar{A} = \frac{{}^t_0X_{11}^2}{(1+\nu)\,{}^t_0X_{11}^2 - \nu}\,{}^0\bar{A}$$

Hence, $\quad {}^t\tau_{11} = \tilde{E}\,{}^t_t\epsilon_{11}^A = \tilde{E}\cdot\frac{1}{2}\,\frac{{}^t_0X_{11}^2 - 1}{{}^t_0X_{11}^2} = \frac{{}^tP}{{}^t\bar{A}}$

$$\therefore \quad \frac{{}^tP}{{}^0\bar{A}} = \frac{\tilde{E}}{2}\,\frac{{}^t_0X_{11}^2 - 1}{(1+\nu)\,{}^t_0X_{11}^2 - \nu} \qquad \text{where} \quad {}^t_0X_{11} = 1 + \frac{{}^tu}{2}$$



6.62

$${}^t_tC_{mnpq} = \frac{{}^t\rho}{{}^0\rho}\, {}^t_0X_{m,i}\, {}^t_0X_{n,j}\, {}^t_0C_{ijrs}\, {}^t_0X_{p,r}\, {}^t_0X_{q,s} \qquad \text{———— (6.187)}$$

$${}^t_0C_{ijrs} = \frac{{}^0\rho}{{}^t\rho}\, {}^0_tX_{i,m}\, {}^0_tX_{j,n}\, {}^t_tC_{mnpq}\, {}^0_tX_{r,p}\, {}^0_tX_{s,q} \qquad \text{———— (6.188)}$$

$${}^t_0X_{11} = \frac{{}^tL}{{}^0L}\ ,\ {}^0_tX_{11} = \frac{{}^0L}{{}^tL}\ ,\ {}^t_0\epsilon_{11} = \frac{1}{2}\left[\left(\frac{{}^tL}{{}^0L}\right)^2 - 1\right],\ {}^t_t\epsilon^A_{11} = \frac{1}{2}\left[1 - \left(\frac{{}^0L}{{}^tL}\right)^2\right],\ \frac{{}^t\rho}{{}^0\rho} = \frac{{}^0L}{{}^tL}$$

(i) $\longrightarrow$ (ii) using (6.188), $\quad {}^t_0C = \left(\frac{{}^tL}{{}^0L}\right)\left(\frac{{}^0L}{{}^tL}\right)^2 {}^t_tC \left(\frac{{}^0L}{{}^tL}\right)^2 = \left(\frac{{}^0L}{{}^tL}\right)^3 {}^t_tC = \left(\frac{{}^0L}{{}^tL}\right)^3 \tilde{E}$

$${}^tP = \frac{{}^t_0C\bar{A}}{2}\left(\frac{{}^tL}{{}^0L}\right)\left[\left(\frac{{}^tL}{{}^0L}\right)^2 - 1\right] = \frac{\tilde{E}\bar{A}}{2}\left[1 - \left(\frac{{}^0L}{{}^tL}\right)^2\right]$$

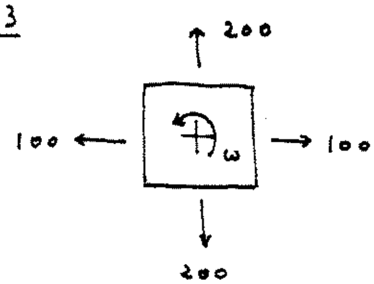
$$\therefore\ {}^tP = \frac{\tilde{E}\bar{A}}{2}\left[1 - \left(\frac{{}^0L}{{}^0L + {}^tu}\right)^2\right]$$

(ii) $\longrightarrow$ (i) using (6.187), $\quad {}^t_tC = \left(\frac{{}^0L}{{}^tL}\right)\left(\frac{{}^tL}{{}^0L}\right)^2 {}^t_0C \left(\frac{{}^tL}{{}^0L}\right)^2 = \left(\frac{{}^tL}{{}^0L}\right)^3 {}^t_0C = \left(\frac{{}^tL}{{}^0L}\right)^3 \tilde{E}$

$${}^tP = \frac{{}^t_tC\bar{A}}{2}\left[1 - \left(\frac{{}^0L}{{}^tL}\right)^2\right] = \frac{\tilde{E}\bar{A}}{2}\left(\frac{{}^tL}{{}^0L}\right)\left[\left(\frac{{}^tL}{{}^0L}\right)^2 - 1\right]$$

$$\therefore\ {}^tP = \frac{\tilde{E}\bar{A}}{2}\left(1 + \frac{{}^tu}{{}^0L}\right)\left[\left(1 + \frac{{}^tu}{{}^0L}\right)^2 - 1\right]$$

6.63



$${}^t_0\underline{X} = {}^t_0\underline{R}\,{}^t_0\underline{U} = {}^t_0\underline{R} = \begin{bmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{bmatrix}$$

$${}^t_0\dot{\underline{X}} = \omega \begin{bmatrix} -\sin\omega t & -\cos\omega t \\ \cos\omega t & -\sin\omega t \end{bmatrix}$$

$${}^t_0\underline{X}^{-1} = \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix}$$

$$\therefore\ {}^t\underline{L} = {}^t_0\dot{\underline{X}}\,{}^t_0\underline{X}^{-1} = \omega \begin{bmatrix} -\sin\omega t & -\cos\omega t \\ \cos\omega t & -\sin\omega t \end{bmatrix} \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$$

$${}^t\underline{D} = \frac{1}{2}\left({}^t\underline{L} + {}^t\underline{L}^T\right) = \underline{0}\ , \quad {}^t\underline{W} = {}^t L = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$$

From the relation of ${}^t\overset{\nabla}{\tau}_{ij} = \underline{0} = {}^t\dot{\tau}_{ij} + {}^t\tau_{ip}\,{}^tW_{pj} + {}^t\tau_{jp}\,{}^tW_{pi}$,

$$\underline{0} = \dot{\underline{\tau}} + \begin{bmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{bmatrix} \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} + \left(\begin{bmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{bmatrix} \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \right)^T$$

$$\therefore \begin{bmatrix} \dot{\tau}_{11} \\ \dot{\tau}_{22} \\ \dot{\tau}_{12} \end{bmatrix} = \begin{bmatrix} -2\omega\tau_{12} \\ 2\omega\tau_{12} \\ \omega(\tau_{11} - \tau_{22}) \end{bmatrix}$$ with ${}^0\tau_{11} = 100$, ${}^0\tau_{22} = 200$, ${}^0\tau_{12} = 0$ at $t = 0$.

Solving these three equations, we obtain

$$\begin{bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} 50(3 - \cos 2\omega t) \\ 50(3 + \cos 2\omega t) \\ -50\sin 2\omega t \end{bmatrix}$$

6.64 For (6.199) $\,{}^t_0\tilde{W} = C_1({}^t_0I_1 - 3) + C_2({}^t_0I_2 - 3)$ with ${}^t_0I_3 = 1$

where C_1, C_2 : material constants

$${}^t_0I_1 = {}^t_0C_{kk}\ , \quad {}^t_0I_2 = \frac{1}{2}\left[({}^t_0I_1)^2 - {}^t_0C_{ij}\,{}^t_0C_{ij}\right], \quad {}^t_0I_3 = \det {}^t_0C$$

$${}^t_0S_{ij} = \frac{\partial {}^t_0\tilde{W}}{\partial {}^t_0C_{mn}} \frac{\partial {}^t_0C_{mn}}{\partial {}^t_0\varepsilon_{ij}} = 2\frac{\partial {}^t_0\tilde{W}}{\partial {}^t_0C_{ij}} \qquad (\quad 2\varepsilon_{ij} = {}^t_0C_{ij} - \delta_{ij})$$

$$= 2\left(C_1 \frac{\partial {}^t_0I_1}{\partial {}^t_0C_{ij}} + C_2 \frac{\partial {}^t_0I_2}{\partial {}^t_0C_{ij}}\right)$$

$$\frac{\partial {}^t_0I_1}{\partial {}^t_0C_{ij}} = \frac{\partial {}^t_0C_{kk}}{\partial {}^t_0C_{ij}} = \delta_{ij}\ , \quad \frac{\partial {}^t_0I_2}{\partial {}^t_0C_{ij}} = \frac{1}{2}\frac{\partial}{\partial {}^t_0C_{ij}}\left[({}^t_0I_1)^2 - {}^t_0C_{kl}\,{}^t_0C_{kl}\right]$$

$$= \frac{1}{2}\left(2\,{}^t_0I_1 \frac{\partial {}^t_0I_1}{\partial {}^t_0C_{ij}} - 2C_{ij}\right) = {}^t_0I_1\delta_{ij} - {}^t_0C_{ij}$$

$$\therefore {}^t_0S_{ij} = 2\left\{C_1\delta_{ij} + C_2\left({}^t_0I_1\delta_{ij} - {}^t_0C_{ij}\right)\right\}$$

$${}^t\tau_{ij} = \frac{1}{\det {}^t_0\underline{X}}\,{}^t_0X_{ik}\,{}^t_0S_{kl}\,{}^t_0X_{jl} = {}^t_0X_{ik}\,{}^t_0S_{kl}\,{}^t_0X_{jl}$$

$$\therefore {}^tp = -\frac{1}{3}{}^t\tau_{ii} = -\frac{1}{3}\,{}^t_0X_{ik}\,{}^t_0S_{kl}\,{}^t_0X_{il} = -\frac{1}{3}\,{}^t_0C_{kl}\,{}^t_0S_{kl}$$

$$= -\frac{2}{3}\,{}^t_0C_{kl}\left[C_1\delta_{kl} + C_2({}^t_0I_1\delta_{kl} - {}^t_0C_{kl})\right]$$

$$= -\frac{2}{3}\left[C_1\,{}^t_0C_{kk} + C_2\left({}^t_0I_1\,{}^t_0C_{kk} - {}^t_0C_{kl}\,{}^t_0C_{kl}\right)\right]$$

$$= -\frac{2}{3}\left(C_1\,{}^t_0I_1 + 2C_2\,{}^t_0I_2\right)$$

Hence when using ${}^t_0\tilde{W}$, we see the pressure is dependent on ${}^t_0I_1$ and ${}^t_0I_2$.

6.64

Now for (6.203), $\,^t_0\tilde{W} = C_1(\,^t_0J_1 - 3) + C_2(\,^t_0J_2 - 3) + \frac{1}{2}\kappa(\,^t_0J_3 - 1)^2$

where $\,^t_0J_1 = \,^t_0I_1(\,^t_0I_3)^{-\frac{1}{3}}$, $\,^t_0J_2 = \,^t_0I_2(\,^t_0I_3)^{-\frac{2}{3}}$, $\,^t_0J_3 = (\,^t_0I_3)^{\frac{1}{2}}$

$$\,^t_0S_{kl} = 2\frac{\partial\,^t_0\tilde{W}}{\partial\,^t_0C_{kl}} = 2\left[C_1\frac{\partial\,^t_0J_1}{\partial\,^t_0C_{kl}} + C_2\frac{\partial\,^t_0J_2}{\partial\,^t_0C_{kl}} + \kappa(\,^t_0J_3 - 1)\frac{\partial\,^t_0J_3}{\partial\,^t_0C_{kl}}\right]$$

$$\frac{\partial\,^t_0J_1}{\partial\,^t_0C_{kl}} = \frac{\partial}{\partial\,^t_0C_{kl}}\left[\,^t_0I_1(\,^t_0I_3)^{-\frac{1}{3}}\right] = \delta_{kl}(\,^t_0I_3)^{-\frac{1}{3}} - \frac{1}{3}\,^t_0I_1(\,^t_0I_3)^{-\frac{4}{3}}(\,^t_0C^{-1}_{kl}\det\,^t_0\underline{C})$$

$$\left(\begin{aligned} &\,^t_0I_3 = \det\,^t_0\underline{C} = \varepsilon^{lmn}\,^t_0C_{pl}\,^t_0C_{qm}\,^t_0C_{rn} \\ &\frac{\partial\,^t_0I_3}{\partial\,^t_0C_{pl}} = \varepsilon^{lmn}\,^t_0C_{qm}\,^t_0C_{rn} = \varepsilon^{lmn}\,^t_0C_{pl}\,^t_0C_{qm}\,^t_0C_{rn}\,^t_0C^{-1}_{pl} = \,^t_0C^{-1}_{pl}\det\,^t_0\underline{C}\end{aligned}\right)$$

$$= \delta_{kl}(\,^t_0I_3)^{-\frac{1}{3}} - \frac{1}{3}\,^t_0I_1(\,^t_0I_3)^{-\frac{1}{3}}\,^t_0C^{-1}_{kl}$$

$$= \left(\delta_{kl} - \frac{1}{3}\,^t_0I_1\,^t_0C^{-1}_{kl}\right)(\,^t_0I_3)^{-\frac{1}{3}}$$

$$\frac{\partial\,^t_0J_2}{\partial\,^t_0C_{kl}} = \frac{\partial}{\partial\,^t_0C_{kl}}\left[\,^t_0I_2(\,^t_0I_3)^{-\frac{2}{3}}\right] = (\,^t_0I_1\delta_{kl} - \,^t_0C_{kl})(\,^t_0I_3)^{-\frac{2}{3}} - \frac{2}{3}\,^t_0I_2(\,^t_0I_3)^{-\frac{5}{3}}(\,^t_0C^{-1}_{kl}\,^t_0I_3)$$

$$= \left(\,^t_0I_1\delta_{kl} - \,^t_0C_{kl} - \frac{2}{3}\,^t_0I_2\,^t_0C^{-1}_{kl}\right)(\,^t_0I_3)^{-\frac{2}{3}}$$

$$\frac{\partial\,^t_0J_3}{\partial\,^t_0C_{kl}} = \frac{\partial}{\partial\,^t_0C_{kl}}\left[(\,^t_0I_3)^{\frac{1}{2}}\right] = \frac{1}{2}(\,^t_0I_3)^{-\frac{1}{2}}(\,^t_0C^{-1}_{kl}\,^t_0I_3) = \frac{1}{2}(\,^t_0I_3)^{\frac{1}{2}}\,^t_0C^{-1}_{kl}$$

$$\therefore\,^t_0S_{kl} = 2\Big[C_1\left(\delta_{kl} - \frac{1}{3}\,^t_0I_1\,^t_0C^{-1}_{kl}\right)(\,^t_0I_3)^{-\frac{1}{3}} + C_2\left(\,^t_0I_1\delta_{kl} - \,^t_0C_{kl} - \frac{2}{3}\,^t_0I_2\,^t_0C^{-1}_{kl}\right)(\,^t_0I_3)^{-\frac{2}{3}} + \kappa(\,^t_0J_3 - 1)\left\{\frac{1}{2}(\,^t_0I_3)^{\frac{1}{2}}\,^t_0C^{-1}_{kl}\right\}\Big]$$

6.64

$$\therefore\ {}^{t}\bar{p} = -\frac{1}{3}\,\frac{1}{\det {}^{t}_{0}\underline{X}}\ {}^{t}_{0}C_{kl}\ {}^{t}_{0}S_{kl}$$

$$= -\frac{1}{3\left({}^{t}_{0}I_3\right)^{\frac{1}{2}}}\ 2\Big[C_1\left({}^{t}_{0}C_{kk} - \tfrac{1}{3}\,{}^{t}_{0}I_1\cdot 3\right)\left({}^{t}_{0}I_3\right)^{-\frac{1}{3}}$$

$$+ C_2\left({}^{t}_{0}I_1\,{}^{t}_{0}C_{kk} - {}^{t}_{0}C_{kl}\,{}^{t}_{0}C_{kl} - \tfrac{2}{3}\,{}^{t}_{0}I_2\cdot 3\right)\left({}^{t}_{0}I_3\right)^{-\frac{2}{3}}$$

$$+ \kappa\left({}^{t}_{0}J_3 - 1\right)\tfrac{1}{2}\left({}^{t}_{0}I_3\right)^{\frac{1}{2}}\cdot 3\Big]$$

$$= -\kappa\left({}^{t}_{0}J_3 - 1\right)$$

6.65 Let L_1, L_2 and L_3 be the principal values of $\underset{\sim}{C}$,

$$\rightarrow {}^t_0\tilde{W} = \sum \frac{\mu_n}{\alpha_n}\left(L_1^{\frac{\alpha_n}{2}} + L_2^{\frac{\alpha_n}{2}} + L_3^{\frac{\alpha_n}{2}} - 3\right), \quad L_1 L_2 L_3 = 1$$

$$^tP = {}_0P({}^t_0\tilde{W}) \quad \text{where } {}_0P = -\frac{2}{3\det {}^t_0\underset{\sim}{X}}\,{}^t_0C_{ij}\frac{\partial}{\partial {}^t_0C_{ij}}$$

$$\therefore {}^tP = {}_0P({}^t_0\tilde{W}) = \frac{1}{2}\sum \mu_n\left[L_1^{\frac{\alpha_n}{2}-1}{}_0P(L_1) + L_2^{\frac{\alpha_n}{2}-1}{}_0P(L_2) + L_3^{\frac{\alpha_n}{2}-1}{}_0P(L_3)\right]$$

$$\text{where } {}_0P(L_a) = -\frac{2}{3\det {}^t_0\underset{\sim}{X}}\,{}^t_0C_{ij}\frac{\partial L_a}{\partial {}^t_0C_{ij}} \qquad \text{——— ①}$$

As L_a satisfies the following characteristic equation (sub- and superscripts are omitted for brevity).

$$L^3 - I_1L^2 + I_2L - I_3 = 0$$

where $I_1 = C_{kk}$, $I_2 = \frac{1}{2}(I_1^2 - C_{ij}C_{ij})$, $I_3 = \det \underset{\sim}{C}$.

Using $\frac{\partial I_1}{\partial C_{ij}} = \delta_{ij}$, $\frac{\partial I_2}{\partial C_{ij}} = I_1\delta_{ij} - C_{ij}$, $\frac{\partial I_3}{\partial C_{ij}} = C_{ij}^{-1}I_3$ and

$(3L^2 - 2I_1L + I_2)\frac{\partial L}{\partial C_{ij}} = \frac{\partial I_1}{\partial C_{ij}}L^2 - \frac{\partial I_2}{\partial C_{ij}}L + \frac{\partial I_3}{\partial C_{ij}}$ we obtain

$$C_{ij}\frac{\partial L}{\partial C_{ij}} = \frac{I_1L^2 - 2I_2L + 3I_3}{3L^2 - 2I_1L + I_2} \qquad \text{——— ②}$$

Using ① and ② we see that this formula results in a pressure as a function of stretches.

Now consider ${}^t_0\overline{W} = {}^t_0\tilde{\tilde{W}} + \frac{1}{2}\kappa({}^t_0J_3 - 1)^2$

$$\text{where } {}^t_0\tilde{\tilde{W}} = \sum \frac{\mu_n}{\alpha_n}\left[\left(L_1^{\frac{\alpha_n}{2}} + L_2^{\frac{\alpha_n}{2}} + L_3^{\frac{\alpha_n}{2}}\right)(L_1L_2L_3)^{-\frac{\alpha_n}{6}} - 3\right]$$

6.65

$${}^t_0\tilde{\tilde{S}}_{ij} = 2\,\frac{\partial\,{}^t_0\tilde{\tilde{W}}}{\partial L_a}\,\frac{\partial L_a}{\partial\,{}^t_0C_{ij}}$$

$$\frac{\partial\,{}^t_0\tilde{\tilde{W}}}{\partial L_1} = \sum \mu_n\left[\frac{1}{3}L_1^{\frac{\alpha_n}{2}-1} - \frac{1}{6}\left(L_2^{\frac{\alpha_n}{2}} + L_3^{\frac{\alpha_n}{2}}\right)L_1^{-1}\right](L_1L_2L_3)^{-\frac{\alpha_n}{6}}$$

with suitable permutations of the La for $\frac{\partial\,{}^t_0\tilde{\tilde{W}}}{\partial L_2}$ and $\frac{\partial\,{}^t_0\tilde{\tilde{W}}}{\partial L_3}$

$$\begin{aligned}\therefore\ {}^t_0C_{ij}\,{}^t_0\tilde{\tilde{S}}_{ij} &= 2\sum \mu_n\left[\frac{1}{3}L_1^{\frac{\alpha_n}{2}-1} - \frac{1}{6}\left(L_2^{\frac{\alpha_n}{2}} + L_3^{\frac{\alpha_n}{2}}\right)L_1^{-1}\right](L_1L_2L_3)^{-\frac{\alpha_n}{6}}\frac{\partial L_1}{\partial\,{}^t_0C_{ij}}\,{}^t_0C_{ij} \\ &+ 2\sum \mu_n\left[\frac{1}{3}L_2^{\frac{\alpha_n}{2}-1} - \frac{1}{6}\left(L_3^{\frac{\alpha_n}{2}} + L_1^{\frac{\alpha_n}{2}}\right)L_2^{-1}\right](L_1L_2L_3)^{-\frac{\alpha_n}{6}}\frac{\partial L_2}{\partial\,{}^t_0C_{ij}}\,{}^t_0C_{ij} \\ &+ 2\sum \mu_n\left[\frac{1}{3}L_3^{\frac{\alpha_n}{2}-1} - \frac{1}{6}\left(L_1^{\frac{\alpha_n}{2}} + L_2^{\frac{\alpha_n}{2}}\right)L_3^{-1}\right](L_1L_2L_3)^{-\frac{\alpha_n}{6}}\frac{\partial L_3}{\partial\,{}^t_0C_{ij}}\,{}^t_0C_{ij}\end{aligned} \quad \text{③}$$

From ② and ③ we have after a lengthy calculation

$${}^t_0C_{ij}\,{}^t_0\tilde{\tilde{S}}_{ij} = 0.$$

And we see the terms under the summation sign do not affect the pressure in expression (6.207).

6.66

(i) For ease of demonstration consider the following deformation:

$$
{}_0^t\underline{X} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ; \quad {}_0^t\underline{C} = \begin{bmatrix} 1 & \gamma & 0 \\ \gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
{}_0^tI_1 = 3 + \gamma^2 , \quad {}_0^tI_2 = 3 + \gamma^2 ; \quad {}_0^t\widetilde{W} = C_1\gamma^2 + C_2\gamma^2
$$

$$
G = \frac{\partial^2\, {}_0^t\widetilde{W}}{\partial\gamma^2} = 2\,(C_1 + C_2) ; \qquad E = 2G\,(1+\nu) = 6\,(C_1 + C_2)
$$

(ii)
$$
{}_0^t\widetilde{W} = \sum_{n=1}^{3} \mu_n/\alpha_n \left[L_1^{\alpha_n/2} + L_2^{\alpha_n/2} + L_3^{\alpha_n/2} - 3 \right] , \quad L_1 L_2 L_3 = 1 ,
$$

and L_i are the eigenvalues of the stretch tensor ${}_0^t\underline{C}$.

$$
L_1 = \frac{2+\gamma^2}{2} - \frac{\left[(2+\gamma^2) - 4 \right]^{1/2}}{2} ; \qquad L_2 = \frac{2+\gamma^2}{2} + \frac{\left[(2+\gamma^2) - 4 \right]^{1/2}}{2} ;
$$

$$
L_3 = 1 ; \qquad \frac{\partial^2 L_1}{\partial\gamma^2} \cong \frac{\partial^2 L_2}{\partial\gamma^2} \cong 1 + \text{h.o.t.} \Rightarrow
$$

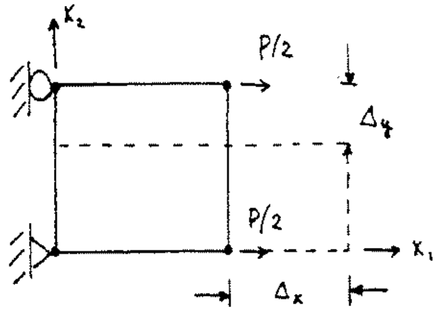
$$
\frac{\partial\, {}_0^t\widetilde{W}}{\partial\gamma} = \sum_{n=1}^{3} \frac{\mu_n}{\alpha_n} \left\{ \frac{\partial L_1}{\partial\gamma} L_1^{\alpha_n/2-1} + \frac{\partial L_2}{\partial\gamma} L_2^{\alpha_n/2-1} \right\} \frac{\alpha_n}{2} ;
$$

$$
G = \frac{\partial^2\, {}_0^t\widetilde{W}}{\partial\gamma^2} = \sum_{n=1}^{3} \frac{\mu_n}{2} \left\{ \frac{\partial^2 L_1}{\partial\gamma^2} L_1^{\alpha_n/2-1} + \frac{\partial^2 L_2}{\partial\gamma^2} L_2^{\alpha_n/2-1} + \left(\frac{\alpha_n}{2} - 1 \right) \times \right.
$$

$$
\left. \times \left[\left(\frac{\partial L_1}{\partial\gamma} \right)^2 L_1^{\alpha_n/2-2} + \left(\frac{\partial L_2}{\partial\gamma} \right)^2 L_2^{\alpha_n/2-2} \right] \right\} \cong \sum_{n=1}^{3} \frac{\mu_n \alpha_n}{2} + \text{h.o.t.} ,
$$

and using $\nu = 0.5$, we have $E = 3G$.

6.67



As κ is very large we assume that the incompressible condition is enforced.

$$2 \cdot 2 = (2+\Delta_x)(2-\Delta_y)$$

$$\therefore \Delta_y = \frac{2\Delta}{2+\Delta} \qquad \text{where } \Delta \equiv \Delta_x$$

$$^t x_1 = \left(1+\frac{\Delta}{2}\right){}^0x_1 = \frac{2+\Delta}{2}\,{}^0x_1 , \quad {}^t x_2 = \left(1-\frac{\Delta_y}{2}\right) = \frac{2}{2+\Delta}\,{}^0x_2 , \quad {}^t x_3 = {}^0x_3$$

$$\therefore \ {}^t_0\underline{X} = \left[\frac{\partial\, {}^t x_i}{\partial\, {}^0 x_j}\right] = \begin{bmatrix} \lambda & & \\ & 1/\lambda & \\ & & 1 \end{bmatrix} \qquad \text{with } \lambda = \frac{2+\Delta}{2}$$

$$ {}^t_0\underline{C} = {}^t_0\underline{X}^T\, {}^t_0\underline{X} = \begin{bmatrix} \lambda^2 & & \\ & 1/\lambda^2 & \\ & & 1 \end{bmatrix}$$

__Mooney - Rivlin model__

$$ {}^t_0\widetilde{W} = C_1\left({}^t_0I_1 - 3\right) + C_2\left({}^t_0I_2 - 3\right)$$

$$ {}^t_0I_1 = {}^t_0C_{kk} \ , \quad {}^t_0I_2 = \frac{1}{2}\left[\left({}^t_0I_1\right)^2 - {}^t_0C_{ij}\,{}^t_0C_{ij}\right]$$

$$ {}^t_0S_{11} = \frac{\partial\, {}^t_0\widetilde{W}}{\partial\, {}^t_0\epsilon_{11}} = C_1\frac{\partial\, {}^t_0I_1}{\partial\, {}^t_0\epsilon_{11}} + C_2\frac{\partial\, {}^t_0I_2}{\partial\, {}^t_0\epsilon_{11}}$$

From ${}^t_0C_{11}\,{}^t_0C_{22}\,{}^t_0C_{33} = 1$ and symmetry,

$$\frac{\partial\, {}^t_0C_{11}}{\partial\, {}^t_0\epsilon_{11}}\,{}^t_0C_{22}\,{}^t_0C_{33} + \frac{\partial\, {}^t_0C_{22}}{\partial\, {}^t_0\epsilon_{11}}\,{}^t_0C_{11}\,{}^t_0C_{33} = 0$$

$$\therefore \ \frac{\partial\, {}^t_0C_{22}}{\partial\, {}^t_0\epsilon_{11}} = -\frac{{}^t_0C_{22}}{{}^t_0C_{11}}\,\frac{\partial\, {}^t_0C_{11}}{\partial\, {}^t_0\epsilon_{11}} = -\frac{2}{\lambda^4}$$

6.67

$$\frac{\partial\,{}_0^tI_1}{\partial\,{}_0^t\epsilon_{11}} = \frac{\partial\,{}_0^tI_1}{\partial\,{}_0^tC_{11}}\frac{\partial\,{}_0^tC_{11}}{\partial\,{}_0^t\epsilon_{11}} + \frac{\partial\,{}_0^tI_1}{\partial\,{}_0^tC_{22}}\frac{\partial\,{}_0^tC_{22}}{\partial\,{}_0^t\epsilon_{11}} = 2\left(1-\frac{1}{\lambda^4}\right)$$

$$\frac{\partial\,{}_0^tI_2}{\partial\,{}_0^t\epsilon_{11}} = \frac{\partial\,{}_0^tI_2}{\partial\,{}_0^tC_{11}}\frac{\partial\,{}_0^tC_{11}}{\partial\,{}_0^t\epsilon_{11}} + \frac{\partial\,{}_0^tI_2}{\partial\,{}_0^tC_{22}}\frac{\partial\,{}_0^tC_{22}}{\partial\,{}_0^t\epsilon_{11}} = 2\left(1-\frac{1}{\lambda^4}\right)$$

$$\therefore\ {}_0^tS_{11} = 2C_1\left(1-\frac{1}{\lambda^4}\right) + 2C_2\left(1-\frac{1}{\lambda^4}\right) = 2(C_1+C_2)\left(1-\frac{1}{\lambda^4}\right)$$

$${}^t\tau_{11} = {}_0^tX_{11}^2\ {}_0^tS_{11} = \lambda^2\ {}_0^tS_{11}$$

$${}^tF = {}^t\tau_{11}\left({}^0A\,\frac{1}{\lambda}\right) = {}^0A\,\lambda\ {}_0^tS_{11} = P$$

$$\Longrightarrow\quad P/{}^0A = 2(C_1+C_2)\left(\lambda-\frac{1}{\lambda^3}\right)$$

Ogden model

$${}_0^t\widetilde{W} = \sum \frac{\mu_n}{\alpha_n}\left(L_1^{\frac{\alpha_n}{2}} + L_2^{\frac{\alpha_n}{2}} + L_3^{\frac{\alpha_n}{2}} - 3\right), \quad L_1L_2L_3 = 1$$

$${}_0^tS_{ij} = \frac{\partial\,{}_0^t\widetilde{W}}{\partial\,{}_0^t\epsilon_{ij}} = 2\frac{\partial\,{}_0^t\widetilde{W}}{\partial\,{}_0^tC_{ij}} = 2\frac{\partial\,{}_0^t\widetilde{W}}{\partial L_a}\frac{\partial L_a}{\partial\,{}_0^tC_{ij}} \qquad a = 1,2,3$$

$${}_0^tS_{11} = \frac{\partial\,{}_0^t\widetilde{W}}{\partial\,{}_0^t\epsilon_{11}} = 2\left(\frac{\partial\,{}_0^t\widetilde{W}}{\partial L_1}\frac{\partial L_1}{\partial\,{}_0^tC_{11}} + \frac{\partial\,{}_0^t\widetilde{W}}{\partial L_2}\frac{\partial L_2}{\partial\,{}_0^tC_{11}}\right)$$

Using $L_1L_2L_3 = 1$ with $L_1 = \lambda^2$, $L_2 = 1/\lambda^2$ and $L_3 = 1$,

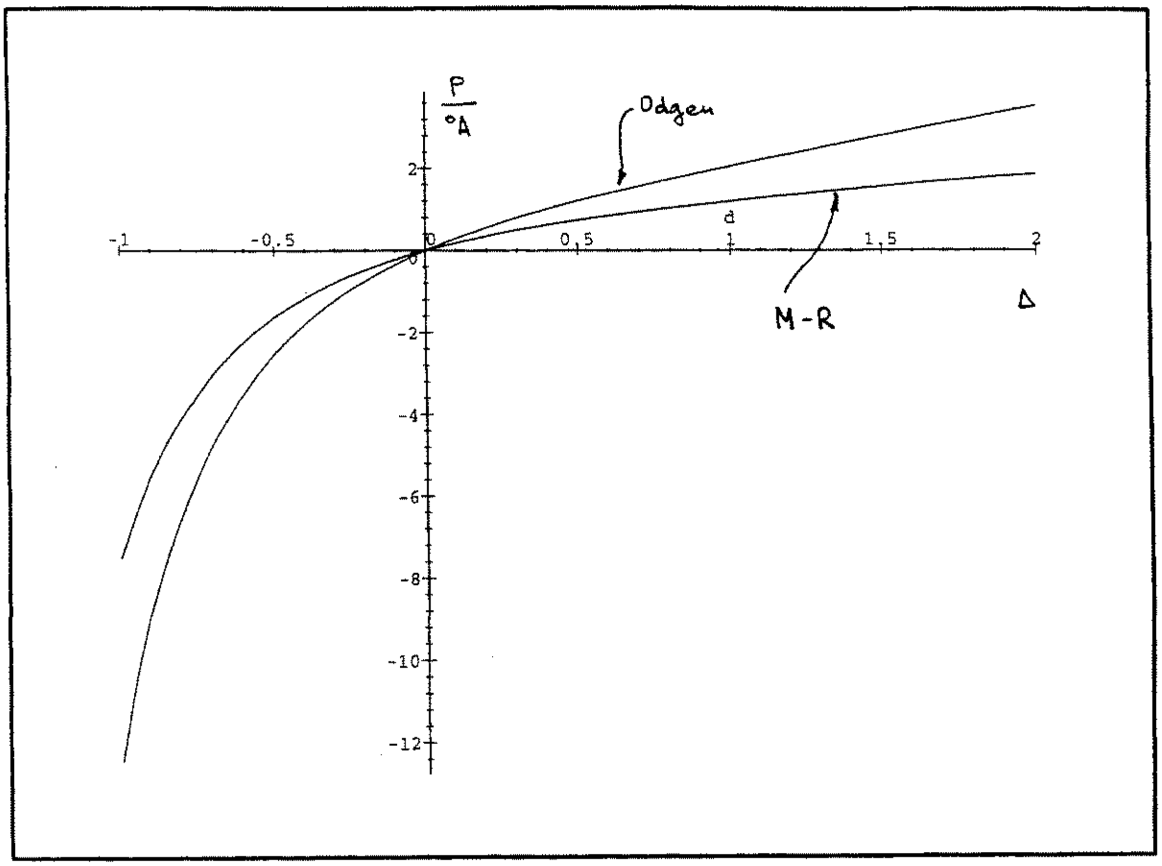
$$\frac{\partial L_1}{\partial\,{}_0^tC_{11}}L_2L_3 + \frac{\partial L_2}{\partial\,{}_0^tC_{11}}L_1L_3 = 0 \qquad \therefore\ \frac{\partial L_2}{\partial\,{}_0^tC_{11}} = -\frac{L_2}{L_1}\frac{\partial L_1}{\partial\,{}_0^tC_{11}} = -\frac{1}{\lambda^4}$$

$$\therefore\ {}_0^tS_{11} = \sum \mu_n\left[L_1^{\frac{\alpha_n}{2}-1} + L_2^{\frac{\alpha_n}{2}-1}\left(-\frac{1}{\lambda^4}\right)\right]$$

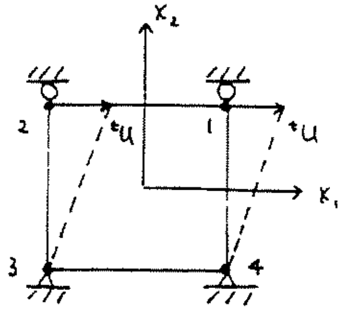
$$= \sum \mu_n\left[\lambda^{\alpha_n-2} - \frac{1}{\lambda^{\alpha_n+2}}\right]$$

$$\Longrightarrow\quad P/{}^0A = \sum \mu_n\left(\lambda^{\alpha_n-1} - \frac{1}{\lambda^{\alpha_n+1}}\right)$$

6.67



6.68



$${}^tx_1 = {}^0x_1 + \frac{1+{}^0x_2}{2}\,{}^tu$$

$${}^tx_2 = {}^0x_2\,, \qquad {}^tx_3 = {}^0x_3$$

$${}^t_0\underline{X} = \left[\frac{\partial\,{}^tx_i}{\partial\,{}^0x_j}\right] = \begin{bmatrix} 1 & \frac{{}^tu}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$${}^t_0\underline{C} = {}^t_0\underline{X}^T\,{}^t_0\underline{X} = \begin{bmatrix} 1 & \frac{{}^tu}{2} & 0 \\ \frac{{}^tu}{2} & 1+\left(\frac{{}^tu}{2}\right)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Using the Mooney-Rivlin model,

$${}^t_0\bar{W} = C_1\left({}^t_0J_1 - 3\right) + C_2\left({}^t_0J_2 - 3\right) + \frac{1}{2}\kappa\left({}^t_0J_3 - 1\right)^2$$

where $\quad {}^t_0J_1 = {}^t_0I_1\left({}^t_0I_3\right)^{-\frac{1}{3}}\,, \quad {}^t_0J_2 = {}^t_0I_2\left({}^t_0I_3\right)^{-\frac{2}{3}}\,, \quad {}^t_0J_3 = \left({}^t_0I_3\right)^{\frac{1}{2}}$

$${}^t_0S_{ij} = \frac{1}{2}\left(\frac{\partial\,{}^t_0\bar{W}}{\partial\,{}^t_0\varepsilon_{ij}} + \frac{\partial\,{}^t_0\bar{W}}{\partial\,{}^t_0\varepsilon_{ji}}\right) = C_1\left({}^t_0J_1\right)^*_{ij} + C_2\left({}^t_0J_2\right)^*_{ij} + \kappa\left({}^t_0J_3 - 1\right)\left({}^t_0J_3\right)^*_{ij}$$

where $\quad (\quad)^*_{ij} = \frac{1}{2}\left(\frac{\partial}{\partial\,{}^t_0\varepsilon_{ij}} + \frac{\partial}{\partial\,{}^t_0\varepsilon_{ji}}\right) = \frac{\partial}{\partial\,{}^t_0C_{ij}} + \frac{\partial}{\partial\,{}^t_0C_{ji}}$

$$\left({}^t_0J_1\right)^*_{ij} = \left({}^t_0I_3\right)^{-\frac{1}{3}}\left({}^t_0I_1\right)^*_{ij} - \frac{1}{3}\left({}^t_0I_1\,{}^t_0I_3^{-\frac{4}{3}}\right)\left({}^t_0I_3\right)^*_{ij}$$

$$\left({}^t_0J_2\right)^*_{ij} = \left({}^t_0I_3\right)^{-\frac{2}{3}}\left({}^t_0I_2\right)^*_{ij} - \frac{2}{3}\left({}^t_0I_2\,{}^t_0I_3^{-\frac{5}{3}}\right)\left({}^t_0I_3\right)^*_{ij}$$

$$\left({}^t_0J_3\right)^*_{ij} = \frac{1}{2}\left({}^t_0I_3\right)^{-\frac{1}{2}}\left({}^t_0I_3\right)^*_{ij}$$

with $\quad \left({}^t_0I_1\right)^*_{ij} = 2\,\delta_{ij} \qquad \left({}^t_0I_2\right)^*_{ij} = 2\left({}^t_0I_1\,\delta_{ij} - {}^t_0C_{ij}\right)$

$$\left({}^t_0I_3\right)^*_{ij} = 2\,{}^0_tC_{ij}\,\det\,{}^t_0\underline{C}\,, \qquad {}^0_tC_{ij} = \left({}^t_0C_{ij}\right)^{-1}$$

6.6B

$$\rightarrow \; {}^t_0S_{ij} = 2\left[C_1\left(\delta_{ij} - \frac{1}{3}\,{}^t_0I_1\,{}^0_tC_{ij}\right) + C_2\left({}^t_0I_1\,\delta_{ij} - {}^t_0C_{ij} - \frac{2}{3}\,{}^t_0I_2\,{}^0_tC_{ij}\right)\right]$$

where

$$\left[{}^0_tC_{ij}\right] = \begin{bmatrix} 1 + \left(\frac{^tu}{2}\right)^2 & \left(\frac{^tu}{2}\right) & 0 \\ -\left(\frac{^tu}{2}\right) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \; {}^t_0S_{11} = -\frac{2}{3}\left(\frac{^tu}{2}\right)^2\left[C_1\left\{4 + \left(\frac{^tu}{2}\right)^2\right\} + C_2\left\{5 + 2\left(\frac{^tu}{2}\right)^2\right\}\right]$$

$${}^t_0S_{22} = -\frac{2}{3}\left(\frac{^tu}{2}\right)^2\left(C_1 + 2C_2\right)$$

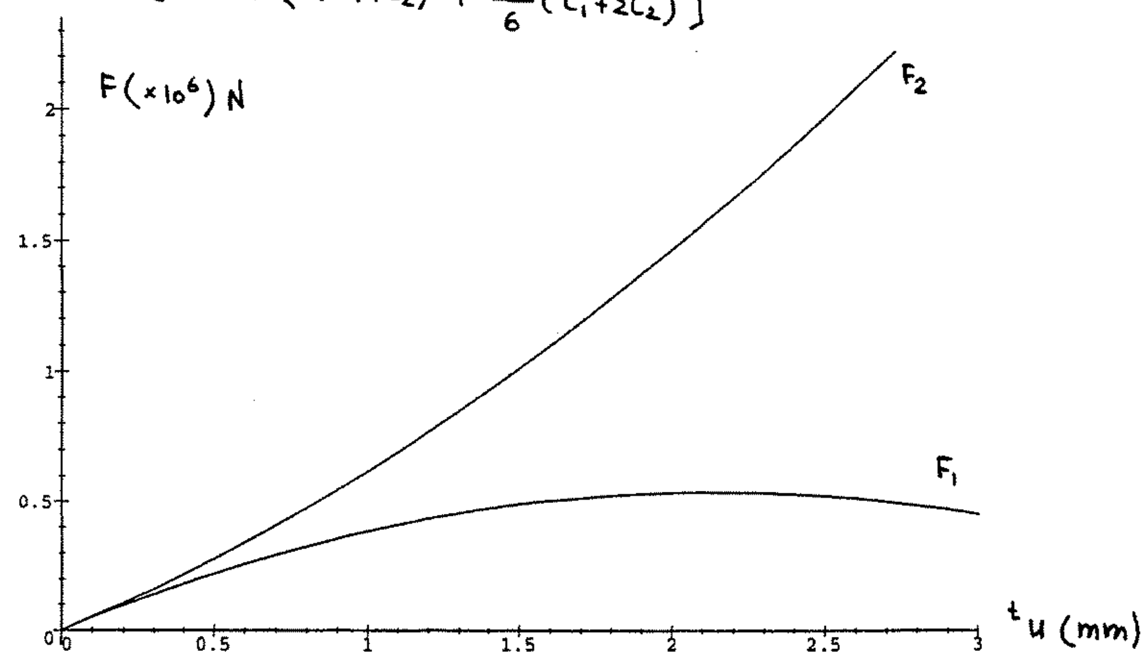
$${}^t_0S_{12} = \frac{2}{3}\left(\frac{^tu}{2}\right)\left[C_1\left\{3 + \left(\frac{^tu}{2}\right)^2\right\} + C_2\left\{3 + 2\left(\frac{^tu}{2}\right)^2\right\}\right]$$

Let F_1 and F_2 be the forces in the x_1 direction at time t for node 1 and 2 respectively.

From $\;{}^t_0\underline{F} = \int_{^0V} {}^t_0\underline{B}_L^T\,{}^t_0\hat{\underline{S}}\; d^0V\;$ we have

$$F_1 = {}^tu\left[\left(C_1 + C_2\right) - \frac{^tu}{6}\left(C_1 + 2C_2\right)\right]$$

$$F_2 = {}^tu\left[\left(C_1 + C_2\right) + \frac{^tu}{6}\left(C_1 + 2C_2\right)\right]$$



6.69 $\quad {}^t_0\tilde{W} = C_1({}^t_0I_1-3) + C_2({}^t_0I_2-3) + C_3({}^t_0I_1-3)^2 + C_4({}^t_0I_1-3)({}^t_0I_2-3) + C_5({}^t_0I_2-3)^2$

with $C_1 = 75$, $C_2 = 25$, $C_3 = 10 = C_4 = C_5$

$${}^tx_1 = \left(1+\frac{\Delta}{{}^0L}\right){}^0x_1\,, \quad \frac{\partial\,{}^tx_1}{\partial\,{}^0x_1} = 1+\frac{\Delta}{{}^0L}$$

Let $${}^t_0\underline{X} = \begin{bmatrix} 1+\frac{\Delta}{{}^0L} & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \gamma \end{bmatrix}, \text{ then } \det {}^t_0\underline{X} = 1 = \left(1+\frac{\Delta}{{}^0L}\right)\gamma^2 \quad \therefore \gamma^2 = 1\Big/\left(1+\frac{\Delta}{{}^0L}\right)$$

$$\therefore \ {}^t_0\underline{X} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1/\sqrt{\lambda} & 0 \\ 0 & 0 & 1/\sqrt{\lambda} \end{bmatrix} \quad \text{where } \lambda = 1+\frac{\Delta}{{}^0L}$$

$${}^t_0\underline{C} = {}^t_0\underline{X}^T\,{}^t_0\underline{X} = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & 1/\lambda & 0 \\ 0 & 0 & 1/\lambda \end{bmatrix} \longrightarrow \ {}^t_0I_1 = {}^t_0C_{kk} = \lambda^2 + \frac{2}{\lambda}$$

$${}^t_0I_2 = \frac{1}{2}\left[({}^t_0I_1)^2 - {}^t_0C_{ij}\,{}^t_0C_{ij}\right] = 2\lambda + \frac{1}{\lambda^2}$$

$${}^t_0S_{11} = \frac{\partial\,{}^t_0\tilde{W}}{\partial\,{}^t_0\epsilon_{11}} = \left\{C_1 + 2C_3({}^t_0I_1-3) + C_4({}^t_0I_2-3)\right\}\frac{\partial\,{}^t_0I_1}{\partial\,{}^t_0\epsilon_{11}}$$
$$+ \left\{C_2 + C_4({}^t_0I_1-3) + 2C_5({}^t_0I_2-3)\right\}\frac{\partial\,{}^t_0I_2}{\partial\,{}^t_0\epsilon_{11}}$$

From ${}^t_0C_{11}\,{}^t_0C_{22}\,{}^t_0C_{33} = 1$ and symmetry,

$$\frac{\partial\,{}^t_0C_{11}}{\partial\,{}^t_0\epsilon_{11}}\,{}^t_0C_{22}\,{}^t_0C_{33} + 2\,\frac{\partial\,{}^t_0C_{22}}{\partial\,{}^t_0\epsilon_{11}}\,{}^t_0C_{11}\,{}^t_0C_{33} = 0$$

$$\therefore \ \frac{\partial\,{}^t_0C_{22}}{\partial\,{}^t_0\epsilon_{11}} = -\frac{1}{2}\,\frac{\partial\,{}^t_0C_{11}}{\partial\,{}^t_0\epsilon_{11}}\,\frac{{}^t_0C_{22}}{{}^t_0C_{11}} = -\frac{1}{\lambda^3}$$

$$\frac{\partial\,{}^t_0I_1}{\partial\,{}^t_0\epsilon_{11}} = \frac{\partial\,{}^t_0I_1}{\partial\,{}^t_0C_{11}}\,\frac{\partial\,{}^t_0C_{11}}{\partial\,{}^t_0\epsilon_{11}} + 2\,\frac{\partial\,{}^t_0I_1}{\partial\,{}^t_0C_{22}}\,\frac{\partial\,{}^t_0C_{22}}{\partial\,{}^t_0\epsilon_{11}} = 2\left(1-\frac{1}{\lambda^3}\right)$$

$$\frac{\partial\,{}^t_0I_2}{\partial\,{}^t_0\epsilon_{11}} = \frac{\partial\,{}^t_0I_2}{\partial\,{}^t_0C_{11}}\,\frac{\partial\,{}^t_0C_{11}}{\partial\,{}^t_0\epsilon_{11}} + 2\,\frac{\partial\,{}^t_0I_2}{\partial\,{}^t_0C_{22}}\,\frac{\partial\,{}^t_0C_{22}}{\partial\,{}^t_0\epsilon_{11}} = 2\left(\frac{1}{\lambda}-\frac{1}{\lambda^4}\right)$$

6.69

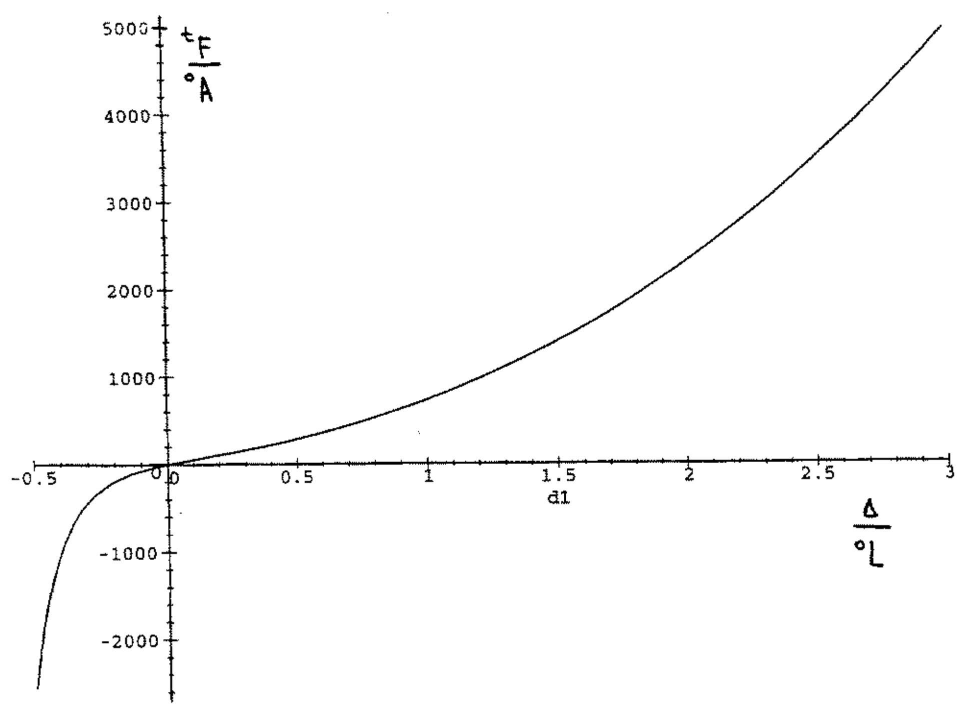
$$\therefore {}_0^tS_{11} = \left\{C_1 + 2C_3\left(\lambda^2 + \frac{2}{\lambda} - 3\right) + C_4\left(2\lambda + \frac{1}{\lambda^2} - 3\right)\right\} 2\left(1 - \frac{1}{\lambda^3}\right)$$
$$+ \left\{C_2 + C_4\left(\lambda^2 + \frac{2}{\lambda} - 3\right) + 2C_5\left(2\lambda + \frac{1}{\lambda^2} - 3\right)\right\} 2\left(\frac{1}{\lambda} - \frac{1}{\lambda^4}\right)$$
$$= 2\Bigg[C_1\left(1 - \frac{1}{\lambda^3}\right) + C_2\left(\frac{1}{\lambda} - \frac{1}{\lambda^4}\right) + 2C_3\left(\lambda^2 + \frac{2}{\lambda} - 3\right)\left(1 - \frac{1}{\lambda^3}\right)$$
$$+ C_4\left\{\left(2\lambda + \frac{1}{\lambda^2} - 3\right)\left(1 - \frac{1}{\lambda^3}\right) + \left(\lambda^2 + \frac{2}{\lambda} - 3\right)\left(\frac{1}{\lambda} - \frac{1}{\lambda^4}\right)\right\}$$
$$+ 2C_5\left(2\lambda + \frac{1}{\lambda^2} - 3\right)\left(\frac{1}{\lambda} - \frac{1}{\lambda^4}\right)\Bigg]$$

$${}^t\tau_{11} = {}_0^tX_{11}^2\, {}_0^tS_{11} = \lambda^2\, {}_0^tS_{11}$$

$${}^tF = {}^t\tau_{11}\left({}^0A\,\frac{1}{\lambda}\right) = {}^0A\,\lambda\, {}_0^tS_{11}$$

$$\therefore {}^tF = 2\,{}^0A\left[20\lambda^3 + 90\lambda^2 - 35\lambda - 105 + \frac{55}{\lambda^2} + \frac{85}{\lambda^3} - \frac{90}{\lambda^4} - \frac{20}{\lambda^5}\right]$$

where $\lambda = 1 + \dfrac{\Delta}{{}^0L}$



6.70 In plane stress analysis, the constitutive relation is given corresponding to the TL formulation and can be written as

$$\begin{bmatrix} {}^t_0S_{11} \\ {}^t_0S_{22} \\ {}^t_0S_{12} \end{bmatrix} = 2C_1\left(\begin{bmatrix}1\\1\\0\end{bmatrix} - ({}^tC_{33})^2\begin{bmatrix}{}^tC_{22}\\{}^tC_{11}\\-{}^tC_{12}\end{bmatrix}\right)$$

$$+2C_2\left({}^tC_{33}\begin{bmatrix}1\\1\\0\end{bmatrix} + \{1-({}^tC_{33})^2({}^tC_{11}+{}^tC_{22})\}\begin{bmatrix}{}^tC_{22}\\{}^tC_{11}\\-{}^tC_{12}\end{bmatrix}\right)$$

in which ${}^tC_{ij} = 2\,{}^t_0\varepsilon_{ij} + \delta_{ij}$

Since the material is incompressible, ${}^tC_{33} = [{}^tC_{11}\,{}^tC_{22} - ({}^tC_{12})^2]^{-1}$

Hence,

$$_0\underline{C} = 4C_1({}^tC_{33})^2\left\{2\,{}^tC_{33}\,{}^t\hat{\underline{C}} + \begin{bmatrix}0&-1&0\\-1&0&0\\0&0&1\end{bmatrix}\right\}$$

$$+4C_2({}^tC_{33})^2\left\{2\,{}^tC_{33}({}^tC_{11}+{}^tC_{22})\,{}^t\hat{\underline{C}} + ({}^tC_{11}+{}^tC_{22})\begin{bmatrix}0&-2&0\\-2&0&0\\0&0&0.5\end{bmatrix}\right.$$

$$\left. + \begin{bmatrix} -2\,{}^tC_{22} & \frac{1}{({}^tC_{33})^2} & {}^tC_{12} \\ \frac{1}{({}^tC_{33})^2} & -2\,{}^tC_{11} & {}^tC_{12} \\ {}^tC_{12} & {}^tC_{12} & -\frac{1}{2({}^tC_{33})^2} \end{bmatrix}\right\}$$

where

$${}^t\hat{\underline{C}} = \begin{bmatrix} ({}^tC_{22})^2 & {}^tC_{11}\,{}^tC_{22} & -{}^tC_{12}\,{}^tC_{22} \\ & ({}^tC_{11})^2 & -{}^tC_{12}\,{}^tC_{11} \\ \text{Symmetric} & & ({}^tC_{12})^2 \end{bmatrix}$$

6.73 Assume uniaxial stress condition in a monotonic loading:

$${}^tS_{ij} = {}^t\sigma_{ij} - {}^t\sigma_m \delta_{ij} \quad \text{with } {}^t\sigma_m = \frac{{}^t\sigma_{ii}}{3} = \frac{{}^t\sigma_{11}}{3}$$

$${}^tS_{11} = {}^t\sigma_{11} - \frac{{}^t\sigma_{11}}{3} = \frac{2}{3}{}^t\sigma_{11}, \quad {}^tS_{22} = -\frac{1}{3}{}^t\sigma_{11} = {}^tS_{33}, \quad \text{other } {}^tS_{ij} = 0$$

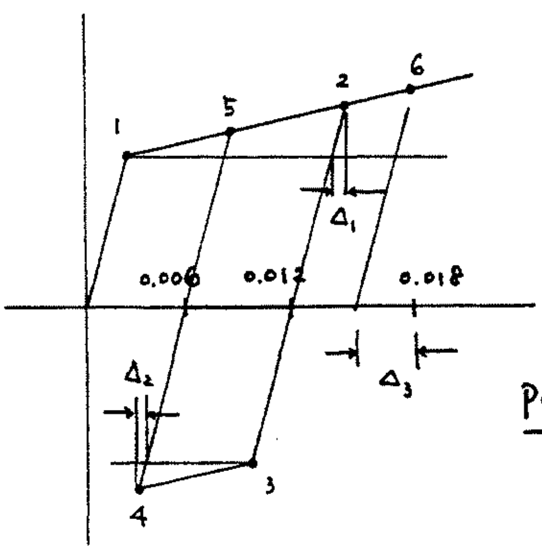
$$\therefore {}^t\bar{\sigma} = \left[\frac{3}{2}\left\{ \left(\frac{2}{3}{}^t\sigma_{11}\right)^2 + \left(-\frac{1}{3}{}^t\sigma_{11}\right)^2 + \left(-\frac{1}{3}{}^t\sigma_{11}\right)^2 \right\} \right]^{1/2} = {}^t\sigma_{11}$$

$$de^P_{11} + de^P_{22} + de^P_{33} = de^P_{11} + 2de^P_{22} = 0, \quad \therefore de^P_{22} = de^P_{33} = -\frac{1}{2}de^P_{11}$$

$$\therefore {}^t\bar{e}^P = \int_0^t d\bar{e}^P = \int_0^t \sqrt{\frac{2}{3}d\underline{e}^P \cdot d\underline{e}^P} = \left[\frac{2}{3}\left\{ \left({}^t\bar{e}^P_{11}\right)^2 + \left(-\frac{1}{2}{}^te^P_{11}\right)^2 \cdot 2 \right\} \right]^{1/2}$$

$$= {}^te^P_{11}$$

Hence the effective stress ${}^t\bar{\sigma}$ and effective plastic strain ${}^t\bar{e}^P$ reduce to the uniaxial stress and corresponding plastic strain.



Path 1-2: $\Delta\bar{e}^P = 0.012$

$\bar{e}^P_1 = 0, \quad \bar{e}^P_2 = 0.012$

$$\sigma_2 - \sigma_y = E_T(0.012 + \Delta_1) = E\Delta_1$$

$$\Delta_1 = 0.6003 \times 10^{-5}$$

$$e_2 = 0.001 + 0.012 + \Delta_1 \doteq 0.013$$

$$\sigma_2 = \sigma_y + E\Delta_1 \doteq 201.2 \text{ MPa}$$

Path 2-3: $\Delta\bar{e}^P = 0, \quad \bar{e}^P_3 = 0.012$

$$\sigma_3 = \sigma_2 - 2\sigma_y = -198.8 \text{ MPa}$$

$$e_3 = e_2 - \frac{2\sigma_y}{E} = 0.011$$

6.73

path 3-4 : $\Delta\bar{e}^P = 0.006$, $\bar{e}^P_4 = 0.018$

$\sigma_3 - \sigma_4 = E_T(0.006 + \Delta_2) = E\Delta_2$, $\Delta_2 = 3.002 \times 10^{-6}$

$e_4 = 0.005$, $\sigma_4 = -199.4$ MPa

Path 4-5 : $\Delta\bar{e}^P = 0$, $\bar{e}^P_5 = 0.018$

$e_5 = e_4 + 2(0.001) = 0.007$

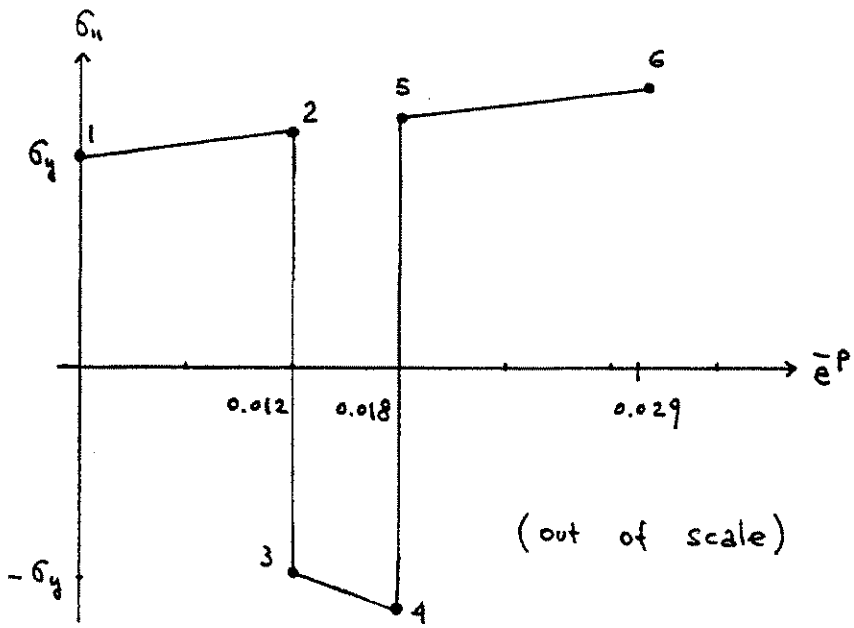
$\sigma_5 \doteq E(0.001) = 200$ MPa

Path 5-6 : $\sigma_6 - \sigma_5 = E_T(e_6 - e_5)$, $\sigma_6 = 201.1$ MPa

$\Delta_6 = \sigma_6 / E \doteq 0.001$

$\Delta\bar{e}^P = (0.018 - 0.001) - 0.006 = 0.011$

$\bar{e}^P_6 = 0.018 + 0.011 = 0.029$



6.74 $\Delta \bar{e}^P = \frac{2}{3}\lambda \, {}^{t+\Delta t}\bar{\sigma} = \dfrac{{}^{t+\Delta t}\bar{\sigma} - {}^{t}\sigma_y}{E_p}$ ← (6.226) and (6.234) —— ①

$(a_E + \lambda)\, {}^{t+\Delta t}\bar{\sigma} = d$ ← (6.230) $\therefore\; a_E \, {}^{t+\Delta t}\bar{\sigma} + \lambda \, {}^{t+\Delta t}\bar{\sigma} = d$ —— ②

From eq. ① and ②, $a_E \, {}^{t+\Delta t}\bar{\sigma} + \dfrac{3}{2} \dfrac{{}^{t+\Delta t}\bar{\sigma} - {}^{t}\sigma_y}{E_p} = d$

$${}^{t+\Delta t}\bar{\sigma}\left(a_E + \frac{3}{2}\frac{1}{E_p}\right) = d + \frac{3\,{}^{t}\sigma_y}{2E_p} \qquad \therefore\; {}^{t+\Delta t}\bar{\sigma} = \frac{2E_p d + 3\,{}^{t}\sigma_y}{2E_p a_E + 3}$$

First, solving for the zero value of the effective stress function will provide the solution for ${}^{t+\Delta t}\bar{\sigma}$ and λ. And the solutions for the current stress state ${}^{t+\Delta t}\underline{S}$ (eq. 6.229) and the incremental plastic strains $\Delta \underline{e}^P$ (eq. 6.225) are obtained. The key step in the solution lies in the calculation of the zero of a function $f(\bar{\sigma}^*)$. For this reason, the algorithm has been termed the ESF algorithm.

6.75 We start from eq. (6.230) : $a^2\, {}^{t+\Delta t}\bar{\sigma}^2 - d^2 = 0$

$$\text{or}\quad a\,{}^{t+\Delta t}\bar{\sigma} - d = 0 \quad , \left(\frac{1+\nu}{E} + \lambda\right) {}^{t+\Delta t}\bar{\sigma} - d = 0,$$

$$\frac{1+\nu}{E}\,{}^{t+\Delta t}\bar{\sigma} + \lambda\,{}^{t+\Delta t}\bar{\sigma} - d = 0, \quad \frac{1+\nu}{E}\,{}^{t+\Delta t}\bar{\sigma} + \frac{3}{2}\,\frac{\Delta\bar{e}^P}{{}^{t+\Delta t}\bar{\sigma}}\,{}^{t+\Delta t}\bar{\sigma} - d = 0$$

(because of (6.228))

$$\frac{2(1+\nu)}{E}\,{}^{t+\Delta t}\bar{\sigma} + 3\left({}^{t+\Delta t}\bar{e}^P - {}^{t}\bar{e}^P\right) - 2d = 0$$

$$\therefore\quad {}^{t+\Delta t}\bar{\sigma} + 3\mu\left({}^{t+\Delta t}\bar{e}^P - {}^{t}\bar{e}^P\right) - 2\mu d = 0 \qquad \left(\frac{E}{2(1+\nu)} = \mu\right)$$

Hence we have $\quad \sigma_y({}^{t+\Delta t}\bar{e}^P) + 3\mu({}^{t+\Delta t}\bar{e}^P - {}^{t}\bar{e}^P) - \bar{\sigma}^E = 0$

$$\text{or}\quad f(e^P_*) = 3\mu(e^P_* - {}^{t}e^P) + \sigma_y(e^P_*) - \bar{\sigma}^E$$

Note that $\bar{\sigma}^E$ is the effective stress corresponding to the trial elastic stress. The reason is as follows:

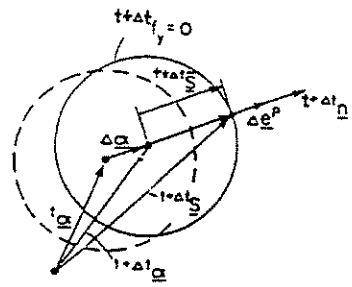
$$(\bar{\sigma}^E)^2 = (2\mu d)^2 = \left(\frac{E}{1+\nu}\,d\right)^2 = \frac{3}{2}\,\frac{E}{1+\nu}\,{}^{t+\Delta t}\underline{e}''\cdot\frac{E}{1+\nu}\,{}^{t+\Delta t}\underline{e}''$$

$$= \frac{3}{2}\,{}^{t+\Delta t}\underline{\underline{S}}^E\cdot{}^{t+\Delta t}\underline{\underline{S}}^E \quad (\text{with } \Delta\underline{e}^P = 0)$$

and we see the state which $\bar{\sigma}^E$ represents is at the point B in Fig. 6.11.

6.76

$$ {}^{t+\Delta t}\underline{S} = \frac{1}{a_E} \left({}^{t+\Delta t}\underline{e}'' - \Delta \underline{e}^P \right) \quad \text{— ①} $$



In the case of kinematic hardening the yield condition is

$$ {}^{t+\Delta t}f_y = \frac{1}{2}\, {}^{t+\Delta t}\tilde{\underline{S}} \cdot {}^{t+\Delta t}\tilde{\underline{S}} - \frac{1}{3}(\sigma_{yv})^2 = 0 \quad \text{— ②} $$

where $ {}^{t+\Delta t}\tilde{\underline{S}} = {}^{t+\Delta t}\underline{S} - {}^{t+\Delta t}\underline{\alpha} $ — ③

The constitutive relations are

$$ \Delta \underline{e}^P = \lambda \left({}^{t+\Delta t}\underline{S} - {}^{t+\Delta t}\underline{\alpha} \right) \quad \text{— ④} $$

$$ \Delta \underline{\alpha} = C\, \Delta \underline{e}^P \qquad \text{where } C = \frac{2}{3}\frac{E E^T}{E - E^T} \quad \text{— ⑤} $$

Using the above approximations we find that the yield condition is satisfied when

$$ {}^{t+\Delta t}\underline{S} = {}^{t}\underline{\alpha} + (1 + C\lambda)\, {}^{t+\Delta t}\tilde{\underline{S}} \quad \text{— ⑥} $$

such that $ \| {}^{t+\Delta t}\tilde{\underline{S}} \| = \sqrt{\frac{2}{3}}\, \sigma_{yv} $

Hence the eq. ①, when using eq. ③ and ④, becomes

$$ {}^{t+\Delta t}\underline{S} = \frac{1}{a_E} \left({}^{t+\Delta t}\underline{e}'' - \lambda\, {}^{t+\Delta t}\tilde{\underline{S}} \right) \quad \text{— ⑦} $$

Then using eq. ⑥ and ⑦ we obtain

$$ {}^{t+\Delta t}\tilde{\underline{S}} = \frac{1}{a_E + (1 + C a_E)\lambda}\, \underline{q} \quad \text{— ⑧} $$

6.76

with $\lambda = \dfrac{\|\underline{g}\| / \|{}^{t+\Delta t}\underset{\sim}{\tilde{S}}\| - a_E}{1 + C a_E}$, $\underline{g} = {}^{t+\Delta t}\underset{\sim}{e}'' - a_E\, {}^{t}\underset{\sim}{\alpha}$

The effective stress function is now obtained from eq. ⑦ by taking the scalar product on both sides,

$$f({}^{t+\Delta t}\bar{\sigma}) = a_E^2\, {}^{t+\Delta t}\bar{\sigma}^2 - d^2 = 0$$

where $d^2 = \frac{3}{2}\underline{P}\cdot\underline{P}$, $\underline{P} = {}^{t+\Delta t}\underset{\sim}{e}'' - \lambda\, {}^{t+\Delta t}\underset{\sim}{\tilde{S}}$

6.77 Eq. (6.229) and (6.252) state

$$ {}^{trat}\underline{S} = \frac{1}{a_E + \lambda}\, {}^{trat}\underline{e}'' , \quad \frac{\partial\, {}^{trat}S_i}{\partial\, {}^{trat}e''_k} = \frac{1}{a_E + \lambda}\delta_{ik} - \frac{1}{(a_E+\lambda)^2}\frac{\partial \lambda}{\partial\, {}^{trat}e''_k}\, {}^{trat}e''_i \qquad \text{—— ①} $$

Hence we need to calculate $\frac{\partial \lambda}{\partial\, {}^{trat}e''_k}$. (From now on we omit the superscript trat for brevity).

Using the flow rule and differentiating

$$ S_{ij}S_{ij} - \frac{2}{3}\sigma_y^2 = 0 , \quad d\sigma_y = \frac{3}{2\sigma_y} S_{ij}\, dS_{ij} \qquad \text{—— ②} $$

From eq. (6.226) and the hardening rule,

$$ d\sigma_y = E_p d(\Delta \bar{e}^P) = E_p \frac{2}{3} d(\lambda \bar{\sigma}) = \frac{2}{3}E_p(\bar{\sigma}\, d\lambda + \lambda\, d\bar{\sigma}) \qquad \text{—— ③} $$

Equations ② and ③ with $\sigma_y = \bar{\sigma}$ give

$$ (1 - \frac{2}{3}E_p\lambda)\, S_{ij}\, dS_{ij} - \frac{4}{9}\bar{\sigma}^2 E_p\, d\lambda = 0 $$

$$ d\lambda = \frac{9}{4}\frac{1}{\bar{\sigma}^2 E_p}(1 - \frac{2}{3}E_p\lambda)\, S_{ij}\, dS_{ij} \qquad \text{—— ④} $$

Considering $S_{ij}\, dS_{ij}$, we have

$$ S_{ij} = \frac{1}{a_E + \lambda} e''_{ij} , \quad dS_{ij} = \frac{1}{a_E+\lambda} de''_{ij} - \frac{1}{(a_E+\lambda)^2} e''_{ij}\, d\lambda = \frac{1}{a_E+\lambda}(de''_{ij} - S_{ij}\, d\lambda) $$

$$ S_{ij}\, dS_{ij} = \frac{1}{a_E+\lambda}(S_{ij}\, de''_{ij} - S_{ij}S_{ij}\, d\lambda) = \frac{1}{a_E+\lambda}(S_{ij}\, de''_{ij} - \frac{2}{3}\bar{\sigma}^2\, d\lambda) $$

Using this, we can simplify the eq. ④ as

$$ d\lambda = \frac{9}{2\bar{\sigma}^2}\, \frac{1 - \frac{2}{3}E_p\lambda}{2E_p a_E + 3}\, S_{ij}\, de''_{ij} $$

6.77

Hence we obtain

$$\frac{\partial \lambda}{\partial e_k''} = A\, S_k \quad \text{or} \quad \frac{\partial \lambda}{\partial e_k''} = A \, \frac{1}{a_E + \lambda}\, e_k''$$

$$\text{where} \quad A = \frac{9}{2\bar{\sigma}^2} \, \frac{1 - \frac{2}{3} E_p \lambda}{2 E_p a_E + 3}$$

Now replacing this in eq. (6.252),

$$\frac{\partial\, {}^{t+\Delta t} S_i}{\partial\, {}^{t+\Delta t} e_k''} = \frac{1}{a_E + \lambda}\, \delta_{ik} - \frac{1}{(a_E + \lambda)^3}\, A\, e_i'' e_k''$$

In matrix notation,

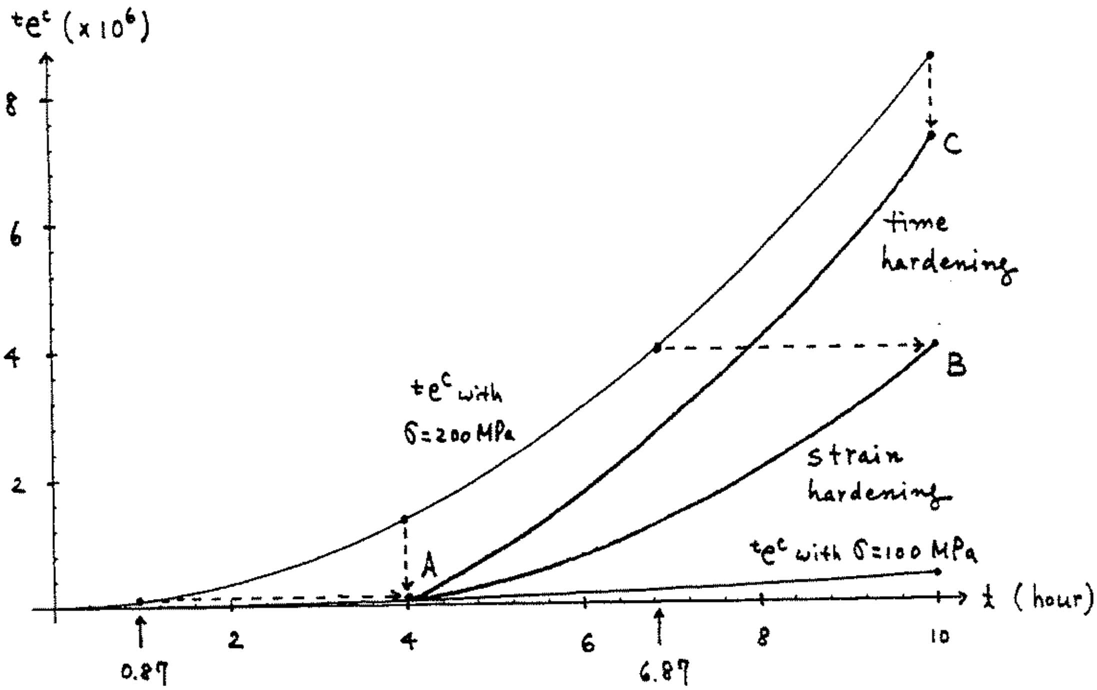
$$\underline{C} = \frac{1}{a_E + \lambda}\, \underline{I} - \frac{A}{(a_E + \lambda)^3}\, \underline{e}'' \underline{e}''$$

where $\underline{I}$ is the 4-th order identity matrix.

6.78 $\quad {}^te^c = a_0\, {}^t\sigma^{a_1} t^{a_2}$

$a_0 = 6.4\times 10^{-18}$, $a_1 = 4.4$, $a_2 = 2.0$

$0 < t < 4^h \quad \sigma = 100$ MPa , $t \geq 4^h \quad \sigma = 200$ MPa



$${}^te^c\Big|_{\sigma=100\text{ MPa}} = 4.038\times 10^{-9}\, t^2$$

$${}^te^c\Big|_{\sigma=200\text{ MPa}} = 8.525\times 10^{-8}\, t^2$$

At point A, $\quad {}^te^c = 6.461\times 10^{-8}$

At point B, $\quad {}^te^c = 4.024\times 10^{-6}$

At point C, $\quad {}^te^c = 7.226\times 10^{-6}$

6.79

$$ {}^{t}\bar{e}^{c} = f_1({}^{t}\bar{\sigma})\, f_2(t)\, f_3({}^{t}\theta) \qquad \text{— (6.265)} $$

$$ \Delta\bar{e}^{c} = \Delta t\, f_1({}^{\tau}\bar{\sigma})\, \dot{f}_2(\tau)\, f_3({}^{\tau}\theta) \qquad \text{— (6.266)} $$

where $$ {}^{\tau}\theta = (1-\alpha)\,{}^{t}\theta + \alpha\,{}^{t+\alpha t}\theta \qquad \text{— (6.267)} $$

$$ \tau = t + \alpha\Delta t $$

Hence for the pseudo-time τ_p, ${}^{t}\bar{e}^{c} + \Delta\bar{e}^{c} = {}^{t+\alpha t}\bar{e}^{c}$

$$ {}^{t}\bar{e}^{c} + \alpha\Delta t\, f_1({}^{\tau}\bar{\sigma})\, \dot{f}_2(\tau_p)\, f_3({}^{\tau}\theta) = f_1({}^{\tau}\bar{\sigma})\, f_2(\tau_p)\, f_3({}^{\tau}\theta) $$

$$ \therefore\ {}^{t}\bar{e}^{c} + f_1({}^{\tau}\bar{\sigma})\, f_3({}^{\tau}\theta) \left[\alpha\Delta t\, \dot{f}_2(\tau_p) - f_2(\tau_p) \right] = 0 $$

6.80 $\quad {}^{t+\Delta t}\underline{S} = \dfrac{{}^{t+\Delta t}\underline{e}'' - (1-\alpha)\Delta t\,{}^{\tau}\gamma\,{}^{t}\underline{S}}{{}^{t+\Delta t}a_E + \alpha\Delta t\,{}^{\tau}\gamma + \lambda}$, $\quad {}^{t+\Delta t}a_E = \dfrac{1+{}^{t+\Delta t}\nu}{{}^{t+\Delta t}E}$ $\quad (\leftarrow 6.268, 270)$

Taking the scalar product on both sides in (6.269),

$$\frac{3}{2}\,{}^{t+\Delta t}\underline{S}\cdot{}^{t+\Delta t}\underline{S} = \frac{3}{2}\,\frac{1}{a^2}\Big[\,{}^{t+\Delta t}\underline{e}''\cdot{}^{t+\Delta t}\underline{e}'' - 2(1-\alpha)\Delta t\,{}^{t+\Delta t}\underline{e}''\cdot{}^{t}\underline{S}\,{}^{\tau}\gamma + \{(1-\alpha)\Delta t\}^2\,{}^{t}\underline{S}\cdot{}^{t}\underline{S}\,{}^{\tau}\gamma^2\,\Big]$$

where $a = {}^{t+\Delta t}a_E + \alpha\Delta t\,{}^{\tau}\gamma + \lambda$

$$\therefore\ {}^{t+\Delta t}\bar{\sigma}^2 = \frac{1}{a^2}\Big[\Big(\frac{3}{2}\,{}^{t+\Delta t}\underline{e}''\cdot{}^{t+\Delta t}\underline{e}''\Big) - \{3(1-\alpha)\Delta t\,{}^{t+\Delta t}\underline{e}''\cdot{}^{t}\underline{S}\}\,{}^{\tau}\gamma + \{(1-\alpha)\Delta t\}^2\,{}^{t}\bar{\sigma}^2\,{}^{\tau}\gamma^2\,\Big]$$

$$a^2\,{}^{t+\Delta t}\bar{\sigma}^2 + \{3(1-\alpha)\Delta t\,{}^{t+\Delta t}\underline{e}''\cdot{}^{t}\underline{S}\}\,{}^{\tau}\gamma - \{(1-\alpha)\Delta t\,{}^{t}\bar{\sigma}\}^2\,{}^{\tau}\gamma^2 - \Big\{\frac{3}{2}\,{}^{t+\Delta t}\underline{e}''\cdot{}^{t+\Delta t}\underline{e}''\Big\} = 0$$

$$\therefore\ a^2\,{}^{t+\Delta t}\bar{\sigma}^2 + b\,{}^{\tau}\gamma - c^2\,{}^{\tau}\gamma^2 - d^2 = 0$$

where $a = {}^{t+\Delta t}a_E + \alpha\Delta t\,{}^{\tau}\gamma + \lambda$, $\ b = 3(1-\alpha)\Delta t\,{}^{t+\Delta t}\underline{e}''\cdot{}^{t}\underline{S}$

$c = (1-\alpha)\Delta t\,{}^{t}\bar{\sigma}$, $\ d^2 = \frac{3}{2}\,{}^{t+\Delta t}\underline{e}''\cdot{}^{t+\Delta t}\underline{e}''$

Now the effective-stress-function is given by

$$f(\bar{\sigma}^*) = a^2(\bar{\sigma}^*)^2 + (b\,{}^{\tau}\gamma - c^2\,{}^{\tau}\gamma^2 - d^2) = 0$$

To calculate the curvature near $\bar{\sigma}^*$ with $f(\bar{\sigma}^*) = 0$ we have to know $\dfrac{\partial f}{\partial \bar{\sigma}^*}$ and $\dfrac{\partial^2 f}{\partial \bar{\sigma}^2}$.

Note that $a = a(\bar{\sigma}^*)$ and ${}^{\tau}\gamma = {}^{\tau}\gamma(\bar{\sigma}^*)$. However, near $\bar{\sigma} = \bar{\sigma}^*$ with $f(\bar{\sigma}^*) = 0$ we consider $f(\bar{\sigma}^*) = a^2(\bar{\sigma}^*)^2 + C$ where C is approximately some constant, and we see that the function has the curvature shown on Fig. 6.14.

6.81

The elastic strain increments are calculated as usually and the viscoelastic strain increments are calculated as:

$$\dot{\epsilon}^{VP} = \hat{\gamma}\left[\sigma - (\sigma_{yv} + E_{VP}\,\epsilon^{VP})\right] \quad \text{with} \quad \epsilon^{E} = \frac{\sigma}{E}$$

i) $E_{VP} = 0$. $\quad \dot{\epsilon}^{VP} = \hat{\gamma}(\sigma - \sigma_{yv})$, $\quad \epsilon = \frac{\sigma}{E} + \int_0^t \hat{\gamma}(\sigma - \sigma_{yv})\,dt$

$$\therefore \ \epsilon = \frac{\sigma}{E} + \hat{\gamma}(\sigma - \sigma_{yv})\,t$$

ii) $E_{VP} \neq 0$. $\quad \frac{d\epsilon^{VP}}{dt} + \hat{\gamma}E_{VP}\epsilon^{VP} = \hat{\gamma}(\sigma - \sigma_{yv})$

Let $\epsilon^{VP} = C_1 \exp(\lambda t) + C_2$, then $\epsilon^{VP}\big|_{t=0} = 0 \quad \therefore C_2 = -C_1$

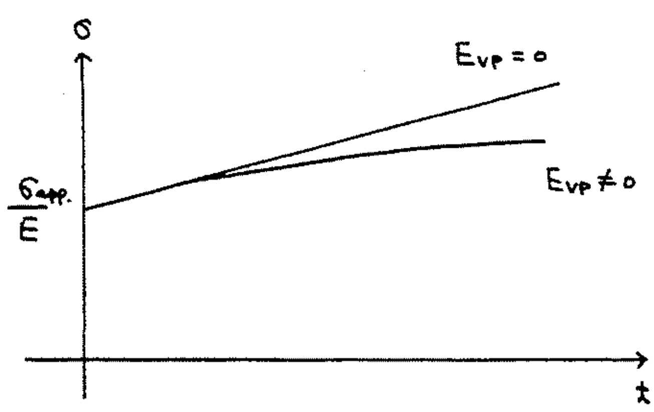
$$\therefore \ \epsilon^{VP} = C_1\left[\exp(\lambda t) - 1\right]$$

$$C_1 \exp(\lambda t)\left(\lambda + \hat{\gamma}E_{VP}\right) - \hat{\gamma}\left\{E_{VP}C_1 + (\sigma - \sigma_{yv})\right\} = 0$$

$$\therefore \ \lambda = -\hat{\gamma}E_{VP}, \quad C_1 = -\frac{\sigma - \sigma_{yv}}{E_{VP}}$$

$$\epsilon^{VP} = \frac{\sigma - \sigma_{yv}}{E_{VP}}\left[1 - \exp(-\hat{\gamma}E_{VP}t)\right]$$

$$\therefore \ \epsilon = \frac{\sigma}{E} + \frac{\sigma - \sigma_{yv}}{E_{VP}}\left[1 - \exp(-\hat{\gamma}E_{VP}t)\right]$$



6.82 For 1-D problem $\ {}^t\bar{\sigma} = \sigma_{11} = \dfrac{2P}{(2\cdot 10^{-3})(1\cdot 10^{-3})} = 40\ \text{MPa} > \sigma_{yv} = 20\ \text{MPa}$.

Hence ${}^t\bar{\sigma}_0$ will approach ${}^t\bar{\sigma}$ (from 20 MPa to 40 MPa), and there will be viscoplastic flow.

$$ {}^t\underline{S}^T = \left[\ \tfrac{2}{3}\sigma_{11} \quad -\tfrac{1}{3}\sigma_{11} \quad -\tfrac{1}{3}\sigma_{11} \quad 0 \quad 0 \quad 0\ \right] $$

$$ d\underline{e}^{VP} = \beta\left(\frac{{}^t\bar{\sigma} - {}^t\bar{\sigma}_0}{{}^t\bar{\sigma}_0}\right)^N \frac{3}{2\,{}^t\bar{\sigma}}\,{}^t\underline{S}\ dt \quad , \quad d\bar{e}^{VP} = \beta\,\frac{{}^t\bar{\sigma} - {}^t\bar{\sigma}_0}{{}^t\bar{\sigma}_0}\,dt \quad \text{——} ① $$

From the material effective stress - viscoplastic strain relationship,

$$ \dot{\bar{e}}^{VP} = {}^t\dot{\bar{\sigma}}_0 / E_{VP} = {}^t\dot{\bar{\sigma}}_0 / 2 \quad \text{——} ② $$

Eq. ① and ② give

$$ \frac{{}^t\dot{\bar{\sigma}}_0}{2} = (10^{-4})\ \frac{40 - {}^t\bar{\sigma}_0}{{}^t\bar{\sigma}_0} $$

Using the Euler backward method,

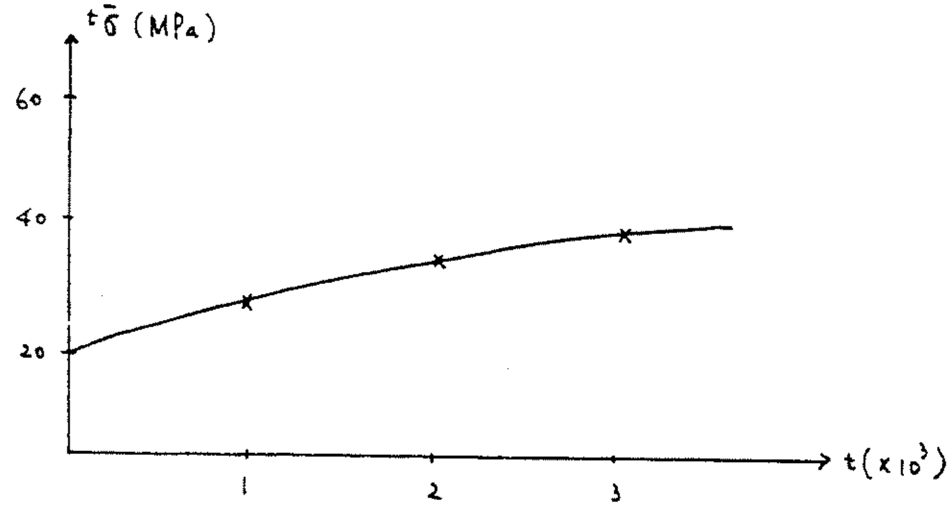
$$ \frac{{}^{t+\Delta t}\bar{\sigma}_0 - {}^t\bar{\sigma}_0}{2\,\Delta t} = (10^{-4})\ \frac{40 - {}^{t+\Delta t}\bar{\sigma}_0}{{}^{t+\Delta t}\bar{\sigma}_0} \quad \text{——} ③ $$

In small strain analysis,

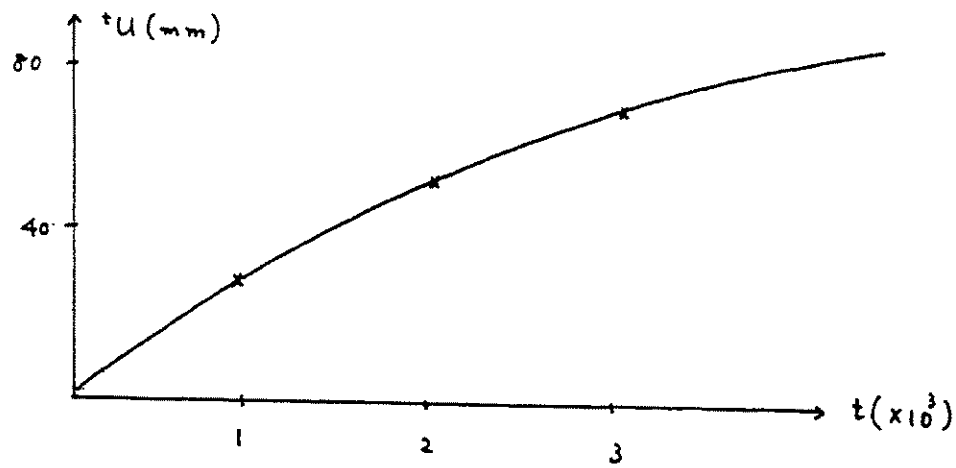
$$ {}^te_{11}^{total} = \frac{{}^tu}{20} = {}^te_{11}^{VP} + {}^te_{11}^{E} = \frac{{}^t\bar{\sigma}_0 - 20}{2} + \frac{40}{20000} \quad \text{——} ④ $$

From eq. ③ and ④, we obtain the viscoplastic response of strain and displacement.

6.82

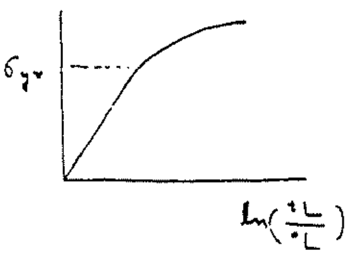


$^t\bar{\sigma}$ (MPa)
60
40
20
1
2
3
$t(\times 10^3)$



^tU (mm)
80
40
1
2
3
$t(\times 10^3)$

6.83



$$\underline{X} = \underline{U} \;,\; \underline{R} = \underline{I}$$

$$\underline{X} = \underline{X}^E \underline{X}^P \;,\; \underline{X}^E = \begin{bmatrix} \lambda_E & & \\ & 1 & \\ & & 1 \end{bmatrix} \;,\; \underline{X}^P = \begin{bmatrix} \lambda_p & & \\ & 1/\sqrt{\lambda_p} & \\ & & 1/\sqrt{\lambda_p} \end{bmatrix}$$

$$\det \underline{X}^E = \lambda_E \;,\; \det \underline{X}^P = 1 \;,\; \det \underline{X} = \lambda_E = J$$

$$\bar{\underline{\tau}} = J \underline{R}^{E^T} \underline{\tau}\, \underline{R}^E \;\longrightarrow\; \bar{\underline{\tau}} = \lambda_E \underline{\tau}$$

$$\underline{L}^P = \dot{\underline{X}}^P \underline{X}^{P-1} = \bar{\underline{L}}^P = \begin{bmatrix} \dot{\lambda}_p \lambda_p^{-1} & & \\ & -\frac{1}{2}\dot{\lambda}_p \lambda_p^{-1} & \\ & & -\frac{1}{2}\dot{\lambda}_p \lambda_p^{-1} \end{bmatrix}$$

$$\underline{L}^E = \dot{\underline{X}}^E \underline{X}^{E-1} = \begin{bmatrix} \dot{\lambda}_E \lambda_E^{-1} & & \\ & 0 & \\ & & 0 \end{bmatrix} \;,\quad \underline{L} = \underline{L}^P + \underline{L}^E$$

$$\bar{\underline{\tau}} \cdot \dot{\underline{E}}^E = \bar{\tau}\,(\dot{\ln \lambda_E}) = \bar{\tau}\,\frac{\dot{\lambda}_E}{\lambda_E} = \lambda_E \tau \frac{\dot{\lambda}_E}{\lambda_E}$$

Now define $\bar{\underline{S}} = 2\mu\, \bar{\underline{E}}^{E'}$, $\bar{\underline{E}}^{E'} = \begin{bmatrix} \frac{2}{3}\ln\lambda_E & & \\ & -\frac{1}{3}\ln\lambda_E & \\ & & -\frac{1}{3}\ln\lambda_E \end{bmatrix}$

$$\bar{\sigma}_n = 3k\, E_n^E \;,\; E_n^E = \ln \lambda_E$$

The effective stress is given by

$$\bar{\sigma} = \frac{1}{\lambda_E}\sqrt{\frac{3}{2}\,\bar{\underline{S}}\cdot\bar{\underline{S}}} = \frac{1}{\lambda_E}\,\bar{\tau} = \tau$$

if $\bar{\sigma} \leq \sigma_{yr}$, the process is elastic,

otherwise, the process is plastic.

6.84 From ${}^{t}_{0}\underline{X} = {}^{t}_{\tau}\underline{X}^{E}\, {}^{\tau}_{0}\underline{X}^{P}$ where τ corresponds to the relaxed hypothetical configuration we have for small strains $\underline{e} = \underline{e}^{E} + \underline{e}^{P}$. Hence in Table 6.10 for small displacements and strains:

$$\underline{X}_*^{E} = \underline{R}_*^{E}\,\underline{U}_*^{E} \simeq \underline{U}_*^{E}, \quad \underline{X}^{P} = \underline{R}^{P}\underline{U}^{P} \simeq \underline{U}^{P},$$

$$\underline{X} = \underline{R}\,\underline{U} \simeq \underline{U}$$

$$\Rightarrow \quad \underline{E}_*^{E} = \ln \underline{U}_*^{E} \simeq \underline{e}_*^{E}, \quad \underline{E}^{P} = \ln \underline{u}^{P} \simeq \underline{e}^{P}, \quad \underline{E} = \ln \underline{u} \simeq \underline{e}$$

Since $\underline{X}_*^{E} = \underline{X}\,(\underline{X}^{P})^{-1}$, i.e., $\underline{U}_*^{E} = \underline{U}\,(\underline{U}^{P})^{-1}$, we have

$$\underline{e}_*^{E} = \underline{e} - \underline{e}^{P}$$

$$tr(\bar{\underline{\tau}}_*) = 3k\, tr(\underline{E}_*^{E}) \quad \Rightarrow \quad tr(\underline{\sigma}_*) = 3k\, tr(\underline{e}_*^{E})$$

$$\bar{\underline{S}}_* = 2\mu\, \underline{E}_*^{E'} \qquad \Rightarrow \quad \underline{\sigma}_*' = 2\mu\, e_*^{E'}$$

$$\bar{\sigma}_* = J^{-1}\sqrt{\tfrac{3}{2}\,\bar{\underline{S}}_* \cdot \bar{\underline{S}}_*} \simeq \sqrt{\tfrac{3}{2}\,\underline{\sigma}_*' \cdot \underline{\sigma}_*'}$$

Plastic solution step:

$$\lambda = \frac{3}{2}\,\frac{{}^{t+\Delta t}\bar{e}^{P} - {}^{t}\bar{e}^{P}}{{}^{t+\Delta t}J\;{}^{t+\Delta t}\bar{\sigma}} \doteq \frac{3}{2}\,\frac{\Delta \bar{e}^{P}}{{}^{t+\Delta t}\bar{\sigma}} \qquad \text{(same as (6.228))}$$

$${}^{t+\Delta t}\underline{S} = \frac{\bar{\underline{S}}_*}{1+2\mu\lambda} = \frac{2\mu\,\underline{E}_*^{E'}}{1+2\mu\lambda} \doteq \frac{\bar{\underline{e}}_*^{E'}}{\frac{1}{2\mu}+\lambda} = \frac{\underline{e}_*' - \underline{e}^{P'}}{\frac{1+\nu}{E}+\lambda}$$

$$\therefore \quad {}^{t+\Delta t}\underline{S} \doteq \frac{1}{\frac{1+\nu}{E}+\lambda}\,\underline{e}'' \qquad \text{(same as (6.229))}$$

6.85 (i) Hencky strain:

$$ {}_0^t\underline{X} = \begin{bmatrix} 1+\frac{{}^tu}{{}^0L} & & \\ & 1 & \\ & & 1 \end{bmatrix}, \quad J = 1+\frac{{}^tu}{{}^0L}, \quad {}_0^t\underline{U} = {}_0^t\underline{X} $$

$$ \underline{E} = \ln {}_0^t\underline{U} = \begin{bmatrix} \ln\left(1+\frac{{}^tu}{{}^0L}\right) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \bar{\underline{\tau}} = J\underline{\tau} $$

$$ \frac{{}^tP}{\bar{A}} = \tau_{11} = J^{-1}\bar{\tau}_{11} = \tilde{E}\,\frac{\ln\left(1+\frac{{}^tu}{{}^0L}\right)}{1+\frac{{}^tu}{{}^0L}} \quad \text{where } \tilde{E} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} $$

(ii) plane stress condition:

As in exercise 6.61,

$$ {}_0^t\underline{X} = \begin{bmatrix} 1+\frac{{}^tu}{{}^0L} & & \\ & 1-\frac{{}^t\Delta}{{}^0L} & \\ & & 1-\frac{{}^t\Delta}{{}^0L} \end{bmatrix}, \quad J = \left(1+\frac{{}^tu}{{}^0L}\right)\left(1-\frac{{}^t\Delta}{{}^0L}\right)^2, \quad {}_0^t\underline{U} = {}_0^t\underline{X} $$

$$ \underline{E} = \ln {}_0^t\underline{U} = \begin{bmatrix} \ln\left(1+\frac{{}^tu}{{}^0L}\right) & & \\ & \ln\left(1-\frac{{}^t\Delta}{{}^0L}\right) & \\ & & \ln\left(1-\frac{{}^t\Delta}{{}^0L}\right) \end{bmatrix} $$

From the plane stress cond. we have $\ln\left(1-\frac{{}^t\Delta}{{}^0L}\right) = -\nu \ln\left(1+\frac{{}^tu}{{}^0L}\right)$

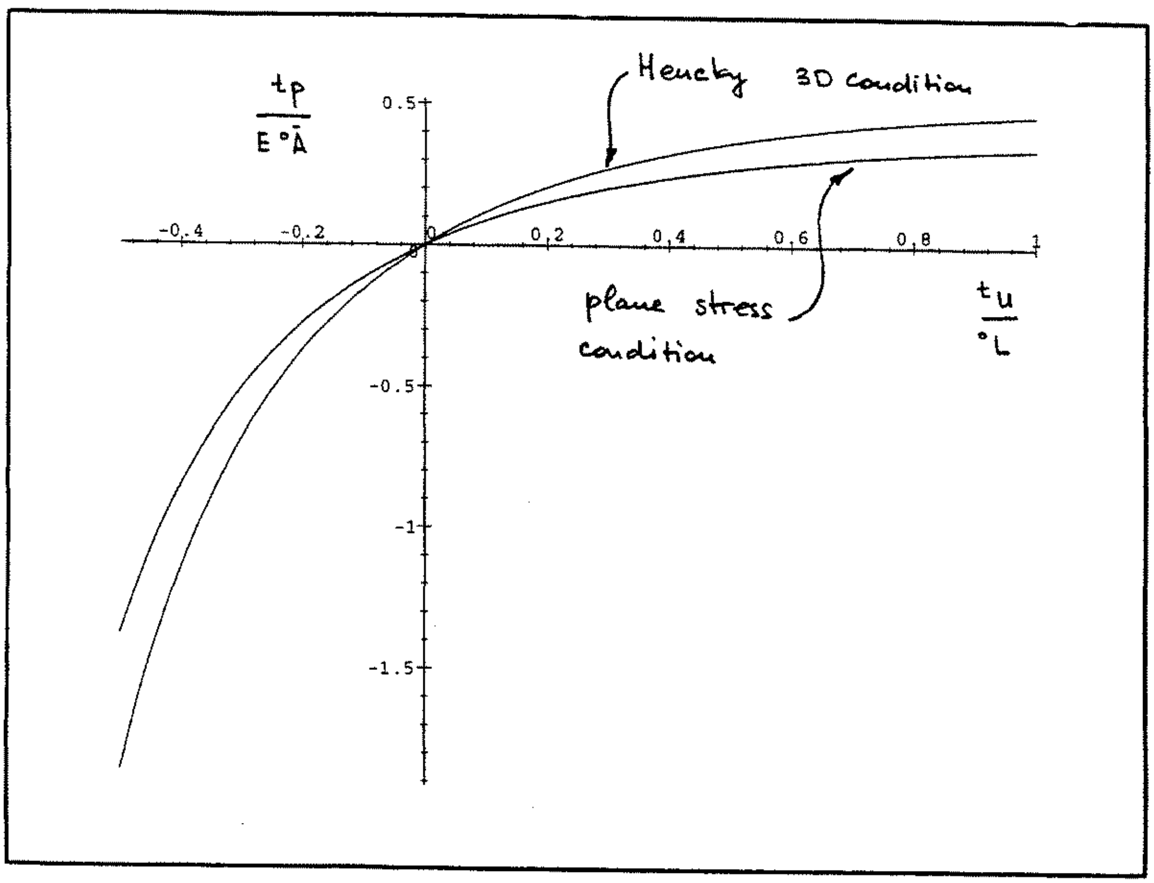
$$ 1-\frac{{}^t\Delta}{{}^0L} = \left(1+\frac{{}^tu}{{}^0L}\right)^{-\nu}, \quad ({}^0L-{}^t\Delta)^2 = ({}^0L)^2\left(1+\frac{{}^tu}{{}^0L}\right)^{-2\nu} $$

6.85

$$\therefore \quad {}^t\bar{A} = {}^0\bar{A}\left(1+\frac{{}^tu}{{}^0L}\right)^{-2\nu}, \quad J = \left(1+\frac{{}^tu}{{}^0L}\right)\left(\frac{{}^0L-{}^t\Delta}{{}^0L}\right)^2 = \left(1+\frac{{}^tu}{{}^0L}\right)^{1-2\nu}$$

Hence $$\frac{{}^tP}{{}^t\bar{A}} = \tau_{11} = J^{-1}\bar{\tau}_{11} = \left(1+\frac{{}^tu}{{}^0L}\right)^{-1+2\nu}\left[E\ln\left(1+\frac{{}^tu}{{}^0L}\right)\right]$$

$$\therefore \quad \frac{{}^tP}{{}^0\bar{A}} = E\,\frac{\ln\left(1+\frac{{}^tu}{{}^0L}\right)}{\left(1+\frac{{}^tu}{{}^0L}\right)}$$



6.86

$$\dot W = \frac{1}{2}\tau_{mn}\left(\dot u_{m,n} + \dot u_{n,m}\right) = \tau_{mn}\dot e_{mn}$$

The rate of increase of internal energy per unit undeformed volume:

$$\dot W_0 = \frac{\rho_0}{\rho}\dot W = J\dot W = J\tau\cdot D^E = \sigma\cdot D^E$$

(τ = Cauchy stress tensor, σ = Kirchhoff stress tensor)

$$\dot U = \dot X X^{-1} = L$$

$$\therefore\ \dot W_0 = \sigma\cdot(\dot X X^{-1}) = \sigma\cdot\frac{1}{2}(\dot X X^{-1} + X^{-T}\dot X^{T})$$

$$\dot C = \dot X^T X + X^T\dot X = \dot U U + U\dot U$$

$$\therefore\ \dot W_0 = \sigma\cdot\frac{1}{2}(X^{-T}\dot C X^{-1}) = \frac{1}{2}(X^{-1}\sigma X^{-T})\cdot\dot C = \frac{1}{2}S\cdot\dot C$$

$$\dot W_0 = \sigma\cdot\frac{1}{2}\left[X^{-T}(U\dot U + \dot U U)X^{-1}\right] = \frac{1}{2}(X^{-1}\sigma X^{-T})\cdot(\dot U U + U\dot U)$$

$$= \frac{1}{2}(U^{-1}\bar\tau U^{-T})\cdot(U\dot U + \dot U U) = \frac{1}{2}\bar\tau\cdot(\dot U U^{-1} + U^{-1}\dot U) \qquad (1)$$

where $\bar\tau = J R^{E^T}\tau R^E$

Now

$$U = R_L^T\Lambda R_L, \quad U^{-1} = R_L^T\Lambda^{-1}R_L$$

$$\dot U = \dot R_L^T\Lambda R_L + R_L^T\dot\Lambda R_L + R_L^T\Lambda\dot R_L$$

$$\dot U U^{-1} = \dot R_L^T R_L + R_L^T\dot\Lambda\Lambda^{-1}R_L + R_L^T\Lambda\dot R_L R_L^T\Lambda^{-1}R_L$$

$$U^{-1}\dot U = R_L^T\Lambda^{-1}R_L\dot R_L^T\Lambda R_L + R_L^T\Lambda^{-1}\dot\Lambda R_L + R_L^T\dot R_L$$

$$\ln U = R_L^T\ln\Lambda R_L$$

$$(\ln\Lambda)^{\cdot} = \Lambda^{-1}\dot\Lambda$$

$$(\ln U)^{\cdot} = \dot R_L^T\ln\Lambda R_L + R_L^T\Lambda^{-1}\dot\Lambda R_L + R_L^T\ln\Lambda\dot R_L$$

(2)

6.86

From the equations in ②,

$$\underline{U}^{-1}\dot{\underline{U}} + \dot{\underline{U}}\,\underline{U}^{-1} = 2\ln\dot{\underline{U}} - 2\dot{\underline{R}}_L^T \ln\underline{\Lambda}\,\underline{R}_L - 2\underline{R}_L^T \ln\underline{\Lambda}\,\dot{\underline{R}}_L \qquad ③$$

$$+ \underline{U}^{-1}\dot{\underline{R}}_L^T\underline{R}_L\underline{U} + \underline{U}\,\underline{R}_L^T\dot{\underline{R}}_L\underline{U}^{-1}$$

$$\left(\because\ \underline{R}_L^T\underline{R}_L = \underline{I}\ ,\ \ \dot{\underline{R}}_L^T\underline{R}_L + \underline{R}_L^T\dot{\underline{R}}_L = \underline{0} \right)$$

Using ① and ③,

$$\dot{W}_o = \bar{\underline{\tau}}\cdot\ln\dot{\underline{U}} - \bar{\underline{\tau}}\cdot\left(\dot{\underline{R}}_L^T\underline{R}_L\ln\underline{U} + \ln\underline{U}\,\underline{R}_L^T\dot{\underline{R}}_L\right) + \frac{1}{2}\bar{\underline{\tau}}\cdot\left(\underline{U}^{-1}\dot{\underline{R}}_L^T\underline{R}_L\underline{U} + \underline{U}\,\underline{R}_L^T\dot{\underline{R}}_L\underline{U}^{-1}\right)$$

$$= \bar{\underline{\tau}}\cdot\ln\dot{\underline{U}} - \left(\bar{\underline{\tau}}\ln\underline{U} - \ln\underline{U}\,\bar{\underline{\tau}}\right)\cdot\dot{\underline{R}}_L^T\underline{R}_L + \frac{1}{2}\left(\underline{U}^{-1}\bar{\underline{\tau}}\,\underline{U} - \underline{U}\,\bar{\underline{\tau}}\,\underline{U}^{-1}\right)\cdot\dot{\underline{R}}_L^T\underline{R}_L$$

$$= \bar{\underline{\tau}}\cdot\ln\dot{\underline{U}}$$

6.87

$$\underline{D} = \text{sym}(\underline{L}) = \text{sym}(\underline{L}^E) + \text{sym}(\underline{L}^P) = \underline{D}^E + \underline{D}^P$$

From Ex. 6.86 $\quad J\underline{\tau}\cdot\underline{D}^E = \bar{\underline{\tau}}\cdot\dot{\underline{E}}^E$.

Similarly,

$$J\underline{\tau}\cdot\underline{D}^P = \frac{J}{2}\underline{\tau}\cdot(\underline{L}^{PT}+\underline{L}^P) = \frac{J}{2}\underline{\tau}\cdot\left[(\underline{X}^E)^{-T}(\bar{\underline{L}}^P)^T(\underline{X}^E)^T + \underline{X}^E\bar{\underline{L}}^P(\underline{X}^E)^{-1}\right]$$

Since $\quad \underline{A}\cdot(\underline{B}\,\underline{C}\,\underline{D}) = \underline{C}\cdot(\underline{B}^T\underline{A}\,\underline{D}^T)$, we have

$$J\underline{\tau}\cdot\underline{D}^P = \frac{J}{2}\left\{\left[(\underline{X}^E)^{-1}\underline{\tau}\,\underline{X}^E\right]\cdot(\bar{\underline{L}}^P)^T + \left[(\underline{X}^E)^T\underline{\tau}\,(\underline{X}^E)^{-T}\right]\cdot\bar{\underline{L}}^P\right\}$$

$$= \frac{1}{2}\left\{\left[(\underline{U}^E)^{-1}\bar{\underline{\tau}}\,\underline{U}^E\right]\cdot(\bar{\underline{L}}^P)^T + \left(\underline{U}^E\bar{\underline{\tau}}\,(U^E)^{-1}\right)\cdot\bar{\underline{L}}^P\right\}$$

Due to the material isotropy, $\bar{\underline{\tau}}$ and $\underline{U}^E$ commute, therefore,

$$J\underline{\tau}\cdot\underline{D}^P = \bar{\underline{\tau}}\cdot\bar{\underline{D}}^P, \quad \text{where } \bar{\underline{D}}^P = \text{sym}(\bar{\underline{L}}^P),$$

and $\quad \underline{L}^P = \underline{X}^E\bar{\underline{L}}^P(\underline{X}^E)^{-1}, \quad \bar{\underline{L}}^P = \dot{\underline{X}}^P(\underline{X}^P)^{-1}$.

6.91

We use the sign convention given in Fig. 6.18.

gap function : $g(\underline{x}, t) = (\underline{x} - \underline{y}^*)^T \underline{n}^*$

This function represents the distance between the two contact points. The contact requirements are that there must be no interpenetration on the contact surfaces.

Of course, the distance $(\underline{x} - \underline{y}^*)^T \underline{n}^*$ must be greater or equal to zero to satisfy the absence of interpenetration on the contact surfaces.

Hence, $g \geq 0$

λ - value

The normal contact traction λ must be compressive on the contact surfaces. With the sign convection λ positive for compression on the contact surfaces. $\lambda \geq 0$.

Normal contact conditions

The physical condition is that, if there is a gap between the two contact surfaces, there can be no contact tractions. On the other hand, if the gap is zero, Contact tractions will be initiated or are on the contact surfaces.

6.91

Mathematically, we therefore have:

if $g > 0$ then $\lambda = 0$, and

if $g = 0$ then $\lambda \geq 0$.

These two conditions can be combined in $g\lambda = 0$.

Hence Equation (6.308) is correct.

Frictional contact condition

1) Sliding condition: If $\dot{u} \neq 0$, then $|t| = \mu\lambda$.

Using (6.309), we have $|\tau| = 1$

2) Sticking condition: if $\dot{u} = 0$, the tangential contact traction is less than or is just reaching the frictional resistance at the contact surfaces, $\mu\lambda$.

Hence, $|t| \leq \mu\lambda$

Using (6.309), we have $|\tau| \leq 1$.

According to the two conditions of sliding contact and sticking contact, we can write Coulomb's law of friction as:

$$|\tau| \leq 1$$

Also, assume $\dot{u} > 0$, then $\underline{\dot{u}}^J \cdot \underline{s}^* > \underline{\dot{u}}^I \cdot \underline{s}^*$, and body J "slides past body I".

Hence, the frictional traction (t) on the body I is positive as is $\dot{u}$. Therefore $|\tau| = 1$ implies $\text{Sign}(\dot{u}) = \text{Sign}(\tau)$

Hence, Eq. (6.311) is correct.

6.92

The principle of virtual work is given in Eq. (6.301), and the constraint function is given by (6.312). Using Eq. (6.314) and the result of Excercise 3.35, we impose the constraint now as:

$$\int_{S^{ID}} \left(\delta\lambda \, w(g,\lambda) + \frac{\delta\lambda \, \lambda}{\alpha} \right) dS = 0 ,$$ where α is the penalty parameter.

The finite element equations are now obtained as those given in Eq's (6.315) to (6.322), with the frictional constraint not included, and the discretization of term $\frac{\delta\lambda\,\lambda}{\alpha}$ added into the equations. This term changes the vector ${}^{t+\Delta t}\underline{F}_k^c$ to

$${}^{t+\Delta t}\underline{F}_k^c = [\, w(g_k, \lambda_k) + \lambda_k/\alpha \,],$$ and the matrix ${}^t\underline{\underline{K}}_{\tau\tau}^c$ accordingly.

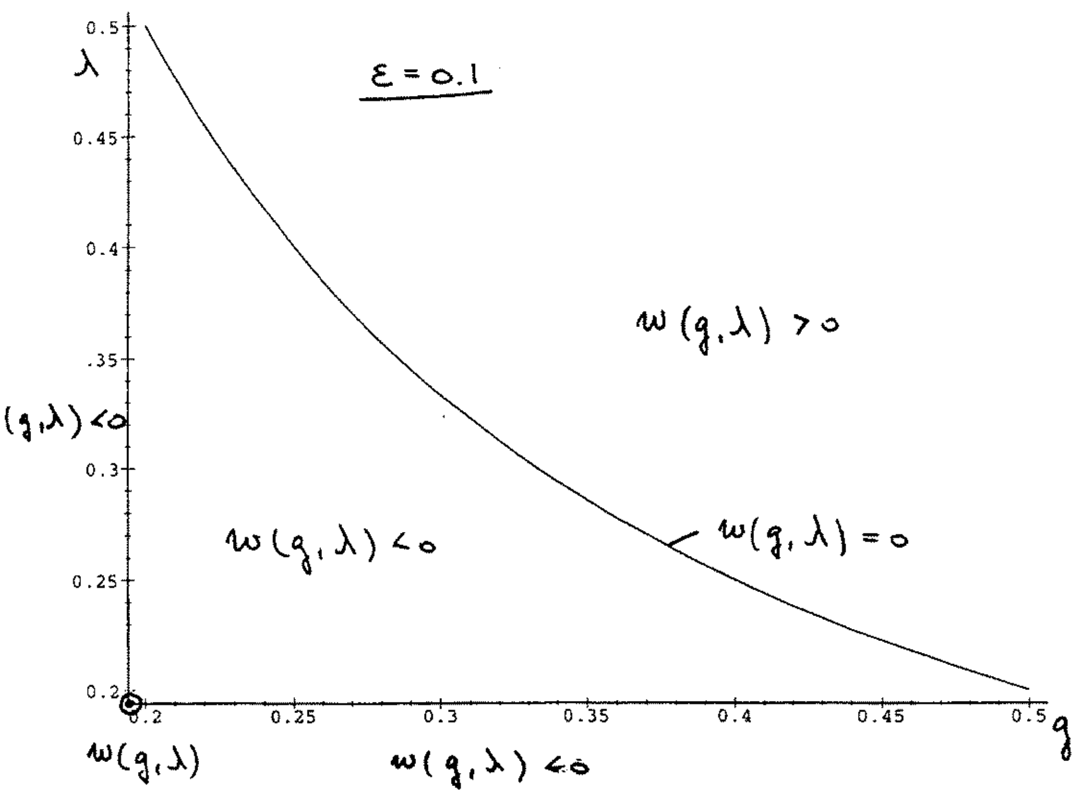
Note that a convenient constraint function $w(g,\lambda)$ should ideally be used to enable the elimination of the λ degrees of freedom (see Excercise 3.35).

6.93

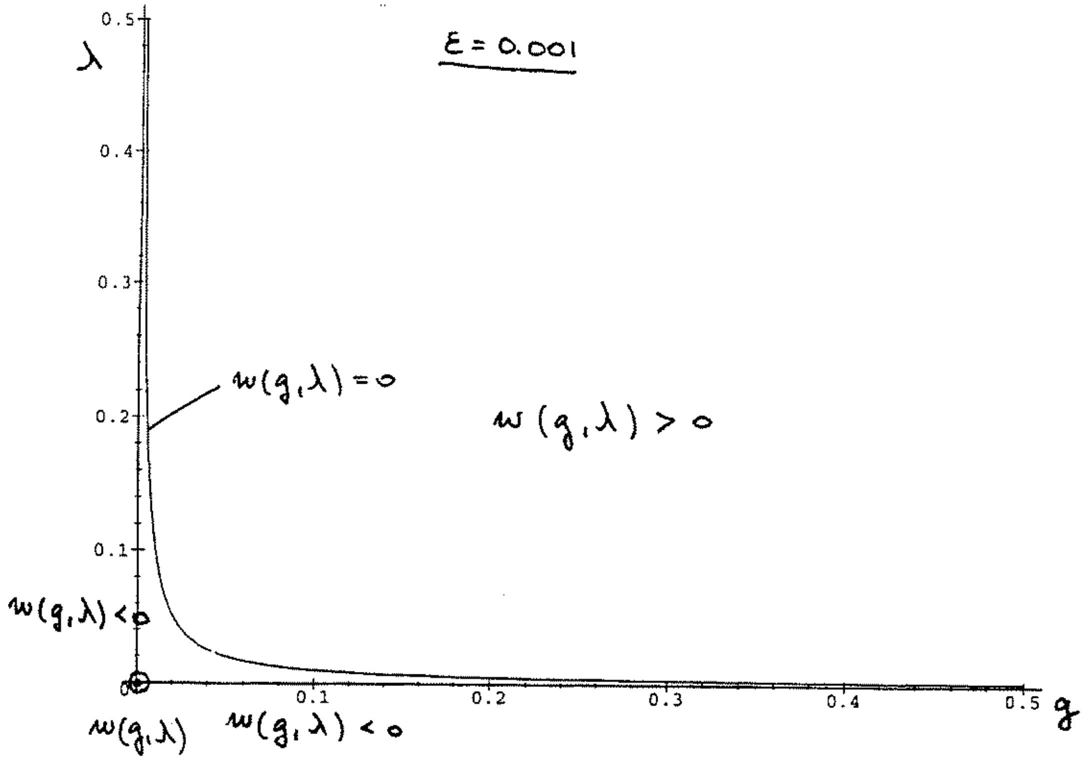
Constraint function.

Consider $w(g, \lambda) = \dfrac{g+\lambda}{2} - \sqrt{\left(\dfrac{g-\lambda}{2}\right)^2 + \varepsilon}$.

The figures show the behaviour of the function for different values ε. Note that with this function if $w(g, \lambda) = 0$, we have $g\lambda = \varepsilon$, and this approximates the function in Fig. 6.19.



6.93



The function $w(g,\lambda)$ is continuous, and the derivatives are:

$$\frac{\partial w}{\partial g} = \frac{1}{2} - \frac{1}{4}\,\frac{2g - 2\lambda}{\sqrt{g^2 - 2g\lambda + \lambda^2 + 4\varepsilon}},$$

$$\frac{\partial w}{\partial \lambda} = \frac{1}{2} - \frac{1}{4}\,\frac{-2g + 2\lambda}{\sqrt{g^2 - 2g\lambda + \lambda^2 + 4\varepsilon}}.$$

These derivatives are used in the solution using Newton-Raphson iteration.

6.94

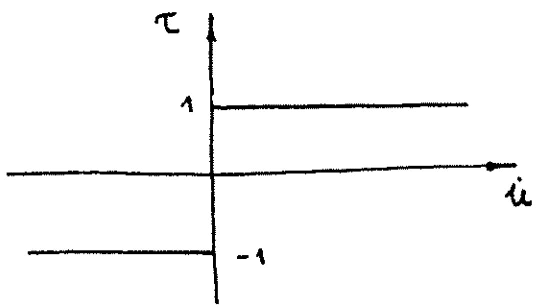
For frictional conditions, the constraint function should satisfy:

$|\tau| \le 1$,

$|\tau| < 1$ implies $\dot{u} = 0$,

$|\tau| = 1$ implies $\text{sign}(\dot{u}) = \text{sign}(\tau)$

Pictorially,

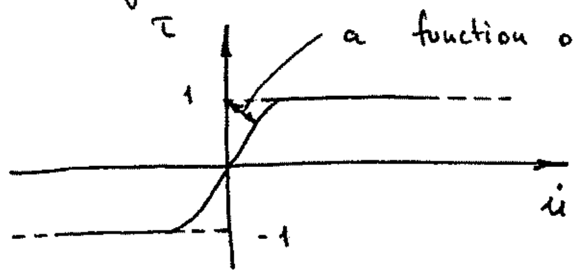


(see Fig. 6.19)

We can define $\tau(\dot{u}) = \dfrac{\text{sign}(\dot{u})\sqrt{|\dot{u}|}}{\varepsilon + \sqrt{|\dot{u}|}}$,

where ε is small but positive number.

This function approximates the analytical value of τ in Fig. 6.19.



A function $v(\tau, \dot{u})$ is now given by constructing the surface obtained by translating a straight line $(1, -1, 1)$ along the curve $(\tau, \dot{u})$.

7.1

$$\int_V \bar{\underline{\theta}}'^T \underline{k}\, \underline{\theta}' dV = \int_V \bar{\theta}\, q^B dV + \int_{S_q} \bar{\theta}^S q^S dS + \sum_i \bar{\underline{\theta}}^i \underline{Q}^i \qquad \text{(eq. 7.7)}$$

where $q^B = 0$, $\underline{Q}^i = \underline{0}$ in this case.

$$\therefore \int_V \bar{\underline{\theta}}'^T \underline{k}\, \underline{\theta}' dV = \int_{S_q} \bar{\underline{\theta}}^S q^S dS \qquad \text{with } \theta = \theta_e \text{ on } S_\theta$$

where $\underline{k} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$, $q^S = k_n \dfrac{\partial \theta}{\partial n}$ on S_q

and $\underline{\theta}'^T = \begin{bmatrix} \dfrac{\partial \theta}{\partial x} & \dfrac{\partial \theta}{\partial y} \end{bmatrix}$

$$\int_V \begin{bmatrix} \dfrac{\partial \bar{\theta}}{\partial x} & \dfrac{\partial \bar{\theta}}{\partial y} \end{bmatrix} (k\underline{I}) \begin{bmatrix} \dfrac{\partial \theta}{\partial x} \\ \dfrac{\partial \theta}{\partial y} \end{bmatrix} dV = \int_V k \left(\frac{\partial \bar{\theta}}{\partial x}\frac{\partial \theta}{\partial x} + \frac{\partial \bar{\theta}}{\partial y}\frac{\partial \theta}{\partial y} \right) dV$$

$$= \int_V k \left[\left\{ \frac{\partial}{\partial x}\left(\bar{\theta} \frac{\partial \theta}{\partial x} \right) + \frac{\partial}{\partial y}\left(\bar{\theta} \frac{\partial \theta}{\partial y} \right) \right\} - \bar{\theta} \left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \right) \right] dV$$

$$= \int_S \bar{\theta} \left(k \frac{\partial \theta}{\partial n} \right) dS - \int_V \bar{\theta}\, k \left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \right) dV = \int_{S_q} \bar{\theta}\, q^S dS$$

As $S = S_\theta \cup S_q$, we obtain

$$\begin{cases} k \left(\dfrac{\partial^2 \theta}{\partial x^2} + \dfrac{\partial^2 \theta}{\partial y^2} \right) = 0 & \text{in } V \\ \theta = \bar{\theta}_e & \text{on } S_\theta \\ k \dfrac{\partial \theta}{\partial n} = q^S & \text{on } S_q \end{cases}$$

7.2 conduction terms:

$$\bar{\theta}'\, {}^{t+\Delta t}q_k \simeq \bar{\theta}'\, {}^{t}q_k + \frac{\partial}{\partial \theta}(\bar{\theta}'\, {}^{t}q_k)\, d\theta$$

$$\therefore \quad \bar{\theta}'\, {}^{t+\Delta t}q_k \simeq -\bar{\theta}'\left[(10+2\,{}^{t}\theta)\frac{\partial\, {}^{t}\theta}{\partial x} + 2\,\frac{\partial\, {}^{t}\theta}{\partial x}\, d\theta + (10+2\,{}^{t}\theta)\frac{\partial\, d\theta}{\partial x}\right]$$

"complete"

Convection terms :

$$\bar{\theta}^{S}\, {}^{t+\Delta t}q_h \simeq \bar{\theta}^{S}\left[{}^{t}q_h + \frac{\partial}{\partial \theta^{S}}({}^{t}q_h)\, d\theta^{S}\right]$$

$$= \bar{\theta}^{S}\left[(2+{}^{t}\theta^{S})({}^{t}\theta_e - {}^{t}\theta^{S}) + \left\{({}^{t}\theta_e - {}^{t}\theta^{S}) - (2+{}^{t}\theta^{S})\right\} d\theta^{S}\right]$$

↑ not included

radiation terms :

$$\bar{\theta}^{S}\, {}^{t+\Delta t}q_r \simeq \bar{\theta}^{S}\left[{}^{t}q_r + \frac{\partial}{\partial \theta^{S}}({}^{t}q_r)\, d\theta^{S}\right]$$

(use °K)

$$= \bar{\theta}^{S}\, h_r\left[\left\{({}^{t}\theta_r)^4 - ({}^{t}\theta^{S})^4\right\} - 4({}^{t}\theta^{S})^3\, d\theta^{S}\right]$$

"complete"

Hence the following terms should be added to the l.h.s. of eq. (b) and (c) in example 7.2 ;

In the modified N.-R. method,

$$-\left[\bar{\theta}^{S}(20-{}^{t}\theta^{S})\,\Delta\theta^{S(i)}\right]\Big|_{x=L}$$

In the full N.-R. iteration

$$-\left[\bar{\theta}^{S}(20-{}^{t+\Delta t}\theta^{S(i-1)})\,\Delta\theta^{S(i)}\right]\Big|_{x=L}$$

7.3 Eq. of the principle of virtual temperatures :

$$\int_V \bar{\underline{\theta}}'^T\, {}^t\underline{k}\, \underline{\theta}'\, dV + \int_{S_c} \bar{\underline{\theta}}^S\, {}^t h\, \underline{\theta}^S\, dS$$

$$= {}^{t+\Delta t}Q + \int_{S_c} \bar{\underline{\theta}}^S\, {}^t h \left({}^{t+\Delta t}\theta_e - {}^t\theta^S\right) dS - \int_V \bar{\underline{\theta}}'^T\, {}^t\underline{k}\, {}^t\underline{\theta}'\, dV$$

$$\therefore \int_0^L \bar{\theta}'\left(60 + {}^{t+\Delta t}\theta^{(i-1)}\right)\Delta\theta'^{(i)}\, dx + \left[\bar{\theta}^S\left(4 + {}^{t+\Delta t}\theta^{S(i-1)}\right)\Delta\theta^{S(i)}\right]\Big|_{x=0}$$

$$= \left[\bar{\theta}^S\left(4 + {}^{t+\Delta t}\theta^{S(i-1)}\right)\left(20 - {}^{t+\Delta t}\theta^{S(i-1)}\right)\right]\Big|_{x=0}$$

$$- \int_0^L \bar{\theta}'\left(60 + {}^{t+\Delta t}\theta^{(i-1)}\right) {}^{t+\Delta t}\theta'^{(i-1)}\, dx \qquad \text{——}(*)$$

Conduction terms :

$$\bar{\theta}'\, {}^{t+\Delta t}q_k \simeq \bar{\theta}'\left[{}^t q_k + \frac{\partial}{\partial\theta}\left({}^t q_k\right) d\theta\right]$$

$$= -\bar{\theta}'\left[\left(60 + {}^t\theta\right)\frac{\partial\, {}^t\theta}{\partial x} + \underbrace{\frac{\partial\, {}^t\theta}{\partial x}}\, d\theta + \left(60 + {}^t\theta\right)\frac{\partial\, d\theta}{\partial x}\right]$$

↖ not included

Convection terms :

$$\bar{\theta}^S\, {}^{t+\Delta t}q_h \simeq \bar{\theta}^S\left[{}^t q_h + \frac{\partial}{\partial\theta^S}\left({}^t q_h\right) d\theta^S\right]$$

$$= \bar{\theta}^S\left[\left(4 + {}^t\theta^S\right)\left(20 - {}^t\theta^S\right) + \left\{\underbrace{\left(20 - {}^t\theta^S\right)} - \left(4 + {}^t\theta^S\right)\right\} d\theta^S\right]$$

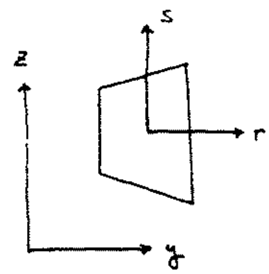
↖ not included

Hence the following terms are to be added to the (l.h.s) of (*).

$$\int_0^L \bar{\theta}'\, {}^{t+\Delta t}\theta'^{(i-1)}\, \Delta\theta^{(i)}\, dx - \left[\bar{\theta}^S\left(20 - {}^{t+\Delta t}\theta^{S(i-1)}\right)\Delta\theta^{S(i)}\right]\Big|_{x=0}$$

7.4 Principle of Virtual Temperatures :

$$\int_V k\bar{\theta}_{,i}(\theta_{,i})\,dV = \int_V \bar{\theta}\, q^B\, dV + \int_{S_q} \bar{\theta}^S q^S\, dS$$



In polar coordinate system,

$$\underline{\nabla} = \frac{\partial}{\partial y}\hat{i}_y + \frac{\partial}{\partial z}\hat{i}_z + \cancel{\frac{1}{y}\frac{\partial}{\partial \phi}}\hat{i}_\phi \qquad \text{(axisymmetric)}$$

$$\therefore (\text{l.h.s.}) = \int_V \left(k\frac{\partial\bar{\theta}}{\partial y}\frac{\partial\theta}{\partial y} + k\frac{\partial\bar{\theta}}{\partial z}\frac{\partial\theta}{\partial z} \right) dV = \int_V \underline{\bar{\theta}}'\,\underline{k}\,\underline{\theta}'\, dV$$

where $\underline{\theta}'^T = \left[\frac{\partial\theta}{\partial y} \quad \frac{\partial\theta}{\partial z} \right]$, $\underline{k} = k\underline{I}$, $\underline{\bar{\theta}}'^T = \left[\frac{\partial\bar{\theta}}{\partial y} \quad \frac{\partial\bar{\theta}}{\partial z} \right]$

Now employing the same interpolation functions as in structural analyses,

$$\underline{H} = \frac{1}{4}\left[(1+r)(1+s) \quad (1-r)(1+s) \quad (1-r)(1-s) \quad (1+r)(1-s) \right]$$

$$\underline{H}^S = \underline{H}\big|_{s=-1} = \left[0 \quad 0 \quad \frac{1-r}{2} \quad \frac{1+r}{2} \right]$$

$$\underline{J} = \begin{bmatrix} 2 & \frac{s}{2} \\ 0 & \frac{5+r}{2} \end{bmatrix} \quad \left(y = 2(6+r),\ z = \tfrac{1}{2}(6+5s+rs) \right),\ \det\underline{J} = 5+r$$

$$\underline{B} = \frac{1}{5+r}\begin{bmatrix} \frac{5+r}{2} & -\frac{s}{2} \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \frac{1+s}{4} & -\frac{1+s}{4} & -\frac{1-s}{4} & \frac{1-s}{4} \\ \frac{1+r}{4} & \frac{1-r}{4} & -\frac{1-r}{4} & -\frac{1+r}{4} \end{bmatrix}$$

$$\underline{k} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

7.4

Conductivity matrix $\underline{K}^k$ (for one radian):

$$\underline{K}^k = \int_V \underline{B}^T \underline{k}\, \underline{B}\, y\, dy\, dz = \int_{-1}^{1}\int_{-1}^{1} \underline{B}^T \underline{k}\, \underline{B}\, (5+r)\{2(6+r)\}\, dr\, ds$$

Convection matrix $\underline{K}^c$:

$$\underline{K}^c = \int_{S_c} h\, \underline{H}^{S^T} \underline{H}^S dS = \int_{-1}^{1} h\, \underline{H}^{S^T} \underline{H}^S \{2(6+r)\}\left(\frac{\sqrt{17}}{2}\right) dr$$

since $dS = y\, dl = y \det \underline{J}^S dr$

$$\det \underline{J}^S = \left[\left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2\right]^{1/2}\Bigg|_{s=-1} = \frac{\sqrt{17}}{2}$$

with $\frac{\partial y}{\partial r} = 2$, $\frac{\partial z}{\partial r} = \frac{s}{2}$

Heat capacity matrix $\underline{C}$:

$$\underline{C} = \int_V \rho c\, \underline{H}^T \underline{H}\, dV = \int_{-1}^{1}\int_{-1}^{1} \rho c\, \underline{H}^T \underline{H}\, \{2(6+r)\}(5+r)\, dr\, ds$$

Heat flow load vector:

$$\underline{Q} = \underline{Q}_B + \underline{Q}^e$$

$$\underline{Q}_B = \int_V \underline{H}^T q^B dV = \int_{-1}^{1}\int_{-1}^{1} \underline{H}^T q^B \{2(6+r)\}(5+r)\, dr\, ds$$

$$\underline{Q}^e = \int_{S_e} h\, \underline{H}^{S^T} \underline{H}^S \hat{\underline{\theta}}_e\, dS = \int_{-1}^{1} h\, \underline{H}^{S^T} \underline{H}^S \hat{\underline{\theta}}_e \{2(6+r)\}\left(\frac{\sqrt{17}}{2}\right) dr$$

7.5 From the interpolation functions in Fig. 5.4

$$h_1 = \frac{1}{4}(1+r)(1+s) - \frac{1}{2}\left\{\frac{1}{2}(1-r^2)(1+s) - \frac{1}{2}(1-r^2)(1-s^2)\right\}$$

$$- \frac{1}{2}\left\{\frac{1}{2}(1-s^2)(1+r) - \frac{1}{2}(1-r^2)(1-s^2)\right\}$$

$$- \frac{1}{4}(1-r^2)(1-s^2)$$

$$\therefore h_1 = \frac{1}{4} rs(1+r)(1+s)$$

$$h_{1,r} = \frac{1}{4}(1+2r)s(1+s) \qquad h_{1,s} = \frac{1}{4} r(1+r)(1+2s)$$

$$\underline{J} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \quad \underline{J}^{-1} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix}, \quad \det \underline{J} = 6$$

$$\underline{H} = [\, h_1 \;\; h_2 \;\; \cdots \;\; h_9 \,]$$

$$\underline{B} = \begin{bmatrix} 1/3 & \\ & 1/2 \end{bmatrix} \begin{bmatrix} h_{1,r} & h_{2,r} & \cdots \\ h_{1,s} & h_{2,s} & \cdots \end{bmatrix} = \begin{bmatrix} \frac{1}{3} h_{1,r} & \cdots \\ \frac{1}{2} h_{1,s} & \cdots \end{bmatrix}$$

$$\therefore K_{11} = (2.0)\int_{-1}^{1}\int_{-1}^{1} \left[\frac{1}{3} h_{1,r} \;\; \frac{1}{2} h_{1,s}\right] k \begin{bmatrix} \frac{1}{3} h_{1,r} \\ \frac{1}{2} h_{1,s} \end{bmatrix} (6)\, dr\, ds = \frac{182}{135} k$$

$$C_{11} = (2.0)\int_{-1}^{1}\int_{-1}^{1} \rho c\, h_1^2\, (6)\, dr\, ds = \frac{64}{75} \rho c\,.$$

7.6 $\left({}^{t}\underline{K}^{k} + {}^{t}\underline{K}^{c} + {}^{t}\underline{K}^{r} \right) \Delta\underline{\theta}^{(i)} = {}^{t+\Delta t}\underline{Q} + {}^{t+\Delta t}\underline{Q}^{c(i-1)} + {}^{t+\Delta t}\underline{Q}^{r(i-1)} - {}^{t+\Delta t}\underline{Q}^{k(i-1)}$

where $\displaystyle {}^{t}\underline{K}^{r} = \int_{S_r} {}^{t}\tilde{\kappa}\, \underline{H}^{sT} \underline{H}^{s}\, dS$ with ${}^{t}\tilde{\kappa} = 4\, {}^{t}h_r ({}^{t}\theta^{s})^3$ (in °K)

$$ {}^{t+\Delta t}\underline{Q}^{r(i-1)} = \int_{S_r} {}^{t+\Delta t}\kappa^{(i-1)} \underline{H}^{sT} \underline{H}^{s} \left({}^{t+\Delta t}\underline{\theta}_r - {}^{t+\Delta t}\underline{\theta}^{(i-1)} \right) dS $$

with ${}^{t+\Delta t}\kappa^{(i-1)} = {}^{t+\Delta t}h_r \left[({}^{t+\Delta t}\theta_r)^2 + ({}^{t+\Delta t}\theta^{s(i-1)})^2 \right] \left({}^{t+\Delta t}\theta_r + {}^{t+\Delta t}\theta^{s(i-1)} \right)$ (in °K)

$$ {}^{t+\Delta t}\underline{\theta}_r^{T} = [\, \theta_r \quad 0 \quad 0 \,] $$

and other matrices are defined in Example 7.4.

Here ${}^{t}\tilde{\kappa} = 4\, {}^{t}h_r ({}^{t}\theta^{s})^3 = 4 h_r ({}^{t}\theta_1)^3$, ${}^{t}\theta^{s} = {}^{t}\theta_1$

$$ {}^{t+\Delta t}\kappa^{(i-1)} = h_r \left[\theta_r^2 + ({}^{t+\Delta t}\theta_1^{(i-1)})^2 \right] \left[\theta_r + {}^{t+\Delta t}\theta_1^{(i-1)} \right] $$

$$ \therefore\ {}^{t}\underline{K}^{r} = \int_{S_r} {}^{t}\tilde{\kappa}\, \underline{H}^{sT} \underline{H}^{s}\, dS = 4 h_r ({}^{t}\theta_1)^3\, \underline{H}^{sT} \underline{H}^{s} $$

with $\underline{H}^{s} = \underline{H}\big|_{r=1} = [\, 1 \quad 0 \quad 0 \,]$

$$ {}^{t+\Delta t}\underline{Q}^{r(i-1)} = \int_{S_r} h_r \left[\theta_r^2 + ({}^{t+\Delta t}\theta_1^{(i-1)})^2 \right] \left[\theta_r + {}^{t+\Delta t}\theta_1^{(i-1)} \right] \underline{H}^{sT} \underline{H}^{s} \left({}^{t+\Delta t}\underline{\theta}_r - {}^{t+\Delta t}\underline{\theta}^{(i-1)} \right) dS $$

$$ = h_r \left[(\theta_r)^4 - ({}^{t+\Delta t}\theta_1^{(i-1)})^4 \right] \underline{H}^{sT} $$

7.15 Basic equations are

$$\rho\left(\frac{\partial v_i}{\partial t} + v_{i,j} v_j\right) = \tau_{ij,j} + f_i^B \qquad \text{— ①}$$

$$\tau_{ij} = -p\,\delta_{ij} + 2\mu e_{ij} \qquad \text{— ②}$$

$$v_{i,i} = 0 \qquad \text{— ③}$$

$$\rho C_p\left(\frac{\partial\theta}{\partial t} + \theta_{,i} v_i\right) = (k\theta_{,i})_{,i} + g^B \qquad \text{— ④}$$

Substituting ② and ③ into ①,

$$\rho\left(\frac{\partial v_i}{\partial t} + v_{i,j} v_j\right) = (-p\delta_{ij} + 2\mu e_{ij})_{,j} + f_i^B$$

$$= -p_{,i} + f_i^B + \mu\,(v_{i,j} + v_{j,i})_{,j}$$

$$= -p_{,i} + f_i^B + \mu\,(v_{i,jj} + \cancel{v_{j,ij}}) \quad (\because v_{j,j} = 0)$$

$$\therefore\ \rho\left(\frac{\partial v_i}{\partial t} + v_{i,j} v_j\right) = -p_{,i} + f_i^B + \mu v_{i,jj} \qquad \Leftarrow (7.65)$$

Multiplying both sides of (7.65) by $L/\rho V^2$,

$$\frac{\partial(v_i/V)}{\partial(tV/L)} + \frac{v_j}{V}\frac{\partial(v_i/V)}{\partial(x_j/L)} = -\frac{\partial(p/\rho V^2)}{\partial(x_i/L)} + f_i^B\frac{L}{\rho V^2} + \frac{\mu}{\rho VL}\frac{\partial^2(v_i/V)}{\partial(x_j/L)^2}$$

$$\therefore\ \frac{\partial v_i^*}{\partial t^*} + v_{i,j}^* v_j^* = -p_{,i}^* + f_i^{B*} + \frac{1}{Re} v_{i,jj}^* \qquad \Leftarrow (7.67)$$

$$\text{where}\quad Re = \rho VL/\mu$$

Similarly from ④ with $L/\rho C_p \Delta\theta V$,

$$\frac{\partial(\theta/\Delta\theta)}{\partial(tV/L)} + \frac{v_i}{V}\frac{\partial(\theta/\Delta\theta)}{\partial(x_i/L)} = \frac{1}{\rho C_p VL}\frac{\partial}{\partial(x_i/L)}\left(k\frac{\partial(\theta/\Delta\theta)}{\partial(x_i/L)}\right) + \frac{g^B L}{\rho C_p \Delta\theta V}$$

or
$$\frac{\partial\left(\frac{\theta-\theta_0}{\Delta\theta}\right)}{\partial(tV/L)} + \frac{v_i}{V}\frac{\partial\left(\frac{\theta-\theta_0}{\Delta\theta}\right)}{\partial(x_i/L)} = \frac{k}{\rho C_p VL}\frac{\partial^2\left(\frac{\theta-\theta_0}{\Delta\theta}\right)}{\partial(x_i/L)^2} + \frac{g^B L}{\rho C_p \Delta\theta V}$$

$$\therefore\ \frac{\partial\theta^*}{\partial t^*} + \theta_{,i}^* v_i^* = g^{B*} + \frac{1}{Pe}\theta_{,ii}^* \quad \text{where}\quad Pe = \frac{\rho C_p VL}{k} \qquad \Leftarrow (7.68)$$

7.16 Applying the Galerkin procedure to (7.57) weighted by $\bar{v}_i$,

$$\int_A \bar{v}_i \left[\rho(\dot{v}_i + v_{i,j} v_j) - \tau_{ij,j} - f_i^B \right] dA = 0$$

$$\int_A \bar{v}_i \rho \dot{v}_i \, dA + \int_A \bar{v}_i \rho v_j v_{i,j} \, dA - \int_A \bar{v}_i \tau_{ij,j} \, dA - \int_A \bar{v}_i f_i^B dA = 0$$

$$\int_A \bar{v}_i \tau_{ij,j} \, dA = \int_A (\bar{v}_i \tau_{ij})_{,j} \, dA - \int_A \bar{v}_{i,j} \tau_{ij} \, dA$$

$$= \int_S (\bar{v}_i \tau_{ij}) n_j \, dS - \int_A \bar{v}_{i,j} (-p\delta_{ij} + 2\mu e_{ij}) \, dA$$

$$= \int_S \bar{v}_i f_i^S \, dS + \int_A \bar{v}_{i,j} \, p \, \delta_{ij} \, dA - \int_A \bar{v}_{i,j} \, \mu (v_{i,j} + v_{j,i}) \, dA$$

$$\therefore \int_A \bar{v}_i \rho \dot{v}_i \, dA + \int_A \bar{v}_i \rho v_j v_{i,j} \, dA - \int_A \bar{v}_{i,j} p \delta_{ij} \, dA + \int_A \bar{v}_{i,j} \mu (v_{i,j} + v_{j,i}) \, dA$$

$$= \int_A \bar{v}_i f_i^B dA + \int_S \bar{v}_i f_i^S dS$$

For $i = 2$,

$$\int_A \bar{v}_2 \rho \dot{v}_2 \, dA + \int_A \bar{v}_2 \rho (v_2 v_{2,2} + v_3 v_{2,3}) \, dA - \int_A \bar{v}_{2,2} \, p \, dA$$

$$+ \int_A \left[\bar{v}_{2,2} \mu \cdot 2 v_{2,2} + \bar{v}_{2,3} \mu (v_{2,3} + v_{3,2}) \right] dA$$

$$= \int_A \bar{v}_2 f_2^B \, dA + \int_S \bar{v}_2 f_2^S \, dS$$

Introducing the interpolation matrix $\underline{H}$ for velocity, and $\tilde{\underline{H}}$ for pressure, we obtain

$$\int_A \underline{H}^T \rho \underline{H} \, dA \; \dot{\hat{\underline{v}}}_2 + \rho \int_A \underline{H}^T (\underline{H} \hat{\underline{v}}_2 H_{,x_2} + \underline{H} \hat{\underline{v}}_3 H_{,x_3}) \, dA \; \hat{\underline{v}}_2$$

$$- \int_A \underline{H}_{,x_2}^T \tilde{\underline{H}} \, dA \; \hat{\underline{p}} + \int_A \left(\underline{H}_{,x_2}^T 2\mu \underline{H}_{,x_2} + \underline{H}_{,x_3}^T \mu \underline{H}_{,x_3} \right) dA \; \hat{\underline{v}}_2$$

$$+ \int_A \underline{H}_{,x_3}^T \mu \underline{H}_{,x_2} \, dA \; \hat{\underline{v}}_3 = \int_A \underline{H}^T f_2^B \, dA + \int_S \underline{H}^{S^T} f_2^S \, dS$$

7.16

Or in matrix form,

$$\underline{M}_{v_2} \dot{\hat{\underline{U}}}_2 + \underline{K}_{vv_2} \hat{\underline{U}}_2 + \underline{K}_{v_2 p} \hat{\underline{P}} + \underline{K}_{\mu v_2 v_2} \hat{\underline{U}}_2 + \underline{K}_{\mu v_2 v_3} \hat{\underline{U}}_3 = \underline{R}_{B_2} + \underline{R}_{S_2}$$

Similarly for $i=3$,

$$\underline{M}_{v_3} \dot{\hat{\underline{U}}}_3 + \underline{K}_{vv_3} \hat{\underline{U}}_3 + \underline{K}_{v_3 p} \hat{\underline{P}} + \underline{K}_{\mu v_3 v_3} \hat{\underline{U}}_3 + \underline{K}_{\mu v_3 v_2} \hat{\underline{U}}_2 = \underline{R}_{B_3} + \underline{R}_{S_3}$$

Note $\underline{K}_{\mu v_3 v_2} = \underline{K}_{\mu v_2 v_3}^T$.

Now from (7.59) weighted by $\bar{p}$, we obtain

$$\int_A \bar{p}\, v_{i,i}\, dA = 0$$

Or in matrix form with $\underline{H}$ and $\tilde{\underline{H}}$ as before,

$$\int_A \tilde{\underline{H}}^T H_{,x_2}\, dA\ \hat{\underline{v}}_2 + \int_A \tilde{\underline{H}}^T \underline{H}_{,x_3}\, dA\ \hat{\underline{v}}_3 = 0$$

$$\underline{K}_{pv_2} \hat{\underline{U}}_2 + \underline{K}_{pv_3} \hat{\underline{U}}_3 = 0$$

Note here $\underline{K}_{pv_2} = \underline{K}_{v_2 p}^T$, $\underline{K}_{pv_3} = \underline{K}_{v_3 p}^T$

Finally from eq. (7.60) weighted with $\bar{\theta}$, using the divergence rule, we obtain

$$\int_A \bar{\theta}\rho c_p \dot{\theta}\, dA + \int_A \bar{\theta}\rho c_p v_i \theta_{,i}\, dA + \int_A \bar{\theta}_{,i} k\, \theta_{,i}\, dA$$
$$= \int_A \bar{\theta} q^B dA + \int_S \bar{\theta} q^S dA$$

$$\int_A \underline{H}^T \rho c_p \underline{H}\, dA\ \dot{\hat{\underline{\theta}}} + \int_A \underline{H}^T \rho c_p (\underline{H}\hat{\underline{v}}_2 H_{,x_2} + \underline{H}\hat{\underline{v}}_3 \underline{H}_{,x_3})\, dA\ \hat{\underline{\theta}}$$
$$+ \int_A (\underline{H}_{,x_2}^T k\, \underline{H}_{,x_2} + \underline{H}_{,x_3}^T k\, \underline{H}_{,x_3})\, dA\ \hat{\underline{\theta}}$$
$$= \int_A \underline{H}^T q^B dA + \int_S \underline{H}^{ST} q^S dS$$

where $\underline{H}$ is used for the velocity and temperature interpolation.

7.16

In matrix form

$$\underset{\sim}{C}\dot{\hat{\underset{\sim}{\theta}}} + K_{v\theta}\hat{\underset{\sim}{\theta}} + \underset{\sim}{K}_{\theta\theta}\hat{\underset{\sim}{\theta}} = Q_B + \underset{\sim}{Q}_s$$

Hence we see that the matrix expressions for the 2-D planar flow conditions given in (7.77) to (7.89) are correct.

7.17 For the steady state case Eq. (7.77) reduces to

$$\underline{k}(\underline{U})\ \underline{U} = \underline{R}$$

where

$$\underline{k}(\underline{U}) = \begin{bmatrix} \underline{K}_{\mu v_2 v_2} + \underline{K}_{v v_2} & \underline{K}_{\mu v_2 v_3} & \underline{K}_{v_2 p} & \underline{0} \\ \underline{K}^T_{\mu v_2 v_3} & \underline{K}_{\mu v_3 v_3} + \underline{K}_{v v_3} & \underline{K}_{v_3 p} & \underline{0} \\ \underline{K}^T_{v_2 p} & \underline{K}^T_{v_3 p} & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{K}_{v\theta} + \underline{K}_{\theta\theta} \end{bmatrix}$$

$$\underline{U}^T = [\ \hat{\underline{v}}_2\ \ \hat{\underline{v}}_3\ \ \hat{\underline{p}}\ \ \hat{\underline{\theta}}\ ]\ , \quad \underline{R} = \begin{bmatrix} \underline{R}_{B_2} + \underline{R}_{S_2} \\ \underline{R}_{B_3} + \underline{R}_{S_3} \\ \underline{0} \\ \underline{Q}_B + \underline{Q}_S \end{bmatrix}$$

Using a full N.-R. iteration we obtain

$$\underline{K}\,\Delta\underline{U} = \underline{R} - \underline{F}(\underline{U}^{(i-1)}) \qquad \text{where} \quad \Delta\underline{U} = \underline{U}^{(i)} - \underline{U}^{(i-1)}$$

$$\underline{F}(\underline{U}) = \underline{k}(\underline{U})\,\underline{U}\ , \qquad \underline{K} = \frac{\partial \underline{F}(\underline{U})}{\partial \underline{U}} = [\ \underline{K}_{ij}\ ]$$

Let

$$F_1 = \int_A \bar{v}_2\, \rho\, (v_2 v_{2,2} + v_3 v_{2,3})\, dA - \int_A \bar{v}_{2,2}\, p\, dA + \int_A [\ \bar{v}_{2,2}\, 2\mu\, v_{2,2} + \bar{v}_{2,3}\, \mu\, (v_{2,3} + v_{3,2})\ ]\, dA$$

$$F_2 = \int_A \bar{v}_3\, \rho\, (v_2 v_{3,2} + v_3 v_{3,3})\, dA - \int_A \bar{v}_{3,3}\, p\, dA + \int_A [\ \bar{v}_{3,2}\, \mu\, (v_{3,2} + v_{2,3}) + \bar{v}_{3,3}\, 2\mu\, v_{3,3}\ ]\, dA$$

$$F_3 = -\int \bar{p}\, (v_{2,2} + v_{3,3})\, dA$$

$$F_4 = \int_A \bar{\theta}\, \rho\, C_p\, (v_2\, \theta_{,2} + v_3\, \theta_{,3})\, dA + \int_A (\bar{\theta}_{,2}\, k\, \theta_{,2} + \bar{\theta}_{,3}\, k\, \theta_{,3})\, dA$$

7.17

Here we use the following :

$$\frac{\partial}{\partial v_{2j}}\left(\bar{v}_2 \rho v_2 v_{2,2}\right) = \frac{\partial \bar{v}_2}{\partial v_{2j}} \rho v_2 v_{2,2} + \bar{v}_2 \rho \frac{\partial v_2}{\partial v_{2j}} + \bar{v}_2 \rho v_2 \frac{\partial^2 v_2}{\partial v_{2j} \partial x_2}$$

$$\frac{\partial \bar{v}_2}{\partial v_{2j}} = \frac{\partial}{\partial v_{2j}}\left(\Sigma h_k \delta v_{2k}\right) = \Sigma h_k \delta\left(\frac{\partial v_{2k}}{\partial v_{2j}}\right) = \Sigma h_k \delta(\delta_{kj}) = 0$$

$$\frac{\partial v_2}{\partial v_{2j}} = \frac{\partial}{\partial v_{2j}}\left(\Sigma h_k v_{2k}\right) = \Sigma h_k \frac{\partial v_{2k}}{\partial v_{2j}} = \Sigma h_k \delta_{kj} = \Sigma h_j = \mathbf{H}$$

$$\frac{\partial^2 v_2}{\partial v_{2j} \partial x_2} = \frac{\partial}{\partial x_2}\left[\frac{\partial}{\partial v_{2j}}\left(\Sigma h_k v_{2k}\right)\right] = \frac{\partial}{\partial x_2}\left[\Sigma h_j\right] = \frac{\partial \mathbf{H}}{\partial x_2}$$

$$\mathbf{K}_{11} = \frac{\partial F_1}{\partial \hat{\mathbf{v}}_2} = \int_A \bar{v}_2 \rho v_{2,2} \mathbf{H}\, dA + \int_A \bar{v}_2 \rho v_2 \frac{\partial \mathbf{H}}{\partial x_2} dA + \int_A \bar{v}_2 \rho v_3 \frac{\partial \mathbf{H}}{\partial x_3} dA$$
$$+ \int_A \bar{v}_{2,2}\, 2\mu \frac{\partial \mathbf{H}}{\partial x_2} dA + \int_A \bar{v}_{2,3}\, \mu \frac{\partial \mathbf{H}}{\partial x_3} dA$$

$$\therefore \mathbf{K}_{11} = \rho \int_A \mathbf{H}^T \mathbf{H}_{,x_2} \hat{\mathbf{v}}_2 \mathbf{H}\, dA + \rho \int_A \mathbf{H}^T \mathbf{H} \hat{\mathbf{v}}_2 \mathbf{H}_{,x_2} dA + \rho \int_A \mathbf{H}^T \mathbf{H} \hat{\mathbf{v}}_3 \mathbf{H}_{,x_3} dA$$
$$+ 2\mu \int_A \mathbf{H}^T_{,x_2} \mathbf{H}_{,x_2} dA + \mu \int_A \mathbf{H}^T_{,x_3} \mathbf{H}_{,x_3} dA$$

(Note that $v_i = \mathbf{H} \hat{\mathbf{v}}_i$, $p = \hat{\mathbf{H}} \hat{\mathbf{p}}$, $\theta = \mathbf{H} \hat{\boldsymbol{\theta}}$.)

Similarly , $\mathbf{K}_{12} = \frac{\partial F_1}{\partial \hat{\mathbf{v}}_3} = \int_A \bar{v}_2 \rho v_{2,3} \mathbf{H}\, dA + \int_A \bar{v}_{2,3}\, \mu \frac{\partial \mathbf{H}}{\partial x_2} dA$

$$= \rho \int_A \mathbf{H}^T \mathbf{H}_{,x_3} \hat{\mathbf{v}}_2 \mathbf{H}\, dA + \mu \int_A \mathbf{H}^T_{,x_3} \mathbf{H}_{,x_2} dA$$

$$\mathbf{K}_{13} = \frac{\partial F_1}{\partial \hat{\mathbf{p}}} = -\int_A \bar{v}_{2,2} \tilde{\mathbf{H}}\, dA = -\int_A \mathbf{H}^T_{,x_2} \tilde{\mathbf{H}}\, dA$$

$$\mathbf{K}_{14} = \frac{\partial F_1}{\partial \hat{\boldsymbol{\theta}}} = \mathbf{0}$$

7.17

$$\underline{K}_{21} = \frac{\partial \underline{F}_2}{\partial \hat{\underline{v}}_2} = \int_A \bar{v}_3 \rho v_{3,2} \underline{H}\, dA + \int_A \bar{v}_{3,2} \mu \frac{\partial \underline{H}}{\partial x_3}\, dA$$

$$= \rho \int_A \underline{H}^T \underline{H}_{,x_2} \hat{v}_3 \underline{H}\, dA + \mu \int_A \underline{H}_{,x_2}^T \underline{H}_{,x_3}\, dA$$

$$\underline{K}_{22} = \frac{\partial \underline{F}_2}{\partial \hat{\underline{v}}_3} = \int_A \bar{v}_3 \rho v_2 \frac{\partial \underline{H}}{\partial x_2}\, dA + \int_A \bar{v}_3 \rho v_{3,3} \underline{H}\, dA + \int_A \bar{v}_3 \rho v_3 \frac{\partial \underline{H}}{\partial x_3}\, dA$$

$$+ \int_A \bar{v}_{3,2} \mu \frac{\partial \underline{H}}{\partial x_2}\, dA + \int_A \bar{v}_{3,3} 2\mu \frac{\partial \underline{H}}{\partial x_3}\, dA$$

$$= \rho \int_A \underline{H}^T \underline{H} \hat{\underline{v}}_2 \underline{H}_{,x_2}\, dA + \rho \int \underline{H}^T \underline{H}_{,x_3} \hat{\underline{v}}_3 \underline{H}\, dA + \rho \int_A \underline{H}^T \underline{H} \hat{\underline{v}}_3 \underline{H}_{,x_3}\, dA$$

$$+ \mu \int_A \underline{H}_{,x_2}^T \underline{H}_{,x_2}\, dA + 2\mu \int_A \underline{H}_{,x_3}^T \underline{H}_{,x_3}\, dA$$

$$\underline{K}_{23} = \frac{\partial \underline{F}_2}{\partial \hat{\underline{p}}} = -\int_A \bar{v}_{3,3} \tilde{\underline{H}}\, dA = -\int_A \underline{H}_{,x_3}^T \tilde{\underline{H}}\, dA$$

$$\underline{K}_{24} = \frac{\partial \underline{F}_2}{\partial \hat{\underline{\Theta}}} = \underline{0}$$

$$\underline{K}_{31} = \frac{\partial \underline{F}_3}{\partial \hat{\underline{v}}_2} = -\int_A \bar{p} \frac{\partial \underline{H}}{\partial x_2}\, dA = -\int_A \tilde{\underline{H}}^T \underline{H}_{,x_2}\, dA$$

$$\underline{K}_{32} = \frac{\partial \underline{F}_3}{\partial \hat{\underline{v}}_3} = -\int_A \bar{p} \frac{\partial \underline{H}}{\partial x_3}\, dA = -\int_A \tilde{\underline{H}}^T \underline{H}_{,x_3}\, dA$$

$$\underline{K}_{33} = \frac{\partial \underline{F}_3}{\partial \hat{\underline{p}}} = 0 \quad , \quad \underline{K}_{34} = \frac{\partial \underline{F}_3}{\partial \hat{\underline{\Theta}}} = \underline{0}$$

$$\underline{K}_{41} = \frac{\partial \underline{F}_4}{\partial \hat{\underline{v}}_2} = \int_A \bar{\Theta} \rho C_p \Theta_{,2} \underline{H}\, dA = \rho C_p \int_A \underline{H}^T \underline{H}_{,x_2} \hat{\underline{\Theta}} \underline{H}\, dA$$

$$\underline{K}_{42} = \frac{\partial \underline{F}_4}{\partial \hat{\underline{v}}_3} = \int_A \bar{\Theta} \rho C_p \Theta_{,3} \underline{H}\, dA = \rho C_p \int_A \underline{H}^T \underline{H}_{,x_3} \hat{\underline{\Theta}} \underline{H}\, dA$$

7.17

$$\underline{K}_{43} = \frac{\partial \underline{F}_4}{\partial \hat{\underline{P}}} = \underline{0}$$

$$\underline{K}_{44} = \frac{\partial \underline{F}_4}{\partial \hat{\underline{\theta}}} = \int_A \bar{\theta} \rho C_p \left(v_2 \frac{\partial \underline{H}}{\partial x_2} + v_3 \frac{\partial \underline{H}}{\partial x_3} \right) dA + \int_A \bar{\theta}_{,2} k \frac{\partial \underline{H}}{\partial x_2} dA + \int_A \bar{\theta}_{,3} k \frac{\partial \underline{H}}{\partial x_3} dA$$

$$= \rho C_p \int_A \underline{H}^T \underline{H} \hat{\underline{v}}_2 \underline{H}_{,x_2} dA + \rho C_p \int_A \underline{H}^T \underline{H} \hat{\underline{v}}_3 \underline{H}_{,x_3} dA + k \int_A \underline{H}_{,x_2}^T \underline{H}_{,x_2} dA + k \int_A \underline{H}_{,x_3}^T \underline{H}_{,x_3} dA$$

7.18 From (7.104), $$f_{i+\frac{1}{2}} = v\left[\theta_i + \frac{\theta_i - \theta_{i+1}}{\exp(P_e^e) - 1}\right]$$

Similarly $$f_{i-\frac{1}{2}} = v\left[\theta_{i-1} + \frac{\theta_{i-1} - \theta_i}{\exp(P_e^e) - 1}\right]$$

Substituting these into (7.103),

$$v\left[\theta_i + \frac{\theta_i - \theta_{i+1}}{\exp(P_e^e) - 1}\right] - v\left[\theta_{i-1} + \frac{\theta_{i-1} - \theta_i}{\exp(P_e^e) - 1}\right] = 0$$

Let $c \equiv \exp(P_e^e) - 1$, then we obtain $(c \neq 0)$

$$(-1-c)\,\theta_{i-1} + (2+c)\,\theta_i - \theta_{i+1} = 0 \qquad \Leftarrow (7.105)$$

$$\text{where} \quad C = \exp(P_e^e) - 1 \qquad \Leftarrow (7.106)$$

7.19 $\displaystyle \int_{-h}^{h} \tilde{h}_i\, v \frac{dh_j}{dx}\theta_j\, dx + \int_{-h}^{h} \frac{d\tilde{h}_i}{dx}\,\alpha\, \frac{dh_j}{dx}\theta_j\, dx = 0 \qquad j = i-1,\ i,\ i+1$

Using the definition of h_i in Fig. 7.6 and $\tilde{h}_i$ in (7.108),

$$\Big\{ \int_{-h}^{0}\left(1+\frac{x}{h}+\frac{\gamma}{2}\right) v \left[-\frac{1}{h} \quad \frac{1}{h} \quad 0\right] dx + \int_{0}^{h}\left(1-\frac{x}{h}-\frac{\gamma}{2}\right) v \left[0 \quad -\frac{1}{h} \quad \frac{1}{h}\right] dx$$

$$+ \int_{-h}^{0}\left(\frac{1}{h}\right)\alpha\left[-\frac{1}{h} \quad \frac{1}{h} \quad 0\right]dx + \int_{0}^{h}\left(-\frac{1}{h}\right)\alpha\left[0 \quad -\frac{1}{h} \quad \frac{1}{h}\right]dx \Big\} \begin{bmatrix}\theta_{i-1}\\ \theta_i \\ \theta_{i+1}\end{bmatrix} = 0$$

$$\Big\{ v\left[-\frac{1}{2}-\frac{\gamma}{2} \quad \frac{1}{2}+\frac{\gamma}{2} \quad 0\right] + v\left[0 \quad -\frac{1}{2}+\frac{\gamma}{2} \quad \frac{1}{2}-\frac{\gamma}{2}\right]$$

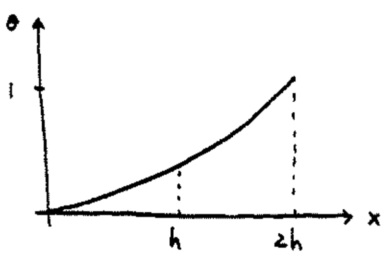
$$+ \alpha\left[-\frac{1}{h} \quad \frac{1}{h} \quad 0\right] + \alpha\left[0 \quad \frac{1}{h} \quad -\frac{1}{h}\right] \Big\} \begin{bmatrix}\theta_{i-1}\\ \theta_i \\ \theta_{i+1}\end{bmatrix} = 0$$

Multiplying both sides of the eqn above by h/α, we obtain

$$\left[-1-\frac{Pe^e}{2}(\gamma+1)\right]\theta_{i-1} + \left[2+\gamma Pe^e\right]\theta_i + \left[-1-\frac{Pe^e}{2}(\gamma-1)\right]\theta_{i+1} = 0 \quad \Leftarrow (7.109)$$

where $Pe^e = vh/\alpha$

Now consider the following case:



$$\frac{\theta - 0}{1-0} = \frac{\exp\left(\frac{Pe}{2h}h\right)-1}{\exp(Pe)-1} \qquad \text{where } Pe = \frac{v\,2h}{\alpha}$$

$$\theta = \frac{\exp\left(\frac{Pe}{2}\right)-1}{\exp(Pe)-1}$$

Taking $\theta_{i-1} = 0$, $\theta_i = \theta$, $\theta_{i+1} = 1$

$$\theta(2+\gamma Pe^e) + \left(-1-\frac{Pe^e}{2}(\gamma-1)\right) = 0 \quad , \quad Pe^e = \frac{vh}{\alpha} = \frac{Pe}{2}$$

$$\frac{\exp\left(\frac{Pe}{2}\right)-1}{\exp(Pe)-1}\left(2+\gamma Pe^e\right) + \left(-1-\frac{Pe^e}{2}(\gamma-1)\right) = 0$$

7.19

When $\exp(P_e) - 1 \neq 0$, $P_e = 2P_e^e$

$$\gamma \left[P_e^e \{\exp(P_e^e) - 1\} - \frac{P_e^e}{2}\{\exp(2P_e^e) - 1\} \right]$$
$$= -2\{\exp(P_e^e) - 1\} + \left(1 - \frac{P_e^e}{2}\right)\{\exp(2P_e^e) - 1\}$$

$$\therefore \gamma = -\frac{2}{P_e^e} + \frac{\exp(P_e^e/2) + \exp(-P_e^e/2)}{\exp(P_e^e/2) - \exp(-P_e^e/2)} = -\frac{2}{P_e^e} + \coth\left(\frac{P_e^e}{2}\right) \Leftarrow (7.110)$$

7.20

The governing ith equation is

$$\underbrace{\int_{-h}^{0}\left(h_i v \frac{d\hat{h}_{i-1}}{dx} + \frac{dh_i}{dx}\alpha\frac{dh_{i-1}}{dx}\right) dx}_{①}\ \theta_{i-1}$$

$$+\underbrace{\int_{-h}^{h}\left(h_i v \frac{d\tilde{h}_i}{dx} + \frac{dh_i}{dx}\alpha\frac{dh_i}{dx}\right) dx}_{②}\ \theta_i$$

$$+\underbrace{\int_{0}^{h}\left(h_i v \frac{d\hat{h}_{i+1}}{dx} + \frac{dh_i}{dx}\alpha\frac{dh_{i+1}}{dx}\right) dx}_{③}\ \theta_{i+1} = 0.$$

$$① = \int_{-h}^{0}\left\{ \left(1+\frac{x}{h}\right) v \left(-\frac{\exp(q(x+h))}{\exp(Pe^e)-1} q\right) - \frac{\alpha}{h^2}\right\} dx$$

$$= v\left(\frac{1}{Pe^e} - \frac{\exp(Pe^e)}{\exp(Pe^e)-1}\right) - \frac{\alpha}{h}.$$

$$② = \int_{-h}^{0}\left\{ \left(1+\frac{x}{h}\right) v \left(\frac{\exp(q(x+h))}{\exp(Pe^e)-1} q\right)\right\} dx + \int_{0}^{h}\left\{ \left(1-\frac{x}{h}\right) v \left(-\frac{\exp(q\cdot x)}{\exp(Pe^e)-1} q\right) + 2\frac{\alpha}{h^2}\right\} dx$$

$$= v\left(1 + \frac{2}{\exp(Pe^e)-1} - \frac{2}{Pe^e}\right) + 2\frac{\alpha}{h}.$$

$$③ = \int_{0}^{h}\left\{ \left(1-\frac{x}{h}\right) v \left(\frac{\exp(q\cdot x)}{\exp(Pe^e)-1} q\right) - \frac{\alpha}{h^2}\right\} dx$$

$$= v\left(\frac{1}{Pe^e} - \frac{1}{\exp(Pe^e)-1}\right) - \frac{\alpha}{h}$$

$$\therefore\ -\exp(Pe^e)\cdot\theta_{i-1} + \left(\exp(Pe^e)+1\right)\theta_i - \theta_{i+1} = 0.$$

7.21 (7.109) states $\left[-1-\frac{Pe^e}{2}(\gamma+1)\right]\theta_{i-1}+(2+\gamma Pe^e)\theta_i+\left[-1-\frac{Pe^e}{2}(\gamma-1)\right]\theta_{i+1}=0$ — ①

where $\gamma = \coth\left(\frac{Pe^e}{2}\right) - \frac{2}{Pe^e} = \frac{\exp(Pe^e)+1}{\exp(Pe^e)-1} - \frac{2}{Pe^e}$ — ②

From ① and ②, and by multiplying both sides by $\{\exp(Pe^e)-1\}/Pe^e$,

$$\left[-1-\{\exp(Pe^e)-1\}\right]\theta_{i-1}+\left[2+\{\exp(Pe^e)-1\}\right]\theta_i-\theta_{i+1}=0$$

$$\therefore [-1-C]\,\theta_{i-1}+[2+C]\,\theta_i-\theta_{i+1}=0 \quad \Longleftarrow \quad (7.105)$$

where $C = \exp(Pe^e)-1$

So we see that (7.109) is identical to (7.105). Also, (7.109) gives the solution of (7.113) because γ in (7.109) is compared to the result of (7.113) to give the exact solution (see exercise 7.20).

7.22

The equation (7.114) is

$$\underbrace{\int_{-h}^{0}\left\{h_i v \frac{dh_{i-1}}{dx} + \frac{dh_i}{dx}(1+\beta)\alpha\frac{dh_{i-1}}{dx}\right\}dx}_{①}\cdot\theta_{i-1}$$

$$+\underbrace{\int_{-h}^{h}\left\{h_i v \frac{dh_i}{dx} + \frac{dh_i}{dx}(1+\beta)\alpha\frac{dh_i}{dx}\right\}dx}_{②}\cdot\theta_i$$

$$+\underbrace{\int_{0}^{h}\left\{h_i v \frac{dh_{i+1}}{dx} + \frac{dh_i}{dx}(1+\beta)\alpha\frac{dh_{i+1}}{dx}\right\}dx}_{③}\cdot\theta_{i+1}.$$

$$① = \int_{-h}^{0}\left\{\left(1+\frac{x}{h}\right)\cdot v\cdot\left(-\frac{1}{h}\right) + \frac{1}{h}(1+\beta)\cdot\alpha\left(-\frac{1}{h}\right)\right\}dx$$

$$= -\left(\frac{v}{2} + (1+\beta)\cdot\alpha\frac{1}{h}\right).$$

$$② = 2\cdot\int_{0}^{h}\left\{\frac{1}{h}(1+\beta)\cdot\alpha\cdot\frac{1}{h}\right\}\cdot dx = 2\cdot(1+\beta)\cdot\alpha\frac{1}{h}.$$

$$③ = \int_{0}^{h}\left\{\left(1-\frac{x}{h}\right)v\frac{1}{h} - \frac{1}{h}(1+\beta)\alpha\frac{1}{h}\right\}dx$$

$$= \frac{v}{2} - (1+\beta)\cdot\alpha\frac{1}{h}.$$

$$\therefore -(1+q)\theta_{i-1} + 2\theta_i - (1-q)\theta_{i+1} = 0$$

where $q = \frac{Pe^e}{2}\frac{1}{1+\beta}$.

If we devide the equation (7.109) by $1+\frac{\gamma Pe^e}{2}$

$$-\left(1+\frac{Pe^e}{2+\gamma Pe^e}\right)\theta_{i-1}+2\theta_i-\left(1-\frac{Pe^e}{2+\gamma Pe^e}\right)\theta_{i+1}=0.$$

Hence by comparing to the equation (7.114), we have

$$\frac{Pe^e}{2+\gamma Pe^e}=\frac{Pe^e}{2(1+\beta)},$$

$$\therefore\ \beta=\frac{\gamma Pe^e}{2}.$$

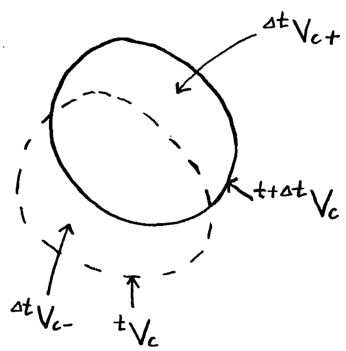
7.23

For a function f (tensor) which is some quantity per unit volume and a control volume V_c,

$$\frac{d}{dt}\int_{{}^tV_c} f\,dV = \int_{{}^tV_c} \frac{\partial f}{\partial t}\,dV + \int_{{}^tS_c} f\,\underline{v}_c\cdot\underline{n}\,dS \quad \cdots (1)$$

where S_c is the closed boundary surface of V_c, $\underline{n}$ is the unit normal vector outward from V_c and $\underline{v}_c$ is the velocity of S_c.

* Proof (1)



$$\frac{d}{dt}\int_{{}^tV_c} f\,dV$$

$$= \lim_{\Delta t\to 0}\frac{\int_{{}^{t+\Delta t}V_c} f(t+\Delta t,\underline{x})\,dV - \int_{{}^tV_c} f(t,\underline{x})\,dV}{\Delta t}$$

Since ${}^{t+\Delta t}V_c = {}^tV_c + {}^{\Delta t}V_{c+} - {}^{\Delta t}V_{c-}$,

$$\frac{d}{dt}\int_{{}^tV_c} f\,dV$$

$$= \int_{{}^tV_c}\frac{\partial f}{\partial t}\,dV + \lim_{\Delta t\to 0}\frac{\int_{{}^{\Delta t}V_{c+}} f(t+\Delta t,\underline{x})\,dV - \int_{{}^{\Delta t}V_{c-}} f(t+\Delta t,\underline{x})\,dV}{\Delta t}$$

$$= \int_{{}^tV_c}\frac{\partial f}{\partial t}\,dV + \int_{{}^tS_c} f\,\underline{v}_c\cdot\underline{n}\,dS.$$

For a material volume V_m of which the closed boundary surface S_m moves with a velocity $\underline{v}_m$, we have from (1):

$$\frac{d}{dt}\int_{{}^tV_m} f\,dV = \int_{{}^tV_m}\frac{\partial f}{\partial t}\,dV + \int_{{}^tS_m} f\,\underline{v}_m\cdot\underline{n}\,dS \quad \cdots (2)$$

For the case when ${}^{t}V_m = {}^{t}V_c$, if we subtract (1) from (2) then we have

$$\frac{d}{dt}\int_{{}^{t}V_m} f\,dV = \frac{d}{dt}\int_{{}^{t}V_c} f\,dV + \int_{{}^{t}S_c} f(\underline{v}_m - \underline{v}_c)\cdot\underline{n}\,dS .$$

i) mass, $f = \rho$

Since $\frac{d}{dt}\int_{{}^{t}V_m} \rho\,dV = 0$,

$$\frac{d}{dt}\int_{{}^{t}V_c} \rho\,dV + \int_{{}^{t}S_c} \rho(\underline{v}_m - \underline{v}_c)\cdot\underline{n}\,dS = 0.$$

ii) momentum, $f = \rho\underline{v}_m$

Since $\frac{d}{dt}\int_{{}^{t}V_m} \rho\underline{v}_m\,dV = \int_{{}^{t}V_m} (\nabla\cdot\underline{\underline{\tau}} + \underline{f}^B)\,dV$,

$$\frac{d}{dt}\int_{{}^{t}V_c} \rho\underline{v}_m\,dV + \int_{{}^{t}S_c} \rho\underline{v}_m(\underline{v}_m - \underline{v}_c)\cdot\underline{n}\,dS = \int_{{}^{t}V_m} (\nabla\cdot\underline{\underline{\tau}} + \underline{f}^B)\,dV.$$

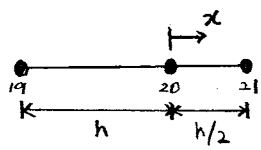
iii) energy, $f = c_p\rho\theta$

Since $\frac{d}{dt}\int_{{}^{t}V_m} c_p\rho\theta\,dV = \int_{{}^{t}V_m} \{\nabla\cdot\nabla(k\theta) + q^B\}\,dV$,

$$\frac{d}{dt}\int_{{}^{t}V_c} c_p\rho\theta\,dV + \int_{{}^{t}S_c} c_p\rho\theta(\underline{v}_m - \underline{v}_c)\cdot\underline{n}\,dS = \int_{{}^{t}V_m} \{\nabla\cdot\nabla(k\theta) + q^B\}\,dV.$$

7.24

(a)



$$h_{19} = -\frac{x}{h}$$

$$h_{20,left} = 1 + \frac{x}{h}$$

$$h_{20,right} = 1 - \frac{2x}{h}$$

$$h_{21} = \frac{2x}{h} \quad \text{and}$$

$$\tilde{h}_{19} = 1 - \frac{\exp(Pe^{e_1}(1+\frac{x}{h})) - 1}{\exp(Pe^{e_1}) - 1}$$

$$\tilde{h}_{20,left} = \frac{\exp(Pe^{e_1}(1+\frac{x}{h})) - 1}{\exp(Pe^{e_1}) - 1}$$

$$\tilde{h}_{20,right} = 1 - \frac{\exp(Pe^{e_2}\frac{2x}{h}) - 1}{\exp(Pe^{e_2}) - 1}$$

$$\hat{h}_{21} = \frac{\exp(Pe^{e_2}\frac{2x}{h}) - 1}{\exp(Pe^{e_2}) - 1}$$

where $Pe^{e_1} = \frac{vh}{\alpha}$ and $Pe^{e_2} = \frac{vh}{2\alpha}$, therefore $Pe^{e_2} = \frac{1}{2}Pe^{e_1}$.

$$\Rightarrow \Big\{ \int_{-h}^{0} (1+\frac{x}{h}) v \Big[-\frac{Pe^{e_1}}{h}\frac{\exp(Pe^{e_1}(1+\frac{x}{h}))}{\exp(Pe^{e_1}) - 1} , \frac{Pe^{e_1}}{h}\frac{\exp(Pe^{e_1}(1+\frac{x}{h}))}{\exp(Pe^{e_1}) - 1} , 0 \Big] dx$$

$$+ \int_{0}^{\frac{h}{2}} (1-\frac{2x}{h}) v \Big[0 , -\frac{2Pe^{e_2}}{h}\frac{\exp(Pe^{e_2}\frac{2x}{h})}{\exp(Pe^{e_2}) - 1} , \frac{2Pe^{e_2}}{h}\frac{\exp(Pe^{e_2}\frac{2x}{h})}{\exp(Pe^{e_2}) - 1} \Big] dx$$

$$+ \int_{-h}^{0} \frac{1}{h}\alpha \Big[-\frac{1}{h} , \frac{1}{h} , 0 \Big] dx$$

$$+ \int_{0}^{\frac{h}{2}} (-\frac{2}{h})\alpha \Big[0 , -\frac{2}{h} , \frac{2}{h} \Big] dx \Big\} \begin{bmatrix} \theta_{19} \\ \theta_{20} \\ \theta_{21} \end{bmatrix} .$$

$$\therefore -\frac{\exp(Pe^{e_1})}{\exp(Pe^{e_1}) - 1}\theta_{19} + \left(\frac{\exp(Pe^{e_1})}{\exp(Pe^{e_1}) - 1} + \frac{1}{\exp(\frac{1}{2}Pe^{e_1}) - 1} \right)\theta_{20}$$

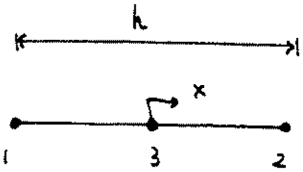
$$- \frac{1}{\exp(\frac{1}{2}Pe^{e_1}) - 1}\theta_{21} = 0.$$

(b) $$\left\{ 2\int_{-h}^{0} (1+x) v \left[-\frac{1}{h},\ \frac{1}{h},\ 0\right] dx + \int_{-h}^{0} \frac{1}{h} \alpha \left[-\frac{1}{h},\ \frac{1}{h},\ 0\right] dx + \int_{0}^{\frac{h}{2}} \left(-\frac{2}{h}\right) \alpha \left[0,\ -\frac{2}{h},\ \frac{2}{h}\right] dx \right\} \begin{bmatrix} \theta_{19} \\ \theta_{20} \\ \theta_{21} \end{bmatrix} = 0$$

$$\therefore\ (-Pe^e - 1)\,\theta_{19} + (Pe^e + 3)\,\theta_{20} - 2\theta_{21} = 0$$

where $Pe^e = \dfrac{vh}{\alpha}$.

7.25



$$h_1 = \frac{1}{2}\left(1 - \frac{2x}{h}\right), \quad h_2 = \frac{1}{2}\left(1 + \frac{2x}{h}\right)$$

$$h_3 = 1 - \left(\frac{2x}{h}\right)^2$$

$$\underset{\sim}{H} = [\, h_1 \quad h_2 \quad h_3 \,], \quad \underset{\sim}{B} = \frac{d\underset{\sim}{H}}{dx} = \left[-\frac{1}{h} \quad \frac{1}{h} \quad -\frac{8x}{h^2} \right]$$

$$\int_{-h/2}^{h/2} h_i\, v \frac{dh_j}{dx}\theta_j \, dx + \int_{-h/2}^{h/2} \frac{dh_i}{dx}\alpha \frac{dh_j}{dx}\theta_j \, dx = 0$$

Simplifying this equation we have

$$\begin{bmatrix} -\frac{v}{2} + \frac{\alpha}{h} & \frac{v}{2} - \frac{\alpha}{h} & \frac{2}{3}v \\ -\frac{v}{2} - \frac{\alpha}{h} & \frac{v}{2} + \frac{\alpha}{h} & -\frac{2}{3}v \\ -\frac{2}{3}v & \frac{2}{3}v & \frac{16}{3}\frac{\alpha}{h} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \underset{\sim}{0}$$

Condensing out θ_3

$$\begin{bmatrix} -\frac{v}{2} + \frac{\alpha}{h} + \frac{1}{12}\frac{v^2 h}{\alpha} & \frac{v}{2} - \frac{\alpha}{h} - \frac{1}{12}\frac{v^2 h}{\alpha} \\ -\frac{v}{2} - \frac{\alpha}{h} - \frac{1}{12}\frac{v^2 h}{\alpha} & \frac{v}{2} + \frac{\alpha}{h} + \frac{1}{12}\frac{v^2 h}{\alpha} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \underset{\sim}{0}$$

Writing the equation for node i

$$\left(-\frac{v}{2} - \frac{\alpha}{h} - \frac{1}{12}\frac{v^2 h}{\alpha}\right)\theta_{i-1} + \left(\frac{2\alpha}{h} + \frac{1}{6}\frac{v^2 h}{\alpha}\right)\theta_i$$

$$+ \left(\frac{v}{2} - \frac{\alpha}{h} - \frac{1}{12}\frac{v^2 h}{\alpha}\right)\theta_{i+1} = 0$$

7.25

Multiplying the equation with $1 \Big/ \left(\frac{\alpha}{h} + \frac{1}{12} \frac{v^2 h}{\alpha} \right)$

$$\left(-1 - \frac{P_e^e}{2 + \frac{1}{6} P_e^{e\,2}} \right) \theta_{i-1} + 2\theta_i + \left(-1 + \frac{P_e^e}{2 + \frac{1}{6} P_e^{e\,2}} \right) \theta_{i+1} = 0$$

Comparing this to eq. (7.116),

$$\gamma = \frac{P_e^e}{2 + \frac{1}{6} P_e^{e\,2}} \quad \text{and} \quad \beta = \frac{P_e^{e\,2}}{12} \quad \text{since} \quad \gamma = \frac{P_e^e}{2} \frac{1}{1+\beta}$$

So this method is similar to an upwinding scheme with the artificial diffusivity $\beta = \frac{1}{12} P_e^{e\,2}$.

7.26

(7.116) becomes

$$\int_S w\left(\underline{v}\phi - \frac{1}{Pe}\nabla\theta\right)\cdot\underline{n}\,dS$$

where $\underline{n}$ is unit normal vector outward the domain.

In the element m, we use

$$w = \begin{cases} 1 & (r,s) \in [0, \frac{h}{2}] \times [0, \frac{h}{2}] \\ 0 & \text{else}, \end{cases}$$

and θ and $\underline{v}$ are interpolated as

$$\theta(r,s) = \frac{r}{h}\frac{s}{h}\theta_1 + (1-\frac{r}{h})\frac{s}{h}\theta_2 + (1-\frac{r}{h})(1-\frac{s}{h})\theta_3 + \frac{r}{h}(1-\frac{s}{h})\theta_4,$$

$$\underline{v}(r,s) = \frac{r}{h}\frac{s}{h}\underline{v}_1 + (1-\frac{r}{h})\frac{s}{h}\underline{v}_2 + (1-\frac{r}{h})(1-\frac{s}{h})\underline{v}_3 + \frac{r}{h}(1-\frac{s}{h})\underline{v}_4.$$

ϕ is interpolated for the flux through ab as

$$\begin{aligned}\phi_{ab}(r,s) = &\left\{ r_1(1-\frac{s}{h}) + r_2\frac{s}{h}\right\}\frac{s}{h}\phi_1 \\ &+\left\{(1-r_1)(1-\frac{s}{h}) + (1-r_2)\frac{s}{h}\right\}\frac{s}{h}\phi_2 \\ &+\left\{(1-r_1)(1-\frac{s}{h}) + (1-r_2)\frac{s}{h}\right\}(1-\frac{s}{h})\phi_3 \\ &+\left\{ r_1(1-\frac{s}{h}) + r_2\frac{s}{h}\right\}(1-\frac{s}{h})\phi_4\end{aligned}$$

where

$$r_1 = \frac{\exp(Pe^1\frac{r}{h})-1}{\exp(Pe^1)-1}, \quad Pe^1 = \left(\frac{\underline{v}_3+\underline{v}_4}{2}\right)\cdot h\hat{\underline{e}}_r\cdot\frac{1}{\alpha},$$

$$r_2 = \frac{\exp(Pe^2\frac{r}{h})-1}{\exp(Pe^2)-1}, \quad Pe^2 = \left(\frac{\underline{v}_1+\underline{v}_2}{2}\right)\cdot h\hat{\underline{e}}_r\cdot\frac{1}{\alpha}.$$

Similarly, ϕ is interpolated for the flux through bc as

$$\phi_{bc}(r,s) = \frac{r}{h}\left\{(1-\frac{r}{h})s_1 + \frac{r}{h}s_2\right\}\phi_1$$
$$+(1-\frac{r}{h})\left\{(1-\frac{r}{h})s_1 + \frac{r}{h}s_2\right\}\phi_2$$
$$+(1-\frac{r}{h})\left\{(1-\frac{r}{h})(1-s_1) + \frac{r}{h}(1-s_2)\right\}\phi_3$$
$$+\frac{r}{h}\left\{(1-\frac{r}{h})(1-s_1) + \frac{r}{h}(1-s_2)\right\}\phi_4$$

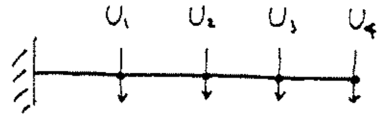
where

$$s_1 = \frac{\exp(Pe^1\frac{s}{h})-1}{\exp(Pe^1)-1}, \quad Pe^1 = \left(\frac{\underline{V}_2+\underline{V}_3}{2}\right)\cdot h\hat{\underline{e}}_s\cdot\frac{1}{\alpha},$$

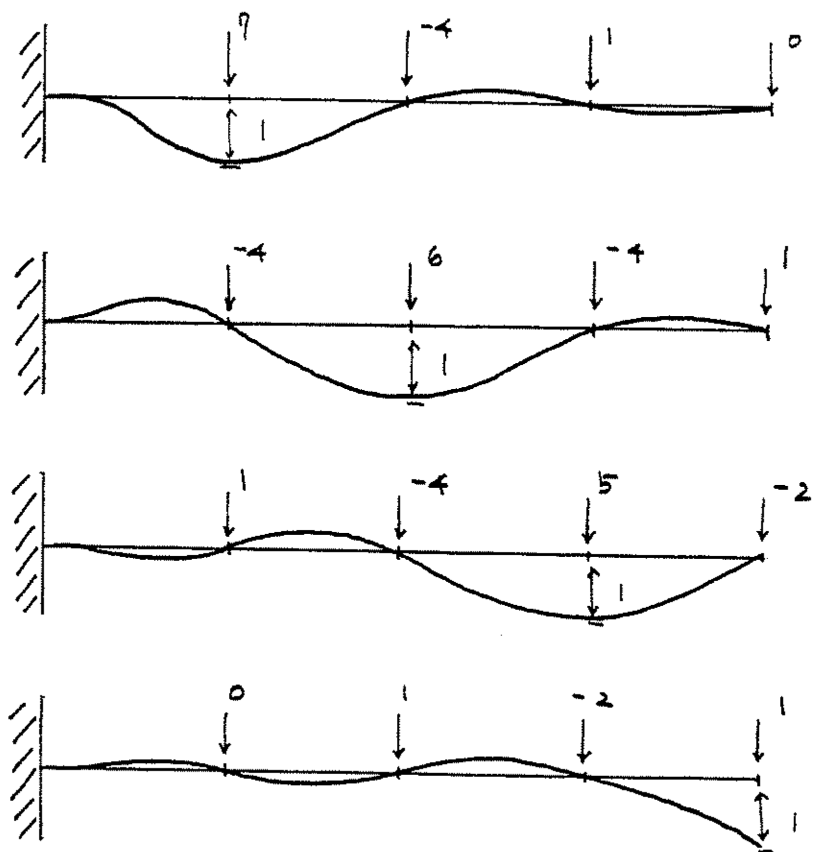
$$s_2 = \frac{\exp(Pe^2\frac{s}{h})-1}{\exp(Pe^2)-1}, \quad Pe^2 = \left(\frac{\underline{V}_1+\underline{V}_4}{2}\right)\cdot h\hat{\underline{e}}_s\cdot\frac{1}{\alpha}.$$

8.1

$$\begin{bmatrix} 7 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 5 & -2 \\ 0 & 1 & -2 & 1 \end{bmatrix} \begin{matrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{matrix}$$



all 4 clamps present

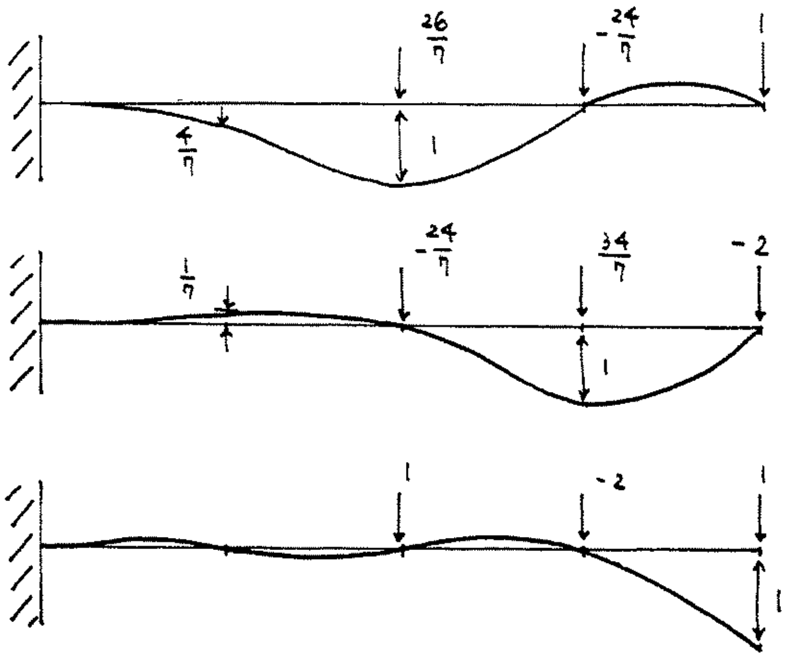


clamp 1 removed

$$\begin{bmatrix} \frac{26}{7} & -\frac{24}{7} & 1 \\ -\frac{24}{7} & \frac{34}{7} & -2 \\ 1 & -2 & 1 \end{bmatrix} \begin{matrix} U_2 \\ U_3 \\ U_4 \end{matrix}$$

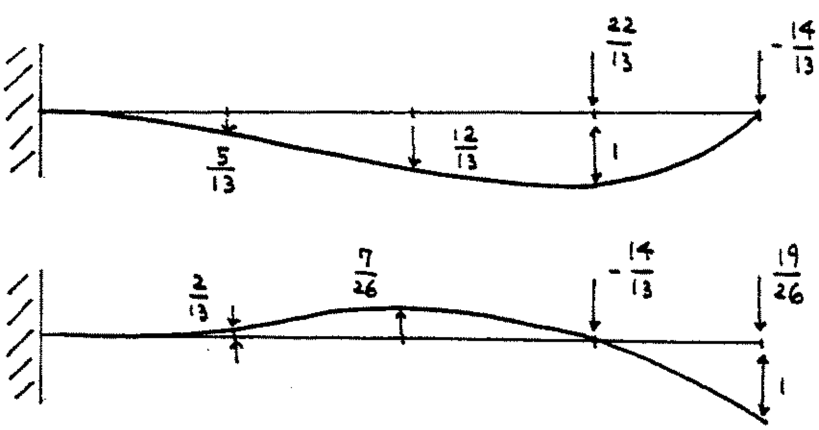
with $U_1 = (4U_2 - U_3)/7$

8.1



clamp 1 and 2 removed

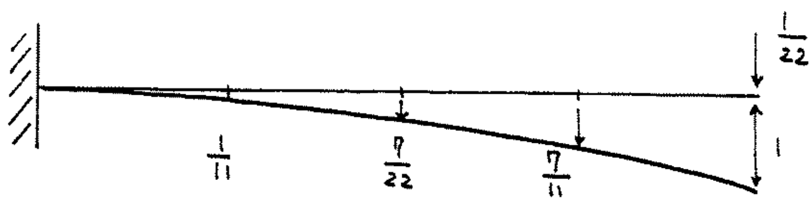
$$\begin{bmatrix} \frac{22}{13} & -\frac{14}{13} \\ -\frac{14}{13} & \frac{19}{26} \end{bmatrix} \begin{matrix} U_3 \\ U_4 \end{matrix} \qquad \text{with} \quad U_2 = \frac{12}{13}U_3 - \frac{7}{26}U_4$$



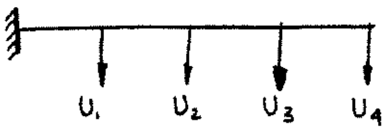
8.1

clamp 1, 2 and 3 removed

$$\left[\frac{1}{22}\right] U_4 \qquad \text{with} \qquad U_3 = \frac{7}{11} U_4$$



8.2



$$\begin{array}{cccc} U_1 & U_2 & U_3 & U_4 \end{array}$$

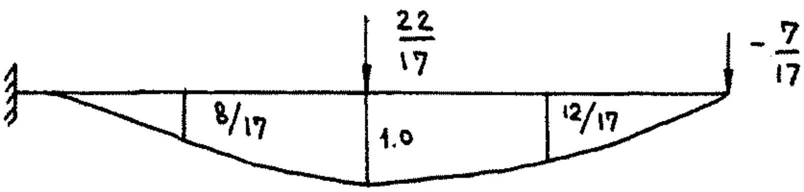
$$\begin{bmatrix} 7 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 5 & -2 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

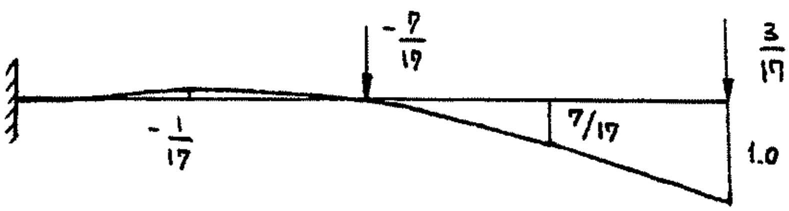
$$U_1 = \frac{1}{17}\left(8U_2 - U_4\right),$$

$$U_3 = \frac{1}{17}\left(12U_2 + 7U_4\right)$$

$$\begin{bmatrix} 7 & -4 & 1 & 0 \\ 1 & -4 & 5 & 2 \\ 0 & 0 & \frac{22}{17} & -\frac{7}{17} \\ 0 & 0 & -\frac{7}{17} & \frac{3}{17} \end{bmatrix}$$

$$\begin{array}{cccc} U_1 & U_3 & U_2 & U_4 \end{array}$$





8.3 When Clamp 2 is removed, the following $\tilde{K}$ is obtained.

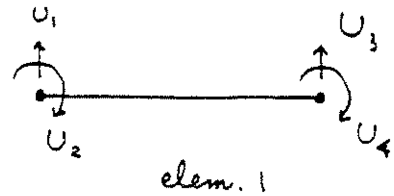
$$\tilde{K} = \begin{bmatrix} \frac{80}{7} & -2 & \frac{3}{7} \\ -2 & \frac{25}{2} & -\frac{15}{2} \\ \frac{3}{7} & -\frac{15}{2} & \frac{251}{14} \end{bmatrix}$$

($\frac{251}{14}$ circled:) different from that measured

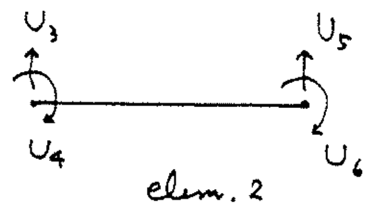
Hence, the measured $\tilde{K}$ is not correct, namely $\tilde{K}_{33}$ was wrongly measured. That means, clamp 4 might not have worked properly.

8.4

$$\underset{\sim}{K}^{(1)} = EI \begin{bmatrix} 12 & -6 & -12 & -6 \\ -6 & 4 & 6 & 2 \\ -12 & 6 & 12 & 6 \\ -6 & 2 & 6 & 4 \end{bmatrix} \quad \begin{matrix} U_1 & U_2 & U_3 & U_4 \end{matrix}$$



$$\underset{\sim}{K}^{(2)} = 2EI \begin{bmatrix} 12 & -6 & -12 & -6 \\ -6 & 4 & 6 & 2 \\ -12 & 6 & 12 & 6 \\ -6 & 2 & 6 & 4 \end{bmatrix} \quad \begin{matrix} U_3 & U_4 & U_5 & U_6 \end{matrix}$$



$$\therefore \underset{\sim}{K} = \sum_m \underset{\sim}{K}^{(m)} = EI \begin{bmatrix} 12 & -6 & -12 & -6 & 0 & 0 \\ -6 & 4 & 6 & 2 & 0 & 0 \\ -12 & 6 & 36 & -6 & -24 & -12 \\ -6 & 2 & -6 & 12 & 12 & 4 \\ 0 & 0 & -24 & 12 & 24 & 12 \\ 0 & 0 & -12 & 4 & 12 & 8 \end{bmatrix} \quad \begin{matrix} U_1 & U_2 & U_3 & U_4 & U_5 & U_6 \end{matrix}$$

From the given boundary conditions,

$$\underset{\sim}{K}' = EI \begin{bmatrix} 12 & 4 \\ 4 & 8 \end{bmatrix} \quad \begin{matrix} U_4 \\ U_6 \end{matrix}$$

Then Using the Gauss elimination to free U_6,

$$EI \begin{bmatrix} 10 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \quad \begin{matrix} U_4 = \Theta \\ U_6 \end{matrix}$$

$$\therefore K_{11} = 10\,EI$$

8.5

$$\begin{bmatrix} 7 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 5 & -2 \\ 0 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 7 & -4 & 1 & 0 \\ -4 & 5 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 6 & -2 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 6 & -2 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \; U_2 = 3, \quad U_1 = \frac{1}{3} U_2 = 1, \quad U_3 = -U_1 + 2U_2 = 5, \quad U_4 = -U_2 + 2U_3 = 7$

$\longrightarrow \quad \underset{\sim}{U}^T = [\, 1 \quad 3 \quad 5 \quad 7 \,]$

8.6

There will be a zero diagonal entry in case 1 after Gauss elimination has been performed for the 4-th degree of freedom. In cases 2 and 3, there will be no zero diagonal elements.

8.7

$$\underset{\sim}{K} = \begin{bmatrix} 7 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 5 & -2 \\ 0 & 1 & -2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 7 & -4 & 1 & 0 \\ 0 & 26/7 & -24/7 & 1 \\ 0 & -24/7 & 34/7 & -2 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 7 & -4 & 1 & 0 \\ 0 & 26/7 & -24/7 & 1 \\ 0 & 0 & 22/13 & -14/13 \\ 0 & 0 & -14/13 & 19/26 \end{bmatrix} \rightarrow \begin{bmatrix} 7 & -4 & 1 & 0 \\ 0 & 26/7 & -24/7 & 1 \\ 0 & 0 & 22/13 & -14/13 \\ 0 & 0 & 0 & 1/22 \end{bmatrix}$$

$$\therefore \underset{\sim}{K} = \underbrace{\begin{bmatrix} 1 & & & \\ -4/7 & 1 & & \\ 1/7 & -12/13 & 1 & \\ 0 & 7/26 & -7/11 & 1 \end{bmatrix}}_{\underset{\sim}{L}} \underbrace{\begin{bmatrix} 7 & & & \\ & 26/7 & & \\ & & 22/13 & \\ & & & 1/22 \end{bmatrix}}_{\underset{\sim}{D}} \underbrace{\begin{bmatrix} 1 & -4/7 & 1/7 & 0 \\ & 1 & -12/13 & 7/26 \\ & & 1 & -7/11 \\ & & & 1 \end{bmatrix}}_{\underset{\sim}{L}^T}$$

$$\det \underset{\sim}{K} = \det(\underset{\sim}{L}\,\underset{\sim}{D}\,\underset{\sim}{L}^T) = \det \underset{\sim}{L} \det \underset{\sim}{D} \det \underset{\sim}{L}^T = \det \underset{\sim}{D} = 2$$

$$\tilde{\underset{\sim}{L}} = \underset{\sim}{L}\,\underset{\sim}{D}^{1/2} = \begin{bmatrix} \sqrt{7} & 0 & 0 & 0 \\ -4/\sqrt{7} & \sqrt{26/7} & 0 & 0 \\ 1/\sqrt{7} & -12\sqrt{2}/\sqrt{91} & \sqrt{22/13} & 0 \\ 0 & \sqrt{7/26} & -7\sqrt{2}/\sqrt{143} & 1/\sqrt{22} \end{bmatrix}$$

8.8 Equation (8.10) to (8.14) give $\underline{K} = \underline{L}\,\underline{S}$

where $\underline{L} = \begin{bmatrix} 1 & & & \\ \ddots & 1 & & \\ & l_{ij} & \ddots & \\ & & & 1 \end{bmatrix}$, $\underline{S} = \begin{bmatrix} S_{11} & \cdots & S_{1i} & \cdots & S_{1n} \\ & \ddots & & & \vdots \\ & & S_{ii} & \cdots & S_{in} \\ & & & \ddots & \vdots \\ & & & & S_{nn} \end{bmatrix}$

Note that the S_{ij} elements off the diagonal are $S_{ij} = k_{ij}^{(i)}$ where the superscript (i) indicates the i-th Gauss elimination step. Then we can factor out a diagonal matrix dividing each row of $\underline{S}$ by its corresponding diagonal element to obtain $\underline{S} = \underline{D}\,\underline{\tilde{S}}$

where $\underline{D} = \begin{bmatrix} S_{11} & & & \underline{0} \\ & \ddots & & \\ & & S_{ii} & \\ \underline{0} & & & \ddots \;\; S_{nn} \end{bmatrix}$ and $\underline{\tilde{S}} = \begin{bmatrix} 1 & \cdots & \tilde{S}_{1i} & \cdots & \tilde{S}_{1n} \\ & \ddots & & & \\ & & 1 & \cdots & \tilde{S}_{in} \\ & & & \ddots & \vdots \\ & & & & 1 \end{bmatrix}$

with $\tilde{S}_{ij} = \dfrac{S_{ij}}{S_{ii}}$ in $\underline{\tilde{S}}$.

Here we see that the S_{ii} are the pivots $k_{ii}^{(i)}$ and $S_{ij} = k_{ij}^{(i)}$. Hence, the $\tilde{S}_{ij} = \dfrac{k_{ij}^{(i)}}{k_{ii}^{(i)}}$ are the Gauss multiplying factors. This means $\underline{\tilde{S}} = \underline{L}^T$.

8.9

$\underline{L}\,\underline{D}\,\underline{L}^T$ - factorization:

$$\underline{L}_1^{-1} = \begin{bmatrix} 1 & & \\ \frac{1}{2} & 1 & \\ 0 & 0 & 1 \end{bmatrix} ; \quad \underline{L}_2^{-1} = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & \frac{2}{3} & 1 \end{bmatrix} ; \quad \underline{L}_3^{-1} = \underline{I}_3 ;$$

$$\underline{L} = \begin{bmatrix} 1 & & \\ -\frac{1}{2} & 1 & \\ 0 & -\frac{2}{3} & 1 \end{bmatrix} ; \qquad \underline{D} = \begin{bmatrix} 2 & & \\ & \frac{3}{2} & \\ & & \frac{4}{3} \end{bmatrix} ;$$

$$\underline{V} = \underline{L}_2^{-1} \underline{L}_1^{-1} \underline{R} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{3} \end{bmatrix}$$

From (8.20) we have $\underline{L}^T \underline{U} = \underline{D}^{-1} \underline{V} \Rightarrow \underline{U} = \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \\ \frac{1}{4} \end{bmatrix}$

Cholesky factor:

$$\tilde{\underline{L}} = \begin{bmatrix} \sqrt{2} & & \\ -\frac{1}{\sqrt{2}} & \frac{\sqrt{6}}{2} & \\ 0 & -\frac{\sqrt{6}}{3} & \frac{2}{\sqrt{3}} \end{bmatrix}$$

8.10

Consider the $\underline{L}\,\underline{D}\,\underline{L}^T$ - factorization of the $\underline{k}$ matrix:

$$\underline{k} = \underline{L}\,\underline{D}\,\underline{L}^T = \begin{bmatrix} 1 & & \\ -\frac{1}{2} & 1 & \\ 0 & -\frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & & \\ & \frac{3}{2} & \\ & & \frac{4}{3}+k \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ & 1 & -\frac{2}{3} \\ & & 1 \end{bmatrix}$$

(a) $\frac{4}{3} + k < 0 \Rightarrow k < -\frac{4}{3}$ means that $\underline{k}$ is an indefinite matrix.

(b) $k = -\frac{4}{3}$ means that $\underline{k}$ is singular, therefore, $\underline{k}^{-1}$ does not exist, and the equations cannot be solved.

8.11

$$\begin{bmatrix} 3 & -1 & 0 \\ -2 & 4 & -1 \\ 0 & -2 & 3 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & -1 & 0 \\ 0 & 10/3 & -1 \\ 0 & -2 & 3 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{—}(*)$$

$$\longrightarrow \begin{bmatrix} 3 & -1 & 0 \\ 0 & 10/3 & -1 \\ 0 & 0 & 12/5 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3/5 \end{bmatrix} \quad \text{—}(**)$$

$$\therefore\ U_3 = \frac{5}{12}\cdot\frac{3}{5} = \frac{1}{4}, \quad U_2 = \frac{3}{10}(U_3+1) = \frac{3}{8}, \quad U_1 = \frac{1}{3}U_2 = \frac{1}{8} \quad \text{—}(***)$$

Here we have

$$\underline{L} = \begin{bmatrix} 1 & & \\ -2/3 & 1 & \\ 0 & -3/5 & 1 \end{bmatrix}, \quad \underline{D} = \begin{bmatrix} 3 & & \\ & 10/3 & \\ & & 12/5 \end{bmatrix} \quad \therefore\ \underline{L}_u = \begin{bmatrix} 1 & -1/3 & 0 \\ & 1 & -3/10 \\ & & 1 \end{bmatrix}$$

We calculate the $\underline{L}$ matrix as in the case of symmetric $\underline{k}$ matrix, which corresponds to eq. (8.10) to (8.15). Now, as $\underline{k}$ is nonsymmetric, $\underline{S} = \underline{D}\,\underline{L}_u$, and $\underline{k} = \underline{L}\,\underline{D}\,\underline{L}_u$, where $\underline{D}\,\underline{L}_u$ is given in (**).

8.12

$$\begin{bmatrix} 4 & -1 & 0 \\ 2 & 6 & -2 \\ 0 & -1 & 4 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 4 & -1 & 0 \\ 0 & 13/2 & -2 \\ 0 & -1 & 4 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1/2 \\ 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 4 & -1 & 0 \\ 0 & 13/2 & -2 \\ 0 & 0 & 48/13 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1/2 \\ -1/13 \end{bmatrix}$$

$$\therefore\ U_3 = \frac{13}{48}\left(-\frac{1}{13}\right) = -\frac{1}{48}\ ,\ U_2 = \frac{2}{13}\left(2U_3 - \frac{1}{2}\right) = -\frac{1}{12}\ ,\ U_1 = \frac{1}{4}(1+U_2) = \frac{11}{48}$$

$$\underline{L} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & -2/13 & 1 \end{bmatrix},\ \underline{D} = \begin{bmatrix} 4 & & \\ & 13/2 & \\ & & 48/13 \end{bmatrix},\ \underline{L}_u = \begin{bmatrix} 1 & -1/4 & 0 \\ & 1 & -4/13 \\ & & 1 \end{bmatrix}$$

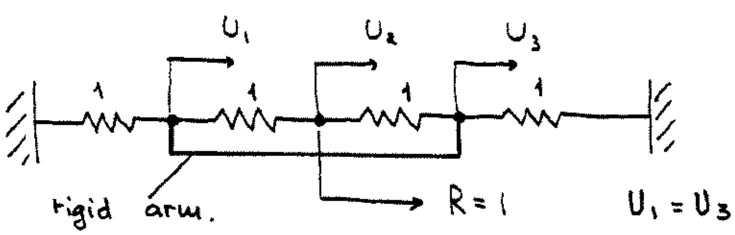
8.13

$$\begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -1 & 0 & -1 \\ 0 & 3/2 & -1 & -1/2 \\ 0 & -1 & 2 & 1 \\ 0 & -1/2 & 1 & -1/2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 2 & -1 & 0 & -1 \\ 0 & 3/2 & -1 & -1/2 \\ 0 & 0 & 4/3 & 2/3 \\ 0 & 0 & 2/3 & -2/3 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -1 & 0 & -1 \\ 0 & 3/2 & -1 & -1/2 \\ 0 & 0 & 4/3 & 2/3 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\therefore \underline{K} = \underline{L}\,\underline{D}\,\underline{L}^T \quad \text{where} \quad \underline{L} = \begin{bmatrix} 1 & & & \\ -1/2 & 1 & & \\ 0 & -2/3 & 1 & \\ -1/2 & -1/3 & 1/2 & 1 \end{bmatrix}, \quad \underline{D} = \begin{bmatrix} 2 & & & \\ & 3/2 & & \\ & & 4/3 & \\ & & & -1 \end{bmatrix}$$

Considering the following model, we see the response is governed by these equations.



8.14

$$\underbrace{\begin{bmatrix} \underline{K} & \underline{K}_\lambda^T \\ \underline{K}_\lambda & \underline{0} \end{bmatrix}}_{\parallel \; \underline{K}_{all}} \begin{bmatrix} \underline{U} \\ \underline{\lambda} \end{bmatrix} = \begin{bmatrix} \underline{R} \\ \underline{R}_\lambda \end{bmatrix} \quad \left(\underline{K} : n \times n , \quad \underline{K}_\lambda : p \times n \right) \text{ — ①}$$

If we can use the solution procedure in (8.10) to (8.20) to solve $\underline{U}$ and $\underline{\lambda}$, $\underline{K}_{all}$ has its inverse matrix $\underline{K}_{all}^{-1}$.

Let $\underline{K}_{all}^{-1} = \begin{bmatrix} \underline{A} & \underline{B} \\ \underline{C} & \underline{D} \end{bmatrix}$, then $\begin{bmatrix} \underline{K} & \underline{K}_\lambda^T \\ \underline{K}_\lambda & \underline{0} \end{bmatrix} \begin{bmatrix} \underline{A} & \underline{B} \\ \underline{C} & \underline{D} \end{bmatrix} = \begin{bmatrix} \underline{I}_n & \underline{0} \\ \underline{0} & \underline{I}_p \end{bmatrix}$

with $\underline{I}_n = n \times n$ and $\underline{I}_p = p \times p$.

That is

$$\underline{K}\,\underline{A} + \underline{K}_\lambda^T \underline{C} = \underline{I}_n \quad \text{— ②}$$

$$\underline{K}\,\underline{B} + \underline{K}_\lambda^T \underline{D} = \underline{0} \quad \text{— ③}$$

$$\underline{K}_\lambda \underline{A} = \underline{0} \quad \text{— ④}$$

$$\underline{K}_\lambda \underline{B} = \underline{I}_p \quad \text{— ⑤}$$

From ② and ③,

$$\underline{A} = \underline{K}^{-1}\left(\underline{I}_n - \underline{K}_\lambda^T \underline{C}\right) \quad \text{— ⑥}$$

$$\underline{B} = -\underline{K}^{-1} \underline{K}_\lambda^T \underline{D} \quad \text{— ⑦}$$

Using ④ to ⑦ we have

$$(\underline{K}_\lambda \underline{K}^{-1} \underline{K}_\lambda^T)\, \underline{C} = \underline{K}_\lambda \underline{K}^{-1}$$

$$(\underline{K}_\lambda \underline{K}^{-1} \underline{K}_\lambda^T)\, \underline{D} = -\underline{I}_p$$

In order to obtain $\underline{C}$ and $\underline{D}$, the matrix $(\underline{K}_\lambda \underline{K}^{-1} \underline{K}_\lambda^T)$ must be invertible. with $\underline{K}_\lambda$ $p \times n$ and $\underline{K}$ symmetric and positive definite. Hence as long as the rows of $\underline{K}_\lambda$ are

8.14

linearly independent with $p<n$, we can obtain $\underset{\sim}{U}$ and $\underset{\sim}{\lambda}$. Note that the condition of $p<n$ is also physically required with p = the number of constraint equations.

8.15

Let the original stiffness matrix be $\underset{\sim}{K}_0$. Now we add the extra stiffness to node 2, the other stiffnesses will be added in the final matrix $\underset{\sim}{K}$.

$$\underset{\sim}{K}_0 = \frac{EA_1}{6L}\begin{bmatrix} 17 & 3 & -20 \\ 3 & 25 & -28 \\ -20 & -28 & 48 \end{bmatrix} \longrightarrow \underset{\sim}{K}_1 = \frac{EA_1}{6L}\begin{bmatrix} 17 & 3 & -20 \\ 3 & 25 & -28 \\ -20 & -28 & \boxed{60} \end{bmatrix}$$

$U_1 \quad U_3 \quad U_2$

Condensing out the d.o.f. U_2,

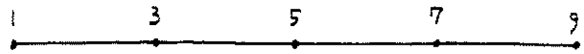
$$\underset{\sim}{K}_2 = \frac{EA_1}{6L}\begin{bmatrix} \frac{31}{3} & -\frac{19}{3} \\ -\frac{19}{3} & \frac{179}{15} \end{bmatrix} \begin{matrix} U_1 \\ U_3 \end{matrix}$$

Using $\underset{\sim}{K}_2$ as a typical stiffness matrix, we have

$$\underset{\sim}{K} = \frac{EA_1}{6L}\left\{\begin{bmatrix} 6 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{31}{3} & -\frac{19}{3} & 0 \\ -\frac{19}{3} & \frac{179}{15} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{62}{3} & -\frac{38}{3} \\ 0 & -\frac{38}{3} & \frac{358}{15} \end{bmatrix} + \begin{bmatrix} 6 & & \\ & 18 & \\ & & 30 \end{bmatrix}\right\}$$

$$= \frac{EA_1}{6L}\begin{bmatrix} \frac{67}{3} & -\frac{19}{3} & 0 \\ -\frac{19}{3} & \frac{253}{5} & -\frac{38}{3} \\ 0 & -\frac{38}{3} & \frac{808}{15} \end{bmatrix}$$

8.16

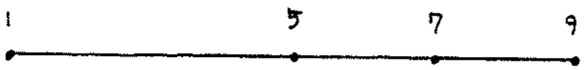


$$\frac{13}{9}\frac{EA_1}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}U_1\\ U_3\end{bmatrix} = \begin{bmatrix}R_1 + 5R_2/12\\ R_3 + 7R_2/12\end{bmatrix} \qquad \text{——— (1)}$$

$$\frac{13}{9}\frac{2EA_1}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}U_3\\ U_5\end{bmatrix} = \begin{bmatrix}R_3 + 5R_4/12\\ R_5 + 7R_4/12\end{bmatrix} \qquad \text{——— (2)}$$

$$\frac{13}{9}\frac{4EA_1}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}U_5\\ U_7\end{bmatrix} = \begin{bmatrix}R_5 + 5R_6/12\\ R_7 + 7R_6/12\end{bmatrix} \qquad \text{——— (3)}$$

$$\frac{13}{9}\frac{8EA_1}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}U_7\\ U_9\end{bmatrix} = \begin{bmatrix}R_7 + 5R_8/12\\ R_9 + 7R_8/12\end{bmatrix} \qquad \text{——— (4)}$$



$$\frac{26}{27}\frac{EA_1}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}U_1\\ U_5\end{bmatrix} = \begin{bmatrix}R_1 + 22R_2/36 + R_3/3 + 5R_4/36\\ 14R_2/36 + 2R_3/3 + 31R_4/36 + R_5\end{bmatrix} \text{ — (5)}$$

(from (1) and (2))

$$\frac{52}{63}\frac{EA_1}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}U_1\\ U_7\end{bmatrix} = \begin{bmatrix}R_1 + 2/3R_2 + 3/7R_3 + 11/42R_4 + 1/7R_5 + 5/84R_6\\ 1/3R_2 + 4/7R_3 + 31/42R_4 + 6/7R_5 + 79/84R_6 + R_7\end{bmatrix}$$

(from (3) and (5)) — (6)

Finally, from (4) and (6), we have

$$\frac{52}{63}\frac{EA_1}{L}\begin{bmatrix}1 & -1 & 0\\ -1 & 15 & -14\\ 0 & -14 & 14\end{bmatrix}\begin{bmatrix}U_1\\ U_7\\ U_9\end{bmatrix} = \begin{bmatrix}R_1 + 2/3R_2 + 3/7R_3 + 11/42R_4 + 1/7R_5 + 5/84R_6\\ 1/3R_2 + 4/7R_3 + 31/42R_4 + 6/7R_5 + 79/84R_6 + R_7\\ R_9 + 7/12R_8\end{bmatrix}$$

8.17

$$\underset{\sim}{K} = \begin{bmatrix} 7 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 5 & -2 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

$$2 - 70\lambda + 87\lambda^2 - 19\lambda^3 + \lambda^4 = 0$$

$\lambda_1 = 0.029658$, $\lambda_2 = 0.98011$,

$\lambda_3 = 5.5156$, $\lambda_4 = 12.475$

$$\underset{\sim}{K}^{(1)} = \begin{bmatrix} 7 & -4 & 1 \\ -4 & 6 & -4 \\ 1 & -4 & 5 \end{bmatrix}$$

$$44 - 74\lambda + 18\lambda^2 - \lambda^3 = 0$$

$\lambda_1^{(1)} = 0.71353$, $\lambda_2^{(1)} = 5.0322$

$\lambda_3^{(1)} = 12.254$

$$\underset{\sim}{K}^{(2)} = \begin{bmatrix} 7 & -4 \\ -4 & 6 \end{bmatrix}$$

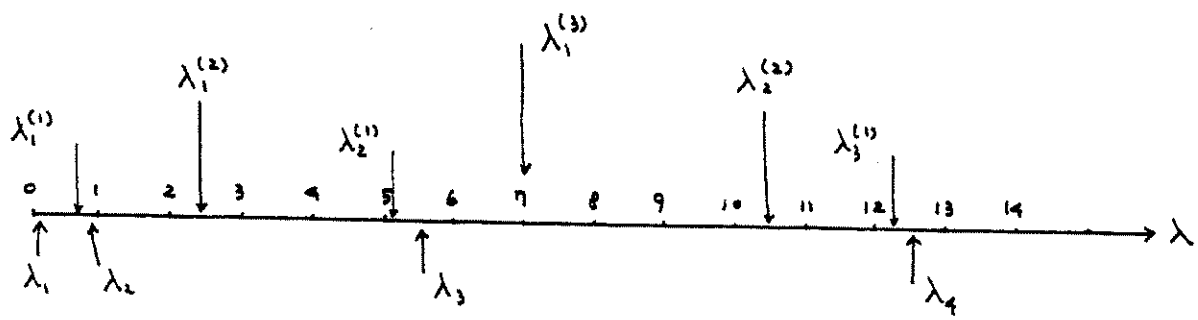
$$26 - 13\lambda + \lambda^2 = 0$$

$\lambda_1^{(2)} = 2.4689$, $\lambda_2^{(2)} = 10.531$

$$\underset{\sim}{K}^{(3)} = [\, 7 \,]$$

$$7 - \lambda = 0$$

$\lambda_1^{(3)} = 7$



$$\lambda_1 < \lambda_1^{(1)} < \lambda_2 < \lambda_2^{(1)} < \lambda_3 < \lambda_3^{(1)} < \lambda_4$$

$$\lambda_1^{(1)} < \lambda_1^{(2)} < \lambda_2^{(1)} < \lambda_2^{(2)} < \lambda_3^{(1)}$$

$$\lambda_1^{(2)} < \lambda_1^{(3)} < \lambda_2^{(2)}$$

Hence we see the Sturm sequence property holds.

8.18

$$\underline{K} = EI \begin{bmatrix} 12 & -6 & -12 & -6 \\ -6 & 4 & 6 & 2 \\ -12 & 6 & 12 & 6 \\ -6 & 2 & 6 & 4 \end{bmatrix}$$

$$60 E^2 I^2 \lambda^2 - 32 EI \lambda^3 + \lambda^4 = 0$$

$$\lambda_1 = \lambda_2 = 0 ,$$

$$\lambda_3 = 2 EI , \quad \lambda_4 = 30 EI$$

$$\underline{K}^{(1)} = EI \begin{bmatrix} 12 & -6 & -12 \\ -6 & 4 & 6 \\ -12 & 6 & 12 \end{bmatrix}$$

$$-24 E^2 I^2 \lambda + 28 EI \lambda^2 - \lambda^3 = 0$$

$$\lambda_1^{(1)} = 0 , \quad \lambda_2^{(1)} = 0.88512 \; EI$$

$$\lambda_3^{(1)} = 27.115 \; EI$$

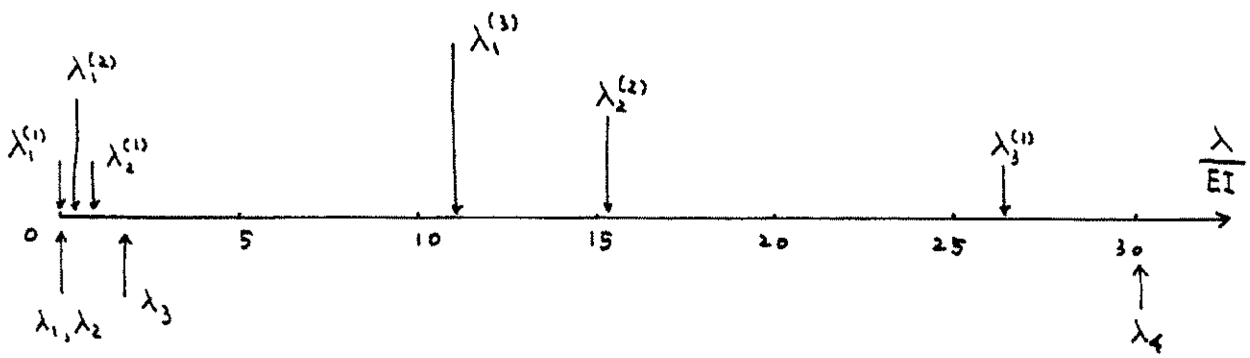
$$\underline{K}^{(2)} = EI \begin{bmatrix} 12 & -6 \\ -6 & 4 \end{bmatrix}$$

$$12 E^2 I^2 - 16 EI \lambda + \lambda^2 = 0$$

$$\lambda_1^{(2)} = 0.78890 \; EI , \quad \lambda_2^{(2)} = 15.211 \; EI$$

$$\underline{K}^{(3)} = EI \, [12]$$

$$\lambda_1^{(3)} = 12 EI$$



$$\begin{cases} \lambda_1 \le \lambda_1^{(1)} \le \lambda_2 < \lambda_2^{(1)} < \lambda_3 < \lambda_3^{(1)} < \lambda_4 \\ \lambda_1^{(1)} < \lambda_1^{(2)} < \lambda_2^{(1)} < \lambda_2^{(2)} < \lambda_3^{(1)} \\ \lambda_1^{(2)} < \lambda_1^{(3)} < \lambda_2^{(2)} \end{cases}$$

The sturm sequence property holds.

8.19 $\det(\underset{\sim}{K} - \lambda \underset{\sim}{I}) = 100k - 40(5+k)\lambda + (30+2k)\lambda^2 - \lambda^3 = 0$

As k becomes bigger,

$$\det(\underset{\sim}{K} - \lambda \underset{\sim}{I}) \sim 100k - 40k\lambda + 2k\lambda^2 - \lambda^3 = 0$$

When $k = 10^8$,

$$\lambda_1 = 2.9289, \quad \lambda_2 = 17.071, \quad \lambda_3 = 2 \times 10^8 \qquad \therefore \text{Cond}(\underset{\sim}{K}) \doteq 0.68 \times 10^8$$

When $k = 10^9$

$$\lambda_1 = 2.9289, \quad \lambda_2 = 17.071, \quad \lambda_3 = 2 \times 10^9 \qquad \therefore \text{Cond}(\underset{\sim}{K}) \doteq 0.68 \times 10^9$$

Hence if $k = \alpha \times 10^8$ with $1 \le \alpha < 10$, $\text{cond}(\underset{\sim}{K}) \sim 10^8$.

8.20 $\det(\underline{K} - \lambda \underline{I}) = 25 - 190\lambda + 131\lambda^2 - 22\lambda^3 + \lambda^4 = 0$

$$\lambda_1 = \frac{7 - 3\sqrt{5}}{2} = 0.14590 \,, \quad \lambda_2 = \frac{15 - 5\sqrt{5}}{2} = 1.9098$$

$$\lambda_3 = \frac{7 + 3\sqrt{5}}{2} = 6.8541 \,, \quad \lambda_4 = \frac{15 + 5\sqrt{5}}{2} = 13.090$$

$$\therefore \text{cond}(\underline{K}) = \frac{\lambda_4}{\lambda_1} = 89.721$$

$\lambda_n^u = \| \underline{K} \|_\infty = 15$

$\lambda_1^l = 0.1460$ (by inverse iteration)

$$\therefore \text{cond}(\underline{K}) \doteq \frac{\lambda_n^u}{\lambda_1^l} = 102.74$$

8.21 Using the Gauss-Seidel iterative method,

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}^{(s+1)} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}^{(s)} + \beta \begin{bmatrix} 1/3 & & \\ & 1/2 & \\ & & 1 \end{bmatrix}$$

$$\times \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & & \\ -1 & 0 & \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}^{(s+1)} - \begin{bmatrix} 3 & -1 & 0 \\ & 2 & -1 \\ & & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}^{(s)} \right\}$$

We iterate as follows:

When $\beta = 2.0$, the solution diverges.

ITER. NO.	U1	U2	U3
1	0.00000	1.00000	2.00000
2	0.66667	2.66667	3.33333
3	1.11111	2.77778	2.22222
4	0.74074	1.18519	0.14815
5	0.04938	0.01235	-0.12346
6	-0.04115	0.82305	1.76955
7	0.58985	2.53635	3.30316
8	1.10105	2.86786	2.43256
9	0.81085	1.37555	0.31855
10	0.10618	0.04918	-0.22019
100000	0.42217	2.20507	3.14364

The error norm is estimated using Eq. (8.68). At the 10th iteration

$$\varepsilon = \frac{\| \underline{u}^{(10)} - \underline{u}^{(9)} \|_2}{\| \underline{u}^{(10)} \|_2} = 6.3991$$

When $\beta = 1.6$, we have a convergent solution after 23 iterations.

ITER. NO.	U1	U2	U3
1	0.00000	0.80000	1.28000
2	0.42667	1.68533	1.92853
3	0.64284	1.84590	1.79632
4	0.59877	1.60854	1.49587
5	0.49862	1.43047	1.39123
6	0.46374	1.42570	1.44638
7	0.48213	1.48738	1.51199
8	0.50400	1.52036	1.52538
9	0.50846	1.51486	1.50854
10	0.50285	1.50020	1.49519
23	0.49999	1.49999	1.50000

At the 10th iteration, with Eq. (8.68) the error norm

$$\varepsilon = \frac{\| \underline{u}^{(10)} - \underline{u}^{(9)} \|_2}{\| \underline{u}^{(10)} \|_2} = 0.0095$$

8.22

We use mathematical induction to prove the orthogonality properties using the given algorithm in (8.72).

(i) From $\alpha_1 = \dfrac{\underline{r}^{(1)T}\underline{r}^{(1)}}{\underline{r}^{(1)T}\underline{K}\,\underline{r}^{(1)}}$, $\underline{r}^{(2)} = \underline{r}^{(1)} - \alpha_1 \underline{K}\,\underline{r}^{(1)}$

we have

$$\underline{r}^{(1)T}\underline{r}^{(2)} = \underline{r}^{(1)T}\underline{r}^{(1)} - \frac{\underline{r}^{(1)T}\underline{r}^{(1)}}{\underline{r}^{(1)T}\underline{K}\,\underline{r}^{(1)}}\underline{r}^{(1)T}\underline{K}\,\underline{r}^{(1)} = 0,$$

and $\underline{p}^{(1)T}\underline{r}^{(2)} = \underline{r}^{(1)T}\underline{r}^{(2)} = 0;$

$$\underline{p}^{(2)T}\underline{K}\,\underline{p}^{(1)} = \left[\underline{r}^{(2)T} + \frac{\underline{r}^{(2)T}\underline{r}^{(2)}}{\underline{r}^{(1)T}\underline{r}^{(1)}}\underline{r}^{(1)T}\right]\left(\underline{r}^{(1)} - \underline{r}^{(2)}\right)\Big/\alpha_1 =$$

$$= \frac{1}{\alpha_1}\left[\underline{r}^{(2)T}\underline{r}^{(2)} - \underline{r}^{(2)T}\underline{r}^{(2)}\right] = 0$$

(ii) Assume

$$\left\{\begin{array}{l} \underline{p}^{(\ell)T}\underline{r}^{(s+1)} = 0, \\ \underline{r}^{(\ell)T}\underline{r}^{(s+1)} = 0, \qquad\qquad 1 \le \ell \le s \\ \underline{p}^{(s+1)T}\underline{K}\,\underline{p}^{(\ell)} = 0, \end{array}\right. \qquad \text{— ①}$$

8.22

From the algorithm we have

$$\underline{r}^{(s+2)} = \underline{r}^{(s+1)} - \alpha_{(s+1)} \underline{k}\, \underline{p}^{(s+1)}$$

$$\underline{p}^{(s+2)} = \underline{r}^{(s+2)} + \beta_{(s+1)} \underline{p}^{(s+1)}, \quad \text{where}$$

$$\alpha_{(s+1)} = \frac{\underline{r}^{(s+1)T} \underline{r}^{(s+1)}}{\underline{p}^{(s+1)T} \underline{k}\, \underline{p}^{(s+1)}}, \quad \beta_{(s+1)} = \frac{\underline{r}^{(s+2)T} \underline{r}^{(s+2)}}{\underline{r}^{(s+1)T} \underline{r}^{(s+1)}}$$

Using ①, we have $\underline{r}^{(s+2)T} \underline{r}^{(\ell)} = \underline{r}^{(s+2)T} \left(\underline{p}^{(\ell)} - \beta_{(\ell-1)} \underline{p}^{(\ell-1)} \right) =$

$$= \left(\underline{r}^{(s+1)T} - \alpha_{(s+1)} \underline{p}^{(s+1)T} \underline{k} \right) \left(\underline{p}^{(\ell)} - \beta_{(\ell-1)} \underline{p}^{(\ell-1)} \right) = 0, \quad \ell = 1..s \quad \text{②}$$

$$\underline{p}^{(s+2)T} \underline{k}\, \underline{p}^{(\ell)} = \underline{r}^{(s+2)T} \left(\underline{r}^{(\ell)} - \underline{r}^{(\ell+1)} \right) / \alpha_\ell =$$

(using ②) $$= -\frac{\underline{r}^{(s+2)T} \underline{r}^{(\ell+1)}}{\alpha_\ell}, \quad \ell = 1..s \quad \text{③}$$

Since $\underline{p}^{(s+1)T} \underline{k}\, \underline{p}^{(s+1)} = \underline{p}^{(s+1)T} \underline{k} \left(\underline{r}^{(s+1)} + \beta_s \underline{p}^{(s)} \right) =$

$$= \underline{p}^{(s+1)T} \underline{k}\, \underline{r}^{(s+1)} \quad \text{④}$$

we have $$\underline{r}^{(s+1)T} \underline{r}^{(s+2)} = \underline{r}^{(s+1)T} \left[\underline{r}^{(s+1)} - \frac{\underline{r}^{(s+1)T} \underline{r}^{(s+1)}}{\underline{p}^{(s+1)T} \underline{k}\, \underline{p}^{(s+1)}} \underline{k}\, \underline{p}^{(s+1)} \right]$$

$$= \underline{r}^{(s+1)T} \left(\underline{r}^{(s+1)} - \underline{r}^{(s+1)} \right) = 0 \quad \text{⑤}$$

8.22

Therefore, from ② and ⑤ we have that

$$\boxed{\underline{r}^{(\ell)T}\,\underline{r}^{(s+2)} = 0 \quad , \quad \ell = 1..\,s+1} \qquad \text{—— ⑥}$$

$$\underline{p}^{(s+2)T}\,\underline{\underline{k}}\;\underline{p}^{(s+1)} = \left[\underline{r}^{(s+2)} + \beta_{(s+1)}\left(\underline{r}^{(s+1)} + \beta_s\,\underline{p}^{(s)}\right)\right]^T \underline{\underline{k}}\,\underline{p}^{(s+1)}$$

$$= \left[\underline{r}^{(s+2)} + \beta_{(s+1)}\,\underline{r}^{(s+1)}\right]^T \underline{\underline{k}}\;\underline{p}^{(s+1)} =$$

$$= \left[\underline{r}^{(s+2)} + \frac{\underline{r}^{(s+2)T}\,\underline{r}^{(s+2)}}{\underline{r}^{(s+1)T}\,\underline{r}^{(s+1)}}\,\underline{r}^{(s+1)}\right]^T \frac{1}{\alpha_{(s+1)}}\left(\underline{r}^{(s+1)} - \underline{r}^{(s+2)}\right) =$$

$$= \frac{1}{\alpha_{(s+1)}}\left(1 - \beta_{(s+1)}\right)\underline{r}^{(s+2)T}\,\underline{r}^{(s+1)} = (\text{using ⑥}) = 0 \quad \text{—— ⑦}$$

Using ⑦ and ③ with ⑥, we obtain that:

$$\boxed{\underline{p}^{(s+2)T}\,\underline{\underline{k}}\;\underline{p}^{(\ell)} = 0 \quad , \quad \ell = 1..\,s+1} \qquad \text{—— ⑧}$$

$$\underline{p}^{(\ell)T}\,\underline{r}^{(s+2)} = \underline{p}^{(\ell)T}\left(\underline{r}^{(s+1)} - \alpha_{(s+1)}\,\underline{\underline{K}}\;\underline{p}^{(s+1)}\right) = 0,$$

$$\ell = 1,..\,s \qquad \text{—— ⑨}$$

$$\underline{p}^{(s+1)T}\,\underline{r}^{(s+2)} = \left(\underline{r}^{(s+1)T} + \beta_s\,\underline{p}^{(s)T}\right)\underline{r}^{(s+2)} =$$

$$= \beta_s\,\underline{p}^{(s)T}\left(\underline{r}^{(s+1)T} - \alpha_{(s+1)}\,\underline{\underline{K}}\;\underline{p}^{(s+1)}\right) = 0 \qquad \text{—— ⑩}$$

Therefore, from ⑨ and ⑩ we have: $\boxed{\underline{p}^{(\ell)T}\,\underline{r}^{(s+2)} = 0 \,,\; \ell = 1..\,s+1}$ —— ⑪

Hence, from ⑥, ⑧, and ⑪ we conclude that (8.73) and (8.74) hold.

8.23 $\underline{U}^{(s+1)} = \underline{U}^{(s)} + \alpha_s \underline{p}^{(s)} = \left\{ \underline{U}^{(s-1)} + \alpha_{s-1} \underline{p}^{(s-1)} \right\} + \alpha_s \underline{p}^{(s)}$

$$= \underline{U}^{(1)} + \left(\alpha_1 \underline{p}^{(1)} + \cdots + \alpha_s \underline{p}^{(s)} \right)$$

Let $\underline{U}^{(1)} = \underline{0}$, then $\underline{U}^{(s+1)} = \alpha_1 \underline{p}^{(1)} + \cdots + \alpha_s \underline{p}^{(s)}$ ——— ①

Consider the total potential $\Pi^{(s+1)}$ given as

$$\Pi^{(s+1)} = \frac{1}{2} \underline{U}^{(s+1)T} \underline{K} \, \underline{U}^{(s+1)} - \underline{U}^{(s+1)T} \underline{R} \qquad ②$$

Also consider the potential $\Pi^{(s+1)}$ given as

$$\Pi^{(s+1)} = \frac{1}{2} \left(\underline{u}^{(s)} + \alpha_s \underline{p}^{(s)} \right)^T \underline{K} \left(\underline{u}^{(s)} + \alpha_s \underline{p}^{(s)} \right) - \left(\underline{u}^{(s)} + \alpha_s \underline{p}^{(s)} \right)^T \underline{R}$$

$$= \frac{1}{2} \underline{u}^{(s)T} \underline{K} \, \underline{u}^{(s)} - \underline{u}^{(s)T} \underline{R} + \alpha_s \underline{p}^{(s)T} \underline{K} \, \underline{u}^{(s)}$$

$$+ \frac{1}{2} \alpha_s^2 \underline{p}^{(s)T} \underline{K} \, \underline{p}^{(s)} - \alpha_s \underline{p}^{(s)T} \underline{R} = \Pi^{(s)} + \alpha_s \underline{p}^{(s)T} \underline{K} \, \underline{u}^{(s)} +$$

$$+ \frac{1}{2} \alpha_s^2 \underline{p}^{(s)T} \underline{K} \, \underline{p}^{(s)} - \alpha_s \underline{p}^{(s)T} \underline{R}.$$

Using the minimum of $\Pi^{(s)}$ in the space spanned by the vectors $\underline{p}^{(1)}, \ldots \underline{p}^{(s-1)}$, we have:

$$\frac{\partial \Pi^{(s+1)}}{\partial \alpha_j} = \frac{\partial \Pi^{(s)}}{\partial \alpha_j} = 0 \quad , \qquad j = 1 \ldots (s-1)$$

Since we have $\dfrac{\partial \Pi^{(s+1)}}{\partial \alpha_s} = \underline{p}^{(s)T} \left(\underline{K} \, \underline{u}^{(s)} - \underline{R} \right) + \alpha_s \underline{p}^{(s)T} \underline{K} \, \underline{p}^{(s)}$

$$= -\left(\underline{r}^{(s)T} + \beta_{(s-1)} \underline{p}^{(s-1)T} \right) \underline{r}^{(s)} + \alpha_s \underline{p}^{(s)T} \underline{K} \, \underline{p}^{(s)}$$

8.23

Using the orthogonality property proved in Excercise 8.22, we obtain:

$$\frac{\partial \Pi^{(s+1)}}{\partial \alpha_s} = \alpha_s \underline{p}^{(s)T} \underline{\underline{K}} \, \underline{p}^{(s)} - \underline{r}^{(s)T} \underline{r}^{(s)}$$

Therefore, if we use $\alpha_s = \dfrac{\underline{r}^{(s)T} \underline{r}^{(s)}}{\underline{p}^{(s)T} \underline{\underline{K}} \, \underline{p}^{(s)}}$,

which is given by the algorithm, we have $\dfrac{\partial \Pi^{(s+1)}}{\partial \alpha_s} = 0$.
Hence, we achieve the minimum of $\Pi^{(s+1)}$ in the Space spanned by the vectors $\underline{p}^{(1)} \ldots \underline{p}^{(s)}$.

8.24 We solve $\tilde{\mathbf{K}}\tilde{\mathbf{U}} = \tilde{\mathbf{R}}$ with $\tilde{\mathbf{K}} = \bar{\mathbf{L}}^{-1}\mathbf{K}(\bar{\mathbf{L}}^{-1})^T$, $\tilde{\mathbf{R}} = \bar{\mathbf{L}}^{-1}\mathbf{R}$

From eq. (8.72), $\tilde{\mathbf{U}}^{(1)} = \bar{\mathbf{L}}^T\mathbf{U}^{(1)}$, $\tilde{\mathbf{r}}^{(1)} = \tilde{\mathbf{R}} - \tilde{\mathbf{K}}\tilde{\mathbf{U}}^{(1)} = \bar{\mathbf{L}}^{-1}\mathbf{r}^{(1)}$

Assign $\mathbf{K}_P = \bar{\mathbf{L}}\bar{\mathbf{L}}^T$, $\mathbf{z}^{(1)} = \mathbf{K}_P^{-1}\mathbf{r}^{(1)} = (\bar{\mathbf{L}}^{-1})^T\tilde{\mathbf{r}}^{(1)}$, $\mathbf{p}^{(1)} = \mathbf{z}^{(1)}$, $\tilde{\mathbf{p}}^{(1)} = \tilde{\mathbf{r}}^{(1)}$

Then, $\mathbf{z}^{(s)} = (\bar{\mathbf{L}}^{-1})^T\tilde{\mathbf{r}}^{(s)} = \mathbf{K}_P^{-1}\mathbf{r}^{(s)}$, $\mathbf{p}^{(s)} = (\bar{\mathbf{L}}^{-1})^T\tilde{\mathbf{p}}^{(s)}$, $\mathbf{r}^{(s)} = \bar{\mathbf{L}}\tilde{\mathbf{r}}^{(s)}$

$$\therefore\ \alpha_s = \frac{\tilde{\mathbf{r}}^{(s)T}\tilde{\mathbf{r}}^{(s)}}{\tilde{\mathbf{p}}^{(s)T}\mathbf{K}\,\tilde{\mathbf{p}}^{(s)}} = \frac{\mathbf{z}^{(s)T}(\bar{\mathbf{L}}\bar{\mathbf{L}}^{-1})\,\mathbf{r}^{(s)}}{\tilde{\mathbf{p}}^{(s)T}(\bar{\mathbf{L}}\bar{\mathbf{L}}^{-1})\mathbf{K}(\bar{\mathbf{L}}^{-1})^T\tilde{\mathbf{p}}^{(s)}} = \frac{\mathbf{z}^{(s)T}\mathbf{r}^{(s)}}{\mathbf{p}^{(s)T}\mathbf{K}\,\mathbf{p}^{(s)}}$$

$$\therefore\ \beta_s = \frac{\tilde{\mathbf{r}}^{(s+1)T}\tilde{\mathbf{r}}^{(s+1)}}{\tilde{\mathbf{r}}^{(s)T}\tilde{\mathbf{r}}^{(s)}} = \frac{\mathbf{z}^{(s+1)T}(\bar{\mathbf{L}}\bar{\mathbf{L}}^{-1})\mathbf{r}^{(s+1)}}{\mathbf{z}^{(s)}(\bar{\mathbf{L}}\bar{\mathbf{L}}^{-1})\,\mathbf{r}^{(s)}} = \frac{\mathbf{z}^{(s+1)T}\mathbf{r}^{(s+1)}}{\mathbf{z}^{(s)}\mathbf{r}^{(s)}}$$

$$\tilde{\mathbf{r}}^{(s+1)} = \tilde{\mathbf{r}}^{(s)} - \alpha_s\tilde{\mathbf{K}}\tilde{\mathbf{p}}^{(s)}$$

$$\bar{\mathbf{L}}\tilde{\mathbf{r}}^{(s+1)} = \bar{\mathbf{L}}\,\tilde{\mathbf{r}}^{(s)} - \alpha_s\bar{\mathbf{L}}\bar{\mathbf{L}}^{-1}\mathbf{K}(\bar{\mathbf{L}}^{-1})^T\,\bar{\mathbf{L}}^T\mathbf{p}^{(s)}$$

$$\therefore\ \mathbf{r}^{(s+1)} = \mathbf{r}^{(s)} - \alpha_s\mathbf{K}\mathbf{p}^{(s)}$$

$$\tilde{\mathbf{p}}^{(s+1)} = \tilde{\mathbf{r}}^{(s+1)} + \beta_s\tilde{\mathbf{p}}^{(s)}$$

$$(\bar{\mathbf{L}}^{-1})^T\tilde{\mathbf{p}}^{(s+1)} = (\bar{\mathbf{L}}^{-1})^T(\bar{\mathbf{L}}^{-1})\,\tilde{\mathbf{r}}^{(s+1)} + \beta_s(\bar{\mathbf{L}}^{-1})^T(\bar{\mathbf{L}})^T\mathbf{p}^{(s)}$$

$$\therefore\ \mathbf{p}^{(s+1)} = \mathbf{z}^{(s+1)} + \beta_s\mathbf{p}^{(s)}$$

8.25

Results:

Starting vectors

U	r	p
0.0000000E+00	0.0000000E+00	0.0000000E+00
0.0000000E+00	0.1000000E+01	0.1000000E+01
0.0000000E+00	0.0000000E+00	0.0000000E+00

ITER= 1 ALPHA= 0.5000000E+00 BETA= 0.5000000E+00 ERR-NORM= 0.1000000E+01

U	r	p
0.0000000E+00	0.5000000E+00	0.5000000E+00
0.5000000E+00	0.0000000E+00	0.5000000E+00
0.0000000E+00	0.5000000E+00	0.5000000E+00

ITER= 2 ALPHA= 0.1000000E+01 BETA= 0.1000000E+01 ERR-NORM= 0.7071068E+00

U	r	p
0.5000000E+00	-0.5000000E+00	0.0000000E+00
0.1000000E+01	0.0000000E+00	0.5000000E+00
0.5000000E+00	0.5000000E+00	0.1000000E+01

ITER= 3 ALPHA= 0.1000000E+01 BETA= 0.0000000E+00 ERR-NORM= 0.5129892E+00

U	r	p
0.5000000E+00	0.0000000E+00	0.0000000E+00
0.1500000E+01	0.0000000E+00	0.0000000E+00
0.1500000E+01	0.0000000E+00	0.0000000E+00

Of course, this should be the case, because we have only three equations (and round-off error is negligible when so few equations are solved).

We use as ERR-NORM the error norm in Eq. (8.68) of the textbook.

8.26

Use the given $\underline{K}_P$

Results:

Starting vectors

U	r	p	z
0.0000000E+00	0.0000000E+00	0.0000000E+00	0.0000000E+00
0.0000000E+00	0.1000000E+01	0.5000000E+00	0.5000000E+00
0.0000000E+00	0.0000000E+00	0.0000000E+00	0.0000000E+00

ITER= 1 ALPHA= 0.1000000E+01 BETA= 0.6666667E+00 ERR-NORM= 0.1000000E+01

U	r	p	z
0.0000000E+00	0.5000000E+00	0.1666667E+00	0.1666667E+00
0.5000000E+00	0.0000000E+00	0.3333333E+00	0.0000000E+00
0.0000000E+00	0.5000000E+00	0.5000000E+00	0.5000000E+00

ITER= 2 ALPHA= 0.3000000E+01 BETA= 0.0000000E+00 ERR-NORM= 0.8583951E+00

U	r	p	z
0.5000000E+00	0.0000000E+00	0.0000000E+00	0.0000000E+00
0.1500000E+01	0.0000000E+00	0.0000000E+00	0.0000000E+00
0.1500000E+01	0.0000000E+00	0.0000000E+00	0.0000000E+00

We use as ERR-NORM the error norm in Eq. (8.68) of the textbook.

8.27 a)

$$\underline{K}_p = \begin{bmatrix} 1 & & & \\ -0.2 & 1 & & \\ -0.2 & 0 & 1 & \\ -0.2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & & & \\ & 2 & & \\ & & 2 & \\ & & & 2 \end{bmatrix} \begin{bmatrix} 1 & -0.2 & -0.2 & -0.2 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{bmatrix}$$

Results:

Starting vectors

U	r	p	z
0.0000000E+00	0.0000000E+00	0.1000000E+00	0.1000000E+00
0.0000000E+00	0.0000000E+00	0.0000000E+00	0.0000000E+00
0.0000000E+00	0.0000000E+00	0.0000000E+00	0.0000000E+00
0.0000000E+00	0.1000000E+01	0.5000000E+00	0.5000000E+00

ITER= 1 ALPHA= 0.1111111E+01 BETA= 0.2469136E-01 ERR-NORM= 0.1000000E+01

U	r	p	z
0.1111111E+00	0.0000000E+00	0.2469136E-01	0.2222222E-01
0.0000000E+00	0.1111111E+00	0.5555556E-01	0.5555556E-01
0.0000000E+00	0.1111111E+00	0.5555556E-01	0.5555556E-01
0.5555556E+00	0.0000000E+00	0.1234568E-01	0.0000000E+00

ITER= 2 ALPHA= 0.1285714E+01 BETA= 0.5204650E-31 ERR-NORM= 0.1791613E+00

U	r	p	z
0.1428571E+00	-0.2453395E-16	-0.3816392E-18	-0.3816392E-18
0.7142857E-01	0.2775558E-16	0.1142439E-16	0.1142439E-16
0.7142857E-01	0.2775558E-16	0.1142439E-16	0.1142439E-16
0.5714286E+00	0.4460718E-17	-0.2230359E-18	-0.2230359E-18

8.27

b) Using the given $\underline{K}_p$, we obtain

Results:

Starting vectors

U	r	p	z
0.0000000E+00	0.0000000E+00	0.0000000E+00	0.0000000E+00
0.0000000E+00	0.0000000E+00	0.0000000E+00	0.0000000E+00
0.0000000E+00	0.0000000E+00	0.0000000E+00	0.0000000E+00
0.0000000E+00	0.1000000E+01	0.5000000E+00	0.5000000E+00

ITER= 1 ALPHA= 0.1000000E+01 BETA= 0.1111111E+00 ERR-NORM= 0.1000000E+01

U	r	p	z
0.0000000E+00	0.5000000E+00	0.1111111E+00	0.1111111E+00
0.0000000E+00	0.0000000E+00	0.5555556E-01	0.5555556E-01
0.0000000E+00	0.0000000E+00	0.0000000E+00	0.0000000E+00
0.5000000E+00	0.0000000E+00	0.5555556E-01	0.0000000E+00

ITER= 2 ALPHA= 0.1125000E+01 BETA= 0.1406250E+00 ERR-NORM= 0.2641353E+00

U	r	p	z
0.1250000E+00	-0.1110223E-15	0.1562500E-01	-0.2467162E-16
0.6250000E-01	0.0000000E+00	0.7812500E-02	-0.1233581E-16
0.0000000E+00	0.1250000E+00	0.6250000E-01	0.6250000E-01
0.5625000E+00	0.0000000E+00	0.7812500E-02	0.0000000E+00

ITER= 3 ALPHA= 0.1142857E+01 BETA= 0.1026324E-30 ERR-NORM= 0.1250000E+00

U	r	p	z
0.1428571E+00	-0.1189525E-16	-0.2643388E-17	-0.2643388E-17
0.7142857E-01	0.0000000E+00	-0.1321694E-17	-0.1321694E-17
0.7142857E-01	-0.2775558E-16	-0.1387779E-16	-0.1387779E-16
0.5714286E+00	-0.2775558E-16	-0.1387779E-16	-0.1387779E-16

We use as ERR-NORM the error norm in Eq. (8.68) of the textbook.

8.28.

Preconditioner (1) : $\underline{K}_P = \begin{bmatrix} 5 & & & \\ & 6 & & \\ & & 6 & \\ & & & 5 \end{bmatrix}$

Results:

Starting vectors

U	r	p	z
0.0000000E+00	0.0000000E+00	0.0000000E+00	0.0000000E+00
0.0000000E+00	0.1000000E+01	0.1666667E+00	0.1666667E+00
0.0000000E+00	0.0000000E+00	0.0000000E+00	0.0000000E+00
0.0000000E+00	0.0000000E+00	0.0000000E+00	0.0000000E+00

ITER= 1 ALPHA= 0.1000000E+01 BETA= 0.1011111E+01 ERR-NORM= 0.1000000E+01

U	r	p	z
0.0000000E+00	0.6666667E+00	0.1333333E+00	0.1333333E+00
0.1666667E+00	0.0000000E+00	0.1685185E+00	0.0000000E+00
0.0000000E+00	0.6666667E+00	0.1111111E+00	0.1111111E+00
0.0000000E+00	-0.1666667E+00	-0.3333333E-01	-0.3333333E-01

ITER= 2 ALPHA= 0.2936536E+01 BETA= 0.1688064E+01 ERR-NORM= 0.8528662E+00

U	r	p	z
0.3915382E+00	0.3621370E+00	0.2975026E+00	0.7242739E-01
0.6615274E+00	0.2037631E-16	0.2844700E+00	0.3396052E-17
0.3262818E+00	-0.9465758E-01	0.1717864E+00	-0.1577626E-01
-0.9788455E-01	0.1133023E+01	0.1703357E+00	0.2266045E+00

ITER= 3 ALPHA= 0.1931788E+01 BETA= 0.7693756E+00 ERR-NORM= 0.5428368E+00

U	r	p	z
0.9662502E+00	-0.6451345E+00	0.9986430E-01	-0.1290269E+00
0.1211063E+01	0.2037631E-16	0.2188643E+00	0.3396052E-17
0.6581368E+00	0.8538545E+00	0.2744773E+00	0.1423091E+00
0.2311680E+00	0.2656436E+00	0.1841809E+00	0.5312873E-01

ITER= 4 ALPHA= 0.6346110E+01 BETA= 0.2214726E-28 ERR-NORM= 0.6285014E+00

U	r	p	z
0.1600000E+01	0.1554312E-14	0.3108624E-15	0.3108624E-15
0.2600000E+01	-0.3326282E-14	-0.5543804E-15	-0.5543804E-15
0.2400000E+01	0.3330669E-14	0.5551115E-15	0.5551115E-15
0.1400000E+01	-0.1831868E-14	-0.3663736E-15	-0.3663736E-15

8.28

Preconditioner (2):

$$\underline{K}_P = \begin{bmatrix} 5 & -4 & & \\ -4 & 6 & & \\ & & 6 & \\ & & & 5 \end{bmatrix}$$

Starting vectors

U	r	p	z
0.0000000E+00	0.0000000E+00	0.2857143E+00	0.2857143E+00
0.0000000E+00	0.1000000E+01	0.3571429E+00	0.3571429E+00
0.0000000E+00	0.0000000E+00	0.0000000E+00	0.0000000E+00
0.0000000E+00	0.0000000E+00	0.0000000E+00	0.0000000E+00

ITER= 1 ALPHA= 0.1000000E+01 BETA= 0.6809524E+00 ERR-NORM= 0.1000000E+01

U	r	p	z
0.2857143E+00	0.0000000E+00	0.1945578E+00	0.0000000E+00
0.3571429E+00	0.0000000E+00	0.2431973E+00	0.0000000E+00
0.0000000E+00	0.1142857E+01	0.1904762E+00	0.1904762E+00
0.0000000E+00	-0.3571429E+00	-0.7142857E-01	-0.7142857E-01

ITER= 2 ALPHA= 0.1304461E+01 BETA= 0.6174613E+00 ERR-NORM= 0.5370929E+00

U	r	p	z
0.5395074E+00	-0.2484688E+00	0.7043817E-01	-0.4969376E-01
0.6743843E+00	0.1987750E+00	0.1501649E+00	-0.1526557E-15
0.2484688E+00	0.2945137E+00	0.1666973E+00	0.4908562E-01
-0.9317580E-01	0.7853699E+00	0.1129696E+00	0.1570740E+00

ITER= 3 ALPHA= 0.1198837E+02 BETA= 0.7772401E+00 ERR-NORM= 0.8163478E+00

U	r	p	z
0.1383946E+01	0.7318379E+00	0.2011149E+00	0.1463676E+00
0.2474616E+01	-0.5854703E+00	0.1167142E+00	0.1582068E-14
0.2246897E+01	0.7771730E-01	0.1425167E+00	0.1295288E-01
0.1261145E+01	0.2072461E+00	0.1292537E+00	0.4144922E-01

ITER= 4 ALPHA= 0.1074282E+01 BETA= 0.2543515E-27 ERR-NORM= 0.7854372E-01

U	r	p	z
0.1600000E+01	0.1110223E-14	-0.1078502E-14	-0.1078502E-14
0.2600000E+01	-0.5440093E-14	-0.1625684E-14	-0.1625684E-14
0.2400000E+01	0.8937295E-14	0.1489549E-14	0.1489549E-14
0.1400000E+01	-0.6605827E-14	-0.1321165E-14	-0.1321165E-14

We use as ERR-NORM the error norm in Eq. (8.68) of the textbook.

8.29

Preconditioner (1) : $\underline{K}_p = \begin{bmatrix} 7 & & & \\ & 6 & & \\ & & 5 & \\ & & & 1 \end{bmatrix}$

Results:

Starting vectors

U	r	p	z
0.0000000E+00	0.0000000E+00	0.0000000E+00	0.0000000E+00
0.0000000E+00	0.0000000E+00	0.0000000E+00	0.0000000E+00
0.0000000E+00	0.0000000E+00	0.0000000E+00	0.0000000E+00
0.0000000E+00	0.1000000E+01	0.1000000E+01	0.1000000E+01

ITER= 1 ALPHA= 0.1000000E+01 BETA= 0.9666667E+00 ERR-NORM= 0.1000000E+01

U	r	p	z
0.0000000E+00	0.0000000E+00	0.0000000E+00	0.0000000E+00
0.0000000E+00	-0.1000000E+01	-0.1666667E+00	-0.1666667E+00
0.0000000E+00	0.2000000E+01	0.4000000E+00	0.4000000E+00
0.1000000E+01	0.0000000E+00	0.9666667E+00	0.0000000E+00

ITER= 2 ALPHA= 0.1709234E+01 BETA= 0.1160056E+01 ERR-NORM= 0.6575343E+00

U	r	p	z
0.0000000E+00	-0.1823183E+01	-0.2604547E+00	-0.2604547E+00
-0.2848723E+00	0.1791749E+01	0.1052820E+00	0.2986248E+00
0.6836935E+00	0.7465619E+00	0.6133349E+00	0.1493124E+00
0.2652259E+01	0.0000000E+00	0.1121388E+01	0.0000000E+00

ITER= 3 ALPHA= 0.2046168E+01 BETA= 0.5065279E+00 ERR-NORM= 0.5014291E+00

U	r	p	z
-0.5329341E+00	0.1514068E+01	0.8436794E-01	0.2162955E+00
-0.6944756E-01	0.1092861E+01	0.2354719E+00	0.1821436E+00
0.1938680E+01	0.4553589E+00	0.4017430E+00	0.9107179E-01
0.4946808E+01	0.4543406E-15	0.5680142E+00	0.4543406E-15

ITER= 4 ALPHA= 0.3002247E+02 BETA= 0.1031579E-25 ERR-NORM= 0.8198423E+00

U	r	p	z
0.2000000E+01	0.4796163E-13	0.6851662E-14	0.6851662E-14
0.7000000E+01	-0.1094680E-12	-0.1824467E-13	-0.1824467E-13
0.1400000E+02	0.1054157E-12	0.2108314E-13	0.2108314E-13
0.2200000E+02	-0.3621046E-13	-0.3621046E-13	-0.3621046E-13

8.29

Preconditioner (2) : $\underline{K}_P = \begin{bmatrix} 7 & -4 & & \\ -4 & 6 & & \\ & & 5 & \\ & & & 1 \end{bmatrix}$

Starting vectors

U	r	p	z
0.0000000E+00	0.0000000E+00	0.0000000E+00	0.0000000E+00
0.0000000E+00	0.0000000E+00	0.0000000E+00	0.0000000E+00
0.0000000E+00	0.0000000E+00	0.0000000E+00	0.0000000E+00
0.0000000E+00	0.1000000E+01	0.1000000E+01	0.1000000E+01

ITER= 1 ALPHA= 0.1000000E+01 BETA= 0.1069231E+01 ERR-NORM= 0.1000000E+01

U	r	p	z
0.0000000E+00	0.0000000E+00	-0.1538462E+00	-0.1538462E+00
0.0000000E+00	-0.1000000E+01	-0.2692308E+00	-0.2692308E+00
0.0000000E+00	0.2000000E+01	0.4000000E+00	0.4000000E+00
0.1000000E+01	0.0000000E+00	0.1069231E+01	0.0000000E+00

ITER= 2 ALPHA= 0.1609226E+01 BETA= 0.4592678E+00 ERR-NORM= 0.6703198E+00

U	r	p	z
-0.2475732E+00	-0.6436904E+00	0.5930822E-02	0.7658741E-01
-0.4332532E+00	0.1463354E+01	0.1713016E+00	0.2949506E+00
0.6436904E+00	0.7373764E+00	0.3311824E+00	0.1474753E+00
0.2720634E+01	0.0000000E+00	0.4910633E+00	0.0000000E+00

ITER= 3 ALPHA= 0.1924111E+02 BETA= 0.9536382E+01 ERR-NORM= 0.8275670E+00

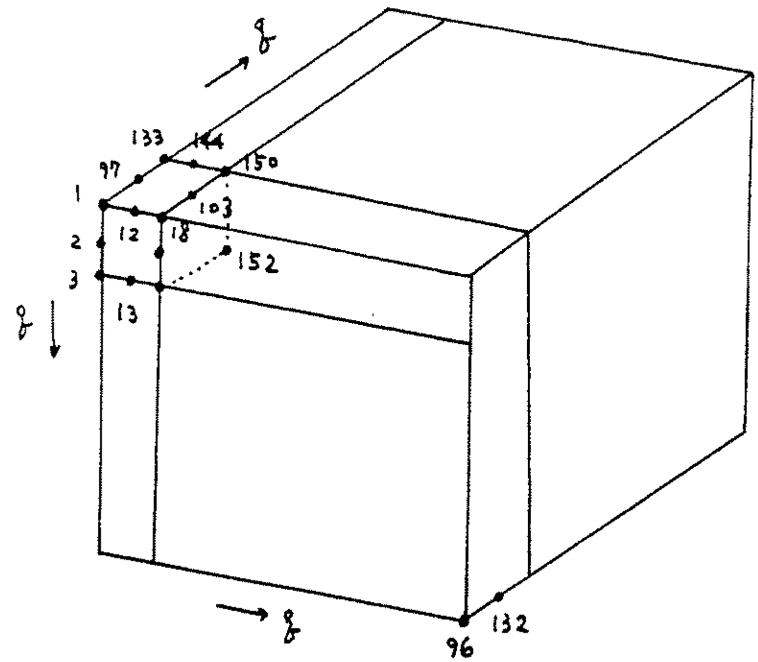
U	r	p	z
-0.1334577E+00	0.5369310E+01	0.1016290E+01	0.9597317E+00
0.2862778E+01	-0.1815709E+01	0.1970800E+01	0.3372030E+00
0.7016006E+01	0.8430076E+00	0.3326883E+01	0.1686015E+00
0.1216923E+02	0.2136192E-14	0.4682967E+01	0.2136192E-14

ITER= 4 ALPHA= 0.2099260E+01 BETA= 0.1881768E-28 ERR-NORM= 0.4774425E+00

U	r	p	z
0.2000000E+01	-0.7993606E-14	-0.2459571E-14	-0.2459571E-14
0.7000000E+01	-0.3996803E-14	-0.2305848E-14	-0.2305848E-14
0.1400000E+02	0.1243450E-13	0.2486900E-14	0.2486900E-14
0.2200000E+02	-0.5321878E-14	-0.5321878E-14	-0.5321878E-14

We use as ERR-NORM the error norm in Eq. (8.68) of the textbook.

8.30



Direct solution

$$m_K = \left[\{(2g+1)(g+1)+(g+1)g\} + (g+1)^2 + \{(2g+1)+(g+1)+3\} \right] \cdot 3 - 1$$

$$= \left[(3g^2+4g+1) + (g^2+2g+1) + \{(3g+2)+3\} \right] \cdot 3 - 1$$

$$\sim 12g^2$$

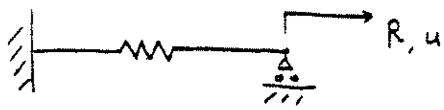
$$n = 3\left[(3g^2+4g+1)\times(g+1) + (g^2+2g+1)\times g \right] \sim 12g^3$$

$$\therefore 2n\,m_K \cong (24g^3)(12g^2) = 288g^5$$

Iterative solution

$$m_K\Big|_{\text{compressed}} = \frac{81}{2}\times 3 \sim 122$$

8.31



$R = 4, \quad F = u + cu^3, \quad c = 0.1$

(a) full Newton-Raphson method

$$\left[1 + 3c\{^{t+\Delta t}u^{(i-1)}\}^2\right]\Delta u^{(i)} = 4 - \left[^{t+\Delta t}u^{(i-1)} + c\{^{t+\Delta t}u^{(i-1)}\}^3\right]$$

ITER	U	ERROR
1	4.00000	.10000E+01
2	2.89655	.44138E+00
3	2.51931	.31282E-01
4	2.47858	.30117E-03
5	2.47814	.34202E-07

initial stress method

$$^0\underline{K}\Delta\underline{U}^{(i)} = \Delta u^{(i)} = 4 - \left[^{t+\Delta t}u^{(i-1)} + c\{^{t+\Delta t}u^{(i-1)}\}^3\right] \quad (\,^0K = 1)$$

The solution diverges.

ITER	U	ERROR
1	0.40000E+01	.10000E+01
2	-0.24000E+01	.25600E+01
3	0.53824E+01	.37854E+01
4	-0.11593E+02	.18010E+02
5	0.15980E+03	.18361E+04
9	0.30947+139	.59859+276

BFGS method

ITER	U(i-1)	dUbar	U(i)	gamma(i)	Kinv(i)
1	0.00000E+00	0.40000E+01	0.40000E+01	0.10400E+02	0.38462E+00
2	0.40000E+01	-0.24615E+01	0.15385E+01	-0.84974E+01	0.28968E+00
3	0.15385E+01	0.60758E+00	0.21460E+01	0.12318E+01	0.49324E+00
4	0.21460E+01	0.42695E+00	0.25730E+01	0.11420E+01	0.37387E+00
5	0.25730E+01	-0.10333E+00	0.24697E+01	-0.30043E+00	0.34395E+00
6	0.24697E+01	0.82684E-02	0.24779E+01	0.23448E-01	0.35262E+00

8.31

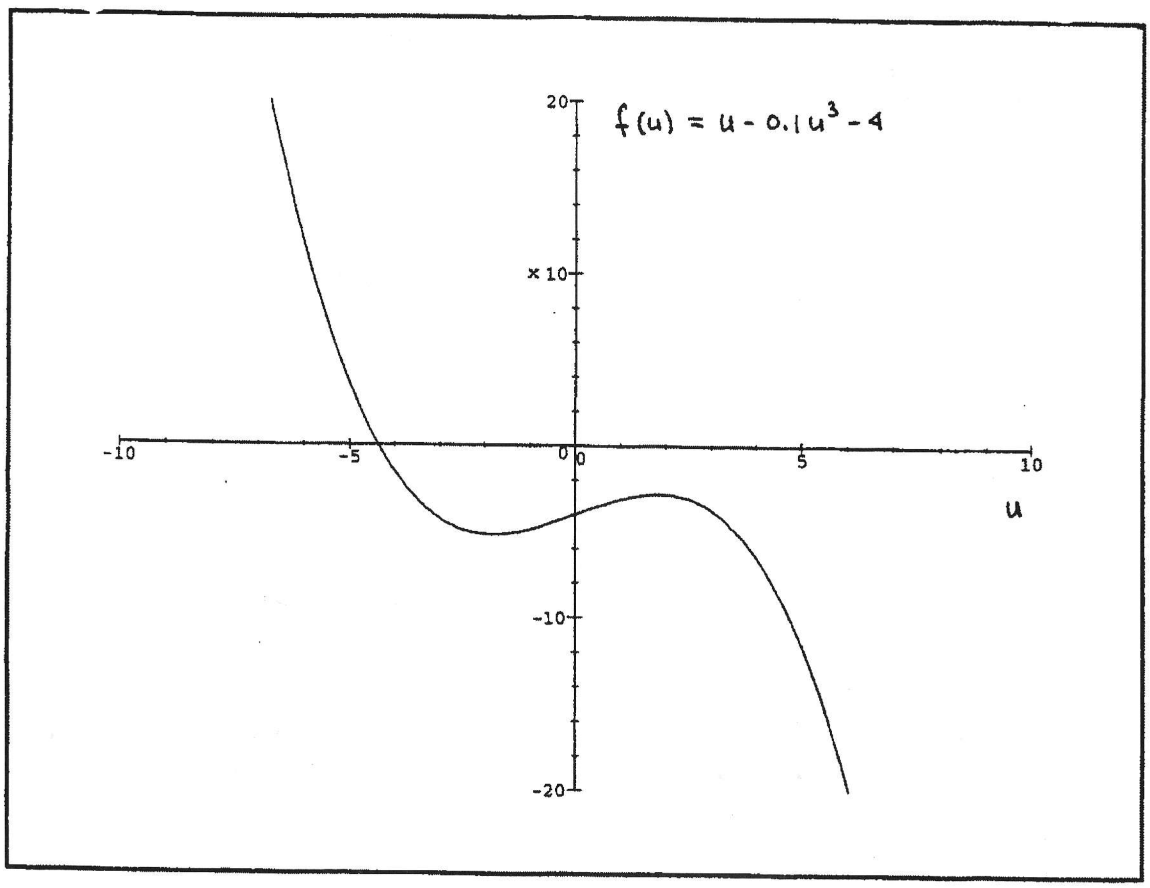
(b) $f(u) = u + cu^3 - 4$

$f'(u) = 1 + 3c\,u^2$

if $c \geq 0$ $\quad f'(u) \geq 1$

if $c < 0$ $\quad f'(u) = [1 + \sqrt{-3c}\,u][1 - \sqrt{-3c}\,u]$

Hence, when $c < 0$ (for example, $c = -0.1$), $f(u)$ looks as shown below, and the full Newton-Raphson method does not converge for starting displacements positive.



8.32 full Newton-Raphson iterative method

$$\left[\frac{1}{L}\cos\frac{{}^{t+\Delta t}u^{(i-1)}}{L}\right]\Delta u^{(i)} = 0.5 - \sin\frac{{}^{t+\Delta t}u^{(i-1)}}{L}$$

ITER	U	ERROR
1	0.50000	.10000E+01
2	0.52344	.19294E-02
3	0.52360	.82477E-07
4	0.52360	.16355E-15

initial stress method

$$ {}^{0}K = \left[\frac{1}{L}\cos\frac{u}{L}\right]_{u=0} = \frac{1}{L}$$

ITER	U	ERROR
1	0.50000E+00	.10000E+01
2	0.52057E+00	.16932E-02
3	0.52320E+00	.27487E-04
4	0.52354E+00	.48710E-06
5	0.52359E+00	.87281E-08
7	0.52360E+00	.28112E-11

BFGS method

$$\left({}^{t+\Delta t}K^{-1}\right)^{(0)} = 1/(1/L) = L$$

ITER	U(i-1)	dUbar	U(i)	gamma(i)	Kinv(i)
1	0.00000E+00	0.50000E+00	0.50000E+00	0.47943E+00	0.10429E+01
2	0.50000E+00	0.21457E-01	0.52146E+00	0.18719E-01	0.11463E+01
3	0.52146E+00	0.21271E-02	0.52358E+00	0.18433E-02	0.11540E+01
4	0.52358E+00	0.14257E-04	0.52360E+00	0.12347E-04	0.11547E+01

8.33

As we use only line searching, the solution shall be obtained solely from:

$$\left| {}^{t+\Delta t}R - {}^{t+\Delta t}F^{(i)} \right| \leq STOL \left| {}^{t+\Delta t}R \right| .$$

Let $\varepsilon = \left| {}^{t+\Delta t}R - {}^{t+\Delta t}F^{(i)} \right| / \left| {}^{t+\Delta t}R \right|$,

and $\Delta \bar{u} = 5$, $STOL = 10^{-4}$. We search for $u^{(i)}$ using the bisection scheme. Here

$$\varepsilon = \left| \frac{4 - (u^{(i)} + 0.1\, u^{(i)3})}{4} \right| , \quad f(u^{(i)}) = 4 - (u^{(i)} + 0.1\, u^{(i)3})$$

$u^{(i+1)} = \frac{1}{2}(u^{(i)} + u_0)$, where u_0 is the latest $u^{(k)}$ $(k = 1 .. i-1)$ carrying the opposite sign of $f(u^{(i)})$.

Starting value

U1	U2
0.0000000E+00	0.5000000E+01

ITER	U	EPS	f(U)
1	0.2500000E+01	0.1562500E-01	-0.6250000E-01
2	0.1250000E+01	0.6386719E+00	0.2554688E+01
3	0.1875000E+01	0.3664551E+00	0.1465820E+01
4	0.2187500E+01	0.1914368E+00	0.7657471E+00
5	0.2343750E+01	0.9219742E-01	0.3687897E+00
6	0.2421875E+01	0.3939486E-01	0.1575794E+00
7	0.2460938E+01	0.1216656E-01	0.4866624E-01
8	0.2480469E+01	0.1658253E-02	-0.6633013E-02
9	0.2470703E+01	0.5271825E-02	0.2108730E-01
10	0.2475586E+01	0.1811213E-02	0.7244850E-02
11	0.2478027E+01	0.7758733E-04	0.3103493E-03

8.34

(a) Using the computer program ADINA, we assign one step to reach the configuration t.

$$\begin{cases} u_1^1 = 0.4 & u_2^1 = 0.3 \\ u_1^2 = 0.3 & u_2^2 = 0.2 \\ u_1^3 = 0.0 & u_2^3 = 0.0 \\ u_1^4 = 0.1 & u_2^4 = 0.0 \end{cases} \qquad \begin{cases} {}^{t}F_1^1 = 0.4478 \cdot 10^5 \\ {}^{t}u_1^1 = 0.4 \end{cases}$$

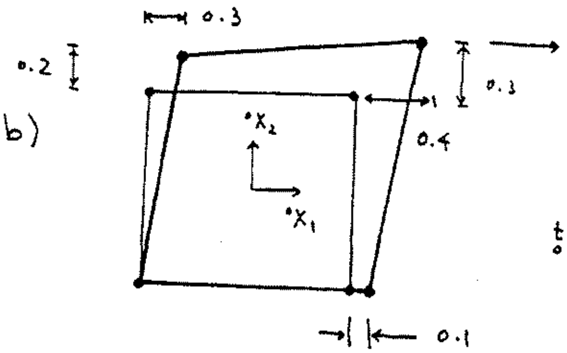
With the application of "restart" in ADINA, at the second step we have:

$$\begin{cases} u_1^1 = 0.4 + 0.02 & u_2^1 = 0.3 \\ u_1^2 = 0.3 & u_2^2 = 0.2 \\ u_1^3 = 0.0 & u_2^3 = 0.0 \\ u_1^4 = 0.1 & u_2^4 = 0.0 \end{cases} \qquad \begin{cases} {}^{t+\Delta t}F_1^1 = 0.4781 \cdot 10^5 \\ {}^{t+\Delta t}u_1^1 = 0.4 + 0.02 \end{cases}$$

$${}^{t}K_{11} \simeq \frac{{}^{t+\Delta t}F_1^1 - {}^{t}F_1^1}{{}^{t+\Delta t}u_1^1 - {}^{t}u_1^1} = 1.515 \cdot 10^5$$

8.34

(b)



$$^t x_1 = {}^0x_1 + \frac{1}{20}(4 + {}^0x_1 + 3\,{}^0x_2)$$

$$^t x_2 = {}^0x_2 + \frac{1}{40}(5 + {}^0x_1)(1 + {}^0x_2)$$

$$^t_0\underline{X} = \left[\frac{\partial\, ^t x_i}{\partial\, ^0 x_j}\right] = \begin{bmatrix} \dfrac{21}{20} & \dfrac{3}{20} \\ \dfrac{1+{}^0x_2}{40} & \dfrac{45+{}^0x_1}{40} \end{bmatrix}$$

$$^t_0\underline{\epsilon} = \frac{1}{2}\left({}^t_0\underline{X}^T\, {}^t_0\underline{X} - \underline{I}\right) = \frac{1}{3200}\begin{bmatrix} 165 + 2\,{}^0x_2 + {}^0x_2^2 & 297 + {}^0x_1 + 45\,{}^0x_2 + {}^0x_1\,{}^0x_2 \\ 297 + {}^0x_1 + 45\,{}^0x_2 + {}^0x_1\,{}^0x_2 & 461 + 90\,{}^0x_1 + {}^0x_1^2 \end{bmatrix}$$

$$^t_0\underline{S} = {}^t_0\underline{C}\; {}^t_0\underline{\epsilon} = {}_0\underline{C}\; {}^t_0\underline{\epsilon} \quad \text{where} \quad {}_0\underline{C} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}$$

$$\therefore\ {}^t_0\underline{S} = \begin{bmatrix} ^t_0S_{11} \\ ^t_0S_{22} \\ ^t_0S_{12} \end{bmatrix} = \begin{bmatrix} 20831 + 1854.4\,{}^0x_1 + 20.604\,{}^0x_1^2 + 137.36\,{}^0x_2 + 68.681\,{}^0x_2^2 \\ 35062 + 6181.3\,{}^0x_1 + 68.681\,{}^0x_1^2 + 41.209\,{}^0x_2 + 20.604\,{}^0x_2^2 \\ 48.078\,(297 + {}^0x_1 + 45\,{}^0x_2 + {}^0x_1\,{}^0x_2) \end{bmatrix}$$

For displacements

$$^tU_1 = \frac{1}{20}(4 + {}^0x_1 + 3\,{}^0x_2)\,, \quad {}^tU_2 = \frac{1}{40}(5 + {}^0x_1)(1 + {}^0x_2)$$

$$U_1 = \frac{1}{4}(1 + {}^0x_1)(1 + {}^0x_2)\,u_1\,, \quad U_2 = 0$$

$${}_0e_{11} = {}_0U_{1,1} + {}^t_0U_{k,1}\,{}_0U_{k,1} = \frac{21}{80}(1 + {}^0x_2)\,u_1$$

$${}_0e_{22} = {}_0U_{2,2} + {}^t_0U_{k,2}\,{}_0U_{k,2} = \frac{3}{80}(1 + {}^0x_1)\,u_1$$

$${}_0e_{12} = \frac{1}{2}\left({}_0U_{1,2} + {}_0U_{2,1} + {}^t_0U_{k,1}\,{}_0U_{k,2} + {}_0U_{k,1}\,{}^t_0U_{k,2}\right)$$

$$= \frac{1}{160}(24 + 21\,{}^0x_1 + 3\,{}^0x_2)\,u_1$$

8.34

$$ {}_0\eta_{11} = \frac{1}{2}\, {}_0u_{k,1}\, {}_0u_{k,1} = \frac{1}{2}\left[\frac{(1+{}^0x_2)^2}{16} u_1^2\right] $$

$$ {}_0\eta_{22} = \frac{1}{2}\, {}_0u_{k,2}\, {}_0u_{k,2} = \frac{1}{2}\left[\frac{(1+{}^0x_1)^2}{16} u_1^2\right] $$

$$ {}_0\eta_{12} = \frac{1}{2}\, {}_0u_{k,1}\, {}_0u_{k,2} = \frac{1}{2}\left[\frac{(1+{}^0x_1)(1+{}^0x_2)}{16} u_1^2\right] $$

$$ \therefore \quad ({}^t_0K_L)_{11} = \int_{-1}^{1}\int_{-1}^{1} \left[\frac{21}{80}(1+{}^0x_2) \quad \frac{3}{80}(1+{}^0x_1) \quad \frac{1}{80}(24+{}^0x_1+3\,{}^0x_2)\right] {}_0\underline{C} \begin{bmatrix} \\ {}'' \\ \\ \end{bmatrix} d\,{}^0x_1\, d\,{}^0x_2 $$

$$ = 122513.7 $$

Similarly,

$$ ({}^t_0K_{NL})_{11} = \int_{-1}^{1}\int_{-1}^{1} \left[\frac{(1+{}^0x_2)^2}{16}\, {}^t_0S_{11} + \frac{(1+{}^0x_1)^2}{16}\, {}^t_0S_{22} + \frac{(1+{}^0x_1)(1+{}^0x_2)}{8}\, {}^t_0S_{12}\right] d\,{}^0x_1\, d\,{}^0x_2 $$

$$ = 27217.7 $$

$$ \therefore \quad ({}^t_0K)_{11} = ({}^t_0K_L)_{11} + ({}^t_0K_{NL})_{11} = 149731.4 $$

8.35 $$(\lambda^{(i)})^2 + \frac{\underline{U}^{(i)T}\underline{U}^{(i)}}{\beta} = (\Delta l)^2 \qquad \text{— ①}$$

with $$\lambda^{(i)} = \lambda^{(i-1)} + \Delta\lambda^{(i)}, \quad \underline{U}^{(i)} = \underline{U}^{(i-1)} + \Delta\bar{\underline{U}}^{(i)} + \Delta\lambda^{(i)}\Delta\bar{\bar{\underline{U}}} \qquad \text{— ②}$$

From ① and ② we obtain the quadratic equation for $\Delta\lambda^{(i)}$.

$$(\beta + \Delta\bar{\bar{\underline{U}}}^T\Delta\bar{\bar{\underline{U}}})(\Delta\lambda^{(i)})^2 + 2\left[\beta\lambda^{(i-1)} + (\underline{U}^{(i-1)} + \Delta\bar{\underline{U}}^{(i)})^T\Delta\bar{\bar{\underline{U}}}\right]\Delta\lambda^{(i)}$$
$$-\left[\beta\left\{(\Delta l)^2 - (\lambda^{(i-1)})^2\right\} - (\underline{U}^{(i-1)} + \Delta\bar{\underline{U}}^{(i)})^T(\underline{U}^{(i-1)} + \Delta\bar{\underline{U}}^{(i)})\right] = 0 \qquad \text{— ③}$$

Eq. ③ has two roots, one is positive and the other one negative. As in the spherical constant arclength scheme the calculation procedure marches forward, we discard the negative root and use the positive root in the practical implementation.

We can write ③ as follows:

$$\left[\sqrt{\beta + \Delta\bar{\bar{\underline{u}}}^T\Delta\bar{\bar{\underline{u}}}}\,\Delta\lambda^{(i)} + c\right]^2 = d^2, \quad \text{where}$$

$$c = \frac{\beta\lambda^{(i-1)} + [\underline{u}^{(i-1)} + \Delta\underline{u}^{(i)}]^T\Delta\bar{\bar{\underline{u}}}}{\sqrt{\beta + \Delta\bar{\bar{\underline{u}}}^T\Delta\bar{\bar{\underline{u}}}}} \Rightarrow$$

$$d^2 = \beta(\Delta l)^2 - \beta(\lambda^{(i-1)})^2 + (\underline{u}^{(i-1)} + \Delta\bar{\underline{u}}^{(i)})^T(\underline{u}^{(i-1)} + \Delta\bar{\underline{u}}^{(i)}) + c^2$$

β has to be selected such that $d^2 > 0$. For incremental loading select $\Delta\lambda^{(i)} = (d-c)/\sqrt{\beta + \Delta\bar{\bar{\underline{u}}}^T\Delta\bar{\bar{\underline{u}}}}$

9.1 $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \ddot{U}_1 \\ \ddot{U}_2 \end{bmatrix} + \begin{bmatrix} 8 & -3 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$ with ${}^0\underline{U} = {}^0\dot{\underline{U}} = \underline{0}$

Solving the eigenproblem of $\underline{K}\underline{\phi} = \omega^2 \underline{M}\underline{\phi}$, we obtain

$$\omega_1 = 1.1514, \quad \omega_2 = 2.9452 \quad \longrightarrow \quad T_1 = \frac{2\pi}{\omega_1} = 5.4569, \quad T_2 = \frac{2\pi}{\omega_2} = 2.1334$$

Let $\Delta t = \frac{T_2}{10} = 0.21$, and ${}^0\ddot{\underline{U}} = \underline{M}^{-1}(-\underline{K}\,{}^0\underline{U} + {}^0\underline{R}) = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$

$$a_0 = \frac{1}{(\Delta t)^2} = 22.676, \quad a_1 = \frac{1}{2\Delta t} = 2.3810, \quad a_2 = 2a_0 = 45.352$$

$$a_3 = \frac{1}{a_2} = 0.02205, \quad {}^{-\Delta t}\underline{U} = {}^0\underline{U} - \Delta t\,{}^0\dot{\underline{U}} + a_3\,{}^0\ddot{\underline{U}} = \begin{bmatrix} 0.2205 \\ 0 \end{bmatrix}$$

Now we consider $\hat{\underline{M}}\,{}^{t+\Delta t}\underline{U} = {}^t\hat{\underline{R}}$

where $\hat{\underline{M}} = a_0\underline{M} = \begin{bmatrix} 22.676 & \\ & 45.352 \end{bmatrix}$, ${}^t\hat{\underline{R}} = \begin{bmatrix} 10 \\ 0 \end{bmatrix} - (\underline{K} - a_2\underline{M})\,{}^t\underline{U} - a_0\underline{M}\,{}^{t-\Delta t}\underline{U}$

The response of the system for $0 \le t \le 4$ is as follows:

N	t+dt	U1	U2
1	0.21000	0.22050E+00	0.00000E+00
2	0.42000	0.80421E+00	0.14586E-01
3	0.63000	0.15471E+01	0.81084E-01
4	0.84000	0.21959E+01	0.24277E+00
5	1.05000	0.25431E+01	0.52831E+00
6	1.26000	0.25040E+01	0.93548E+00
7	1.47000	0.21463E+01	0.14258E+01
8	1.68000	0.16609E+01	0.19323E+01
9	1.89000	0.12862E+01	0.23783E+01
10	2.10000	0.12134E+01	0.26995E+01
11	2.31000	0.15107E+01	0.28630E+01
12	2.52000	0.20947E+01	0.28739E+01
13	2.73000	0.27610E+01	0.27698E+01
14	2.94000	0.32606E+01	0.26041E+01
15	3.15000	0.33954E+01	0.24244E+01
16	3.36000	0.30941E+01	0.22555E+01
17	3.57000	0.24405E+01	0.20923E+01
18	3.78000	0.16438E+01	0.19060E+01
19	3.99000	0.96031E+00	0.16604E+01
20	4.20000	0.59868E+00	0.13318E+01

9.2 As in exercise 9.1, we use $\Delta t = 0.21$ and $\delta = 0.5$, $\alpha = 0.25$,

with ${}^{0}\underline{U} = {}^{0}\dot{\underline{U}} = \underline{0}$, ${}^{0}\ddot{\underline{U}} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$

$\rightarrow$ $a_0 = 90.703$ $a_1 = 9.5238$ $a_2 = 19.048$ $a_3 = a_4 = 1$

$a_5 = 0$ $a_6 = 0.105 = a_7$

Considering $\hat{\underline{K}}\, {}^{t+\Delta t}\underline{U} = {}^{t+\Delta t}\hat{\underline{R}}$

where $$\hat{\underline{K}} = \begin{bmatrix} 8 & -3 \\ -3 & 4 \end{bmatrix} + 90.703 \begin{bmatrix} 1 & \\ & 2 \end{bmatrix} = \begin{bmatrix} 98.703 & -3 \\ -3 & 185.41 \end{bmatrix}$$

$${}^{t+\Delta t}\hat{\underline{R}} = \begin{bmatrix} 10 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & \\ & 2 \end{bmatrix} \left(90.703\, {}^{t}\underline{U} + 19.048\, {}^{t}\dot{\underline{U}} + {}^{t}\ddot{\underline{U}} \right)$$

Performing the calculations we obtain

t+dt	U1	U2
0.21000	0.20273E+00	0.32803E-02
0.42000	0.74594E+00	0.24908E-01
0.63000	0.14571E+01	0.95383E-01
0.84000	0.21153E+01	0.25108E+00
1.05000	0.25269E+01	0.51802E+00
1.26000	0.25909E+01	0.89818E+00
1.47000	0.23320E+01	0.13633E+01
1.68000	0.18889E+01	0.18587E+01
1.89000	0.14636E+01	0.23163E+01
2.10000	0.12475E+01	0.26721E+01
2.31000	0.13527E+01	0.28832E+01
2.52000	0.17704E+01	0.29381E+01
2.73000	0.23724E+01	0.28571E+01
2.94000	0.29551E+01	0.26827E+01
3.15000	0.33098E+01	0.24644E+01
3.36000	0.32962E+01	0.22417E+01
3.57000	0.28921E+01	0.20325E+01
3.78000	0.22030E+01	0.18305E+01
3.99000	0.14270E+01	0.16118E+01
4.20000	0.78817E+00	0.13485E+01

9.2

Now we use a larger Δt ($> \Delta t_{cr} = 0.68$) to see how the response changes. Let $\Delta t = 1.00$ and follow the same procedure as for $\Delta t = 0.21$, then we have the following response, which is not stable.

t+dt	U1	U2
1.00000	0.17778E+01	0.44444E+00
2.00000	0.28444E+01	0.18963E+01
3.00000	0.14828E+01	0.30571E+01
4.00000	0.20283E+01	0.21014E+01
5.00000	0.23951E+01	0.32745E+00
6.00000	0.37621E+00	-0.84367E-01
7.00000	0.88305E+00	0.62396E+00
8.00000	0.31128E+01	0.18141E+01
9.00000	0.21640E+01	0.29036E+01
10.00000	0.12572E+01	0.22962E+01
11.00000	0.22980E+01	0.37123E+00

9.3

Consider the following time stepping equations:

$$\left.\begin{aligned} {}^{t+\tau}\ddot{\underline{u}} &= {}^{t}\ddot{\underline{u}} + \frac{\tau}{\theta\Delta t}\left({}^{t+\theta\Delta t}\ddot{\underline{u}} - {}^{t}\ddot{\underline{u}}\right) \\ {}^{t+\tau}\dot{\underline{u}} &= {}^{t}\dot{\underline{u}} + \tau\left[\left(1-\frac{\tau}{\theta\Delta t}\delta\right){}^{t}\ddot{\underline{u}} + \frac{\tau}{\theta\Delta t}\delta\,{}^{t+\theta\Delta t}\ddot{\underline{u}}\right] \\ {}^{t+\tau}\underline{u} &= {}^{t}\underline{u} + \tau\,{}^{t}\dot{\underline{u}} + \tau^2\left[\left(\frac{1}{2}-\frac{\tau}{\theta\Delta t}\alpha\right){}^{t}\ddot{\underline{u}} + \frac{\tau}{\theta\Delta t}\alpha\,{}^{t+\theta\Delta t}\ddot{\underline{u}}\right] \end{aligned}\right\} - (1)$$

where $0 \le \tau \le \theta\Delta t$

We have the cases:

i) $\delta = \frac{1}{2}$, $\alpha = \frac{1}{6}$; equations (1) give the Wilson θ method (see page 777 of the textbook)

Note that the equilibrium equations used are:

$$\underline{M}\,{}^{t+\theta\Delta t}\ddot{\underline{u}} + \underline{C}\,{}^{t+\theta\Delta t}\dot{\underline{u}} + \underline{K}\,{}^{t+\theta\Delta t}\underline{u} = {}^{t+\theta\Delta t}\bar{\underline{R}},$$

where $\quad {}^{t+\theta\Delta t}\bar{\underline{R}} = {}^{t}\underline{R} + \theta\left({}^{t+\Delta t}\underline{R} - {}^{t}\underline{R}\right)$

ii) $\alpha = \frac{1}{4}$, $\delta = \frac{1}{2}$, $\theta = 1$; equations (1) give the trapezoidal rule. Now we use with eq. (1),

$$\underline{M}\,{}^{t+\Delta t}\ddot{\underline{u}} + \underline{C}\,{}^{t+\Delta t}\dot{\underline{u}} + \underline{K}\,{}^{t+\Delta t}\underline{u} = {}^{t+\Delta t}\underline{R}$$

The computational scheme can be stated as follows:

A. Initial calculations:

1. Form the stiffness matrix $\underline{K}$, mass matrix $\underline{M}$, and damping matrix $\underline{C}$.
2. Initialize ${}^{0}\underline{u}$, ${}^{0}\dot{\underline{u}}$, ${}^{0}\ddot{\underline{u}}$.
3. Select the time step Δt and calculate

9.3

integration constants (for Newmark method use $\theta = 1$)

$$a_0 = \frac{1}{\alpha(\theta\Delta t)^2}, \quad a_1 = \frac{\delta}{\alpha(\theta\Delta t)}, \quad a_2 = \frac{1}{\alpha(\theta\Delta t)},$$

$$a_3 = (\theta\Delta t)\left(\frac{\delta}{2\alpha} - 1\right), \quad a_4 = \frac{a_0}{\theta}, \quad a_5 = -\frac{a_2}{\theta},$$

$$a_6 = 1 - \frac{1}{2\alpha\theta}, \quad a_7 = (1-\delta)\Delta t, \quad b_3 = \delta\Delta t,$$

$$b_1 = \frac{1}{2\alpha} - 1, \quad b_2 = \frac{\delta}{\alpha} - 1, \quad a_8 = \left(\frac{1}{2} - \alpha\right)\Delta t^2, \quad b_4 = \alpha\Delta t^2$$

4. Form the effective stiffness matrix $\hat{\underline{K}} = \underline{K} + a_0\underline{M} + a_1\underline{C}$

5. Triangularize $\hat{\underline{K}} = \underline{L}\,\underline{D}\,\underline{L}^T$

B. For each time step:

1. Calculate effective loads at time $t+\theta\Delta t$:

$$^{t+\theta\Delta t}\hat{\underline{R}} = {}^{t}\underline{R} + \theta\left({}^{t+\Delta t}\underline{R} - {}^{t}\underline{R}\right) + \underline{M}\left(a_0\,{}^{t}\underline{u} + a_2\,{}^{t}\dot{\underline{u}} + b_1\,{}^{t}\ddot{\underline{u}}\right)$$
$$+ \underline{C}\left(a_1\,{}^{t}\underline{u} + b_2\,{}^{t}\dot{\underline{u}} + a_3\,{}^{t}\ddot{\underline{u}}\right)$$

2. Solve for displacements at time $t+\theta\Delta t$:

$$\underline{L}\,\underline{D}\,\underline{L}^T\;{}^{t+\theta\Delta t}\underline{u} = {}^{t+\theta\Delta t}\hat{\underline{R}}$$

3. Calculate displacements, velocities, and accelerations:

$$^{t+\Delta t}\ddot{\underline{u}} = a_4\left({}^{t+\theta\Delta t}\underline{u} - {}^{t}\underline{u}\right) + a_5\,{}^{t}\dot{\underline{u}} + a_6\,{}^{t}\ddot{\underline{u}},$$

$$^{t+\Delta t}\dot{\underline{u}} = {}^{t}\dot{\underline{u}} + a_7\,{}^{t}\ddot{\underline{u}} + b_3\,{}^{t+\Delta t}\ddot{\underline{u}},$$

$$^{t+\Delta t}\underline{u} = {}^{t}\underline{u} + \Delta t\,{}^{t}\dot{\underline{u}} + a_8\,{}^{t}\ddot{\underline{u}} + b_4\,{}^{t+\Delta t}\ddot{\underline{u}}$$

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9.4

For the first sub-step, we use

$$^{t+(2-\sqrt{2})\Delta t}\dot{\underline{U}} = {}^{t}\dot{\underline{U}} + \frac{^{t}\ddot{\underline{U}} + {}^{t+(2-\sqrt{2})\Delta t}\ddot{\underline{U}}}{2}(2-\sqrt{2})\Delta t,$$

$$^{t+(2-\sqrt{2})\Delta t}\underline{U} = {}^{t}\underline{U} + \frac{^{t}\dot{\underline{U}} + {}^{t+(2-\sqrt{2})\Delta t}\dot{\underline{U}}}{2}(2-\sqrt{2})\Delta t.$$

Then, with the finite element equilibrium equations at time $t+(2-\sqrt{2})\Delta t$, we obtain the effective stiffness matrix $\hat{\underline{K}}_1$:

$$\hat{\underline{K}}_1 = \underline{K} + \frac{4}{(2-\sqrt{2})^2\Delta t^2}\underline{M} + \frac{2}{(2-\sqrt{2})\Delta t}\underline{C}.$$

Similarly, with the finite element equilibrium equations at time $t+\Delta t$, if we use for the second sub-step

$$^{t+\Delta t}\dot{\underline{U}} = \frac{\sqrt{2}-1}{(2-\sqrt{2})\Delta t}{}^{t}\underline{U} - \frac{1}{(\sqrt{2}-1)(2-\sqrt{2})\Delta t}{}^{t+(2-\sqrt{2})\Delta t}\underline{U} + \frac{\sqrt{2}}{(\sqrt{2}-1)\Delta t}{}^{t+\Delta t}\underline{U},$$

$$^{t+\Delta t}\ddot{\underline{U}} = \frac{\sqrt{2}-1}{(2-\sqrt{2})\Delta t}{}^{t}\dot{\underline{U}} - \frac{1}{(\sqrt{2}-1)(2-\sqrt{2})\Delta t}{}^{t+(2-\sqrt{2})\Delta t}\dot{\underline{U}} + \frac{\sqrt{2}}{(\sqrt{2}-1)\Delta t}{}^{t+\Delta t}\dot{\underline{U}}$$

we have the effective stiffness $\hat{\underline{K}}_2$:

$$\hat{\underline{K}}_2 = \underline{K} + \frac{2}{(\sqrt{2}-1)^2\Delta t^2}\underline{M} + \frac{\sqrt{2}}{(\sqrt{2}-1)\Delta t}\underline{C}.$$

$$\therefore\ \hat{\underline{K}}_1 = \hat{\underline{K}}_2$$

9.5

Here we have $\Delta t_{cr} = \frac{2}{\omega} = \frac{4}{\sqrt{10}}$,

$\Delta t = 1.01 \; \Delta t_{cr} = \frac{4.04}{\sqrt{10}}$

The problem can be solved in different ways, but one approach is to use the central difference method in the form:

$$\begin{bmatrix} {}^{t+\Delta t}u \\ {}^{t}u \end{bmatrix} = \begin{bmatrix} 2-\omega^2\Delta t & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} {}^{t}u \\ {}^{t-\Delta t}u \end{bmatrix} = \underline{A} \begin{bmatrix} {}^{t}u \\ {}^{t-\Delta t}u \end{bmatrix}$$

(see also page 804 of the textbook).

From $\det(\underline{A} - \lambda \underline{I}_2) = 0$, we have:

$$[(2-\omega^2\Delta t) - \lambda](-\lambda) + 1 = 0 \Rightarrow$$

$$\Rightarrow \quad \lambda^2 - (2-\omega^2\Delta t)\lambda + 1 = 0.$$

The spectral radius of $\underline{A}$ is

$$\rho(\underline{A}) = \max_{i=1,2} |\lambda_i| = 1.327$$

Let n be the number of time step. Then we can estimate the time step to reach overflow from $(\rho(\underline{A}))^n \, {}^0u = 10^{30}$, or

$$n \simeq \frac{\log(10^{30}/10^{-12})}{\log(\rho(\underline{A}))} = 342$$

9.6 $\quad U_1 \leftarrow$ CDM, $\quad U_2 \leftarrow$ Trapezoidal

$$\begin{bmatrix} 1 & \\ & 2 \end{bmatrix}\begin{bmatrix} \ddot{U}_1 \\ \ddot{U}_2 \end{bmatrix} + \begin{bmatrix} 8 & -3 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}, \quad {}^0U = {}^0\dot{U} = 0$$

D.o.f. 1 : ${}^t\ddot{U}_1 + 8\,{}^tU_1 - 3\,{}^tU_2 = 10$ —— ①

D.o.f. 2 : ${}^{t+\Delta t}\ddot{U}_2 - 3\,{}^{t+\Delta t}U_1 + 4\,{}^{t+\Delta t}U_2 = 0$ —— ②

Using CDM for ①, ${}^t\ddot{U}_1 = \left({}^{t+\Delta t}U_1 - 2\,{}^tU_1 + {}^{t-\Delta t}U_1\right)/\Delta t^2$ —— ③

With the trapezoidal rule for ②,

$$\left.\begin{aligned} {}^{t+\Delta t}\dot{U}_2 &= {}^t\dot{U}_2 + \frac{\Delta t}{2}\left({}^t\ddot{U}_2 + {}^{t+\Delta t}\ddot{U}_2\right) \\ {}^{t+\Delta t}U_2 &= {}^tU_2 + {}^t\dot{U}_2\,\Delta t + \frac{\Delta t^2}{4}\left({}^t\ddot{U}_2 + {}^{t+\Delta t}\ddot{U}_2\right) \end{aligned}\right\} \quad \text{—— ④}$$

The initial conditions are

$${}^0U_1 = {}^0\dot{U}_1 = {}^0U_2 = {}^0\dot{U}_2 = {}^0\ddot{U}_2 = 0, \quad {}^0\ddot{U}_1 = 10$$

From eq. (9.7) $\quad {}^{-\Delta t}U_1 = {}^0U_1 - \Delta t\,{}^0\dot{U}_1 + \frac{\Delta t^2}{2}\,{}^0\ddot{U}_1 = 5\,\Delta t^2$

Now for each time step we use ① and ③ to solve for ${}^{t+\Delta t}U_1$, then ② and ④ to solve for ${}^{t+\Delta t}U_2$ implicitly, i.e.,

<u>d.o.f. 1</u>

$${}^t\ddot{U}_1 = 10 - 8\,{}^tU_1 + 3\,{}^tU_2$$

$${}^{t+\Delta t}U_1 = \Delta t^2\,{}^t\ddot{U}_1 + 2\,{}^tU_1 - {}^{t-\Delta t}U_1$$

<u>d.o.f. 2</u>

$${}^{t+\Delta t}U_2 = \frac{1}{1+\Delta t^2}\left[\frac{3\Delta t^2}{4}\,{}^{t+\Delta t}U_1 + {}^tU_2 + \Delta t\,{}^t\dot{U}_2 + \frac{\Delta t^2}{4}\,{}^t\ddot{U}_2\right]$$

$${}^{t+\Delta t}\ddot{U}_2 = 3\,{}^{t+\Delta t}U_1 - 4\,{}^{t+\Delta t}U_2$$

$${}^{t+\Delta t}\dot{U}_2 = {}^t\dot{U}_2 + \frac{\Delta t}{2}\,{}^t\ddot{U}_2 + \frac{\Delta t}{2}\,{}^{t+\Delta t}\ddot{U}_2$$

9.6

N	t+dt	U1	U2
1	0.21000	0.22050E+00	0.69850E-02
2	0.42000	0.80513E+00	0.45575E-01
3	0.63000	0.15527E+01	0.15338E+00
4	0.84000	0.22138E+01	0.36299E+00
5	1.05000	0.25829E+01	0.68503E+00
6	1.26000	0.25724E+01	0.10989E+01
7	1.47000	0.22407E+01	0.15550E+01
8	1.68000	0.17652E+01	0.19873E+01
9	1.89000	0.13709E+01	0.23328E+01
10	2.10000	0.12425E+01	0.25495E+01
11	2.31000	0.14541E+01	0.26270E+01
12	2.52000	0.19413E+01	0.25870E+01
13	2.73000	0.25268E+01	0.24716E+01
14	2.94000	0.29888E+01	0.23270E+01
15	3.15000	0.31453E+01	0.21863E+01
16	3.36000	0.29224E+01	0.20591E+01
17	3.57000	0.23818E+01	0.19313E+01
18	3.78000	0.16975E+01	0.17750E+01
19	3.99000	0.10901E+01	0.15636E+01
20	4.20000	0.74601E+00	0.12873E+01

9.7 $\underline{K} = \begin{bmatrix} 8 & -3 \\ -3 & 4 \end{bmatrix}$, $\underline{M} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $\underline{R} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$, ${}^0\underline{U} = {}^0\dot{\underline{U}} = \underline{0}$

Considering the eigenproblem $\underline{K}\underline{\phi} = \omega^2 \underline{M}\underline{\phi}$

$$\omega_1^2 = \frac{10 - 3\sqrt{6}}{2} = 1.3258 \;, \quad \underline{\phi}_1 = \begin{bmatrix} \dfrac{\sqrt{6} - 2}{2\sqrt{3 - \sqrt{6}}} \\[2ex] \dfrac{\sqrt{6} + 2}{2\sqrt{6 + 2\sqrt{6}}} \end{bmatrix} = \begin{bmatrix} 0.30291 \\ 0.67389 \end{bmatrix}$$

$$\omega_2^2 = \frac{10 + 3\sqrt{6}}{2} = 8.6742 \;, \quad \underline{\phi}_2 = \begin{bmatrix} -\dfrac{\sqrt{6} + 2}{2\sqrt{3 + \sqrt{6}}} \\[2ex] \dfrac{\sqrt{6} - 2}{2\sqrt{6 - 2\sqrt{6}}} \end{bmatrix} = \begin{bmatrix} -0.95302 \\ 0.21419 \end{bmatrix}$$

Using $\underline{U} = \underline{\Phi}\underline{X}$ where $\underline{\Phi} = [\underline{\phi}_1 \, \underline{\phi}_2]$,

$$\ddot{\underline{X}} + \begin{bmatrix} \omega_1^2 & \\ & \omega_2^2 \end{bmatrix} \underline{X} = \begin{bmatrix} 5(\sqrt{6}-2)/\sqrt{3-\sqrt{6}} \\ -5(\sqrt{6}+2)/\sqrt{3+\sqrt{6}} \end{bmatrix} = \begin{bmatrix} 3.0291 \\ -9.5302 \end{bmatrix}$$

From ${}^0\underline{U} = {}^0\dot{\underline{U}} = \underline{0}$, $\underline{X} = \underline{0}$ and $\dot{\underline{X}} = \underline{0}$

$$\therefore \quad \underline{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \dfrac{r_1}{\omega_1^2}(1 - \cos\omega_1 t) \\[2ex] \dfrac{r_2}{\omega_2^2}(1 - \cos\omega_2 t) \end{bmatrix} = \begin{bmatrix} 2.2848\,(1 - \cos 1.1514t) \\ -1.0987\,(1 - \cos 2.9452t) \end{bmatrix}$$

$$\therefore \quad \underline{U} = \underline{\Phi}\underline{X} = \begin{bmatrix} 1.7391 - 0.69207\cos(1.1514t) - 1.0471\cos(2.9452t) \\ 1.3044 - 1.5397\cos(1.1514t) + 0.23532\cos(2.9452t) \end{bmatrix}$$

9.8 $\begin{bmatrix} 1 & \\ & 2 \end{bmatrix}\begin{bmatrix} \ddot{U}_1 \\ \ddot{U}_2 \end{bmatrix} + \begin{bmatrix} 8 & -3 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \underline{R}$, ${}^0\underline{U} = {}^0\underline{\dot{U}} = \underline{0}$

(a) First consider the eigenproblem of $\underline{K}\underline{\phi} = \lambda \underline{M}\underline{\phi}$

$$P(\lambda) = \det(\underline{K} - \lambda\underline{M}) = 2\lambda^2 - 20\lambda + 23 = 0$$

$$\therefore \lambda_1 = \frac{10 - 3\sqrt{6}}{2}, \quad \lambda_2 = \frac{10 + 3\sqrt{6}}{2}$$

Using the eigenvalues λ_1 and λ_2, we obtain the eigenvectors $\underline{\phi}_1$ and $\underline{\phi}_2$ which are M-orthonormal as

$$\underline{\phi}_1 = \begin{bmatrix} 0.30291 \\ 0.67389 \end{bmatrix}, \quad \underline{\phi}_2 = \begin{bmatrix} -0.95302 \\ 0.21419 \end{bmatrix}$$

A load vector that will excite only the second mode of the system corresponds to any vector proportional to $\underline{M}\underline{\phi}_2$, i.e.,

$$\underline{R} = c\,\underline{M}\underline{\phi}_2 = c\begin{bmatrix} -0.9530 \\ 0.4284 \end{bmatrix}; \quad c = \text{constant}$$

(b) Any ${}^0\underline{\dot{U}}$ proportional to $\underline{\phi}_1$ will excite only the first mode

$${}^0\underline{\dot{U}}_{1st} = c\begin{bmatrix} 0.30291 \\ 0.67389 \end{bmatrix}; \quad c = \text{constant}$$

9.9 $\ddot{x} + 2\xi\omega\dot{x} + \omega^2 x = \sin\hat{\omega}t$ with $\xi = 0.2$ ——— (*)

The complete solution of the equation consists of the homogeneous and particular solutions, x_h and x_p, respectively.

$x = x_h + x_p$ where $x_h = e^{-\xi\omega t}(A_1 \cos\bar{\omega}t + A_2 \sin\bar{\omega}t)$ when $\xi < 1$

$\bar{\omega} = \sqrt{1-\xi^2}\,\omega$

A_1 and A_2 : determined from initial conditions.

Let $x_p = X\sin(\hat{\omega}t - \phi)$ and substituting x_p into (*), we obtain

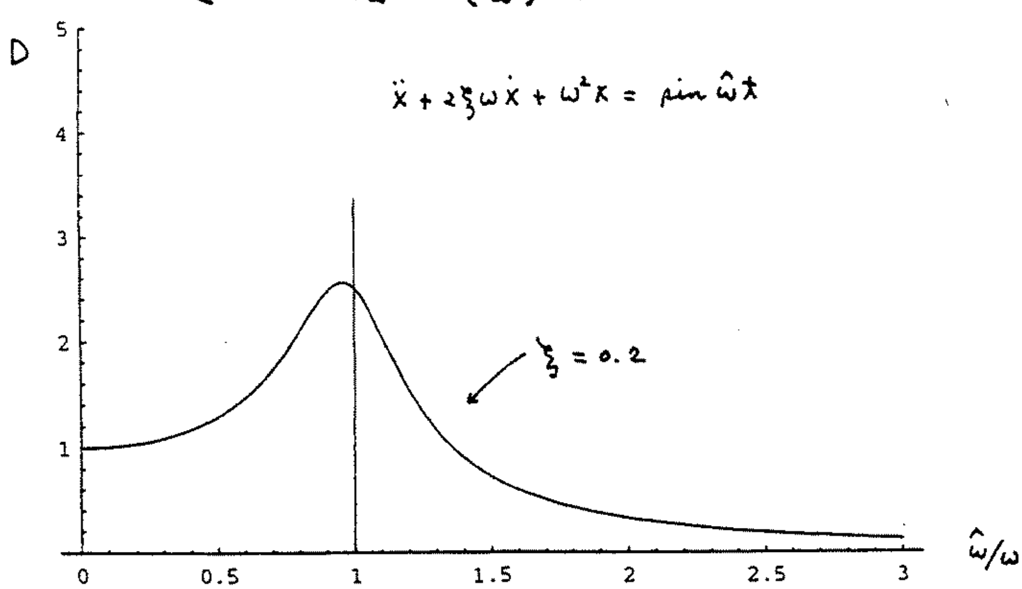
$$X = \frac{1/\omega^2}{\left[\left\{1-\left(\frac{\hat{\omega}}{\omega}\right)^2\right\}^2 + \left(2\xi\frac{\hat{\omega}}{\omega}\right)^2\right]^{1/2}}, \quad \phi = \tan^{-1}\frac{2\xi}{1-\left(\frac{\hat{\omega}}{\omega}\right)^2}$$

By definition, the dynamic load factor D is

$$D = \frac{X}{1/\omega^2} = 1\Big/\left[\left\{1-\left(\frac{\hat{\omega}}{\omega}\right)^2\right\}^2 + \left(2\xi\frac{\hat{\omega}}{\omega}\right)^2\right]^{1/2}$$

Hence we have the plot for $\xi = 0.2$ as shown:

$$D = 1\Big/\left[1 - 1.84\left(\frac{\hat{\omega}}{\omega}\right)^2 + \left(\frac{\hat{\omega}}{\omega}\right)^4\right]^{1/2}$$



9.10 Using only the 1^{st} mode we have

$$U(t) = \underline{\Phi}_1 \left[\left(\alpha_1 \sin \omega_1 t + \beta_1 \cos \omega_1 t \right) + \frac{1}{\omega_1} \int_0^t (3.029) \sin \omega_1 (t-\tau)\, d\tau \right]$$

$$= \underline{\Phi}_1 \left[\left(\alpha_1 \sin \omega_1 t + \beta_1 \cos \omega_1 t \right) + \frac{1}{\omega_1^2}(3.029)(1 - \cos \omega_1 t) \right]$$

$$\dot{U}(t) = \underline{\Phi}_1 \left[\left(\alpha_1 \omega_1 \cos \omega_1 t - \beta_1 \omega_1 \sin \omega_1 t \right) + \frac{1}{\omega_1^2}(3.029)\, \omega_1 \sin \omega_1 t \right]$$

From the b.c.'s,

$$\dot{U}(0) = \underline{\Phi}_1 \alpha_1 \omega_1 = \underline{0} \qquad \therefore \alpha_1 = 0$$

$$U(0) = \underline{\Phi}_1 \beta_1 = \underline{0} \qquad \therefore \beta_1 = 0$$

$$\rightarrow U(t) = \underline{\Phi}_1 \left[\frac{1}{\omega_1^2}(3.029)(1 - \cos \omega_1 t) \right]$$

The static correction is

$$\Delta \underline{R} = \underline{R} - r_1 (\underline{M}\,\underline{\Phi}_1) = \begin{bmatrix} 10 \\ 0 \end{bmatrix} - (3.029) \begin{bmatrix} 0.3029 \\ 1.3478 \end{bmatrix} = \begin{bmatrix} 9.0825 \\ -4.0825 \end{bmatrix}$$

9.11 Let $\underset{\sim}{C} = \alpha \underset{\sim}{M} + \beta \underset{\sim}{K}$, then $\underset{\sim}{\phi}_i^T (\alpha \underset{\sim}{M} + \beta \underset{\sim}{K}) \underset{\sim}{\phi}_i = 2\omega_i \xi_i$

$$\therefore \quad \alpha + \beta \omega_1^2 = 2\omega_1 \xi_1 \quad , \quad \alpha + \beta \omega_2^2 = 2\omega_2 \xi_2$$

$$\longrightarrow \alpha = \frac{2\omega_1\omega_2(\omega_2\xi_1 - \omega_1\xi_2)}{\omega_2^2 - \omega_1^2} \quad , \quad \beta = \frac{2(\omega_2\xi_2 - \omega_1\xi_1)}{\omega_2^2 - \omega_1^2}$$

Using $\omega_1 = 1.15142$, $\omega_2 = 2.94521$, $\xi_1 = 0.02$, $\xi_2 = 0.08$

$$\alpha = -0.0306509, \quad \beta = 0.057859$$

$$\therefore \quad \underset{\sim}{C} = \alpha \underset{\sim}{M} + \beta \underset{\sim}{K} = \begin{bmatrix} 0.432221 & -0.173577 \\ -0.173577 & 0.170134 \end{bmatrix}$$

9.12 Using $\alpha + \beta \omega_i^2 = 2\omega_i \xi_i$ as in exercise 9.11, with $\omega_1 = 1.2$, $\omega_4 = 3.1$, $\xi_1 = 0.04$ and $\xi_4 = 0.10$,

$$\alpha = 0.00364, \quad \beta = 0.0641 \quad \text{where } \underline{C} = \alpha \underline{M} + \beta \underline{K}$$

The other damping ratios are obtained from

$$\xi_i = \frac{\alpha + \beta \omega_i^2}{2\omega_i} = \frac{0.00364 + 0.0641\,\omega_i^2}{2\omega_i}$$

$$\rightarrow \xi_2 = 0.0745, \quad \xi_3 = 0.0936, \quad \xi_5 = 0.157, \quad \xi_6 = 0.324$$

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9.13

The results of time integration scheme would be:

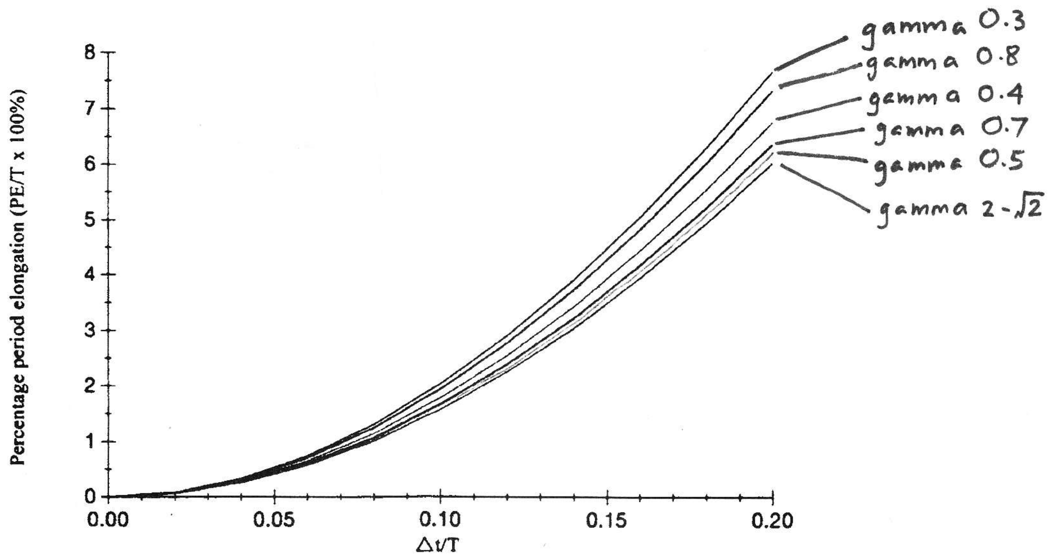
$u = e^{-\xi \varpi t} \cos(\varpi t)$ (1.1)

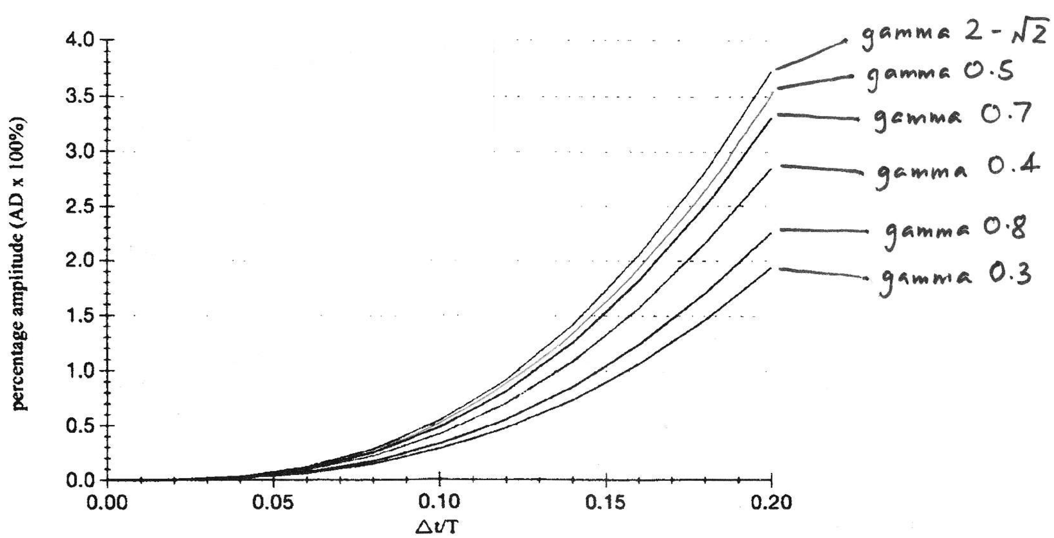
Hence percentage period elongations and amplitude decays are

$PE/T(100\%) = 100(\frac{1}{\varpi} - 1)$ (1.2)

$AD(100\%) = 100(1 - e^{-2\xi\pi})$ (1.3)

In order to obtain the amplitude decay parameter ξ, and frequency ϖ, we should first analyze the initial value problem with Bathe method. Then we could obtain both parameters with the help the least-square-fitting algorithm (post-processing software such as Abscissa will have the fitting module)

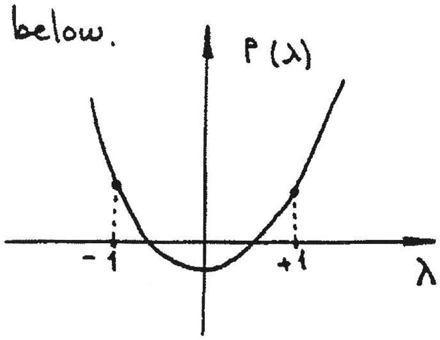




9.14 $\begin{bmatrix} \dfrac{2-\omega^2\Delta t^2}{1+\xi\omega\Delta t} & -\dfrac{1-\xi\omega\Delta t}{1+\xi\omega\Delta t} \\ 1 & 0 \end{bmatrix} \underline{u} = \lambda \underline{u}$

$$p(\lambda) = \left(\frac{2-\omega^2\Delta t^2}{1+\xi\omega\Delta t} - \lambda\right)(-\lambda) + \frac{1-\xi\omega\Delta t}{1+\xi\omega\Delta t} = \lambda^2 - \frac{2-\omega^2\Delta t^2}{1+\xi\omega\Delta t}\lambda + \frac{1-\xi\omega\Delta t}{1+\xi\omega\Delta t} = 0$$

$p(\lambda)$ is a second order polynomial. We want that the roots λ_1 and λ_2 satisfy $|\lambda_i| \le 1$, $i=1,2$. Hence, we want the polynomial to look like shown below.



To see under what condition $p(\lambda)$ looks like shown, we evaluate:

$$p(-1) = 1 - \frac{2-\omega^2\Delta t^2}{1+\xi\omega\Delta t} + \frac{1-\xi\omega\Delta t}{1+\xi\omega\Delta t} = \frac{\omega^2\Delta t^2}{1+\xi\omega\Delta t} \ge 0 \quad \text{— ①}$$

$$p(1) = 1 + \frac{2-\omega^2\Delta t^2}{1+\xi\omega\Delta t} + \frac{1-\xi\omega\Delta t}{1+\xi\omega\Delta t} = \frac{4-\omega^2\Delta t^2}{1+\xi\omega\Delta t} \ge 0 \quad \text{— ②}$$

To ensure that both roots lie between -1 and 1, we need the condition $0 \le \omega^2\Delta t^2 \le 4$

$\Rightarrow \Delta t_{cr} = \dfrac{2}{\omega_n}$, and $\Delta t < \Delta t_{cr} = \dfrac{2}{\omega_n}$

9.15

- Houbolt method

$$\underline{A} \approx \begin{bmatrix} 1.26652\times10^{-9} & -1.01321\times10^{-9} & 2.53303\times10^{-10} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$|\lambda_1| \approx 6.32189\times10^{-4}, \quad |\lambda_2| = |\lambda_3| \approx 6.32990\times10^{-4}$$

$$\therefore \rho(\underline{A}) \approx 0.$$

- Newmark method ($\delta = \frac{1}{2}$, $\alpha = \frac{1}{4}$)

$$\underline{A} \approx \begin{bmatrix} -1 & -4/\Delta t & -4/\Delta t \\ 5.06606\times10^{-10}\Delta t & -1 & -2/\Delta t \\ 2.53303\times10^{-10}\Delta t & 1.01321\times10^{-9}\Delta t & 1.01321\times10^{-9} \end{bmatrix}$$

$$|\lambda_1| \approx 0, \quad |\lambda_2| = |\lambda_3| \approx 1$$

$$\therefore \rho(\underline{A}) \approx 1.$$

- Bathe method

$$\underline{A} \approx \begin{bmatrix} -7.09248\times10^{-9} & 7.95775\times10^{-5}\omega & 4.81276\times10^{-9}\omega^2 \\ -1.01321\times10^{-9}\Delta t & -1.19052\times10^{-8} & 1.59155\times10^{-5}\omega \\ 1.79655\times10^{-18}\Delta t^2 & -1.26651\times10^{-9}\Delta t & -4.81276\times10^{-9} \end{bmatrix}$$

$$|\lambda_1| \approx 0, \quad |\lambda_2| = |\lambda_3| \approx 7.95775\times10^{-5}$$

$$\therefore \rho(\underline{A}) \approx 0.$$

9.16

• Houbolt method

N	t+Ndt	X(t-dt)	X(t)	X(t+dt)
0	0.00000	1.00000E+00	8.09017E-01	3.09017E-01
1	1.00000	8.09017E-01	3.09017E-01	-2.88537E-01
2	2.00000	3.09017E-01	-2.88537E-01	-7.80753E-01
3	3.00000	-2.88537E-01	-7.80753E-01	-1.01913E+00
4	4.00000	-7.80753E-01	-1.01913E+00	-9.44212E-01
5	5.00000	-1.01913E+00	-9.44212E-01	-5.95162E-01
6	6.00000	-9.44212E-01	-5.95162E-01	-9.10713E-02
7	7.00000	-5.95162E-01	-9.10713E-02	4.09674E-01
8	8.00000	-9.10713E-02	✓ 4.09674E-01	7.58938E-01
9	9.00000	4.09674E-01	✓ 7.58938E-01	8.62259E-01
10	10.00000	7.58938E-01	✓ 8.62259E-01	7.03702E-01
11	11.00000	8.62259E-01	✓ 7.03702E-01	3.45924E-01
12	12.00000	7.03702E-01	3.45924E-01	-9.30890E-02
13	13.00000	3.45924E-01	-9.30890E-02	-4.78306E-01

Using an interpolating polynomial of order 3, say,

$f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$ with the data given at

$N = 8, 9, 10$ and 11, we have

$a_0 \approx -11.172$, $a_1 \approx 1.8917$, $a_2 \approx -0.043305$ and $a_3 \approx -0.0026556$.

$f'(t_{max}) = 0 \Rightarrow t_{max} \approx 10.9044$,

$f(t_{max}) \approx 0.86345$.

$\therefore$ Percentage period elongation $\approx \dfrac{10.9044 - 10}{10} \times 100 = 9.044$.

Percentage amplitude decay $\approx \dfrac{1 - 0.86345}{1} \times 100 = 13.655$.

· Newmark method ($\delta = 1/2$ and $\alpha = 1/4$)

N	t+Ndt	X	X'	X"
0	0.00000	1.00000E+00	0.00000E+00	-3.94784E-01
1	1.00000	8.20340E-01	-3.59321E-01	-3.23857E-01
2	2.00000	3.45914E-01	-5.89530E-01	-1.36562E-01
3	3.00000	-2.52805E-01	-6.07909E-01	9.98035E-02
4	4.00000	-7.60687E-01	-4.07854E-01	3.00307E-01
5	5.00000	-9.95238E-01	-6.12483E-02	3.92904E-01
6	6.00000	-8.72179E-01	3.07365E-01	3.44323E-01
7	7.00000	-4.35729E-01	5.65536E-01	1.72019E-01
8	8.00000	1.57288E-01	6.20498E-01	-6.20948E-02
9	9.00000	6.93788E-01	4.52502E-01	-2.73896E-01
10	10.00000	✓ 9.80995E-01	✓ 1.21913E-01	✓ -3.87281E-01
11	11.00000	9.15711E-01	-2.52482E-01	✓ -3.61508E-01
12	12.00000	5.21393E-01	-5.36155E-01	-2.05838E-01
13	13.00000	-6.02725E-02	-6.27176E-01	2.37946E-02

Using (9.80), we have

$$0 = {}^{10}\dot{X} + \frac{\Delta t}{2}\left({}^{10}\ddot{X} + {}^{11}\ddot{X}\right)$$

where we set $t = 10$, ${}^{t+\Delta t}\ddot{X} \approx {}^{11}\ddot{X}$ and ${}^{t+\Delta t}X = 0$.

$\therefore \Delta t \approx 0.32563 \Rightarrow t_{max} \approx 10.326.$

From (9.81), we have

$${}^{10+\Delta t}X = {}^{10}X + \Delta t \cdot {}^{10}\dot{X} + \frac{\Delta t^2}{4}\left({}^{10}\ddot{X} + {}^{11}\ddot{X}\right) \approx 1.00085 \approx 1$$

$\therefore$ Percentage period elongation $\approx \dfrac{10.326 - 10}{10} \times 100 = 3.26.$

Percentage amplitude decay $\approx 0.$

• Bathe method

N	t+Ndt	X	X'	X"
0	0.00000	1.00000E+00	0.00000E+00	-3.94784E-01
1	1.00000	8.14444E-01	-3.64028E-01	-3.21530E-01
2	2.00000	3.27651E-01	-5.92962E-01	-1.29351E-01
3	3.00000	-2.79914E-01	-6.02208E-01	1.10506E-01
4	4.00000	-7.83267E-01	-3.88569E-01	3.09222E-01
5	5.00000	-9.96225E-01	-3.13358E-02	3.93294E-01
6	6.00000	-8.40264E-01	3.37133E-01	3.31723E-01
7	7.00000	-3.73480E-01	5.80456E-01	1.47444E-01
8	8.00000	2.31057E-01	6.08706E-01	-9.12177E-02
9	9.00000	7.49468E-01	4.11646E-01	-2.95878E-01
10	10.00000	∨ 9.89976E-01	∨ 6.24350E-02	∨ -3.90827E-01
11	11.00000	8.63851E-01	-3.09530E-01	∨ -3.41035E-01
12	12.00000	4.18143E-01	-5.66561E-01	-1.65076E-01
13	13.00000	-1.81869E-01	-6.13648E-01	7.17990E-02

Using (9.90), we have

$$''\ddot{X} = \frac{1}{\Delta t}\,{}^{10}\dot{X} - \frac{4}{\Delta t}\,{}^{10+\Delta t/2}\dot{X} \qquad \text{①}$$

where we set $t=10$, ${}^{t+\Delta t}\ddot{X} \approx {}''\dot{X}$ and ${}^{t+\Delta t}\dot{X} = 0$.

From (9.86), (9.87) and (9.88), we have

$${}^{10+\Delta t/2}\dot{X} = \frac{1}{1+\omega^2(\frac{\Delta t}{4})^2}\left[\left\{1-\omega^2\left(\frac{\Delta t}{4}\right)^2\right\}{}^{10}\dot{X} + \frac{\Delta t}{4}\,{}^{10}\ddot{X} - \omega^2\frac{\Delta t}{4}\,{}^{10}X\right]. \qquad \text{②}$$

From ① and ②, we obtain

$$\Delta t \approx 0.16661 \quad \Rightarrow \quad t_{max} \approx 10.167$$

Using (9.89), we have

$${}^{10+\Delta t}X = {}^{10}X + \frac{\Delta t}{3}\left({}^{10}\dot{X} + {}^{10+\Delta t/2}\dot{X}\right) \approx 0.9951.$$

∴ Percentage period elongation $\approx \frac{10.167-10}{10} \times 100 = 1.67$

Percentage amplitude decay $\approx \frac{1-0.9951}{1} \times 100 = 0.49$.

∴ In each case the values calculated here are close to those given in Fig. 9.6.

9.25 $\underline{C}\dot{\underline{\theta}} + \underline{K}\underline{\theta} = \underline{Q} \quad , \quad \underline{\theta}|_{t=0} = {}^{\circ}\underline{\theta}$ ——— ①

(i) Assume $\underline{\theta} = \underline{\phi} e^{-\lambda t}$ and $\underline{Q} = \underline{0}$, then $\dot{\underline{\theta}} = -\lambda \underline{\phi} e^{-\lambda t}$

$\therefore \quad \underline{C}(-\lambda \underline{\phi} e^{-\lambda t}) + \underline{K}\underline{\phi} e^{-\lambda t} = \underline{0} \qquad \therefore \quad \underline{K}\underline{\phi} = \lambda \underline{C}\underline{\phi}$ ——— ②

(ii) Let $\underline{\phi}^T \underline{K} \underline{\phi} = \underline{\Lambda}$ and $\underline{\phi}^T \underline{C} \underline{\phi} = \underline{I}$ where $\underline{\Lambda}$ is a diagonal matrix listing the eigenvalues of ②. Using $\underline{\theta} = \underline{\Phi}\underline{\Lambda}$ where $\underline{\Phi} = [\underline{\phi}_1 \ \underline{\phi}_2 \cdots]$ we obtain from ①:

$$\dot{\underline{X}} + \underline{\Lambda}\underline{X} = \underline{\Phi}^T \underline{Q}$$ ——— ③

Considering the ith equation in ③, we obtain:

$$\dot{x}_i + \lambda x_i = g_i$$ ——— ④

Now we obtain the solution of ① by using the response of ④ and $\underline{\theta} = \underline{\Phi}\underline{X}$.

(iii) $\underline{C} = \begin{bmatrix} \frac{1}{2} & \\ & 1 \end{bmatrix}$, $\underline{K} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$, $\underline{Q} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

The eigensolutions of $\underline{K}\underline{\phi} = \lambda \underline{C}\underline{\phi}$ are

$$\lambda_1 = \frac{5-\sqrt{17}}{2} = 0.43845 \ , \qquad \underline{\phi}_1 = \begin{bmatrix} 0.52191 \\ 0.92941 \end{bmatrix}$$

$$\lambda_2 = \frac{5+\sqrt{17}}{2} = 4.56155 \ , \qquad \underline{\phi}_2 = \begin{bmatrix} -1.31438 \\ 0.36905 \end{bmatrix}$$

Let $\underline{\Phi} = [\underline{\phi}_1 \ \underline{\phi}_2]$, then $\underline{\Phi}^T \underline{Q} = \underline{g} = \begin{bmatrix} 0.52191 \\ -1.31438 \end{bmatrix}$

9.25

We now consider $\dot{\underline{X}} + \underline{\Lambda}\,\underline{X} = \underline{g}$ where $\underline{X}^T = [\,x_1 \;\; x_2\,]$.

$$ {}^\circ\underline{X} = \underline{\Phi}^T\underline{C}\,{}^\circ\underline{\theta} = \begin{bmatrix} 0.26096 & 0.92941 \\ -0.65719 & 0.36905 \end{bmatrix} \begin{bmatrix} {}^\circ\theta_1 \\ {}^\circ\theta_2 \end{bmatrix} = \begin{bmatrix} {}^\circ x_1 \\ {}^\circ x_2 \end{bmatrix} $$

As $x_i = \dfrac{g_i}{\lambda_i} + \left({}^\circ x_i - \dfrac{g_i}{\lambda_i}\right)\cos\lambda_i t$

$$ \underline{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.1904 + (-1.1904 + 0.26096\,{}^\circ\theta_1 + 0.92941\,{}^\circ\theta_2)\cos\lambda_1 t \\ -0.28814 + (0.28814 - 0.65719\,{}^\circ\theta_1 + 0.36905\,{}^\circ\theta_2)\cos\lambda_2 t \end{bmatrix} $$

$$ \therefore\; \underline{\theta} = \underline{\Phi}\,\underline{X} = \begin{bmatrix} 1 + \cos\lambda_1 t\,(-0.62127 + 0.13620\,{}^\circ\theta_1 + 0.48507\,{}^\circ\theta_2) \\ \qquad + \cos\lambda_2 t\,(-0.37873 + 0.86380\,{}^\circ\theta_1 - 0.48507\,{}^\circ\theta_2) \\ 1 + \cos\lambda_1 t\,(-1.1063 + 0.24254\,{}^\circ\theta_1 + 0.86380\,{}^\circ\theta_2) \\ \qquad + \cos\lambda_2 t\,(0.10634 - 0.24254\,{}^\circ\theta_1 + 0.13620\,{}^\circ\theta_2) \end{bmatrix} $$

where $\lambda_1 = 0.43845$, $\lambda_2 = 4.5616$

9.26 Consider the general system in heat transfer analysis

$$\underset{\sim}{C}\dot{\underset{\sim}{\theta}} + \underset{\sim}{K}\underset{\sim}{\theta} = \underset{\sim}{Q}$$

Let $\underset{\sim}{\theta}^p$ be the response predicted by mode superposition using p modes; an error measure ϵ^p as in structural analysis is

$$\epsilon^p(t) = \frac{\| \underset{\sim}{Q}(t) - [\underset{\sim}{C}\dot{\underset{\sim}{\theta}}^p(t) + \underset{\sim}{K}\underset{\sim}{\theta}^p(t)] \|_2}{\| \underset{\sim}{Q}(t) \|_2}$$

where we assume $\| \underset{\sim}{Q}(t) \|_2 \neq 0$

Here we may say that ϵ^p is a measure of the part of the external load vector that has not been included in the mode superposition analysis. Since $\underset{\sim}{Q} = \sum_{i=1}^{n} f_i \underset{\sim}{C}\underset{\sim}{\phi}_i$, we can evaluate

$$\Delta\underset{\sim}{Q} = \underset{\sim}{Q} - \sum_{i=1}^{p} f_i (\underset{\sim}{C}\underset{\sim}{\phi}_i)$$

For a properly modeled problem the response to $\Delta\underset{\sim}{Q}$ can be approximated neglecting the term $\underset{\sim}{C}\dot{\underset{\sim}{\theta}}$. Therefore, a good correction $\Delta\theta$ to the mode superposition solution $\underset{\sim}{\theta}^p$ can be obtained from

$$\underset{\sim}{K}\Delta\underset{\sim}{\theta}(t) = \Delta\underset{\sim}{Q}(t)$$

where $\Delta\underset{\sim}{\theta}$ is the "static correction".

9.27 $\underline{C}\dot{\underline{\theta}} + \underline{K}\underline{\theta} = \underline{Q}$

We first show that for any finite element assemblage we have

$$\lambda_n \le \max\left(\lambda_N^{(m)}\right)$$

where $\underline{K}\underline{\phi}_n = \lambda_n \underline{C}\underline{\phi}_n$ of the complete assemblage and

$$\underline{K}^{(m)}\underline{\phi}_N^{(m)} = \lambda_N^{(m)}\underline{C}^{(m)}\underline{\phi}_N^{(m)} \quad \text{of element } m;$$

n and N denote the largest eigenvalues in each case.

Using the Rayleigh quotient,

$$\lambda_n = \frac{\underline{\phi}_n^T \sum_m \underline{K}^{(m)}\underline{\phi}_n}{\underline{\phi}_n^T \sum_m \underline{C}^{(m)}\underline{\phi}_n} = \frac{\sum_m U^{(m)}}{\sum_m L^{(m)}}$$

where $U^{(m)} = \underline{\phi}_n^T \underline{K}^{(m)}\underline{\phi}_n$, $\underline{L}^{(m)} = \underline{\phi}_n^T \underline{C}^{(m)}\underline{\phi}_n$

For a single element

$$\rho^{(m)} = \frac{\underline{\phi}_n^T \underline{K}^{(m)}\underline{\phi}_n}{\underline{\phi}_n^T \underline{C}^{(m)}\underline{\phi}_n} = \frac{U^{(m)}}{L^{(m)}}$$

Since $\rho^{(m)} \le \lambda_N^{(m)}$, $U^{(m)} \le \lambda_N^{(m)} L^{(m)}$

and

$$\lambda_n \le \frac{\sum_m \lambda_N^{(m)} L^{(m)}}{\sum_m L^{(m)}} \le \max\left(\lambda_N^{(m)}\right)\frac{\sum_m L^{(m)}}{\sum_m L^{(m)}} = \max\left(\lambda_N^{(m)}\right).$$

Hence we see that in the Euler forward method,

$$\frac{2}{\max\left(\lambda_N^{(m)}\right)} \le \Delta t_{cr}.$$

9.27

Considering now 1-D analysis using 2-node elements with length Δx,

$$\underline{K} = \frac{k}{\Delta x}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \underline{C} = \rho c \frac{\Delta x}{2}\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$$

$$\therefore \; p(\lambda) = \begin{vmatrix} \Delta_k - \lambda\Delta_c & -\Delta_k \\ -\Delta_k & \Delta_k - \lambda\Delta_c \end{vmatrix} = \Delta_c \lambda \left(\Delta_c\lambda - 2\Delta_k\right) = 0$$

where $\Delta_k = \frac{k}{\Delta x}, \quad \Delta_c = \rho c \frac{\Delta x}{2}$

$$\lambda_N = \frac{2\Delta_k}{\Delta_c} = \frac{4(k/\rho c)}{(\Delta x)^2} = \frac{4a}{(\Delta x)^2}$$

Hence $\Delta t_{cr} = \frac{2}{\lambda_N} = \frac{(\Delta x)^2}{2a}$

9.28

$$\underline{M}_v \dot{\hat{\underline{v}}} + (\underline{K}_{\mu vv} + \underline{K}_{vv}) \hat{\underline{v}} + \underline{K}_{vp} \hat{\underline{p}} = \underline{R}_B + \underline{R}_S \qquad (7.74)$$

$$\underline{K}_{vp}^T \hat{\underline{v}} = \underline{0} \qquad (7.75)$$

$$\underline{C} \dot{\hat{\underline{\theta}}} + (\underline{K}_{v\theta} + \underline{K}_{\theta\theta}) \hat{\underline{\theta}} = \underline{Q}_B + \underline{Q}_S \qquad (7.76)$$

$$\begin{bmatrix} \underline{M}_v & \underline{0} & \underline{0} \\ \underline{0} & & \underline{0} \\ \underline{0} & \underline{0} & \underline{C} \end{bmatrix} \begin{bmatrix} \dot{\hat{\underline{v}}} \\ \dot{\hat{\underline{p}}} \\ \dot{\hat{\underline{\theta}}} \end{bmatrix} + \begin{bmatrix} \underline{K}_{\mu vv} + \underline{K}_{vv} & \underline{K}_{vp} & \underline{0} \\ \underline{K}_{vp}^T & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{K}_{v\theta} + \underline{K}_{\theta\theta} \end{bmatrix} \begin{bmatrix} \hat{\underline{v}} \\ \hat{\underline{p}} \\ \hat{\underline{\theta}} \end{bmatrix} = \begin{bmatrix} \underline{R}_B + \underline{R}_S \\ \underline{0} \\ \underline{Q}_B + \underline{Q}_S \end{bmatrix}$$

Now calculate ${}^{t+\Delta t}\hat{\underline{v}}$, ${}^{t+\Delta t}\hat{\underline{p}}$ and ${}^{t+\Delta t}\hat{\underline{\theta}}$ using

$${}^{t}\dot{\hat{\underline{v}}} = \frac{1}{\Delta t}\left({}^{t+\Delta t}\hat{\underline{v}} - {}^{t}\hat{\underline{v}}\right), \quad {}^{t+\Delta t}\dot{\hat{\underline{p}}} = \frac{1}{\Delta t}\left({}^{t+\Delta t}\hat{\underline{p}} - {}^{t}\hat{\underline{p}}\right)$$

$${}^{t}\dot{\hat{\underline{\theta}}} = \frac{1}{\Delta t}\left({}^{t+\Delta t}\hat{\underline{\theta}} - {}^{t}\hat{\underline{\theta}}\right)$$

$$\underline{M}_v {}^{t}\dot{\hat{\underline{v}}} + \bar{\underline{K}}_{vv} {}^{t}\hat{\underline{v}} + \underline{K}_{vp} {}^{t+\Delta t}\hat{\underline{p}} = {}^{t}\underline{R} \qquad \text{——— ①}$$

$$\underline{K}_{vp}^T \, {}^{t+\Delta t}\underline{v} = \underline{0} \qquad \text{——— ②}$$

$$\underline{C} \, {}^{t}\dot{\hat{\underline{\theta}}} + \underline{K}_\theta \hat{\underline{\theta}} = {}^{t}\underline{Q} \qquad \text{——— ③}$$

where $\bar{\underline{K}}_{vv} = \underline{K}_{\mu vv} + \underline{K}_{vv}$, $\underline{K}_\theta = \underline{K}_{v\theta} + \underline{K}_{\theta\theta}$

$\underline{R} = \underline{R}_B + \underline{R}_S$ and $\underline{Q} = \underline{Q}_B + \underline{Q}_S$

From ①

$$\frac{\underline{M}_v}{\Delta t} {}^{t+\Delta t}\hat{\underline{v}} = {}^{t}\underline{R} - \left[-\frac{\underline{M}_v}{\Delta t} {}^{t}\hat{\underline{v}} + \bar{\underline{K}}_{vv} {}^{t}\hat{\underline{v}} + \underline{K}_{vp} {}^{t+\Delta t}\hat{\underline{p}} \right] \qquad \text{——— ④}$$

Then we iterate ④ and ② until the convergence is assessed, and we solve ③ explicitly.

10.1

$$\underline{K}\,\underline{\phi} = \lambda \underline{M}\,\underline{\phi} \qquad \text{where} \quad \underline{K} = \begin{bmatrix} 6 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad \underline{M} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

(a) $p(\lambda) = \det\left(\underline{K} - \lambda \underline{M}\right) = 40 - 73\lambda + 26\lambda^2 - 2\lambda^3 = 0$

$\therefore \quad \lambda_1 = 0.72446, \quad \lambda_2 = 2.9652, \quad \lambda_3 = 9.3104$

For λ_1,

$$\begin{bmatrix} 4.5511 & -1 & 0 \\ -1 & 2.5511 & -1.7245 \\ 0 & -1.7245 & 1.2755 \end{bmatrix} \underline{\phi}_1 = \underline{0} \quad \text{and} \quad \underline{\phi}_1^T \underline{M}\,\underline{\phi}_1 = 1$$

$\therefore \ \underline{\phi}_1^T = [\,0.085347 \quad 0.38842 \quad 0.52512\,]$

Similarly for λ_2 and λ_3 with $\underline{\phi}_2^T \underline{M}\,\underline{\phi}_2 = \underline{\phi}_3^T \underline{M}\,\underline{\phi}_3 = 1$,

$\underline{\phi}_2^T = [\,-0.69814 \quad -0.048619 \quad 0.19974\,]$

$\underline{\phi}_3^T = [\,0.072912 \quad -0.92020 \quad 1.2978\,]$

We now show that $\underline{\phi}_i^T \underline{M}\,\underline{\phi}_j = \delta_{ij}$

When $i = j$, automatically satisfied.

$\underline{\phi}_1^T \underline{M}\,\underline{\phi}_2 = \underline{\phi}_2^T \underline{M}\,\underline{\phi}_1 = 0$

$\underline{\phi}_1^T \underline{M}\,\underline{\phi}_3 = \underline{\phi}_3^T \underline{M}\,\underline{\phi}_1 = 0$

$\underline{\phi}_2^T \underline{M}\,\underline{\phi}_3 = \underline{\phi}_3^T \underline{M}\,\underline{\phi}_2 = 0$ "O.K."

(b) Let $\underline{v}_1^T = [\,\alpha \quad \alpha \quad \alpha\,]$, then from $\underline{v}_1^T \underline{M}\,\underline{v}_1 = 1$, $\alpha = 1/\sqrt{7}$

$\therefore \quad \underline{v}_1^T = \frac{1}{\sqrt{7}} [\,1 \quad 1 \quad 1\,]$

10.1

Now with $\bar{\underline{v}}_2^T = [\,\beta \;\; -\beta \;\; \beta\,]$ we find out $\underline{v}_2 = \bar{\underline{v}}_2 - \gamma\,\underline{v}_1$ such that $\underline{v}_1^T \underline{M}\, \underline{v}_2 = 0$.

$$\underline{v}_1^T \underline{M}\, \underline{v}_2 = \underline{v}_1^T \underline{M}\, \bar{\underline{v}}_2 - \gamma\, \underline{v}_1^T \underline{M}\, \underline{v}_1 = \underline{v}_1^T \underline{M}\, \bar{\underline{v}}_2 - \gamma = 0$$

$$\gamma = \underline{v}_1^T \underline{M}\, \bar{\underline{v}}_2 = \frac{\beta}{\sqrt{7}} \qquad \therefore\; \underline{v}_2 = \frac{\beta}{7}\begin{bmatrix} 6 \\ -8 \\ 6 \end{bmatrix}$$

Normalizing $\underline{v}_2$ by $\underline{v}_2^T \underline{M}\, \underline{v}_2 = 1$ we have

$$\beta = \frac{\sqrt{7}}{2\sqrt{5}} \qquad \therefore\; \underline{v}_2 = \frac{1}{\sqrt{35}}\begin{bmatrix} 3 \\ -4 \\ 3 \end{bmatrix}$$

Note that $\underline{v}_1$ and $\underline{v}_2$ are not eigenvectors because they do not satisfy $\underline{K}\,\underline{v}_i = \lambda\, \underline{M}\, \underline{v}_i$ where $i = 1, 2$, that is,

$$\underline{K}\,[\,\underline{v}_1 \;\; \underline{v}_2\,] = \begin{bmatrix} \frac{5}{\sqrt{7}} & \frac{22}{\sqrt{35}} \\ \frac{2}{\sqrt{7}} & -\frac{22}{\sqrt{35}} \\ \frac{1}{\sqrt{7}} & \frac{10}{\sqrt{35}} \end{bmatrix}, \qquad \underline{M}\,[\,\underline{v}_1 \;\; \underline{v}_2\,] = \begin{bmatrix} \frac{2}{\sqrt{7}} & \frac{6}{\sqrt{35}} \\ \frac{3}{\sqrt{7}} & -\frac{5}{\sqrt{35}} \\ \frac{2}{\sqrt{7}} & -\frac{1}{\sqrt{35}} \end{bmatrix}$$

(In Section 11.2.5 we introduce the Gram-Schmidt orthogonalization, which we used here to obtain $\underline{v}_2$)

10.2 $\lambda_1 = 0.72446$, $\lambda_2 = 2.9652$, $\lambda_3 = 9.3104$

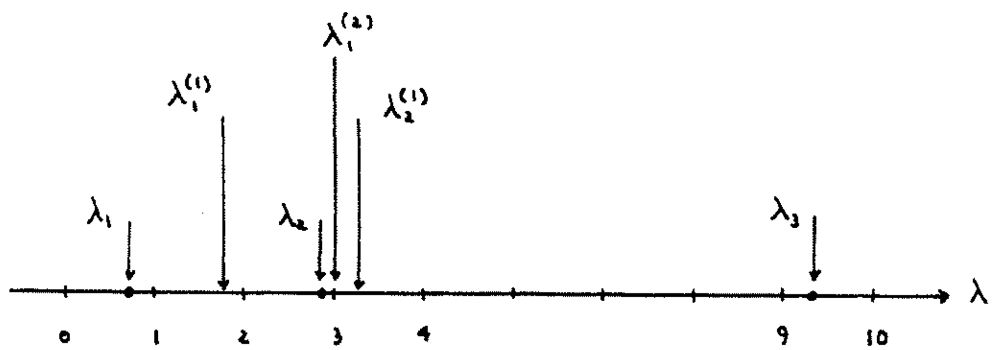
1st associated constraint problem

$$\begin{bmatrix} 6 & -1 \\ -1 & 4 \end{bmatrix} \underline{\phi}^{(1)} = \lambda^{(1)} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \underline{\phi}^{(1)}, \quad p^{(1)}(\lambda^{(1)}) = 23 - 20\lambda + 4\lambda^2 = 0$$

$$\therefore \lambda_1^{(1)} = \frac{5 - \sqrt{2}}{2} = 1.7929, \quad \lambda_2^{(1)} = \frac{5 + \sqrt{2}}{2} = 3.2071$$

2nd associated constraint problem

$$[\,6\,]\, \underline{\phi}^{(2)} = \lambda^{(2)} [\,2\,]\, \underline{\phi}^{(2)}, \quad \lambda_1^{(2)} = 3$$



$$\therefore \quad \lambda_1 < \lambda_1^{(1)} < \lambda_2 < \lambda_2^{(1)} < \lambda_3$$

$$\lambda_1^{(1)} < \lambda_1^{(2)} < \lambda_2^{(1)}$$

Hence the separation property holds.

10.3

$$\underline{K}\,\underline{\phi} = \lambda \underline{M}\,\underline{\phi} \quad \text{where} \quad \underline{K} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad \underline{M} = \begin{bmatrix} 1 & & \\ & 2 & \\ & & \frac{3}{2} \end{bmatrix}$$

(a) $$p(\lambda) = \det(\underline{K} - \lambda \underline{M}) = \frac{3}{2}\left(6 - 15\lambda + 10\lambda^2 - 2\lambda^3\right) = 0$$

$$\therefore \lambda_1 = \frac{3-\sqrt{3}}{2} = 0.63398, \quad \lambda_2 = 2, \quad \lambda_3 = \frac{3+\sqrt{3}}{2} = 2.3660$$

For λ_1,

$$\begin{bmatrix} 1.3660 & -1 & 0 \\ -1 & 0.73205 & 0 \\ 0 & 0 & 2.0490 \end{bmatrix} \underline{\phi}_1 = \underline{0} \quad \text{with} \quad \underline{\phi}_1^T \underline{M}\,\underline{\phi}_1 = 1$$

$$\underline{\phi}_1^T = \begin{bmatrix} 0.45970 & 0.62796 & 0 \end{bmatrix}$$

Similarly, $$\underline{\phi}_2^T = \begin{bmatrix} 0 & 0 & 0.81650 \end{bmatrix}$$

$$\underline{\phi}_3^T = \begin{bmatrix} 0.88807 & -0.32506 & 0 \end{bmatrix}$$

1st associated constraint problem:

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \underline{\phi}^{(1)} = \lambda^{(1)} \begin{bmatrix} 1 & \\ & 2 \end{bmatrix} \underline{\phi}^{(1)}$$

$$p^{(1)}(\lambda^{(1)}) = 3 - 6\lambda^{(1)} + 2\lambda^{(1)2} = 0$$

$$\lambda_1^{(1)} = \frac{3-\sqrt{3}}{2} = \lambda_1, \quad \lambda_2^{(1)} = \frac{3+\sqrt{3}}{2} = \lambda_3$$

2nd associated constraint problem:

$$[\,2\,]\,\underline{\phi}^{(2)} = \lambda^{(2)}\,[\,1\,]\,\underline{\phi}^{(2)}, \quad \lambda_1^{(2)} = 2 = \lambda_2$$

10.3

(b) Let $\underline{v}_1 = \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\bar{\underline{v}}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

From $\underline{v}_1^T \underline{M}\, \underline{v}_1 = 1$, $\alpha = \frac{\sqrt{2}}{3}$

$\therefore \underline{v}_1^T = \frac{\sqrt{2}}{3} [\ 1 \quad 1 \quad 1\ ]$

Then $\tilde{\underline{v}}_2 = \bar{\underline{v}}_2 - \gamma \underline{v}_1$ such that $\underline{v}_1^T \underline{M}\, \tilde{\underline{v}}_2 = 0$

$\underline{v}_1^T \underline{M}\, \tilde{\underline{v}}_2 = \underline{v}_1^T \underline{M}\, \bar{\underline{v}}_2 - \gamma = 0 \quad \therefore \gamma = \underline{v}_1^T \underline{M}\, \bar{\underline{v}}_2 = \frac{\sqrt{2}}{6}$

$$\rightarrow \tilde{\underline{v}}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \frac{\sqrt{2}}{3} \cdot \frac{1}{9} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 8 \\ -10 \\ 8 \end{bmatrix}$$

Normalizing $\tilde{\underline{v}}_2$ to obtain $\underline{v}_2$ we have

$$\underline{v}_2 = \frac{1}{3\sqrt{10}} \begin{bmatrix} 4 \\ -5 \\ 4 \end{bmatrix}$$

Note that the two vectors $\underline{v}_1$ and $\underline{v}_2$ are M-orthogonal but are not eigenvectors because they do not satisfy

$$\underline{K}\, \underline{v}_i = \lambda \underline{M}\, \underline{v}_i \quad \text{where } i = 1, 2 .$$

10.4 $\underline{K}\,\underline{\Phi} = \lambda \underline{M}\,\underline{\Phi}$ where $\underline{K} = \begin{bmatrix} 6 & -1 \\ -1 & 4 \end{bmatrix}$, $\underline{M} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$

$\underline{M}\,\underline{\Phi} = (\frac{1}{\lambda})\underline{K}\,\underline{\Phi}$, $p(\lambda) = \det\left(\underline{M} - \frac{1}{\lambda}\underline{K}\right) = \frac{1}{\lambda}\left(\frac{23}{\lambda} - 8\right)$, $\lambda_1 = \frac{23}{8}$, $\lambda_2 = \infty$.

For λ_1, $\frac{1}{23}\begin{bmatrix} -2 & 8 \\ 8 & -32 \end{bmatrix}\underline{\phi}_1 = \underline{0}$ with $\underline{\Phi}_1^T \underline{M}\,\underline{\Phi}_1 = 1$ $\therefore$ $\underline{\Phi}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/4\sqrt{2} \end{bmatrix}$

For λ_2, we see $\underline{\Phi}_2^T = [\,0 \quad 1\,]$

Now applying a shift $\rho = 3$ on $\underline{K}$,

$\hat{\underline{K}}\,\underline{\Psi} = \mu \underline{M}\,\underline{\Psi}$ where $\hat{\underline{K}} = \underline{K} - \rho\underline{M} = \begin{bmatrix} 0 & -1 \\ -1 & 4 \end{bmatrix}$

$\underline{M}\,\underline{\Psi} = (\frac{1}{\mu})\hat{\underline{K}}\,\underline{\Psi}$, $p(\mu) = \det\left(\underline{M} - \frac{1}{\mu}\hat{\underline{K}}\right) = -\frac{1}{\mu}\left(8 + \frac{1}{\mu}\right) = 0$

$\therefore$ $\mu_1 = -\frac{1}{8}$, $\mu_2 = \infty$

Hence the eigenvalues have decreased by $\rho = 3$, while the eigenvectors are the same.

10.5

$$\underline{M} = \underline{L}\,\underline{D}\,\underline{L}^T = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 & & \\ & 2 & \\ & & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ & 1 & \frac{1}{2} \\ & & 1 \end{bmatrix}$$

$$\tilde{\underline{L}}_M = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & & \\ & \sqrt{2} & \\ & & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \sqrt{2} & & \\ 0 & \sqrt{2} & \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\Rightarrow \tilde{\underline{K}} = \tilde{\underline{L}}_M^{-1}\,\underline{K}\,\underline{L}_M^{-T} = \begin{bmatrix} 3 & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & 2 & -3 \\ \frac{1}{2} & -3 & 8 \end{bmatrix}$$

$$\Rightarrow \tilde{\underline{K}}\,\tilde{\underline{\Phi}} = \lambda\,\tilde{\underline{\Phi}}$$ (done by Cholesky factorization)

10.6 (a) $\underline{\Phi} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $\underline{\Lambda} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$

$$\underline{K} = \underline{\Phi}\, \underline{\Lambda}\, \underline{\Phi}^T = \frac{1}{2} \begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix}$$

The matrix $\underline{K}$ obtained here is unique.

(b) In this case, we have the following equations:

$$\underline{\Phi}^T \underline{M}\, \underline{\Phi} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} , \quad \underline{\Phi}^T \underline{K}\, \underline{\Phi} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

These equations form a system of 6 equations to be solved for the unknowns k_{11}, k_{12}, k_{22}, m_{11}, m_{12}, m_{22}.

The matrices $\underline{K}$ and $\underline{M}$ are obtained after solving these equations:

$$\underline{K} = \begin{bmatrix} 4 & -2 \\ -2 & 3 \end{bmatrix} ; \qquad \underline{M} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} ,$$

and they are also unique.

10.7
$$\begin{bmatrix} 6 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix} \underline{\phi} = \lambda \begin{bmatrix} 0 & & \\ & 2 & \\ & & 1 \end{bmatrix} \underline{\phi}$$

(i) $\underline{K}_a \underline{\phi}_a = \lambda \underline{M}_a \underline{\phi}_a$ where $\underline{K}_a = \underline{K}_{aa} - \underline{K}_{ac} \underline{K}_{cc}^{-1} \underline{K}_{ca}$

Arranging columns and rows

$$\begin{bmatrix} 4 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 6 \end{bmatrix} \begin{bmatrix} \underline{\phi}_a \\ \\ \underline{\phi}_c \end{bmatrix} = \lambda \begin{bmatrix} 2 & & \\ & 1 & \\ & & 0 \end{bmatrix} \begin{bmatrix} \underline{\phi}_a \\ \\ \underline{\phi}_c \end{bmatrix}$$

$$\underline{K}_a = \begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \end{bmatrix} \left[\frac{1}{6} \right] [-1 \quad 0] = \begin{bmatrix} 23/6 & -1 \\ -1 & 2 \end{bmatrix}$$

Next we solve the equations

$$\begin{bmatrix} 6 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix} [\underline{V}_1 \quad \underline{V}_2] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \quad \underline{V}_1^T = \left[\frac{1}{20} \quad \frac{3}{10} \quad \frac{3}{20} \right], \quad \underline{V}_2^T = \left[\frac{1}{40} \quad \frac{3}{20} \quad \frac{23}{40} \right]$$

Hence

$$\underline{F}_a = \begin{bmatrix} \frac{3}{10} & \frac{3}{20} \\ \frac{3}{20} & \frac{23}{40} \end{bmatrix}, \quad \text{and} \quad \underline{K}_a = \underline{F}_a^{-1} = \begin{bmatrix} 23/6 & -1 \\ -1 & 2 \end{bmatrix}$$

10.8

$$\begin{bmatrix} 6 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix} \underline{\phi} = \lambda \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \underline{\phi} \quad , \quad \underline{\psi}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} , \underline{\psi}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\underline{\Psi} = [\underline{\psi}_1 \ \underline{\psi}_2] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} , \quad \underline{\tilde{K}} = \underline{\Psi}^T \underline{K} \underline{\Psi} = \begin{bmatrix} 8 & 4 \\ 4 & 16 \end{bmatrix} , \underline{\tilde{M}} = \underline{\Psi}^T \underline{M} \underline{\Psi} = \begin{bmatrix} 7 & 1 \\ 1 & 3 \end{bmatrix}$$

We now solve the eigenproblem $\underline{\tilde{K}} \underline{x} = \rho \underline{\tilde{M}} \underline{x}$ where $\underline{x}^T = [x_1 \ x_2]$ and obtain

$$\rho_1 = \frac{16 - 2\sqrt{29}}{5} = 1.0459$$

For ρ_1, $(\underline{\tilde{K}} - \rho_1 \underline{\tilde{M}}) \underline{x}_1 = \underline{0}$ with $\underline{x}_1^T \underline{\tilde{M}} \underline{x}_1 = 1$

$$\therefore \ \underline{x}_1^T = [0.45623 \quad -0.10478]$$

$$\rightarrow \ \underline{\bar{\phi}}_1 = \underline{\Psi} \underline{x}_1 = \begin{bmatrix} 0.35145 \\ 0.56100 \\ 0.35145 \end{bmatrix}$$

10.9 Exact eigenpairs: $(\lambda_1, \underline{\phi}_1)$, $(\lambda_2, \underline{\phi}_2)$

$$\underline{K}\underline{\phi}_1 = \lambda_1 \underline{M}\,\underline{\phi}_1 \quad , \quad \underline{K}\underline{\phi}_2 = \lambda_2 \underline{M}\,\underline{\phi}_2$$

Let $\underline{\Psi} = [\,\underline{\psi}_1 \;\; \underline{\psi}_2\,]$ where $\underline{\psi}_1 = \underline{\phi}_1 + 2\underline{\phi}_2$, $\underline{\psi}_2 = 3\underline{\phi}_1 - \underline{\phi}_2$

Then consider the eigenproblem $\tilde{\underline{K}}\,\underline{x} = \mu \tilde{\underline{M}}\,\underline{x}$

where $\quad \tilde{\underline{K}} = \underline{\Psi}^T \underline{K}\,\underline{\Psi} \quad$ and $\quad \tilde{\underline{M}} = \underline{\Psi}^T \underline{M}\,\underline{\Psi}$

$$\tilde{\underline{K}} = \begin{bmatrix} \underline{\psi}_1^T \underline{K}\,\underline{\psi}_1 & \underline{\psi}_1^T \underline{K}\,\underline{\psi}_2 \\ \underline{\psi}_2^T \underline{K}\,\underline{\psi}_1 & \underline{\psi}_2^T \underline{K}\,\underline{\psi}_2 \end{bmatrix} \quad , \quad \tilde{\underline{M}} = \begin{bmatrix} \underline{\psi}_1^T \underline{M}\,\underline{\psi}_1 & \underline{\psi}_1^T \underline{M}\,\underline{\psi}_2 \\ \underline{\psi}_2^T \underline{M}\,\underline{\psi}_1 & \underline{\psi}_2^T \underline{M}\,\underline{\psi}_2 \end{bmatrix}$$

$$\begin{aligned} \underline{\psi}_1^T \underline{K}\,\underline{\psi}_1 &= \left(\underline{\phi}_1^T + 2\underline{\phi}_2^T\right)\underline{K}\left(\underline{\phi}_1 + 2\underline{\phi}_2\right) \\ &= \underline{\phi}_1^T \underline{K}\underline{\phi}_1 + 2\left(\underline{\phi}_1^T \underline{K}\underline{\phi}_2 + \underline{\phi}_2^T \underline{K}\underline{\phi}_1\right) + 4\,\underline{\phi}_2^T \underline{K}\underline{\phi}_2 \\ &= \lambda_1 + 4\lambda_2 \end{aligned}$$

Similarly, $\quad \underline{\psi}_2^T \underline{K}\,\underline{\psi}_1 = \underline{\psi}_1^T \underline{K}\,\underline{\psi}_2 = 3\lambda_1 - 2\lambda_2$

$$\underline{\psi}_2^T \underline{K}\,\underline{\psi}_2 = 9\lambda_1 + \lambda_2$$

$$\begin{aligned} \underline{\psi}_1^T \underline{M}\,\underline{\psi}_1 &= \left(\underline{\phi}_1^T + 2\underline{\phi}_2^T\right)\underline{M}\left(\underline{\phi}_1 + 2\underline{\phi}_2\right) = \\ &= \underline{\phi}_1^T \underline{M}\,\underline{\phi}_1 + 2\left(\underline{\phi}_1^T \underline{M}\underline{\phi}_2 + \underline{\phi}_2^T \underline{M}\,\underline{\phi}_1\right) + 4\,\underline{\phi}_2^T \underline{M}\,\underline{\phi}_2 \\ &= 1 + 0 + 4 = 5 \end{aligned}$$

Similarly, $\quad \underline{\psi}_1^T \underline{M}\,\underline{\psi}_2 = \underline{\psi}_2^T \underline{M}\,\underline{\psi}_1 = 1 \quad , \quad \underline{\psi}_2^T \underline{M}\,\underline{\psi}_2 = 10$

$$\therefore \quad \tilde{\underline{K}} = \begin{bmatrix} \lambda_1 + 4\lambda_2 & 3\lambda_1 - 2\lambda_2 \\ 3\lambda_1 - 2\lambda_2 & 9\lambda_1 + \lambda_2 \end{bmatrix} , \quad \tilde{\underline{M}} = \begin{bmatrix} 5 & 1 \\ 1 & 10 \end{bmatrix}$$

10.9

$$\det(\tilde{\underline{K}} - \mu \tilde{\underline{M}}) = 49 \{ \mu^2 - (\lambda_1 + \lambda_2)\mu + \lambda_1 \lambda_2 \}$$
$$= 49 \; (\mu - \lambda_1)(\mu - \lambda_2) = 0$$

$$\therefore \; \mu_1 = \lambda_1 \; , \; \mu_2 = \lambda_2 \qquad \text{——— } ①$$

For $\mu_1 = \lambda_1$, $(\tilde{\underline{K}} - \mu_1 \tilde{\underline{M}}) \underline{x}_1 = \underline{0}$ with $\underline{x}_1^T \underline{M} \underline{x}_1 = 1$

$$\therefore \; x_1^T = \left[\frac{1}{7} \quad \frac{2}{7} \right]$$

Similarly for $\mu_2 = \lambda_2$, $x_2^T = \left[-\frac{3}{7} \quad \frac{1}{7} \right]$

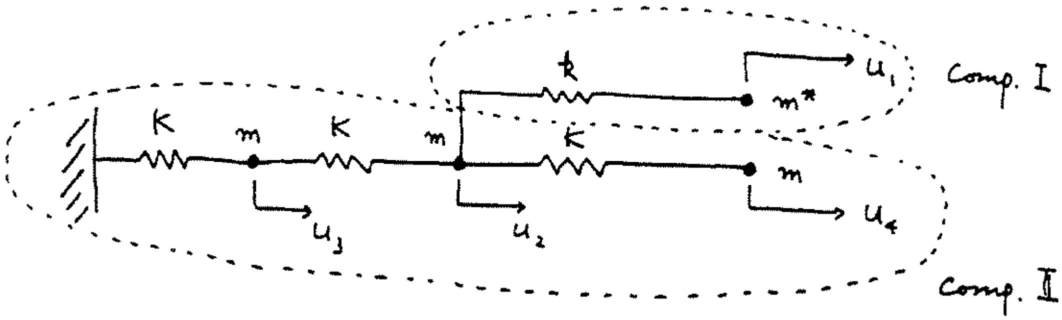
Hence $\bar{\underline{\phi}}_1 = \frac{1}{7} \underline{\psi}_1 + \frac{2}{7} \underline{\psi}_2 = \frac{1}{7}(\underline{\phi}_1 + 2\underline{\phi}_2) + \frac{2}{7}(3\underline{\phi}_1 - \underline{\phi}_2) = \underline{\phi}_1$

$$\bar{\underline{\phi}}_2 = -\frac{3}{7} \underline{\psi}_1 + \frac{1}{7} \underline{\psi}_2 = -\frac{3}{7}(\underline{\phi}_1 + 2\underline{\phi}_2) + \frac{1}{7}(3\underline{\phi}_1 - \underline{\phi}_2) = -\underline{\phi}_2$$

$$\text{or} \quad \bar{\underline{\phi}}_2 = \underline{\phi}_2 \qquad \text{—— } ②$$

From ① and ②, the exact eigenpairs are obtained.

10.10



(a)

$$\underset{\sim}{K} = \begin{bmatrix} k & -k & 0 & 0 \\ -k & 2K+k & -K & -K \\ 0 & -K & 2K & 0 \\ 0 & -K & 0 & K \end{bmatrix} = \begin{bmatrix} 1 & -1 & & \\ -1 & 21 & -10 & -10 \\ & -10 & 20 & \\ & -10 & & 10 \end{bmatrix}$$

$u_1 \quad u_2 \quad u_3 \quad u_4$

$$\underset{\sim}{M} = \begin{bmatrix} m^* & & & \\ & m & & \\ & & m & \\ & & & m \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 2 & & \\ & & 2 & \\ & & & 2 \end{bmatrix}$$

$$\underset{\sim}{K}\underset{\sim}{\Phi} = \lambda \underset{\sim}{M}\underset{\sim}{\Phi}, \quad \det(\underset{\sim}{K} - \lambda \underset{\sim}{M}) = 4(250 - 600\lambda + 365\lambda^2 - 53\lambda^3 + 2\lambda^4)$$

$$\therefore \quad \lambda_1 = 0.65008, \quad \underset{\sim}{\Phi}_1 = \begin{bmatrix} 0.73695 \\ 0.25788 \\ 0.13790 \\ 0.29641 \end{bmatrix}$$

(b) $\underset{\sim}{K}_{I} = [1], \quad \underset{\sim}{M}_{I} = [1] \Rightarrow \lambda_1^{I} = 1, \quad \underset{\sim}{\phi}_1^{I} = [1]$

$$\underset{\sim}{K}_{II} = \begin{bmatrix} 20 & 0 \\ 0 & 10 \end{bmatrix}, \quad \underset{\sim}{M}_{II} = \begin{bmatrix} 2 & \\ & 2 \end{bmatrix} \quad \Rightarrow$$

10.10

$$\lambda_1^{II} = 5, \qquad \underline{\phi}_1^{II} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \end{bmatrix};$$

$$\lambda_2^{II} = 10, \qquad \underline{\phi}_2^{II} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \end{bmatrix};$$

Then, $\underline{R}$ in (10.95) is:
$$\underline{R} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

Now, performing a Ritz analysis given in (10.79) to (10.84),

$$\underline{K}\,\underline{\Psi} = \underline{R} \quad \Rightarrow \quad \underline{\Psi} = \begin{bmatrix} 1.3414 & 0.2 \\ 0.3414 & 0.2 \\ 0.1707 & 0.1 \\ 0.4121 & 0.2 \end{bmatrix}$$

$$\tilde{\underline{K}} = \underline{\Psi}^T \underline{K}\,\underline{\Psi} = \begin{bmatrix} 1.633 & 0.341 \\ 0.341 & 0.2 \end{bmatrix}; \quad \tilde{\underline{M}} = \underline{\Psi}^T \underline{M}\,\underline{\Psi} = \begin{bmatrix} 2.43 & 0.604 \\ 0.604 & 0.22 \end{bmatrix}$$

Solving $\tilde{\underline{K}}\,\underline{x} = \rho\,\tilde{\underline{M}}\,\underline{x}$, we obtain the approximation to the smallest eigenvalue as: $\rho = 0.6519$

10.11 Consider the eigenproblem $\underline{A}\,\underline{x} = \lambda\,\underline{x}$ and the inverse iteration $\underline{K}\,\bar{\underline{\Phi}} = \underline{M}\,\hat{\underline{\Phi}}$. Let $\underline{M} = \underline{S}\,\underline{S}^T$ and $\underline{K} = \underline{S}\,\underline{A}\,\underline{S}^T$, then we have $\underline{A}\,\underline{x} = \underline{S}^T\hat{\underline{\Phi}}$ when we define $\underline{x} = \underline{S}^T\bar{\underline{\Phi}}$.

$$\underline{x}^T\underline{x} = (\underline{S}^T\bar{\underline{\Phi}})^T(\underline{S}^T\bar{\underline{\Phi}}) = \bar{\underline{\Phi}}^T\underline{S}\,\underline{S}^T\bar{\underline{\Phi}} = \bar{\underline{\Phi}}^T\underline{M}\,\bar{\underline{\Phi}}$$

$$\underline{x}^T\underline{A}\,\underline{x} = (\underline{S}^T\bar{\underline{\Phi}})^T(\underline{S}^T\hat{\underline{\Phi}}) = \bar{\underline{\Phi}}^T\underline{S}\,\underline{S}^T\hat{\underline{\Phi}} = \bar{\underline{\Phi}}^T\underline{M}\,\hat{\underline{\Phi}} = \bar{\underline{\Phi}}^T\underline{K}\,\bar{\underline{\Phi}}$$

$$\underline{x}^T\underline{A}^T\underline{A}\,\underline{x} = (\underline{S}^T\hat{\underline{\Phi}})^T(\underline{S}^T\hat{\underline{\Phi}}) = \hat{\underline{\Phi}}^T\underline{S}\,\underline{S}^T\hat{\underline{\Phi}} = \hat{\underline{\Phi}}^T\underline{M}\,\hat{\underline{\Phi}}$$

$$\frac{\underline{x}^T\underline{A}\,\underline{x}}{\underline{x}^T\underline{x}} = \frac{\bar{\underline{\Phi}}^T\underline{K}\,\bar{\underline{\Phi}}}{\bar{\underline{\Phi}}^T\underline{M}\,\bar{\underline{\Phi}}} = \rho(\bar{\underline{\Phi}}), \qquad \frac{\underline{x}^T\underline{A}^T\underline{A}\,\underline{x}}{\underline{x}^T\underline{x}} = \frac{\hat{\underline{\Phi}}^T\underline{M}\,\hat{\underline{\Phi}}}{\bar{\underline{\Phi}}^T\underline{M}\,\bar{\underline{\Phi}}}$$

$$\therefore \min_i |\lambda_i - \rho(\bar{\underline{\Phi}})| \le \left[\frac{\hat{\underline{\Phi}}^T\underline{M}\,\hat{\underline{\Phi}}}{\bar{\underline{\Phi}}^T\underline{M}\,\bar{\underline{\Phi}}} - \{\rho(\bar{\underline{\Phi}})\}^2\right]^{1/2}$$

Now from $\dfrac{\hat{\underline{\Phi}}^T\underline{M}\,\hat{\underline{\Phi}}}{\bar{\underline{\Phi}}^T\underline{M}\,\bar{\underline{\Phi}}} \ge \lambda_i^2$ we have

$$\min_i \left|\frac{\lambda_i - \rho(\bar{\underline{\Phi}})}{\lambda_i}\right| \le \left[1 - \frac{\{\rho(\bar{\underline{\Phi}})\}^2}{\dfrac{\hat{\underline{\Phi}}^T\underline{M}\,\hat{\underline{\Phi}}}{\bar{\underline{\Phi}}^T\underline{M}\,\bar{\underline{\Phi}}}}\right]^{1/2}$$

10.12 Let $\hat{\underline{\Phi}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, then using $\underline{K}\,\bar{\underline{\Phi}} = \underline{M}\,\hat{\underline{\Phi}}$, $\bar{\underline{\Phi}} = \begin{bmatrix} 11/20 \\ 13/10 \\ 33/20 \end{bmatrix}$

and $$\rho(\bar{\underline{\Phi}}) = \frac{\bar{\underline{\Phi}}^T \underline{K}\,\bar{\underline{\Phi}}}{\bar{\underline{\Phi}}^T \underline{M}\,\bar{\underline{\Phi}}} = \frac{40}{53} = 0.75472$$

Hence $$\underline{r}_M = \underline{K}\bar{\underline{\Phi}} - \bar{\lambda}\underline{M}\bar{\underline{\Phi}} = \begin{bmatrix} 62/53 \\ -11/53 \\ -12/53 \end{bmatrix};$$

$$\underline{r} = \underline{S}^{-1}\underline{r}_M = \begin{bmatrix} 0.827 \\ -0.147 \\ -0.173 \end{bmatrix},$$ where $\underline{S}^{-1} \equiv \tilde{\underline{L}}_M^{-1}$ in Excercise 10.5.

$\|\underline{r}_M\| = 0.86$. Using (10.101), we must have $\min_i |\lambda_i - \bar{\lambda}| \leq 0.86$

From eq. (10.106),

$$\min_i |\lambda_i - 0.75472| \leq \left[0.63651 - (0.75472)^2\right]^{1/2} = 0.25867$$

$$\left(\because \; (\hat{\underline{\Phi}}^T\underline{M}\hat{\underline{\Phi}})/(\bar{\underline{\Phi}}^T\underline{M}\bar{\underline{\Phi}}) = 0.63651 \right)$$

Now evaluate the error measure (10.108)

$$\epsilon = \frac{\|\underline{K}\bar{\underline{\Phi}} - \bar{\lambda}\underline{M}\bar{\underline{\Phi}}\|_2}{\|\underline{K}\bar{\underline{\Phi}}\|_2} = \frac{1.2095}{4.1231} = 0.29334$$

The exact error is: $\min_i |\lambda_i - \lambda| = 0.03$

11.1
$$\begin{bmatrix} 6 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 12 \end{bmatrix} \underline{\phi} = \lambda \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \underline{\phi} \quad , \quad \underline{x}_1^T = [1 \quad 1 \quad 1]$$

(a)
$$\underline{y}_1 = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}, \quad \underline{\bar{x}}_2 = \begin{bmatrix} 11/20 \\ 13/10 \\ 33/20 \end{bmatrix}, \quad \underline{\bar{y}}_2 = \begin{bmatrix} 11/10 \\ 17/4 \\ 59/20 \end{bmatrix}$$

$$\rho(\underline{\bar{x}}_2) = \frac{\underline{\bar{x}}_2^T \underline{y}_1}{\underline{\bar{x}}_2^T \underline{\bar{y}}_2} = 0.75472, \quad \underline{y}_2 = \begin{bmatrix} 0.33170 \\ 1.28157 \\ 0.88956 \end{bmatrix}$$

$$\underline{\bar{x}}_3 = \begin{bmatrix} 0.14437 \\ 0.53451 \\ 0.71204 \end{bmatrix}, \quad \underline{\bar{y}}_3 = \begin{bmatrix} 0.28874 \\ 1.7810 \\ 1.2466 \end{bmatrix} \Rightarrow \rho(x_3) = 0.72625 \approx \lambda_1$$

(b)
$$\underline{y}_1 = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}, \quad \underline{\bar{x}}_2 = \begin{bmatrix} 5/2 \\ 1 \\ 0 \end{bmatrix}, \quad \underline{\bar{y}}_2 = \begin{bmatrix} 14 \\ 3/2 \\ -1 \end{bmatrix}$$

$$\rho(\underline{\bar{x}}_2) = \frac{\underline{\bar{x}}_2^T \underline{\bar{y}}_2}{\underline{\bar{x}}_2^T \underline{y}_1} = 2.5172, \quad \underline{y}_2 = \begin{bmatrix} 3.6766 \\ 0.39392 \\ -0.26261 \end{bmatrix}$$

$$\underline{\bar{x}}_3 = \begin{bmatrix} 1.8383 \\ 0.65653 \\ -0.91915 \end{bmatrix}, \quad \underline{\bar{y}}_3 = \begin{bmatrix} 10.373 \\ 1.7070 \\ -2.4948 \end{bmatrix}, \quad \rho(\underline{\bar{x}}_3) = 3.0974$$

Note that in this case $\rho(\bar{\underline{x}}_3)$ is not a good approximation to λ_3. Hence, more iterations are needed to obtain a reasonable approximation. After 7 iterations one finds : $\rho(\bar{\underline{x}}_8) = 9.3083 \sim \lambda_3 = 9.3104$

11.2

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 6 & -1 \\ 0 & -1 & 8 \end{bmatrix} \underline{\phi} = \lambda \begin{bmatrix} 1 & & \\ & 1/2 & \\ & & 2 \end{bmatrix} \underline{\phi} \quad , \quad \underline{x}_1^T = [1 \quad 1 \quad 1]$$

(a)

$$\underline{y}_1 = \begin{bmatrix} 1 \\ 1/2 \\ 2 \end{bmatrix} , \quad \underline{\bar{x}}_2 = \begin{bmatrix} 53/86 \\ 10/43 \\ 12/43 \end{bmatrix} , \quad \underline{\bar{y}}_2 = \begin{bmatrix} 53/86 \\ 5/43 \\ 24/43 \end{bmatrix}$$

$$\rho(\underline{\bar{x}}_2) = 2.2942 , \qquad \underline{y}_2 = \begin{bmatrix} 0.82163 \\ 0.15503 \\ 0.74412 \end{bmatrix}$$

$$\underline{\bar{x}}_3 = \begin{bmatrix} 0.47211 \\ 0.12258 \\ 0.10834 \end{bmatrix} , \quad \underline{\bar{y}}_3 = \begin{bmatrix} 0.47211 \\ 0.061289 \\ 0.21667 \end{bmatrix}$$

$$\rho(\underline{\bar{x}}_3) = 1.9203 \sim \lambda_1 , \qquad \underline{y}_3 = \begin{bmatrix} 0.93699 \\ 0.12164 \\ 0.43003 \end{bmatrix}$$

(b)

$$\underline{y}_1 = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} , \quad \underline{\bar{x}}_2 = \begin{bmatrix} 1 \\ 8 \\ 7/2 \end{bmatrix} , \quad \underline{\bar{y}}_2 = \begin{bmatrix} -6 \\ 87/2 \\ 20 \end{bmatrix}$$

$$\rho(\underline{\bar{x}}_2) = 7.1652 , \qquad \underline{y}_2 = \begin{bmatrix} -0.79126 \\ 5.7366 \\ 2.6375 \end{bmatrix}$$

$$\underline{\bar{x}}_3 = \begin{bmatrix} -0.79126 \\ 11.473 \\ 1.3188 \end{bmatrix} , \quad \underline{\bar{y}}_3 = \begin{bmatrix} -13.056 \\ 68.312 \\ -0.92313 \end{bmatrix} , \quad \rho(\underline{\bar{x}}_3) = 11.339 \sim \lambda_3$$

11.3 $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} \underline{\phi} = \lambda \underline{\phi}$; $\lambda_1 = 1$, $\underline{\phi}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\lambda_2 = 2$, $\underline{\phi}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$\underline{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\quad \tilde{\underline{x}}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \alpha_1 \underline{\phi}_1 - \alpha_2 \underline{\phi}_2$

Using (11.64), $\alpha_i = \underline{\phi}_i^T \underline{x}_1$ $\quad \therefore \alpha_1 = 1/\sqrt{3}$, $\alpha_2 = 0$

$\therefore \tilde{\underline{x}}_1^T = \begin{bmatrix} \frac{2}{3} & \frac{4}{3} & \frac{2}{3} \end{bmatrix}$

Using $\gamma \tilde{\underline{x}}_1^T \tilde{\underline{x}}_1 = 1$, $\underline{\phi}_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

To check that $\underline{\phi}_3$ is the eigenvector and evaluate λ_3.

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \frac{1}{\sqrt{6}} = \frac{1}{\sqrt{6}} \begin{bmatrix} 4 \\ 8 \\ 4 \end{bmatrix} = 4 \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Hence, $\lambda_3 = 4$.

11.4 $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix} \underline{\phi} = \lambda \begin{bmatrix} 1/2 & & \\ & 1 & \\ & & 1/2 \end{bmatrix} \underline{\phi}$

with $\underline{\phi}_1^T = \frac{1}{\sqrt{2}} [\, 1 \quad 1 \quad 1 \,]$, $\underline{\phi}_3^T = \frac{1}{\sqrt{2}} [\, 1 \quad -1 \quad 1 \,]$

Let $\underline{x}_1^T = [-1 \quad 1 \quad 1]$ where we note that we must start with a vector $\underline{x}_1$ that is not in the space spanned by $\underline{\phi}_1$ and $\underline{\phi}_3$.

$\tilde{\underline{x}}_1 = \underline{x}_1 - \alpha_1 \underline{\phi}_1 - \alpha_3 \underline{\phi}_3$ where $\alpha_i = \underline{\phi}_i^T \underline{M}\, \underline{x}_1 \quad i = 1, 3$

$\therefore \; \alpha_1 = \sqrt{2}/2, \quad \alpha_3 = -\sqrt{2}/2 , \qquad \tilde{\underline{x}}_1^T = [-1 \quad 0 \quad 1]$

As $\tilde{\underline{x}}_1$ is a multiple of $\underline{\phi}_2$, using $\tilde{\underline{x}}_1^T \underline{M}\, \tilde{\underline{x}}_1 = 1$

$\therefore \; \underline{\phi}_2^T = \quad [-1 \quad 0 \quad 1]$

From $\underline{\phi}_2^T \underline{K}\, \underline{\phi}_2 = \lambda_2$, $\lambda_2 = 4$

The other eigenvalues are:

$\lambda_1 = \underline{\phi}_1^T \underline{K}\, \underline{\phi}_1 = 2$

$\lambda_3 = \underline{\phi}_3^T \underline{K}\, \underline{\phi}_3 = 6$

11.5 1st case

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \underline{\phi} = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underline{\phi}$$

$$\cos\theta = \frac{1}{\sqrt{2}}, \quad \sin\theta = \frac{1}{\sqrt{2}} \qquad \underline{P}_1 = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\underline{P}_1^T \underline{K} \underline{P}_1 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = \underline{\Lambda} = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}$$

$$\underline{\Phi} = \underline{P}_1 = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

2nd case

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \underline{\phi} = \lambda \begin{bmatrix} 1 & \\ & 2 \end{bmatrix} \underline{\phi}$$

$$\bar{k}_{11}^{(1)} = k_{11}^{(1)} m_{12}^{(1)} - m_{11}^{(1)} k_{12}^{(1)} = 1, \quad \bar{k}_{22}^{(1)} = k_{22}^{(1)} m_{12}^{(1)} - m_{22}^{(1)} k_{12}^{(1)} = 2$$

$$\bar{k}^{(1)} = k_{11}^{(1)} m_{22}^{(1)} - k_{22}^{(1)} m_{11}^{(1)} = 3, \quad x = \frac{3+\sqrt{17}}{2}, \quad \gamma = -\frac{\sqrt{17}-3}{4}, \quad \alpha = \frac{\sqrt{17}-3}{2}$$

$$\underline{P}_1 = \begin{bmatrix} 1 & \alpha \\ \gamma & 1 \end{bmatrix} = \begin{bmatrix} 1 & (\sqrt{17}-3)/2 \\ -(\sqrt{17}-3)/4 & 1 \end{bmatrix}$$

$$\underline{P}_1^T \underline{K} \underline{P}_1 = \begin{bmatrix} (17+\sqrt{17})/8 & 0 \\ 0 & 17-4\sqrt{17} \end{bmatrix}, \quad \underline{P}_1^T \underline{M} \underline{P}_1 = \begin{bmatrix} (17-3\sqrt{17})/4 & 0 \\ 0 & (17-3\sqrt{17})/2 \end{bmatrix}$$

$$\therefore \quad \underline{\Lambda} = \begin{bmatrix} 2(17-4\sqrt{17})/(17-3\sqrt{17}) & \\ 0 & (17+\sqrt{17})/(34-6\sqrt{17}) \end{bmatrix} = \begin{bmatrix} 0.21922 & \\ 0 & 2.2808 \end{bmatrix}$$

$$\underline{\Phi} = \begin{bmatrix} 0.36805 & 0.82841 \\ 0.65719 & -0.26100 \end{bmatrix}$$

11.6 $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \underline{\Phi} = \lambda \begin{bmatrix} 1 & & \\ & 2 & \\ & & 3/2 \end{bmatrix} \underline{\Phi}$

For $i=1,\ j=2$ $\quad \bar{k}_{11}^{(1)} = 1,\quad \bar{k}_{22}^{(1)} = 2,\quad \bar{k}^{(1)} = 2$

$$x = 1+\sqrt{3},\quad \gamma = -\frac{\sqrt{3}-1}{2},\quad \alpha = \sqrt{3}-1$$

$$\underline{P}_1 = \begin{bmatrix} 1 & \alpha & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \sqrt{3}-1 & 0 \\ -(\sqrt{3}-1)/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{P}_1^T \underline{K} \underline{P}_1 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 12-6\sqrt{3} & 0 \\ 0 & 0 & 3 \end{bmatrix},\quad \underline{P}_1^T \underline{M} \underline{P}_1 = \begin{bmatrix} 3-\sqrt{3} & 0 & 0 \\ 0 & 6-2\sqrt{3} & 0 \\ 0 & 0 & 3/2 \end{bmatrix}$$

$$\longrightarrow \quad \underline{\Lambda} = \begin{bmatrix} \frac{3-\sqrt{3}}{2} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{3+\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} 0.63398 & & \\ & 2 & \\ & & 2.3660 \end{bmatrix}$$

$$\underline{\Phi} = \begin{bmatrix} 0.45970 & 0 & 0.88807 \\ 0.62796 & 0 & -0.32506 \\ 0 & 0.81650 & 0 \end{bmatrix}$$

11.7

We start with eq.'s (11.86) and (11.87) in the textbook:

$$\begin{cases} \alpha k_{ii}^{(k)} + (1+\alpha\gamma) k_{ij}^{(k)} + \gamma k_{jj}^{(k)} = 0 & \text{--- ①} \\ \alpha m_{ii}^{(k)} + (1+\alpha\gamma) m_{ij}^{(k)} + \gamma m_{jj}^{(k)} = 0 & \text{--- ②} \end{cases}$$

From ① and ②

$$\gamma = -\frac{\alpha k_{ii}^{(k)} + k_{ij}^{(k)}}{\alpha k_{ij}^{(k)} + k_{jj}^{(k)}} = -\frac{\alpha m_{ii}^{(k)} + m_{ij}^{(k)}}{\alpha m_{ij}^{(k)} + m_{jj}^{(k)}} \quad \text{--- ③}$$

$$\therefore \quad \alpha^2 \left[k_{ii}^{(k)} m_{ij}^{(k)} - k_{ij}^{(k)} m_{ii}^{(k)} \right] + \alpha \left[k_{ii}^{(k)} m_{jj}^{(k)} - k_{jj}^{(k)} m_{ii}^{(k)} \right] + \left[k_{ij}^{(k)} m_{jj}^{(k)} - k_{jj}^{(k)} m_{ij}^{(k)} \right] = 0$$

or

$$\bar{k}_{ii}^{(k)} \alpha^2 + \bar{k}^{(k)} \alpha - \bar{k}_{jj}^{(k)} = 0$$

where

$$\bar{k}_{ii}^{(k)} = k_{ii}^{(k)} m_{ij}^{(k)} - k_{ij}^{(k)} m_{ii}^{(k)}$$

$$\bar{k}^{(k)} = k_{ii}^{(k)} m_{jj}^{(k)} - k_{jj}^{(k)} m_{ii}^{(k)}$$

$$\bar{k}_{jj}^{(k)} = k_{jj}^{(k)} m_{ij}^{(k)} - k_{ij}^{(k)} m_{jj}^{(k)}$$

Now, multiplying this equation by $\left(-\frac{1}{\alpha^2}\right)$, we obtain

$$\alpha = \frac{\bar{k}_{jj}^{(k)}}{X} \quad \text{with} \quad X = \frac{\bar{k}^{(k)}}{2} + \operatorname{sgn}\left(\bar{k}^{(k)}\right) \sqrt{\left(\frac{\bar{k}^{(k)}}{2}\right)^2 + \bar{k}_{ii}^{(k)} \bar{k}_{jj}^{(k)}} \quad \text{--- ④}$$

Following the same procedure, we also obtain

$$\gamma = -\frac{\bar{k}_{ii}^{(k)}}{X}$$

11.8 $\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \underline{\Phi} = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \underline{\Phi}$

$$\underline{M} = \begin{bmatrix} 1 & \\ & 2 \end{bmatrix} = \underline{S}\,\underline{S}^T = \begin{bmatrix} 1 & \\ & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & \\ & \sqrt{2} \end{bmatrix}$$

Transforming the eigenproblem into a standard form we obtain

$$\tilde{\underline{K}}\,\tilde{\underline{\Phi}} = \lambda \tilde{\underline{\Phi}} \quad \text{where} \quad \tilde{\underline{K}} = \underline{S}^{-1}\underline{K}\,\underline{S}^{-T} = \begin{bmatrix} 2 & -1/\sqrt{2} \\ -1/\sqrt{2} & 1/2 \end{bmatrix}, \quad \tilde{\underline{\Phi}} = \underline{S}^T \underline{\Phi}$$

Now using the QR iteration

1st iteration $\sin\theta = -\frac{1}{3}, \quad \cos\theta = \frac{2\sqrt{2}}{3}$

$$\underline{P}_{2,1} = \begin{bmatrix} 2\sqrt{2}/3 & 1/3 \\ -1/3 & 2\sqrt{2}/3 \end{bmatrix}, \quad \underline{P}_{2,1}^T \tilde{\underline{K}} = \begin{bmatrix} 3/\sqrt{2} & -5/6 \\ 0 & 1/3\sqrt{2} \end{bmatrix} = \underline{R}_1$$

$$\underline{Q}_1 = \underline{P}_{2,1}, \quad \tilde{\underline{K}}_2 = \underline{R}_1\underline{Q}_1 = \begin{bmatrix} 2.2778 & -0.078567 \\ -0.078567 & 0.22222 \end{bmatrix}$$

2nd iteration $\sin\theta = -0.034473, \quad \cos\theta = 0.99941$

$$\underline{P}_{2,1} = \begin{bmatrix} 0.99941 & 0.034473 \\ -0.034473 & 0.99941 \end{bmatrix}$$

$$\underline{P}_{2,1}^T \tilde{\underline{K}}_2 = \begin{bmatrix} 2.2791 & -0.086181 \\ 0 & 0.21938 \end{bmatrix} = \underline{R}_2$$

$$\underline{Q}_2 = \underline{P}_{2,1}, \quad \underline{Q}_1\,\underline{Q}_2 = \begin{bmatrix} 0.93076 & 0.36564 \\ -0.36564 & 0.93076 \end{bmatrix}$$

$$\tilde{\underline{K}}_3 = \underline{R}_2\underline{Q}_2 = \begin{bmatrix} 2.2808 & -0.0075626 \\ -0.0075626 & 0.21925 \end{bmatrix}$$

11.8

Hence $\lambda_1 = 0.21925$ $\qquad \tilde{\underline{\phi}}_1 = \begin{bmatrix} 0.36564 \\ 0.93076 \end{bmatrix}$

$\lambda_2 = 2.2808$ $\qquad \tilde{\underline{\phi}}_2 = \begin{bmatrix} 0.93076 \\ -0.36564 \end{bmatrix}$

From $\tilde{\underline{\phi}} = \underline{S}^T \underline{\phi}$, $\quad \underline{\phi}_1 = \begin{bmatrix} 0.36564 \\ 0.65815 \end{bmatrix}, \quad \underline{\phi}_2 = \begin{bmatrix} 0.93076 \\ -0.25854 \end{bmatrix}$

<u>11.9</u>

$$\underline{k} = \begin{bmatrix} 6 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \qquad \underline{M} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Using (11.119): $\mu_{k+1} = \mu_k - \dfrac{p(\mu_k)}{p(\mu_k) - p(\mu_{k-1})}(\mu_k - \mu_{k-1})$

$\mu_1 = 0$ $\quad p(\mu_1) = 40$

$\mu_2 = 1$ $\quad p(\mu_2) = -9$

$\mu_3 = 0.8163$ $\quad p(\mu_3) = -3.3526$

$\mu_4 = 0.70725$ $\quad p(\mu_4) = 0.668$

$\mu_5 = 0.72537$ $\quad p(\mu_5) = -0.0358$

$\mu_6 = 0.72445$ $\Rightarrow$ $\lambda_1 = 0.72445$

$$p^{(1)}(\lambda) = \frac{p(\lambda)}{\lambda - \lambda_1}.$$

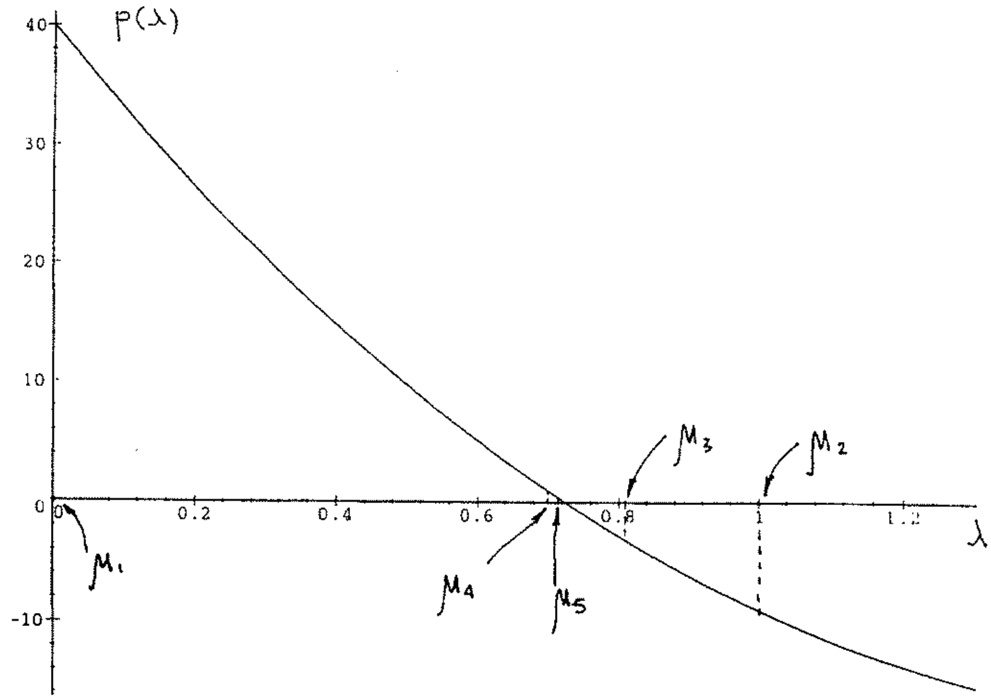
Let $\mu_1 = 2$ $\quad p^{(1)}(\mu_1) = -14.111$

$\mu_2 = 3$ $\quad p^{(1)}(\mu_2) = 0.4394$

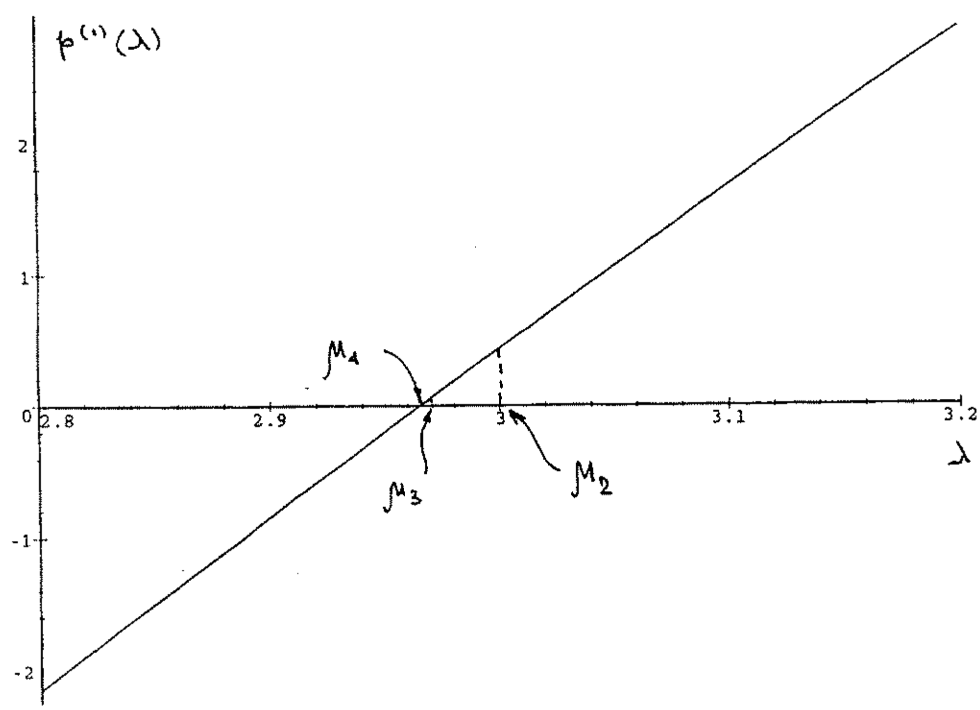
$\mu_3 = 2.9698$ $\quad p^{(1)}(\mu_3) = 0.058595$

$\mu_4 = 2.9652$ $\Rightarrow$ $\lambda_2 = 2.9652$

11.9

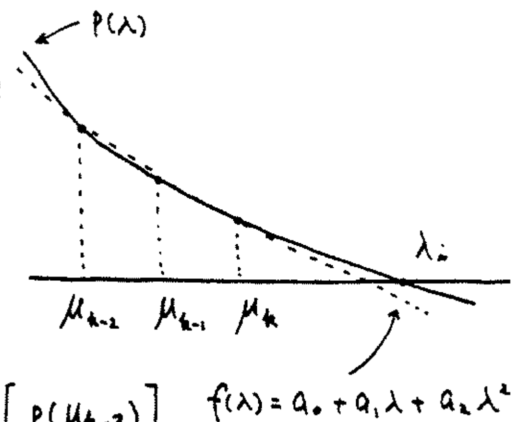


p(λ)
40
30
20
10
0
-10
0
0.2
0.4
0.6
0.8
1
1.2
λ
μ1
μ2
μ3
μ4
μ5



p(1)(λ)
2
1
0
-1
-2
2.8
2.9
3
3.1
3.2
λ
μ2
μ3
μ4

11.10 Let $p(\lambda) = \underset{\sim}{K} - \lambda \underset{\sim}{M}$ and we assume that $p(\lambda)$ can be approximated by $f(\lambda) = a_0 + a_1\lambda + a_2\lambda^2$ near $\lambda = \lambda_i$.



Then $\underset{\sim}{A}\,\underset{\sim}{U} = \hat{\underset{\sim}{P}}$

where $\underset{\sim}{U} = [a_0 \quad a_1 \quad a_2]$

$$\underset{\sim}{A} = \begin{bmatrix} 1 & \mu_{k-2} & \mu_{k-2}^2 \\ 1 & \mu_{k-1} & \mu_{k-1}^2 \\ 1 & \mu_k & \mu_k^2 \end{bmatrix}, \quad \hat{\underset{\sim}{P}} = \begin{bmatrix} P(\mu_{k-2}) \\ P(\mu_{k-1}) \\ P(\mu_k) \end{bmatrix}$$

Now we obtain the approximation μ_{k+1} to λ_i by using $f(\mu_{k+1}) = 0$,

hence, $$\mu_{k+1} = \frac{-a_1 + \sqrt{a_1^2 + 4a_0a_2}}{2a_2} \qquad \text{where} \quad \mu_{k+1} > \mu_k$$

Using this scheme we can get the solution of $p(\lambda) = 0$ in exercise 11.1.

11.11 $\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \underline{\phi} = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \underline{\phi}$

$$\mu = 0, \quad \underline{K} - \mu \underline{M} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \\ -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ & \frac{1}{2} \end{bmatrix} \quad \therefore \lambda_1 > 0$$

$$\mu = 1, \quad \underline{K} - \mu \underline{M} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ & -1 \end{bmatrix} \quad \therefore \lambda_1 < 1$$

$$\mu = 2 : \quad \underline{K} - \mu \underline{M} = \begin{bmatrix} 0 & -1 \\ -1 & -3 \end{bmatrix} \Rightarrow \text{interchanges are needed :}$$

(Interchange rows and columns).

$$\begin{bmatrix} -3 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & \\ \frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} -3 & -1 \\ & \frac{1}{3} \end{bmatrix} \Rightarrow \lambda_2 > 2 ;$$

$$\mu_3 = 3 ; \quad \underline{K} - \mu \underline{M} = \begin{bmatrix} -1 & -1 \\ -1 & -5 \end{bmatrix} = \begin{bmatrix} 1 & \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ & -4 \end{bmatrix} \Rightarrow \lambda_2 <$$

$$\Rightarrow \begin{cases} 0 < \lambda_1 < 1 \\ 2 < \lambda_2 < 3 \end{cases}$$

Next, use the secant iteration to calculate λ_1 and λ_2

Let $\mu_1 = 0$, $\mu_2 = 1$. Then $p(\mu_1) = 1$, $p(\mu_2) = -2$,

$\mu_3 = 0.333$.

$p(\mu_3) = -0.44322 \longrightarrow \mu_4 = 0.14311$,

$p(\mu_4) = 0.3254 \longrightarrow \mu_5 = 0.22351$,

11.11

$$p(\mu_5) = -0.01763 \longrightarrow \mu_6 = 0.21938.$$

$$p^{(1)}(\lambda) = \frac{p(\lambda)}{\lambda - \lambda_1}, \quad \text{where} \quad p(\lambda) = \det(\underline{K} - \lambda \underline{I}).$$

Let $\mu_1 = 2$ and $\mu_2 = 3$. Then

$$p^{(1)}(\mu_1) = \frac{p(\mu_1)}{\mu_1 - \lambda_1} = -0.5616, \quad p^{(1)}(\mu_2) = 1.4385$$

$\Rightarrow \mu_3 = 2.2808$

Hence, $\lambda_1 \doteq 0.21938$, $\lambda_2 \doteq 2.2808$

11.12 $\underline{K}\,\underline{\phi} = \lambda \underline{M}\,\underline{\phi}$ where $\underline{K} = \begin{bmatrix} 6 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix}$, $\underline{M} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Let $\mu = -1$,

$$\underline{K} - \mu \underline{M} = \begin{bmatrix} 8 & -1 & 0 \\ -1 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & & \\ -1/8 & 1 & \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8 & -1 & 0 \\ & 47/8 & 0 \\ & & 3 \end{bmatrix}$$

$\therefore \lambda_1 > -1$

$$\mu = 0, \quad \underline{K} - \mu \underline{M} = \begin{bmatrix} 6 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & \\ -1/6 & 1 & \\ 0 & -6/23 & 1 \end{bmatrix} \begin{bmatrix} 6 & -1 & \\ & 23/6 & -1 \\ & & 40/23 \end{bmatrix}$$

$\therefore \lambda_1 > 0$

$$\mu = 1, \quad \underline{K} - \mu \underline{M} = \begin{bmatrix} 4 & -1 & \\ -1 & 2 & -2 \\ & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ -1/4 & 1 & \\ 0 & -8/7 & 1 \end{bmatrix} \begin{bmatrix} 4 & -1 & \\ & 7/4 & -2 \\ & & -9/7 \end{bmatrix}$$

$\therefore \lambda_1 < 1$

Now using the secant iteration to calculate λ_1,

$\mu_1 = -1, \quad \mu_2 = 0, \quad p(-1) = 141, \quad p(0) = 40$

$$\mu_3 = -\frac{40}{40-141}\{0-(-1)\} = 0.39604$$

Similarly, $p(0.39604) = 15.043, \qquad \mu_4 = 0.63475$

$p(0.63475) = 3.6272, \qquad \mu_5 = 0.71060$

$p(0.71060) = 0.53725, \qquad \mu_6 = 0.72379$

$p(0.72379) = 0.025670, \qquad \mu_7 = 0.72445$

$\therefore \quad \lambda_1 = \mu_7 = 0.72445$

11.13

$$\left\{\begin{aligned}
&\underline{x}_1 = \frac{\underline{x}}{\gamma} \quad \text{where} \quad \gamma = (\underline{x}^T \underline{M}\, \underline{x})^{1/2} \\
&\underline{x}_2 = \frac{\tilde{\underline{x}}_1}{\beta_1} = \frac{1}{\beta_1}(\bar{\underline{x}}_1 - \alpha_1 \underline{x}_1) \quad \text{where} \quad \underline{K}\,\bar{\underline{x}}_1 = \underline{M}\,\underline{x}_1,\ \alpha_1 = \bar{\underline{x}}_1^T \underline{M}\,\underline{x}_1, \\
&\qquad \beta_1 = (\tilde{\underline{x}}_1^T \underline{M}\, \tilde{\underline{x}}_1)^{1/2} \\
&\underline{x}_3 = \frac{\tilde{\underline{x}}_2}{\beta_2} = \frac{1}{\beta_2}(\bar{\underline{x}}_2 - \alpha_2 \underline{x}_2 - \beta_1 \underline{x}_1) \quad \text{where} \quad \underline{K}\,\bar{\underline{x}}_2 = \underline{M}\,\underline{x}_2, \\
&\qquad \alpha_2 = \bar{\underline{x}}_2^T \underline{M}\,\underline{x}_2,\ \beta_1 = (\tilde{\underline{x}}_1^T \underline{M}\, \tilde{\underline{x}}_1)^{1/2}
\end{aligned}\right.$$

First we show that $\underline{x}_i^T \underline{M}\,\underline{x}_j = \delta_{ij} \quad i,j = 1,2,3$.

$$\underline{x}_1^T \underline{M}\,\underline{x}_1 = \frac{\underline{x}^T}{\gamma} \underline{M} \frac{\underline{x}}{\gamma} = \frac{\underline{x}^T \underline{M}\,\underline{x}}{\gamma^2} = 1$$

$$\underline{x}_2^T \underline{M}\,\underline{x}_2 = \frac{\tilde{\underline{x}}_1^T}{\beta_1} \underline{M} \frac{\tilde{\underline{x}}_1}{\beta_1} = \frac{\tilde{\underline{x}}_1^T M\, \tilde{\underline{x}}_1}{\beta_1^2} = 1$$

$$\underline{x}_3^T \underline{M}\,\underline{x}_3 = \frac{\tilde{\underline{x}}_2^T}{\beta_2} \underline{M} \frac{\tilde{\underline{x}}_2}{\beta_2} = \frac{\tilde{\underline{x}}_2^T \underline{M}\, \tilde{\underline{x}}_2}{\beta_2^2} = 1$$

$$\underline{x}_2^T \underline{M}\,\underline{x}_1 = \frac{1}{\beta_1}(\bar{\underline{x}}_1^T - \alpha_1 \underline{x}_1^T)\underline{M}\,\underline{x}_1 = \frac{1}{\beta_1}(\bar{\underline{x}}_1^T \underline{M}\,\underline{x}_1 - \alpha_1 \underline{x}_1^T \underline{M}\,\underline{x}_1) = 0$$

$$(\bar{\underline{x}}_1^T \underline{M}\,\underline{x}_1 = \alpha_1, \quad \underline{x}_1^T \underline{M}\,\underline{x}_1 = 1)$$

$$\begin{aligned}
\underline{x}_3^T \underline{M}\,\underline{x}_2 &= \frac{1}{\beta_2}(\bar{\underline{x}}_2 - \alpha_2 \underline{x}_2 - \beta_1 \underline{x}_1)^T \underline{M}\,\underline{x}_2 \\
&= \frac{1}{\beta_2}(\bar{\underline{x}}_2^T \underline{M}\,\underline{x}_2 - \alpha_2\, \underline{x}_2^T \underline{M}\,\underline{x}_2 - \beta_1\, \underline{x}_1^T \underline{M}\,\underline{x}_2) \\
&= \frac{1}{\beta_2}(\alpha_2 - \alpha_2 \cdot 1 - \beta_1 \cdot 0) = 0
\end{aligned}$$

$$\begin{aligned}
\underline{x}_3^T \underline{M}\,\underline{x}_1 &= \frac{1}{\beta_2}(\bar{\underline{x}}_2 - \alpha_2 \underline{x}_2 - \beta_1 \underline{x}_1)^T \underline{M}\,\underline{x}_1 \\
&= \frac{1}{\beta_2}(\bar{\underline{x}}_2^T \underline{M}\,\underline{x}_1 - \alpha_2\, \underline{x}_2^T \underline{M}\,\underline{x}_1 - \beta_1\, \underline{x}_1^T \underline{M}\,\underline{x}_1)
\end{aligned}$$

11.13

$$= \frac{1}{\beta_2}\left(\bar{\underline{x}}_2^T \underline{M}\, \underline{x}_1 - \beta_1 \right) = 0$$

$$\left(\begin{array}{ll} \text{because} & \beta_1 \underline{x}_2 = \tilde{\underline{x}}_1 , \quad \beta_1 = \underline{x}_2^T \underline{M}\, \tilde{\underline{x}}_1 \\ & \beta_1 = \underline{x}_2^T \underline{M}\left(\bar{\underline{x}}_1 - \alpha_1 \underline{x}_1 \right) = \underline{x}_2^T \underline{M}\, \bar{\underline{x}}_1 = \bar{\underline{x}}_2^T \underline{M}\, \underline{x}_1 \end{array} \right)$$

Hence we see that $\underline{x}_1$, $\underline{x}_2$ and $\underline{x}_3$ are $\underline{M}$-orthonormal. Now we assume that for $i, j = 1, 2, \cdots, k, k+1, k+2$, $\underline{x}_i^T \underline{M}\, \underline{x}_j = \delta_{ij}$ holds. And we show that the M-orthonormality still holds for $i = 1, \cdots, k+3$.

By definition $\underline{x}_{k+3}^T \underline{M}\, \underline{x}_{k+3} = 1$.

$$\underline{x}_{k+3}^T \underline{M}\, \underline{x}_{k+2} = \left(\frac{\tilde{\underline{x}}_{k+2}}{\beta_{k+2}} \right)^T \underline{M}\, \underline{x}_{k+2} = \frac{1}{\beta_{k+2}} \left(\bar{\underline{x}}_{k+2} - \alpha_{k+2} \underline{x}_{k+2} - \beta_{k+1} \underline{x}_{k+1} \right)^T \underline{M}\, \underline{x}_{k+2}$$

$$= \frac{1}{\beta_{k+2}} \left(\bar{\underline{x}}_{k+2}^T \underline{M}\, \underline{x}_{k+2} - \alpha_{k+2} \underline{x}_{k+2}^T \underline{M}\, \underline{x}_{k+2} - \beta_{k+1} \underline{x}_{k+1}^T \underline{M}\, \underline{x}_{k+2} \right)$$

$$= \frac{1}{\beta_{k+2}} \left(\alpha_{k+2} - \alpha_{k+2} \cdot 1 - 0 \right) = 0 .$$

$$\underline{x}_{k+3}^T \underline{M}\, \underline{x}_{k+1} = \frac{1}{\beta_{k+2}} \left(\bar{\underline{x}}_{k+2}^T \underline{M}\, \underline{x}_{k+1} - \alpha_{k+2} \underline{x}_{k+2}^T \underline{M}\, \underline{x}_{k+1} - \beta_{k+1} \underline{x}_{k+1}^T \underline{M}\, \underline{x}_{k+1} \right)$$

$$= \frac{1}{\beta_{k+2}} \left(\bar{\underline{x}}_{k+2}^T \underline{M}\, \underline{x}_{k+1} - \beta_{k+1} \right) = 0$$

$$\left(\begin{array}{ll} \text{because} & \underline{x}_{k+2} = \tilde{\underline{x}}_{k+1} / \beta_{k+1} , \\ \multicolumn{2}{l}{\beta_{k+1} = \underline{x}_{k+2}^T \underline{M}\, \tilde{\underline{x}}_{k+1} = \underline{x}_{k+2}^T \underline{M} \left(\bar{\underline{x}}_{k+1} - \alpha_{k+1} \underline{x}_{k+1} - \beta_k \underline{x}_k \right)} \\ \multicolumn{2}{l}{\quad = \underline{x}_{k+2}^T M\, \bar{\underline{x}}_{k+1} = \bar{\underline{x}}_{k+2}^T M\, \underline{x}_{k+1}} \end{array} \right)$$

11.13

$$\underline{x}_{k+3}^T \underline{M}\, \underline{x}_k = \frac{1}{\beta_{k+2}}\left(\bar{\underline{x}}_{k+2}^T \underline{M}\, \underline{x}_k - \alpha_{k+2}\, \underline{x}_{k+2}^T \underline{M}\, \underline{x}_k - \beta_{k+1}\, \underline{x}_{k+1}^T \underline{M}\, \underline{x}_k\right)$$

$$= \frac{1}{\beta_{k+2}}\left(\bar{\underline{x}}_{k+2}^T \underline{M}\, \bar{\underline{x}}_k\right) = \frac{1}{\beta_{k+2}}\underbrace{\left(\underline{x}_{k+2}^T M\, \bar{\underline{x}}_k\right)}_{\Downarrow}$$

$$\frac{1}{\beta_{k+1}}\left(\bar{\underline{x}}_{k+1} - \alpha_{k+1}\underline{x}_{k+1} - \beta_k \underline{x}_k\right)^T \underline{M}\, \bar{\underline{x}}_k$$

$$= \frac{1}{\beta_{k+1}}\left(\bar{\underline{x}}_{k+1}^T \underline{M}\, \bar{\underline{x}}_k - \alpha_{k+1}\, \underline{x}_{k+1}^T \underline{M}\, \bar{\underline{x}}_k - \beta_k\, \underline{x}_k^T \underline{M}\, \bar{\underline{x}}_k\right)$$

$$= \frac{1}{\beta_{k+1}}\left(\bar{\underline{x}}_{k+1}^T \underline{M}\, \bar{\underline{x}}_k - \alpha_{k+1}\beta_k - \alpha_k \beta_k\right)$$

$$= \frac{1}{\beta_{k+1}}\left[\bar{\underline{x}}_{k+1}^T \underline{M}\left(\beta_k \underline{x}_{k+1} + \alpha_k \underline{x}_k + \beta_{k-1}\underline{x}_{k-1}\right) - \alpha_{k+1}\beta_k - \alpha_k \beta_k\right]$$

$$= \frac{\beta_{k-1}}{\beta_{k+1}}\left(\bar{\underline{x}}_{k+1}^T \underline{M}\, \bar{\underline{x}}_{k-1}\right) = 0$$

$\therefore\ \underline{x}_{k+3}^T \underline{M}\, \underline{x}_k = 0$

Similarly we have $\underline{x}_{k+3}^T \underline{M}\, \underline{x}_{k-1} = \cdots = \underline{x}_{k+3}^T \underline{M}\, \underline{x}_1 = 0$

Therefore we see that the vectors $\underline{x}_i$ are $\underline{M}$ - orthonormal.

11.14 Let the error vector $\underline{f}_i$ be:

$$\underline{f}_i = \beta_i \underline{x}_{i+1} - (\overline{\underline{x}}_i - \alpha_i \underline{x}_i - \beta_{i-1} \underline{x}_{i-1}) \quad \text{—— ①}$$

Then, $$\beta_i \underline{x}_{i+1} = \underline{K}^{-1} \underline{M} \underline{x}_i - \alpha_i \underline{x}_i - \beta_{i-1} \underline{x}_{i-1} + \underline{f}_i \quad \text{—— ②}$$

where $$\alpha_i = \underline{x}_i^T \underline{M} \underline{K}^{-1} \underline{M} \underline{x}_i \quad \text{—— ③}$$

Suppose that we have performed j Lanczos steps and that loss of orthogonality in the Lanczos vectors is bounded by ε; i.e.,

$$|\underline{x}_i^T \underline{M} \underline{x}_k| \le \varepsilon \quad ; \quad 1 \le i \le j, \quad 1 \le k \le j \quad ; \quad i \ne k \quad \text{—— ④}$$

$$|\underline{x}_i^T \underline{M} \underline{x}_i| = 1 \quad ; \quad 1 \le i \le j \quad \text{—— ⑤}$$

Using ②, we have

$$|\underline{x}_i^T \underline{M} \underline{x}_{j+1}| = |\underline{x}_i^T \underline{M} \frac{1}{\beta_j} (\underline{K}^{-1} \underline{M} \underline{x}_j - \alpha_j \underline{x}_j - \beta_{j-1} \underline{x}_{j-1} + \underline{f}_j)| \quad \text{—— ⑥}$$

$$= \frac{1}{\beta_j} | \underline{x}_j^T \underline{M} (\beta_i \underline{x}_{i+1} + \alpha_i \underline{x}_i + \beta_{i-1} \underline{x}_{i-1} - f_i) - \alpha_j \underline{x}_i^T \underline{M} \underline{x}_j - \beta_{j-1} \underline{x}_i^T \underline{M} \underline{x}_{j-1} + \underline{x}_i^T \underline{M} \underline{f}_j | \quad \text{—— ⑦}$$

When $i \le j-1$

Using ③, ④, ⑤ and ⑦, we have

$$|\underline{x}_i^T \underline{M} \underline{x}_{j+1}| \le \begin{cases} \frac{1}{\beta_j} \{ (\beta_i + \beta_{i-1} + \beta_{j-1} + |\alpha_i - \alpha_j|) \varepsilon + \kappa_1 \} & (i \le j-2) \\ \frac{1}{\beta_j} \{ (\beta_{j-2} + |\alpha_i - \alpha_j|) \varepsilon + \kappa_1 \} & (i = j-1) \end{cases} \quad \text{—— ⑧}$$

where $\kappa_1 = |\underline{x}_j^T \underline{M} \underline{f}_i| + |\underline{x}_i^T \underline{M} \underline{f}_j|$

11.14

When $i = j$

using ③, ④, ⑤ and ⑥, we have

$$|\underline{x}_i^T \underline{M}\, \underline{x}_{j+1}| \le \frac{1}{\beta_j}(\beta_{j-1}\varepsilon + K_2) \qquad \text{— ⑨}$$

where $K_2 = |\underline{x}_j^T \underline{M}\, \underline{f}_j|$

Hence ignoring the smaller terms, K_1 in ⑧ and K_2 in ⑨, we obtain

$$|\underline{x}_i^T \underline{M}\, \underline{x}_{j+1}| \lesssim \frac{f_{ij}}{\beta_j}\varepsilon$$

where

$$f_{ij} = \begin{cases} |\alpha_i - \alpha_j| + \beta_{j-1} + \beta_{i-1} + \beta_i & (i \le j-2) \\ |\alpha_i - \alpha_j| + \beta_{j-2} & (i = j-1) \\ \beta_{j-1} & (i = j) \end{cases}$$

11.15

$$\begin{bmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \underline{\phi} = \lambda \begin{bmatrix} 2 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \underline{\phi}$$

Let the starting vector $\underline{x}^T = [\,1 \quad 1 \quad 1 \quad 1\,]$

$$\gamma = (\underline{x}^T \underline{M}\, \underline{x})^{1/2} = \sqrt{5}\,, \qquad \underline{x}_1^T = \frac{\underline{x}^T}{\gamma} = \frac{1}{\sqrt{5}} [\,1 \quad 1 \quad 1 \quad 1\,]$$

$$\bar{\underline{x}}_1 = \begin{bmatrix} 0.36590 \\ 0.56918 \\ 1.4636 \\ 1.9108 \end{bmatrix}, \qquad \alpha_1 = \bar{\underline{x}}_1^T \underline{M}\, \underline{x}_1 = 2.0909$$

$$\tilde{\underline{x}}_1 = \begin{bmatrix} -0.56918 \\ -0.36590 \\ 0.52853 \\ 0.97574 \end{bmatrix}, \qquad \beta_1 = (\tilde{\underline{x}}_1^T \underline{M}\, \tilde{\underline{x}}_1)^{1/2} = 1.4189$$

$$\underline{x}_2 = \frac{\tilde{\underline{x}}_1}{\beta_1} = \begin{bmatrix} -0.40115 \\ -0.25788 \\ 0.37249 \\ 0.68768 \end{bmatrix}, \qquad \bar{\underline{x}}_2 = \begin{bmatrix} -0.14587 \\ 0.21881 \\ 1.2790 \\ 1.9667 \end{bmatrix}$$

$$\tilde{\underline{x}}_2 = \begin{bmatrix} -0.022462 \\ 0.071521 \\ -0.059372 \\ 0.032775 \end{bmatrix}, \qquad \begin{aligned} \alpha_2 &= 1.8895 \\ \beta_2 &= 0.10355 \end{aligned}$$

11.15

Approximation is done for $\underline{K}\,\underline{\phi} = \lambda \underline{M}\,\underline{\phi}$ by cond

$\underline{T}_2\,\underline{s} = \nu\,\underline{s}$ where $\nu = \frac{1}{\rho}$

$\rho_1 = \frac{1}{\nu_2} = 0.29303$, $s_1 = \begin{bmatrix} 0.73171 \\ 0.68161 \end{bmatrix}$

$\bar{\underline{\phi}}_1 = \underline{X}_2\,\underline{s}_1 = \begin{bmatrix} 0.048795 \\ 0.14888 \\ 0.58485 \\ 0.80284 \end{bmatrix}$ where $\underline{X}_2 = [\underline{x}_1 \;\; \underline{x}_2]$

The error bound calculation (11.144) gives here

$|\lambda_1^{-1} - \nu_2| = 0.00161 < 0.0705$

Note that the exact solution for the smallest eigenvalue and corresponding eigenvector is given by

$\underline{\phi}_1 = \begin{bmatrix} 0.048720 \\ 0.16634 \\ 0.56792 \\ 0.80315 \end{bmatrix}$, $\lambda_1 = 0.29289$

11.16

$$\begin{bmatrix} 2 & -1 & & \\ -1 & 1 & 1/4 & \\ & 1/4 & 1 & -1 \\ & & -1 & 2 \end{bmatrix} \underline{\phi} = \lambda \begin{bmatrix} 1 & & & \\ & 1/2 & & \\ & & 1/2 & \\ & & & 1 \end{bmatrix} \underline{\phi}$$

Let the starting vector $\underline{x}^T = [\,1 \;\; 1 \;\; 1 \;\; 1\,]$

$$\gamma = (\underline{x}^T \underline{M}\, \underline{x})^{1/2} = \sqrt{3}, \quad \underline{x}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \bar{\underline{x}}_1 = \sqrt{3} \begin{bmatrix} 7/18 \\ 4/9 \\ 4/9 \\ 7/18 \end{bmatrix}$$

$$\alpha_1 = (\bar{\underline{x}}_1^T \underline{M}\, \underline{x}_1) = \frac{11}{9}, \quad \tilde{\underline{x}}_1 = \bar{\underline{x}}_1 - \alpha_1 \underline{x}_1 = \sqrt{3} \begin{bmatrix} -1/54 \\ 1/27 \\ 1/27 \\ -1/54 \end{bmatrix}$$

$$\beta_1 = (\tilde{\underline{x}}_1^T \underline{M}\, \tilde{\underline{x}}_1)^{1/2} = \frac{\sqrt{2}}{18}, \quad \underline{x}_2 = \frac{\tilde{\underline{x}}_1}{\beta_1} = \sqrt{6} \begin{bmatrix} -1/6 \\ 1/3 \\ 1/3 \\ -1/6 \end{bmatrix}$$

$$\bar{\underline{x}}_2 = (\underline{K}^{-1} \underline{M})\, \underline{x}_2 = \sqrt{6} \begin{bmatrix} -1/36 \\ 1/9 \\ 1/9 \\ -1/36 \end{bmatrix}, \quad \alpha_2 = \bar{\underline{x}}_2^T \underline{M}\, \underline{x}_2 = \frac{5}{18}$$

$$\therefore \quad \underline{T}_2 = \begin{bmatrix} \frac{11}{9} & \frac{\sqrt{2}}{18} \\ \frac{\sqrt{2}}{18} & \frac{5}{18} \end{bmatrix}$$

Approximation to the eigenvalues of $\underline{K}\underline{\phi} = \lambda \underline{M}\underline{\phi}$ is done by

11.16

considering $\underline{I}_2 \underline{s} = \frac{1}{\rho} \underline{s}$ $\quad \therefore \rho_1 = 0.8136, \quad \rho_2 = 3.6861$

and $\quad \underline{s}_1 = \begin{bmatrix} 12.103 \\ 1 \end{bmatrix} \qquad \underline{s}_2 = \begin{bmatrix} -0.08262 \\ 1 \end{bmatrix}$

Eigenvectors are obtained by $\bar{\underline{\Phi}}_1 = \underline{X}\,\underline{s}_1$, $\bar{\underline{\Phi}}_2 = \underline{X}\,\underline{s}_2$

where $\underline{X} = [\underline{x}_1 \quad \underline{x}_2]$

$$\therefore \quad \bar{\underline{\Phi}}_1 = \begin{bmatrix} 6.5796 \\ 7.8044 \\ 7.8044 \\ 6.5796 \end{bmatrix}, \quad \bar{\underline{\Phi}}_2 = \begin{bmatrix} -0.4559 \\ 0.7688 \\ 0.7688 \\ -0.4559 \end{bmatrix}$$

The analytical solution for the problem is

$$\lambda_1 = 0.3139 \qquad \underline{\phi}_1^T = [\,-1 \quad -1.6861 \quad 1.6861 \quad 1\,]$$

$$\lambda_2 = 0.8137 \qquad \underline{\phi}_2^T = [\,1 \quad 1.1862 \quad 1.1862 \quad 1\,]$$

$$\lambda_3 = 3.1861 \qquad \underline{\phi}_3^T = [\,-1 \quad 1.1862 \quad -1.1862 \quad 1\,]$$

$$\lambda_4 = 3.6861 \qquad \underline{\phi}_4^T = [\,1 \quad -1.6861 \quad -1.6861 \quad 1\,]$$

From the results obtained we see that instead of having approximations to the two smallest eigenvalues we **obtained the 2nd and the 4th eigenvalues exactly.** That is because the starting vector $\underline{x}^T = [\,1 \quad 1 \quad 1 \quad 1\,]$ is orthogonal to the eigenvectors corresponding to the 1st and 3rd eigenvalues.

Therefore it is necessary that the starting vector be changed — we choose $\underline{X}^T = [\,1 \quad 0 \quad 1 \quad 1\,]$ Following the same

11.16

steps but with the new starting vector.

$$\underline{T}_2\, \underline{s} = \nu\, \underline{s} \qquad \text{where} \qquad \underline{T}_2 = \begin{bmatrix} \frac{6}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{581}{270} \end{bmatrix}$$

$$\rho_1 = 0.4095, \quad \rho_2 = 1.099$$

$$\underline{s}_2 = \begin{bmatrix} 0.4351 \\ 0.9004 \end{bmatrix}, \quad \underline{s}_1 = \begin{bmatrix} 0.9004 \\ -0.4351 \end{bmatrix}$$

$$\bar{\underline{\Phi}}_1 = \underline{X}\, \underline{s}_2 = \begin{bmatrix} -0.3892 \\ 0 \\ 1.447 \\ 0.9726 \end{bmatrix}, \quad \bar{\underline{\Phi}}_2 = \underline{X}\, \underline{s}_1 = \begin{bmatrix} 0.8905 \\ 0 \\ 1.057 \\ 1.286 \end{bmatrix}$$

$$\text{where} \quad \underline{X} = [\, \underline{x}_1 \;\; \underline{x}_2 \,]$$

The error bounds are from (11.144)

$$|\lambda_1^{-1} - \nu_2| = 0.7440 < 1.128$$

$$|\lambda_2^{-1} - \nu_1| = 0.3189 < 0.5452$$

11.18 In subspace iteration, we solve the eigenproblem of the projected operators:

$$\underline{K}_{k+1}\underline{Q}_{k+1} = \underline{M}_{k+1}\underline{Q}_{k+1}\underline{\Lambda}_{k+1} \quad \text{—— ①}$$

where $\underline{K}_{k+1} = \bar{\underline{X}}_{k+1}^T \underline{K}\, \bar{\underline{X}}_{k+1}$, $\underline{M}_{k+1} = \bar{\underline{X}}_{k+1}^T \underline{M}\, \bar{\underline{X}}_{k+1}$ —— ②

And then find an improved approximation to the eigenvectors.

From ① we have $\underline{Q}_{k+1}^T \underline{M}_{k+1} \underline{Q}_{k+1} = \underline{I}$

or $\underline{Q}_{k+1}^T (\bar{\underline{X}}_{k+1}^T \underline{M}\, \bar{\underline{X}}_{k+1}) \underline{Q}_{k+1} = \underline{I}$

$$\therefore\ \underline{X}_{k+1}^T \underline{M}\, \underline{X}_{k+1} = \underline{I} \quad \text{—— ③}$$

From ① and ③

$$\underline{Q}_{k+1}^T \underline{K}_{k+1} \underline{Q}_{k+1} = \underline{Q}_{k+1}^T \underline{M}_{k+1} \underline{Q}_{k+1} \underline{\Lambda}_{k+1}$$

$$\underline{Q}_{k+1}^T (\bar{\underline{X}}_{k+1}^T \underline{K}\, \bar{\underline{X}}_{k+1}) \underline{Q}_{k+1} = \underline{\Lambda}_{k+1}$$

$$\therefore\ \underline{X}_{k+1}^T \underline{K}\, \underline{X}_{k+1} = \underline{\Lambda}_{k+1}$$

11.19

$$\begin{bmatrix} 6 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix} \underline{\phi} = \lambda \begin{bmatrix} 2 & & \\ & 2 & 1 \\ & 1 & 1 \end{bmatrix} \underline{\phi}$$

Let $\underline{X}_1 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$, then $\bar{\underline{X}}_2 = \begin{bmatrix} 0.55 & -0.4 \\ 1.3 & -0.4 \\ 1.65 & -0.2 \end{bmatrix}$

$$\underline{K}_2 = \begin{bmatrix} 8.3 & -2.4 \\ -2.4 & 1.2 \end{bmatrix}, \quad \underline{M}_2 = \begin{bmatrix} 10.9975 & -2.73 \\ -2.73 & 0.84 \end{bmatrix}$$

$$\underline{\Lambda}_2 = \begin{bmatrix} 0.72861 & \\ & 3.2294 \end{bmatrix}, \quad \underline{Q}_2 = \begin{bmatrix} 0.36293 & 0.58213 \\ 0.25364 & 2.4692 \end{bmatrix}$$

$$\underline{X}_2 = \begin{bmatrix} 0.098157 & -0.66749 \\ 0.37035 & -0.23090 \\ 0.54811 & 0.46667 \end{bmatrix}$$

Similarly, $\bar{\underline{X}}_3 = \begin{bmatrix} 0.12176 & -0.22748 \\ 0.53423 & -0.029919 \\ 0.72634 & 0.10293 \end{bmatrix}$

$$\underline{K}_3 = \begin{bmatrix} 1.3795 & 0.011318 \\ 0.011318 & 0.32781 \end{bmatrix}, \quad \underline{M}_3 = \begin{bmatrix} 1.9041 & 0.020656 \\ 0.020656 & 0.10972 \end{bmatrix}$$

$$\underline{\Lambda}_3 = \begin{bmatrix} 0.72449 & \\ & 2.9930 \end{bmatrix}, \quad \underline{Q}_3 = \begin{bmatrix} 0.72457 & -0.035334 \\ 0.010762 & 3.0219 \end{bmatrix}$$

$$\underline{X}_3 = \begin{bmatrix} 0.085774 & -0.68175 \\ 0.38677 & -0.10929 \\ 0.52740 & 0.28539 \end{bmatrix}$$

11.19

$$\bar{X}_4 = \begin{bmatrix} 0.11792 & -0.23437 \\ 0.53598 & -0.022720 \\ 0.72507 & 0.076687 \end{bmatrix}$$

$$K_4 = \begin{bmatrix} 1.3803 & 0.00034116 \\ 0.00034116 & 0.33624 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 1.9054 & 0.00060250 \\ 0.00060250 & 0.11329 \end{bmatrix}$$

$$\Lambda_4 = \begin{bmatrix} 0.72446 & \\ & 2.9680 \end{bmatrix}, \quad Q_4 = \begin{bmatrix} -0.72446 & -0.0010057 \\ 0.0044940 & 2.9710 \end{bmatrix}$$

$$X_4 = \begin{bmatrix} -0.086483 & -0.69644 \\ -0.38840 & -0.068042 \\ -0.52494 & 0.22711 \end{bmatrix}$$

Hence

$$\lambda_1 \doteq 0.72446 \qquad \phi_1 \doteq \begin{bmatrix} 0.086483 \\ 0.38840 \\ 0.52494 \end{bmatrix}$$

$$\lambda_2 \doteq 2.9680 \qquad \phi_2 \doteq \begin{bmatrix} -0.69644 \\ -0.068042 \\ 0.22711 \end{bmatrix}$$

Using (11.156) we check

$$\left[1 - \frac{(0.72446)^2}{0.52486} \right]^{1/2} = 0.0062; \qquad \left[1 - \frac{(2.9680)^2}{8.8268} \right]^{1/2} = 0.045$$

11.20 In the subspace iteration we have after $(k-1)$ iterations

$$\underline{K}\,\bar{\underline{X}}_k = \underline{M}\,\underline{X}_{k-1} \qquad \text{——— ①}$$

$$\underline{X}_k = \bar{\underline{X}}_k\,\underline{Q}_k \quad \text{or} \quad \bar{\underline{\phi}} = \bar{\underline{X}}_k\,\underline{q}_i^{(k)} \qquad \text{——— ②}$$

Hence
$$\rho(\bar{\underline{\phi}}) = \frac{\bar{\underline{\phi}}^T \underline{K}\,\bar{\underline{\phi}}}{\bar{\underline{\phi}}^T \underline{M}\,\bar{\underline{\phi}}} = \frac{\lambda_i^{(k)}}{1} = \lambda_i^{(k)}$$

Considering $\hat{\underline{\phi}}^T \underline{M}\,\hat{\underline{\phi}}$, we see there is "some" $\hat{\underline{\phi}}$ in the subspace $\underline{X}_{k-1}$ which gives $\bar{\underline{\phi}}$, that is,

$$\hat{\underline{\phi}}^T(\underline{M}\,\hat{\underline{\phi}}) = \hat{\underline{\phi}}^T(\underline{K}\,\bar{\underline{\phi}}) = \bar{\underline{\phi}}^T \underline{K}\,\underline{M}^{-1}\underline{K}\,\bar{\underline{\phi}}$$

$$\left(\text{because } \underline{K}\,\bar{\underline{\phi}} = \underline{M}\,\hat{\underline{\phi}} \text{ and } \hat{\underline{\phi}} = \underline{M}^{-1}\underline{K}\,\bar{\underline{\phi}}\right)$$

$$= \left(\underline{q}_i^{(k)T}\,\bar{\underline{X}}_k^T\right)\underline{K}\,\underline{M}^{-1}\underline{K}\left(\bar{\underline{X}}_k\,\underline{q}_i^{(k)}\right) \qquad (\text{use eq. ②})$$

$$= \underline{q}_i^{(k)T}\,\underline{X}_{k-1}^T\,\underline{M}\,\underline{M}^{-1}\underline{M}\,\underline{X}_{k-1}\,\underline{q}_i^{(k)} \qquad (\text{use eq. ①})$$

$$= \underline{q}_i^{(k)T}\left(\underline{X}_{k-1}^T\,\underline{M}\,\underline{X}_{k-1}\right)\underline{q}_i^{(k)} \qquad (\underline{X}_{k-1}^T\,\underline{M}\,\underline{X}_{k-1} = \underline{I})$$

$$= \underline{q}_i^{(k)T}\,\underline{q}_i^{(k)}$$

Therefore using eq. (10.107)

$$\left[1 - \frac{\{\rho(\bar{\underline{\phi}})\}^2}{\dfrac{\hat{\underline{\phi}}^T \underline{M}\,\hat{\underline{\phi}}}{\bar{\underline{\phi}}^T \underline{M}\,\bar{\underline{\phi}}}}\right]^{1/2} = \left[1 - \frac{(\lambda_i^{(k)})^2}{\underline{q}_i^{(k)T}\,\underline{q}_i^{(k)}}\right]^{1/2} \le \text{tol}$$

Part B

Solutions to Three Exercises Using the ADINA System

Solution to exercise 4.45

A reasonable finite element mesh with its loading and boundary conditions is shown in Figure 4.45-1. Due to symmetry we analyze only one quarter of the plate. Since the analysis is linear, we apply a unit pressure load and all computed stresses are proportional to the load.

The elements are 9/3 u/p isoparametric elements. These elements are necessary for the solution with $\nu = 0.499$, and are effective for the solution with $\nu = 0.3$ as well.

The computed stresses for $\nu = 0.499$ along the top half of section A-A are shown in Figure 4.45-2. As expected, a stress concentration is observed at the hole. The computed stress concentration factor is 4.75 based on the applied stress and is 2.85 based on the nominal stress (the nominal stress is the total applied force divided by the area of the plate at section A-A). These definitions of the stress concentration factor can be found in R.E. Peterson, *Stress Concentration Factors*, John Wiley and Sons, 1974. The well-known stress concentration factor of 3 is not applicable in this problem because the hole is relatively large compared to the plate and the loads are applied relatively close to the hole. Observe that the zz stress component is nearly zero at the hole and outer boundary, as it must be in order to match the boundary conditions. The shear stress component must be zero due to symmetry and this is also observed in the finite element solution.

Band plots of the solution are shown in Figures 4.45-3 and 4.45-4. The stresses are interpolated from the integration points to the element nodes prior to smoothing (see Fig. 4.16, textbook); this is what is meant by the term RST CALC that appears in the plots. In the lower half of Figure 4.45-4, we plot the unsmoothed effective stress and pressure as repeating isobands (see Section 4.3.6, textbook). As these isobands are quite continuous, we conclude that an accurate solution has been obtained.

The computed stresses for $\nu = 0.3$ are shown in Figures 4.45-5, 4.45-6 and 4.45-7. Again, an accurate solution has been obtained. The primary difference in the solutions is in the xx stress component, which is smaller. This makes sense since the out-of-plane stress should be nearly proportional to ν.

When we use the displacement-based 9-node finite elements instead of the 9/3 elements, we obtain an accurate solution for $\nu = 0.3$ (not shown here), but an inaccurate solution for $\nu = 0.499$. This solution is shown in Figures 4.45-8 and 4.45-9. The source of the inaccuracy is seen to be the pressure, as the shear stress and effective stress are still reasonably accurate.

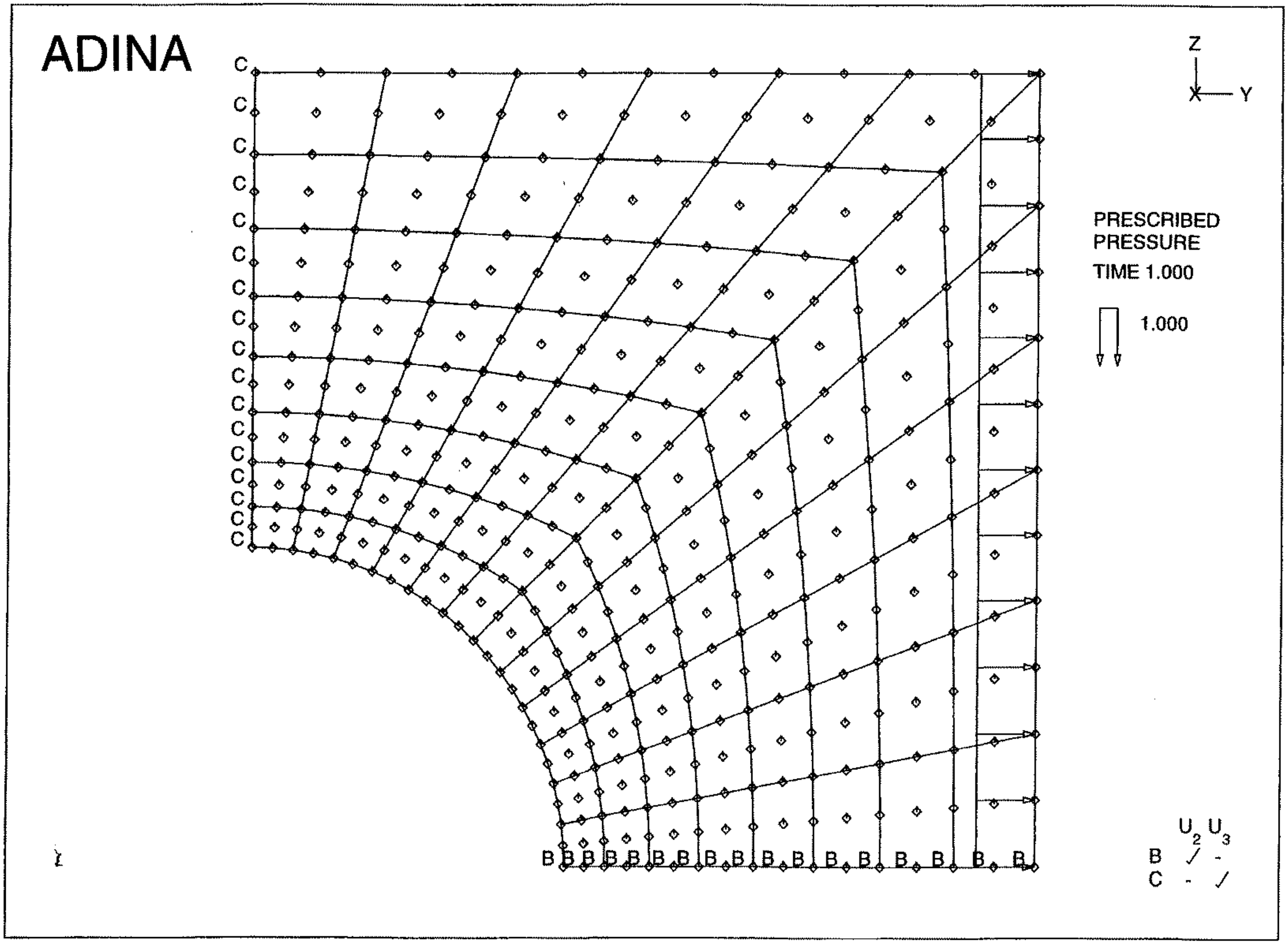
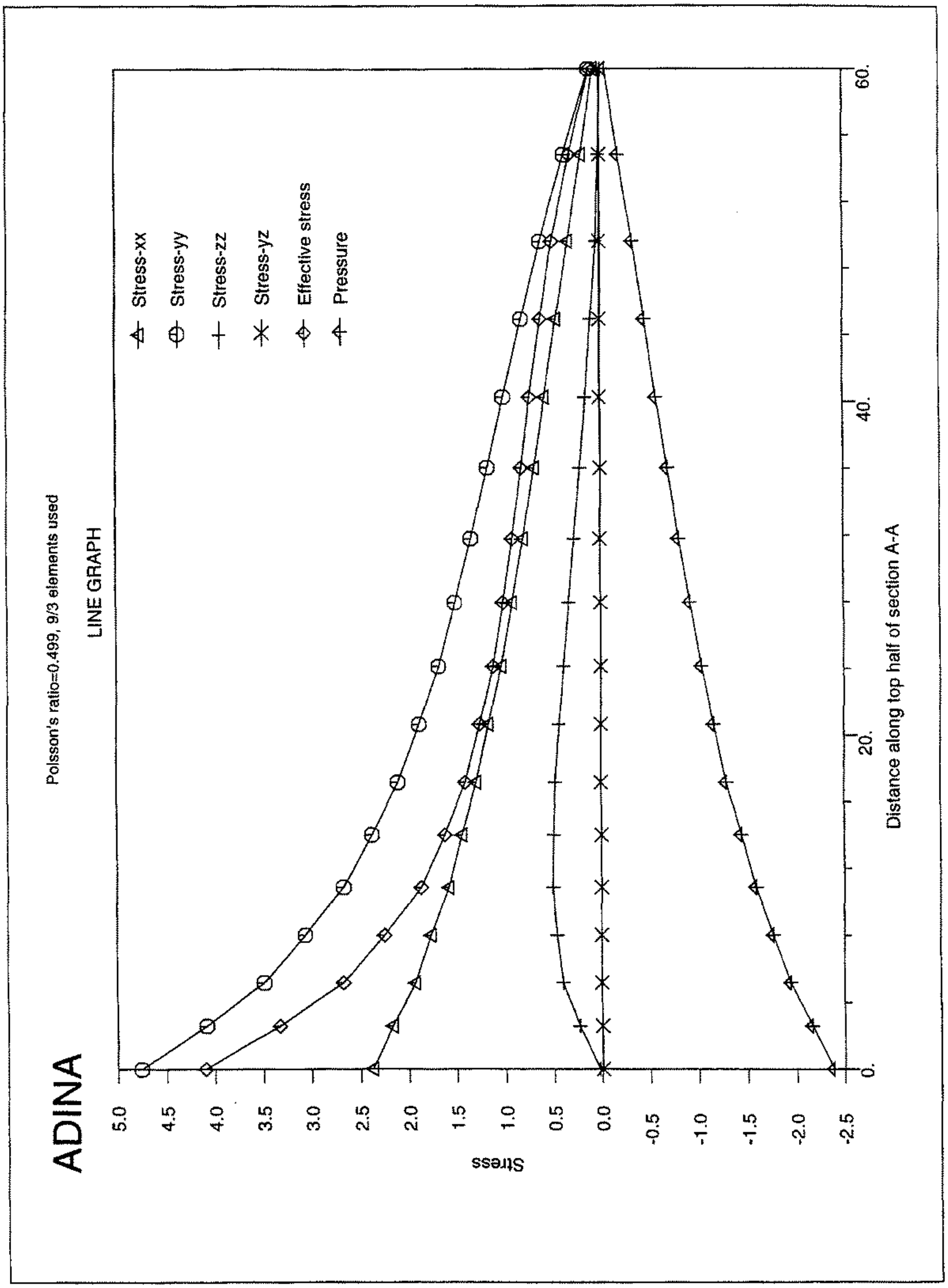
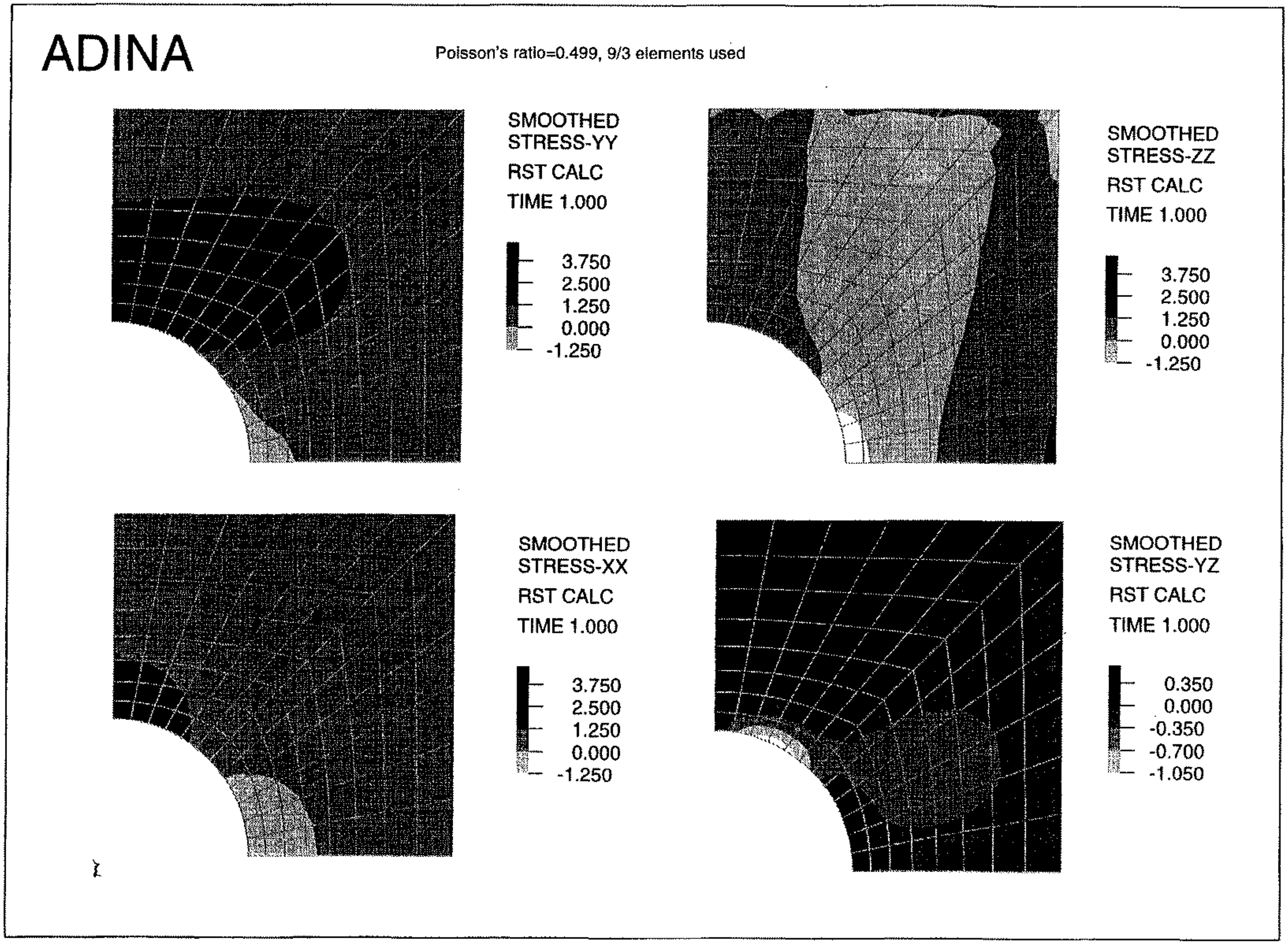


Figure 4.45-1



ADINA
Poisson's ratio=0.499, 9/3 elements used
LINE GRAPH
Stress-xx
Stress-yy
Stress-zz
Stress-yz
Effective stress
Pressure
Stress
5.0
4.5
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
-2.0
-2.5
0.
20.
40.
60.
Distance along top half of section A-A

Figure 4.45-2



ADINA
Poisson's ratio=0.499, 9/3 elements used
SMOOTHED
STRESS-YY
RST CALC
TIME 1.000
3.750
2.500
1.250
0.000
-1.250
SMOOTHED
STRESS-ZZ
RST CALC
TIME 1.000
3.750
2.500
1.250
0.000
-1.250
SMOOTHED
STRESS-XX
RST CALC
TIME 1.000
3.750
2.500
1.250
0.000
-1.250
SMOOTHED
STRESS-YZ
RST CALC
TIME 1.000
0.350
0.000
-0.350
-0.700
-1.050

Figure 4.45-3

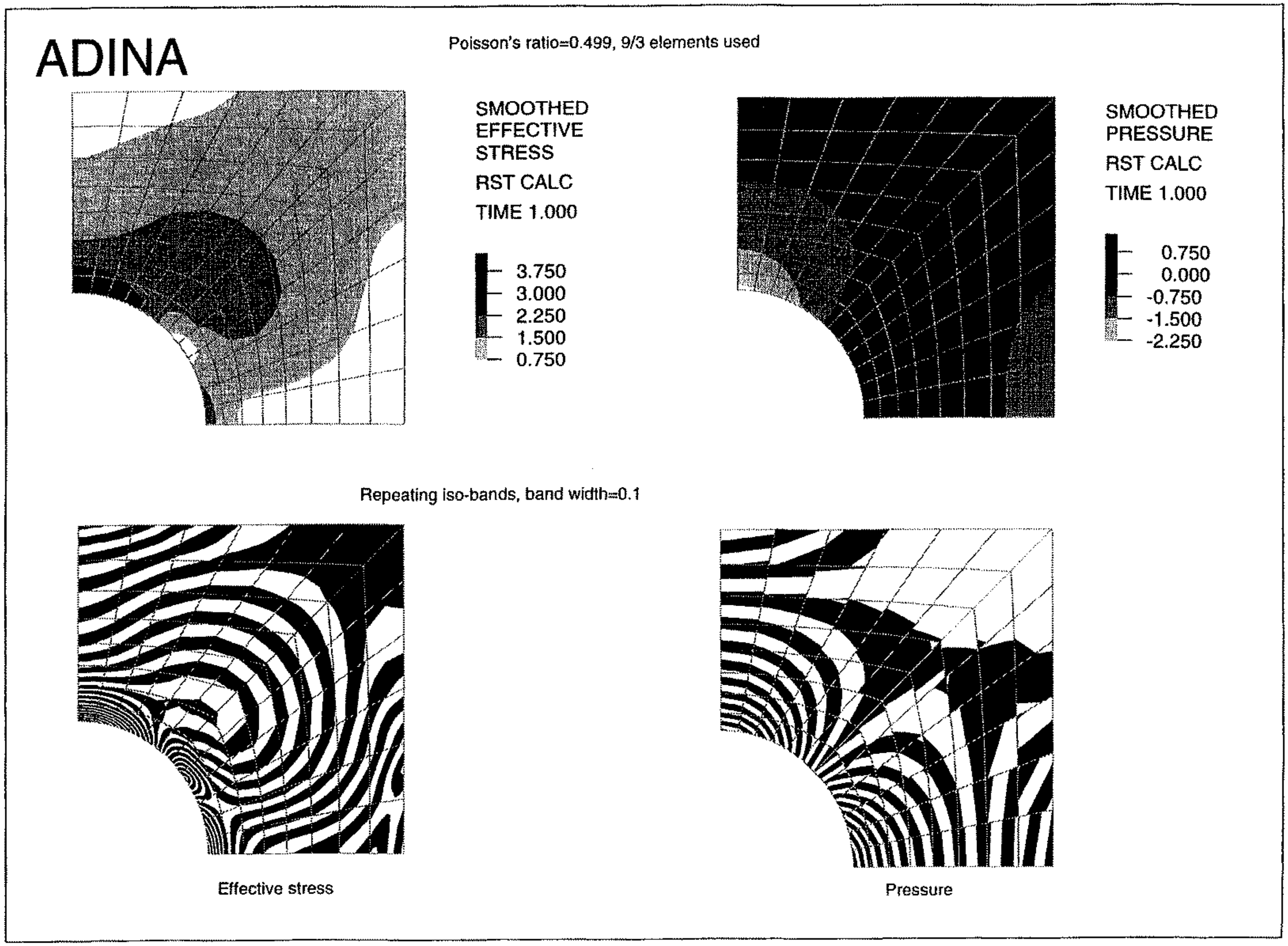

ADINA
Poisson's ratio=0.499, 9/3 elements used
SMOOTHED
EFFECTIVE
STRESS
RST CALC
TIME 1.000
3.750
3.000
2.250
1.500
0.750
SMOOTHED
PRESSURE
RST CALC
TIME 1.000
0.750
0.000
-0.750
-1.500
-2.250
Repeating iso-bands, band width=0.1
Effective stress
Pressure

Figure 4.45-4

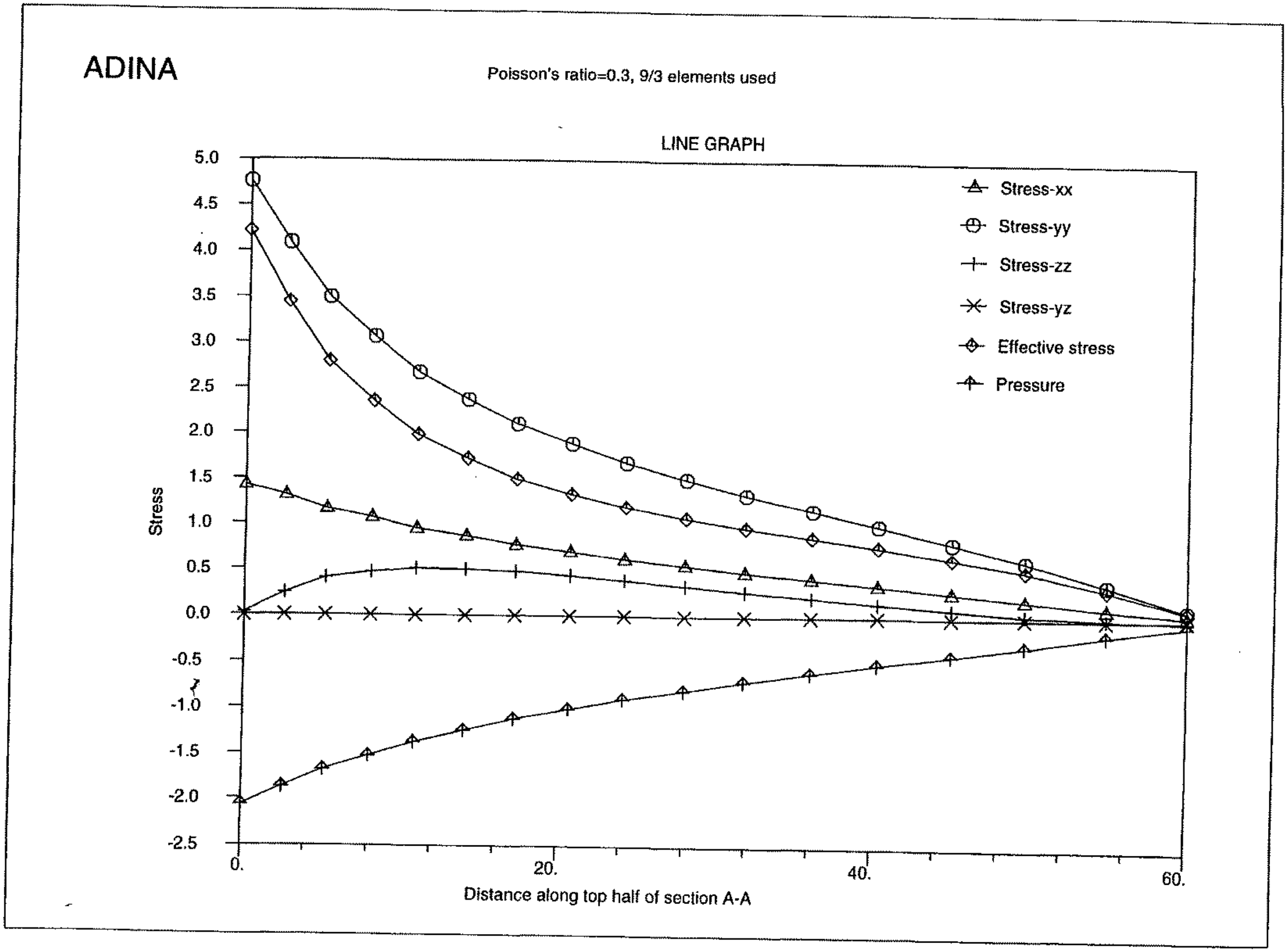

ADINA
Poisson's ratio=0.3, 9/3 elements used
LINE GRAPH
Stress
5.0
4.5
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
-2.0
-2.5
0.
20.
40.
60.
Distance along top half of section A-A
Stress-xx
Stress-yy
Stress-zz
Stress-yz
Effective stress
Pressure

Figure 4.45-5

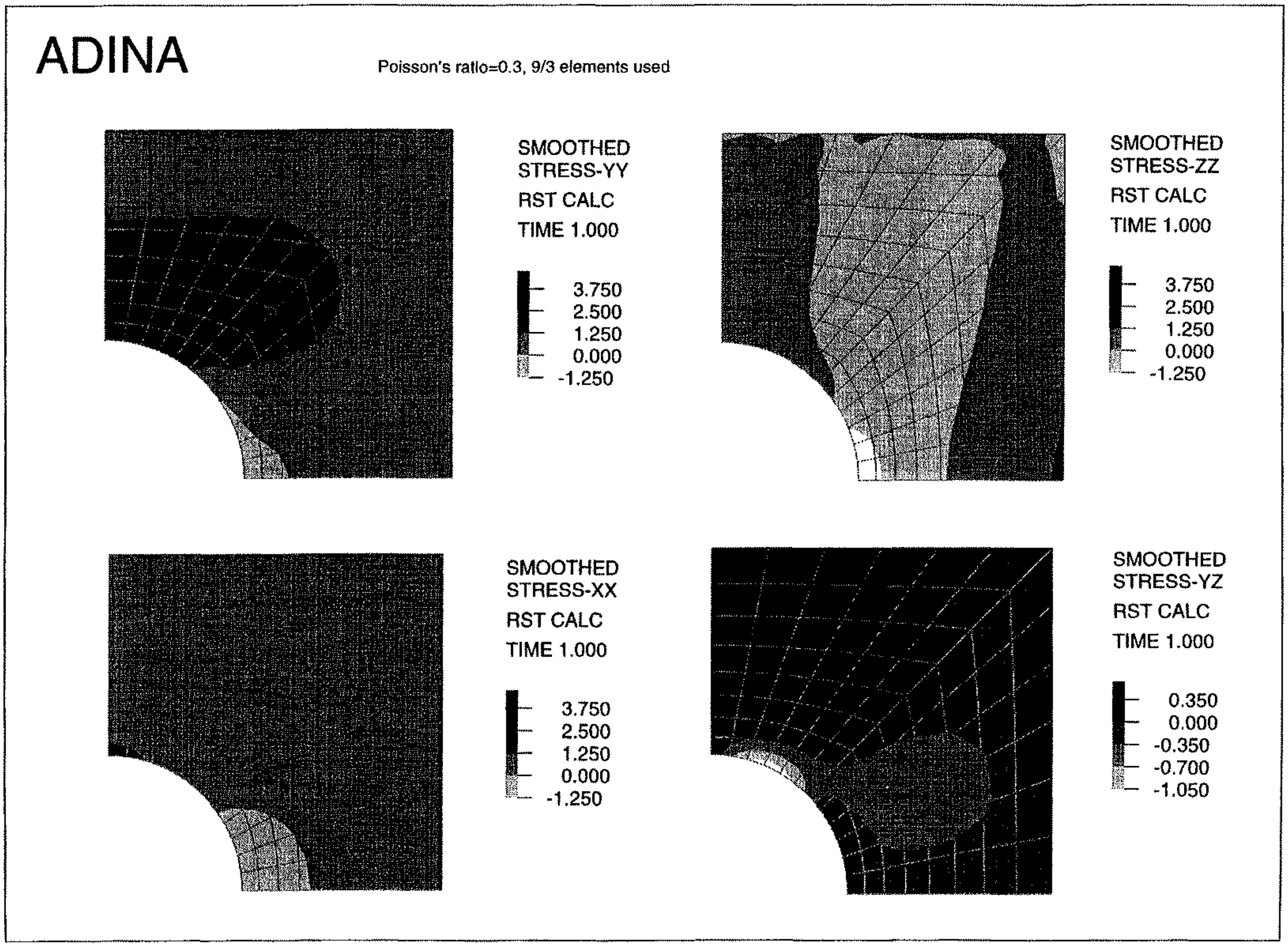

ADINA
Poisson's ratio=0.3, 9/3 elements used
SMOOTHED
STRESS-YY
RST CALC
TIME 1.000
3.750
2.500
1.250
0.000
-1.250
SMOOTHED
STRESS-ZZ
RST CALC
TIME 1.000
3.750
2.500
1.250
0.000
-1.250
SMOOTHED
STRESS-XX
RST CALC
TIME 1.000
3.750
2.500
1.250
0.000
-1.250
SMOOTHED
STRESS-YZ
RST CALC
TIME 1.000
0.350
0.000
-0.350
-0.700
-1.050

Figure 4.45-6

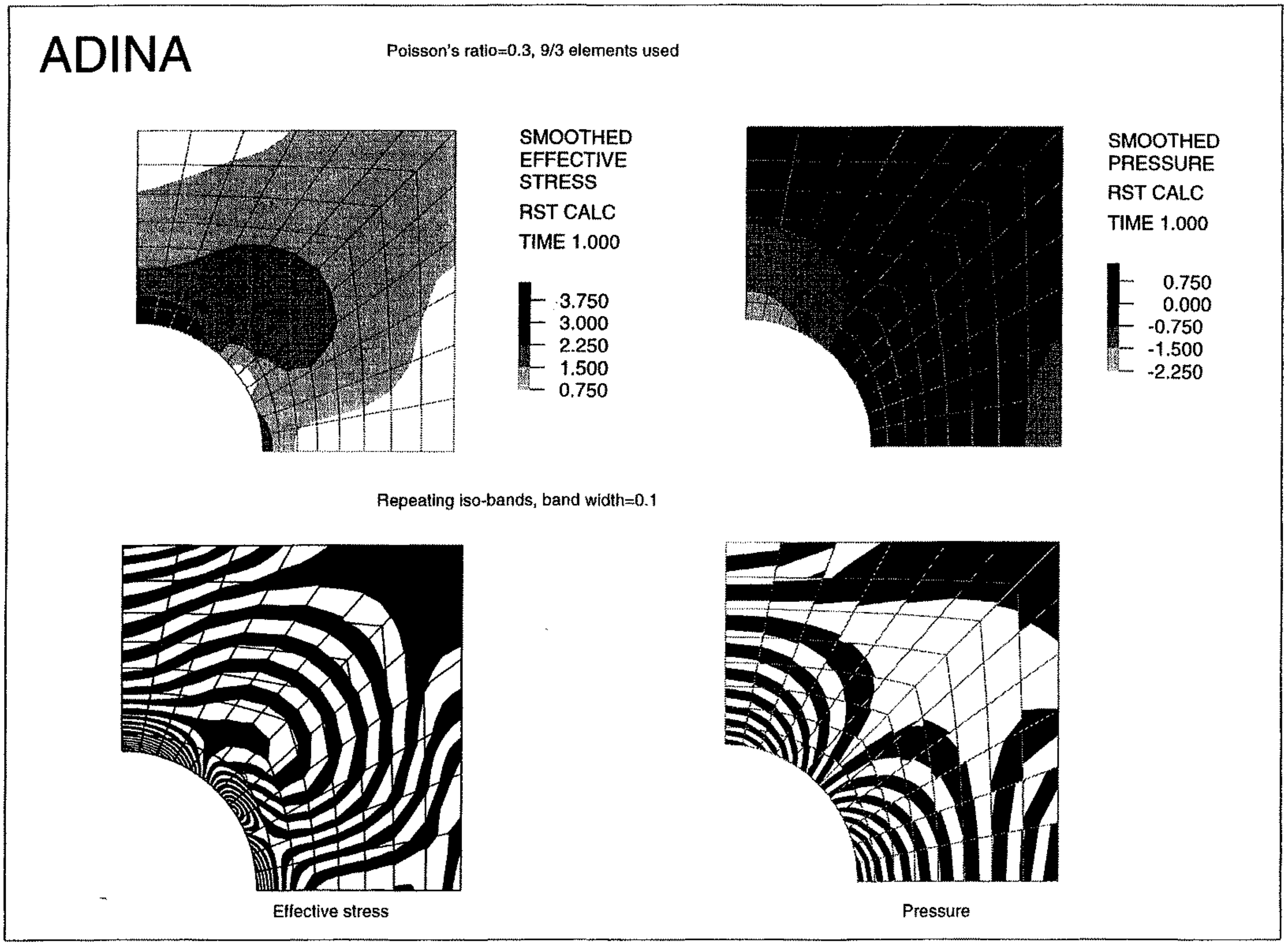

ADINA
Poisson's ratio=0.3, 9/3 elements used
SMOOTHED
EFFECTIVE
STRESS
RST CALC
TIME 1.000
3.750
3.000
2.250
1.500
0.750
SMOOTHED
PRESSURE
RST CALC
TIME 1.000
0.750
0.000
-0.750
-1.500
-2.250
Repeating iso-bands, band width=0.1
Effective stress
Pressure

Figure 4.45-7

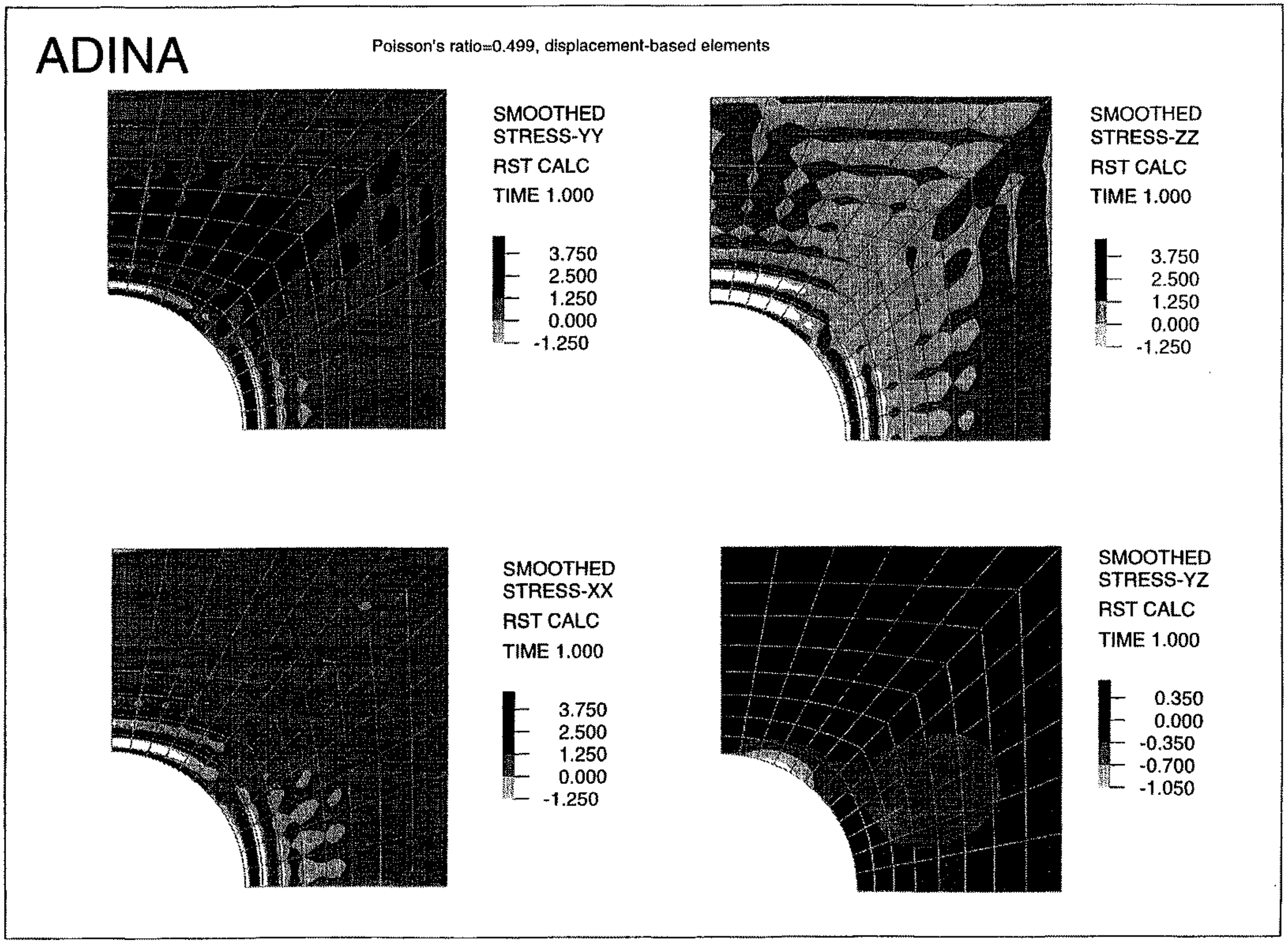

ADINA
Poisson's ratio=0.499, displacement-based elements
SMOOTHED
STRESS-YY
RST CALC
TIME 1.000
3.750
2.500
1.250
0.000
-1.250
SMOOTHED
STRESS-ZZ
RST CALC
TIME 1.000
3.750
2.500
1.250
0.000
-1.250
SMOOTHED
STRESS-XX
RST CALC
TIME 1.000
3.750
2.500
1.250
0.000
-1.250
SMOOTHED
STRESS-YZ
RST CALC
TIME 1.000
0.350
0.000
-0.350
-0.700
-1.050

Figure 4.45-8

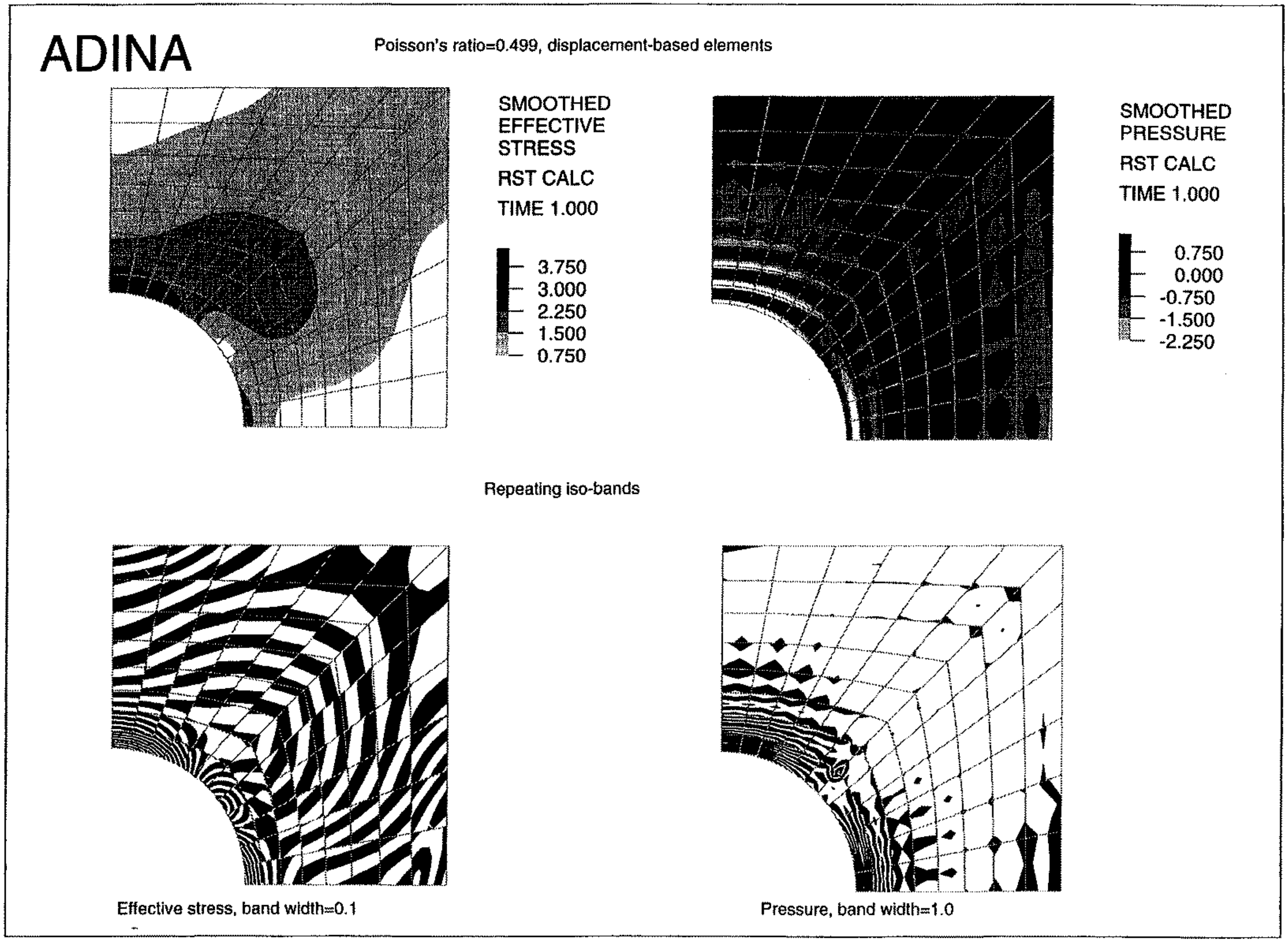

ADINA
Poisson's ratio=0.499, displacement-based elements
SMOOTHED
EFFECTIVE
STRESS
RST CALC
TIME 1.000
3.750
3.000
2.250
1.500
0.750
SMOOTHED
PRESSURE
RST CALC
TIME 1.000
0.750
0.000
-0.750
-1.500
-2.250
Repeating iso-bands
Effective stress, band width=0.1
Pressure, band width=1.0

Figure 4.45-9

Solution to exercise 6.89

The first mesh we consider is shown in Figure 6.89-1, along with its loading and boundary conditions. 9/3 u/p elements are used in this mesh. The effect of using displacement-based elements in this mesh is presented later. We choose to enter all coordinates in mm rather than in cm so that we do not have to convert the moduli and yield stress from MPa (recall that $1\,MPa = 1\,\frac{N}{mm^2}$). Isotropic hardening is assumed.

The force-deflection relationship computed with this mesh is shown in Figure 6.89-2. Also shown in this figure is the effective plastic strain corresponding to $\Delta = \frac{L}{2}$ and a plot of bending stress vs. applied displacement. In this plot, we compare the computed bending stresses at the built-in end of the beam, top and bottom fibers, against approximations to this stress. In one approximation, we estimate the bending stress from the applied moment and beam theory, limit load assumptions:

$$\sigma = \frac{4FL'}{bh^2}; \quad F = \text{tip load}; \quad b = \text{width}, \ h = \text{height of beam};$$

$$\begin{aligned} L' &= \text{moment arm} \\ &= L + {}^{t}u_y \big|_{\text{at tip}} \end{aligned}$$

In another approximation, we show the yield stress for a plane strain bar in tension (which is slightly larger than the input yield stress of 200 MPa because a bar in tension under plane strain conditions yields at a larger stress than a bar in tension under plane stress conditions).

The computed bending stresses compare quite well with the approximations.

Figure 6.89-3 shows the effect of using 4 elements through the thickness instead of two elements. The force-deflection curve is virtually identical. The spike in the bending stress curve in the coarse model disappears when we refine the mesh.

Figure 6.89-4 shows the effective stress and pressure for $\Delta = \frac{L}{1000}$, $\Delta = \frac{L}{10}$ and $\Delta = \frac{L}{2}$ for the finer mesh. The stress state evolves from a linear distribution of stress through the thickness to a constant magnitude stress state.

Figure 6.89-5 shows the effect of using displacement-based elements in the coarser model. The displacement-based elements are stiffer than the 9/3 elements and give very inaccurate stress predictions.

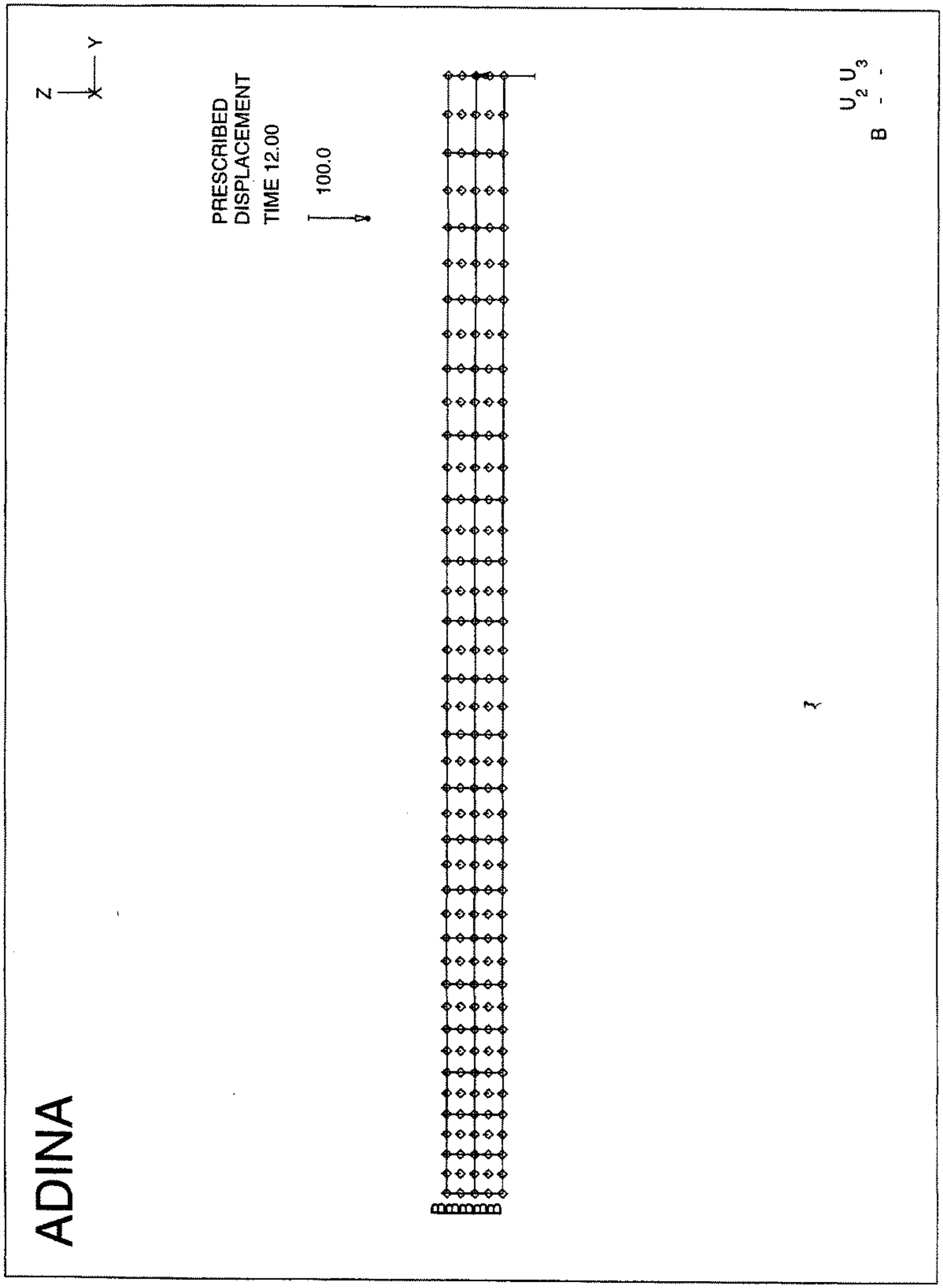


Figure 6.89-1

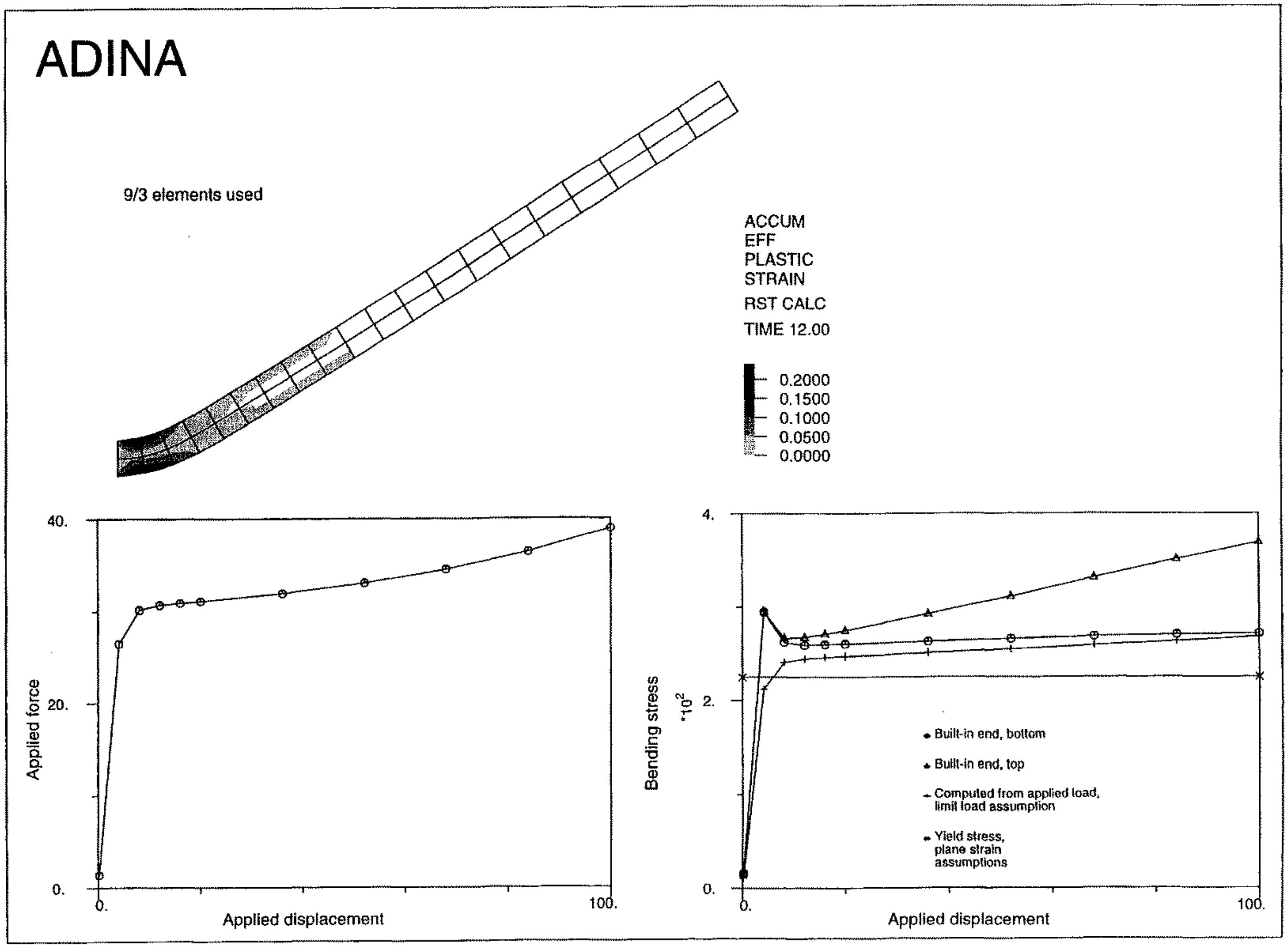

ADINA
9/3 elements used
ACCUM
EFF
PLASTIC
STRAIN
RST CALC
TIME 12.00
0.2000
0.1500
0.1000
0.0500
0.0000
40.
20.
0.
Applied force
0.
100.
Applied displacement
4.
2.
0.
Bending stress
*10^2
Built-in end, bottom
Built-in end, top
Computed from applied load, limit load assumption
Yield stress, plane strain assumptions
0.
100.
Applied displacement

Figure 6.89-2

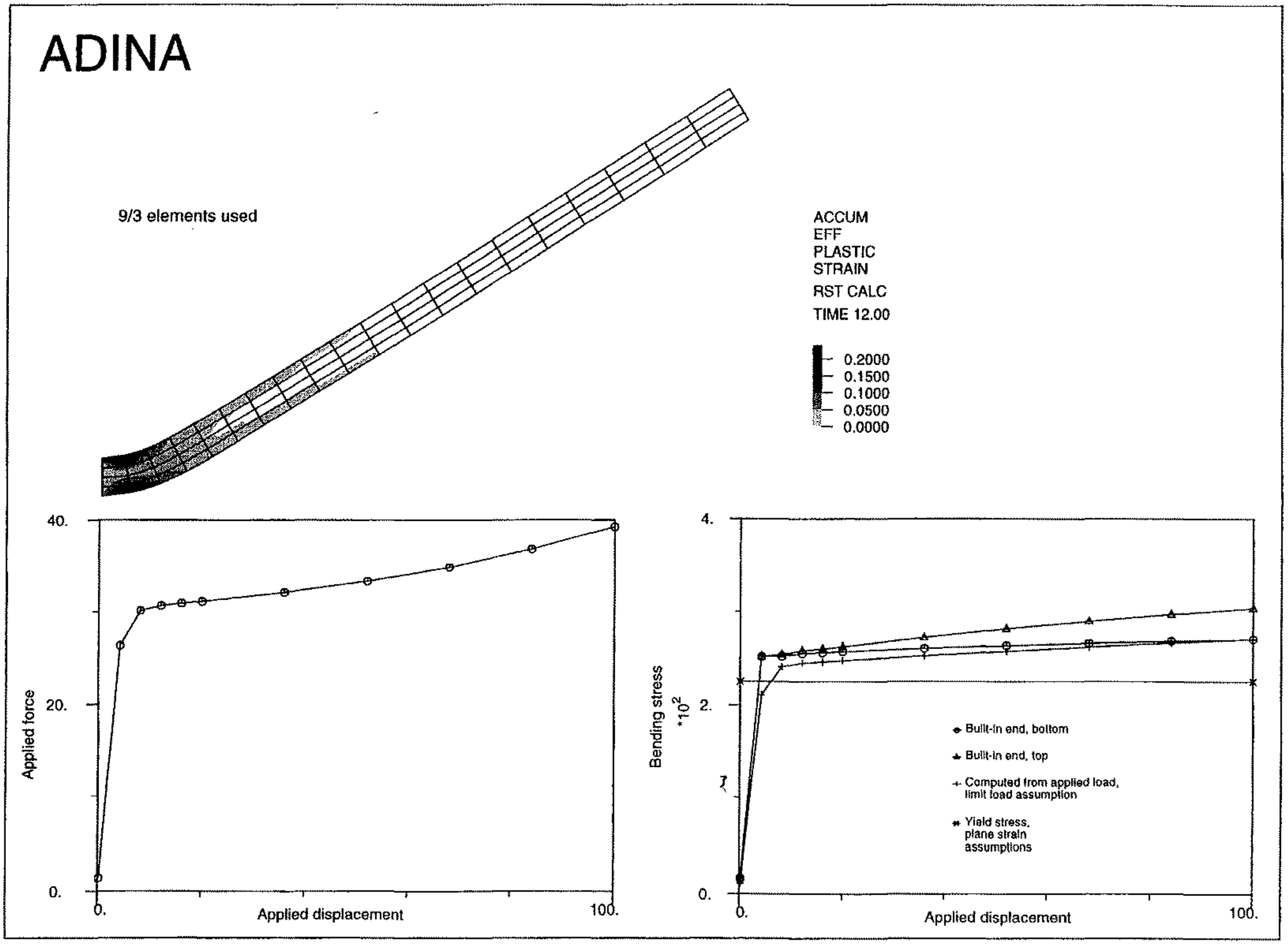

ADINA
9/3 elements used
ACCUM
EFF
PLASTIC
STRAIN
RST CALC
TIME 12.00
0.2000
0.1500
0.1000
0.0500
0.0000
Applied force
40.
20.
0.
0.
100.
Applied displacement
Bending stress
*10^2
4.
2.
0.
0.
100.
Applied displacement
Built-in end, bottom
Built-in end, top
Computed from applied load, limit load assumption
Yield stress, plane strain assumptions

Figure 6.89-3

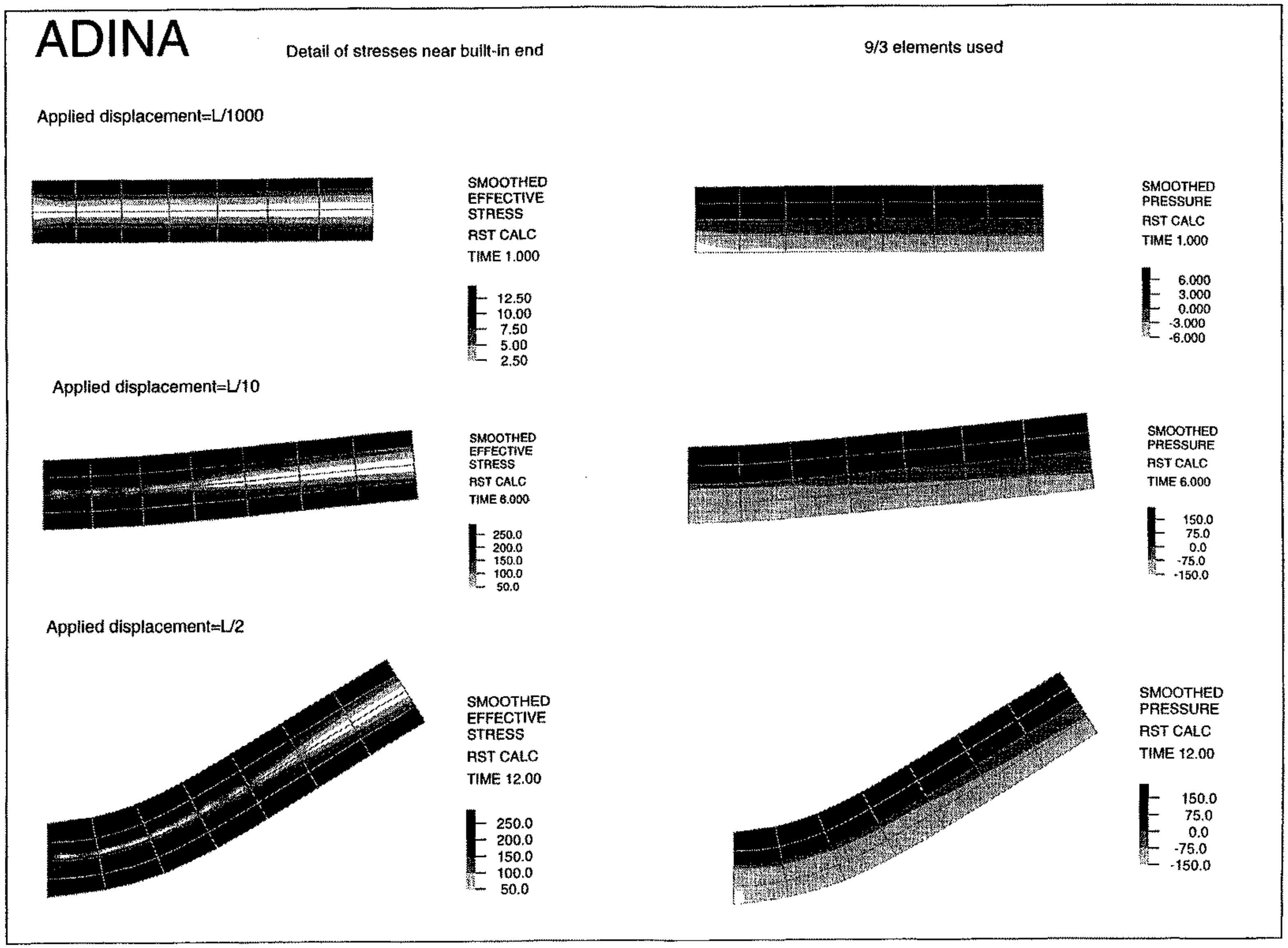

ADINA
Detail of stresses near built-in end
9/3 elements used
Applied displacement=L/1000
SMOOTHED EFFECTIVE STRESS
RST CALC
TIME 1.000
12.50
10.00
7.50
5.00
2.50
SMOOTHED PRESSURE
RST CALC
TIME 1.000
6.000
3.000
0.000
-3.000
-6.000
Applied displacement=L/10
SMOOTHED EFFECTIVE STRESS
RST CALC
TIME 6.000
250.0
200.0
150.0
100.0
50.0
SMOOTHED PRESSURE
RST CALC
TIME 6.000
150.0
75.0
0.0
-75.0
-150.0
Applied displacement=L/2
SMOOTHED EFFECTIVE STRESS
RST CALC
TIME 12.00
250.0
200.0
150.0
100.0
50.0
SMOOTHED PRESSURE
RST CALC
TIME 12.00
150.0
75.0
0.0
-75.0
-150.0

Figure 6.89-4

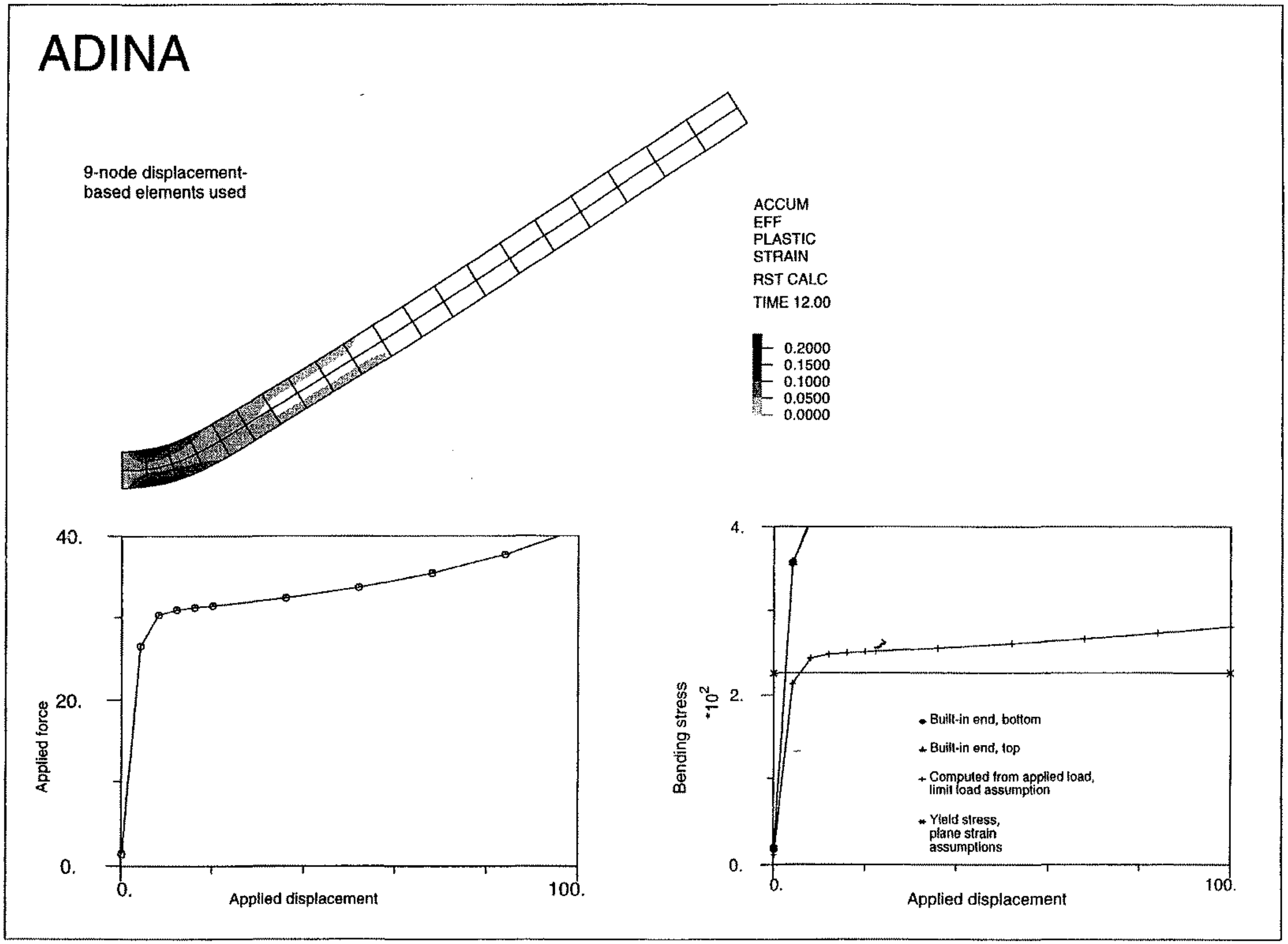

ADINA
9-node displacement-
based elements used
ACCUM
EFF
PLASTIC
STRAIN
RST CALC
TIME 12.00
0.2000
0.1500
0.1000
0.0500
0.0000
40.
20.
0.
Applied force
0.
100.
Applied displacement
4.
2.
0.
Bending stress
*10^2
Built-in end, bottom
Built-in end, top
Computed from applied load,
limit load assumption
Yield stress,
plane strain
assumptions
0.
100.
Applied displacement

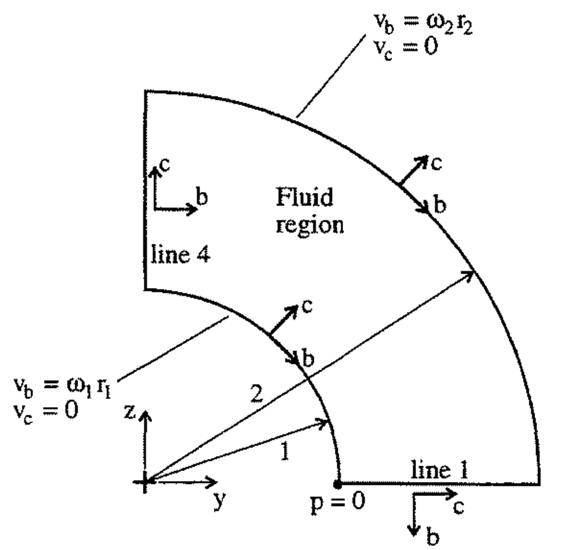
Figure 6.89-5

Solution to exercise 7.28

From symmetry, we can analyze any circumferential section of the fluid region provided that we assign the appropriate boundary conditions. We choose to analyze a 90 degree section as shown in Figure 7.28-1. The skew systems assigned to the boundary of the fluid region are for convenience in assigning the prescribed velocities and symmetry boundary conditions. Because the flow is incompressible, we must specify the pressure at one point in the model; we choose to specify zero pressure at a point on the inner cylinder.

A reasonable finite element mesh and the corresponding solution are shown in Figure 7.28-2. The elements are 3-node triangular fluid elements (see ref. K.J. Bathe, H. Zhang, M.H. Wang, textbook). The flow field is one-dimensional in nature, as expected. Figure 7.28-3 shows the finite element and analytical solutions for the tangential velocity, shear stress, pressure and vorticity along a radial line. Excellent agreement is observed.

It is possible to choose a less structured finite element mesh, such as the one shown in Figure 7.28-4. The results for this mesh are shown in Figure 7.28-5 and reasonable agreement is observed.



The velocities and pressure for each node on line 4 are constrained to be equal to the velocities and pressure for each node on line 1.

Figure 7.28-1

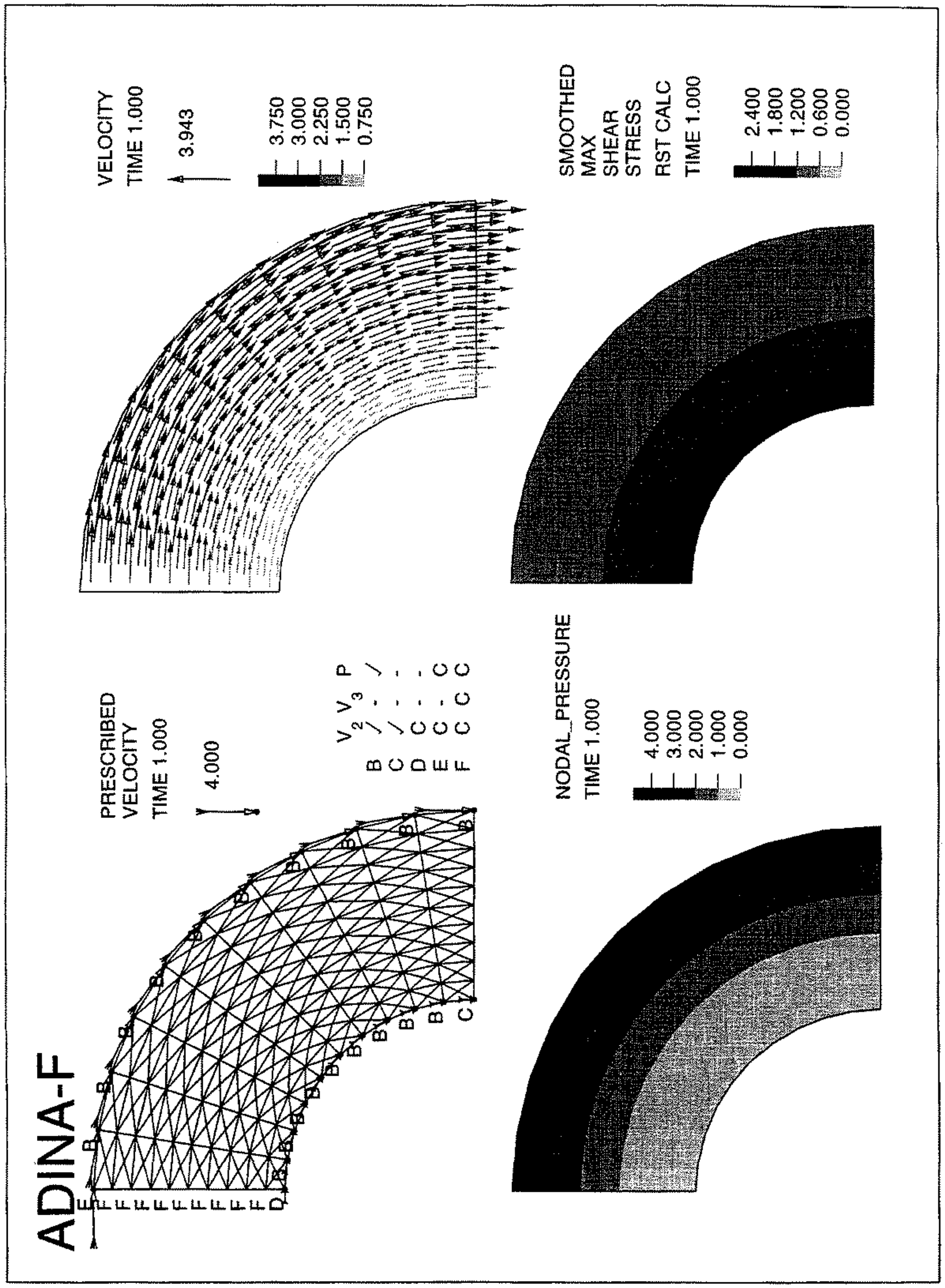
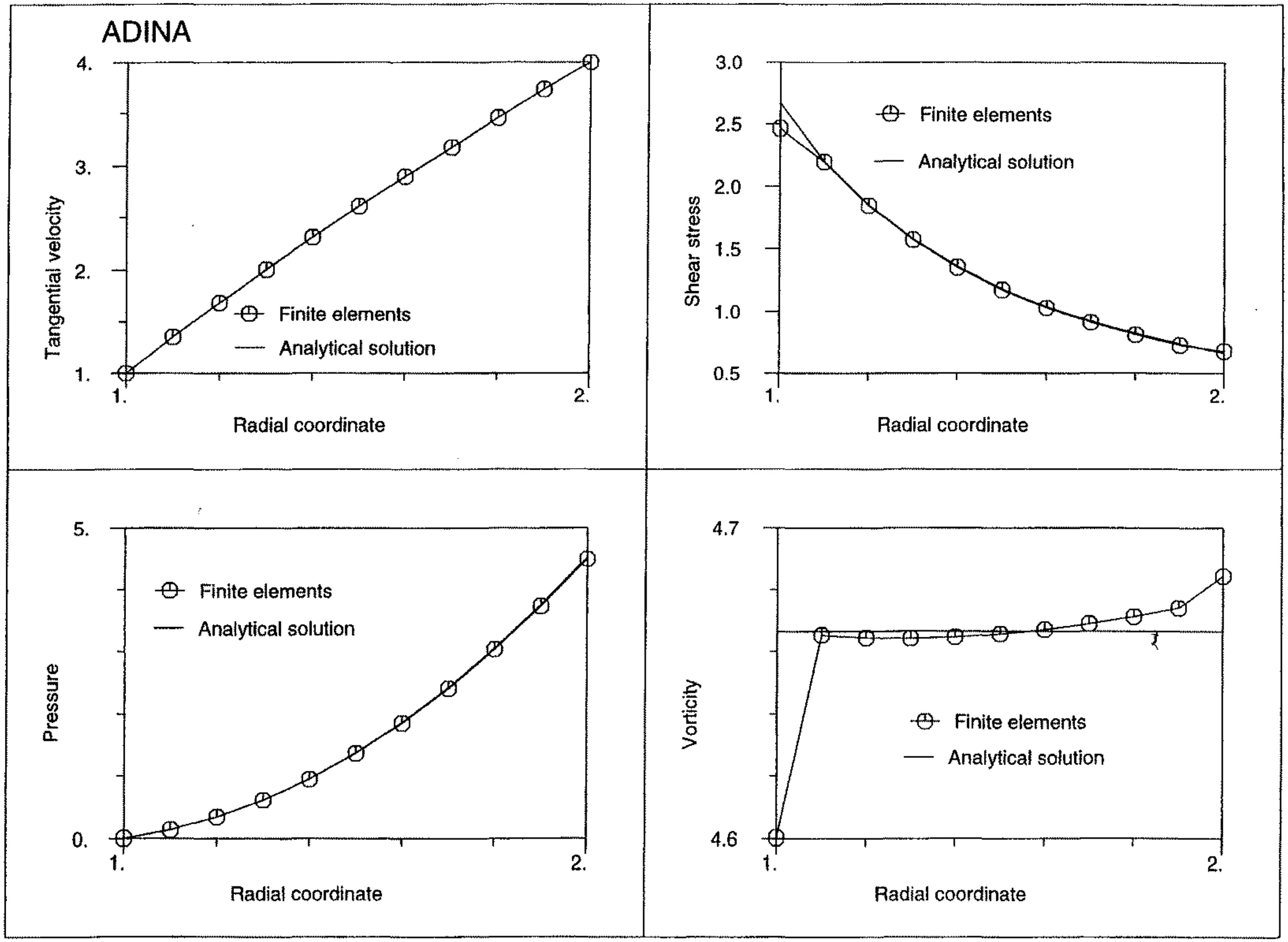

ADINA-F
PRESCRIBED
VELOCITY
TIME 1.000
4.000
V2 V3 P
B / - /
C / - -
D C - -
E C - C
F C C C
VELOCITY
TIME 1.000
3.943
3.750
3.000
2.250
1.500
0.750
NODAL_PRESSURE
TIME 1.000
4.000
3.000
2.000
1.000
0.000
SMOOTHED
MAX
SHEAR
STRESS
RST CALC
TIME 1.000
2.400
1.800
1.200
0.600
0.000

Figure 7.28-2



ADINA
Tangential velocity
4.
3.
2.
1.
1.
2.
Finite elements
Analytical solution
Radial coordinate
Shear stress
3.0
2.5
2.0
1.5
1.0
0.5
1.
2.
Finite elements
Analytical solution
Radial coordinate
Pressure
5.
0.
1.
2.
Finite elements
Analytical solution
Radial coordinate
Vorticity
4.7
4.6
1.
2.
Finite elements
Analytical solution
Radial coordinate

Figure 7.28-3

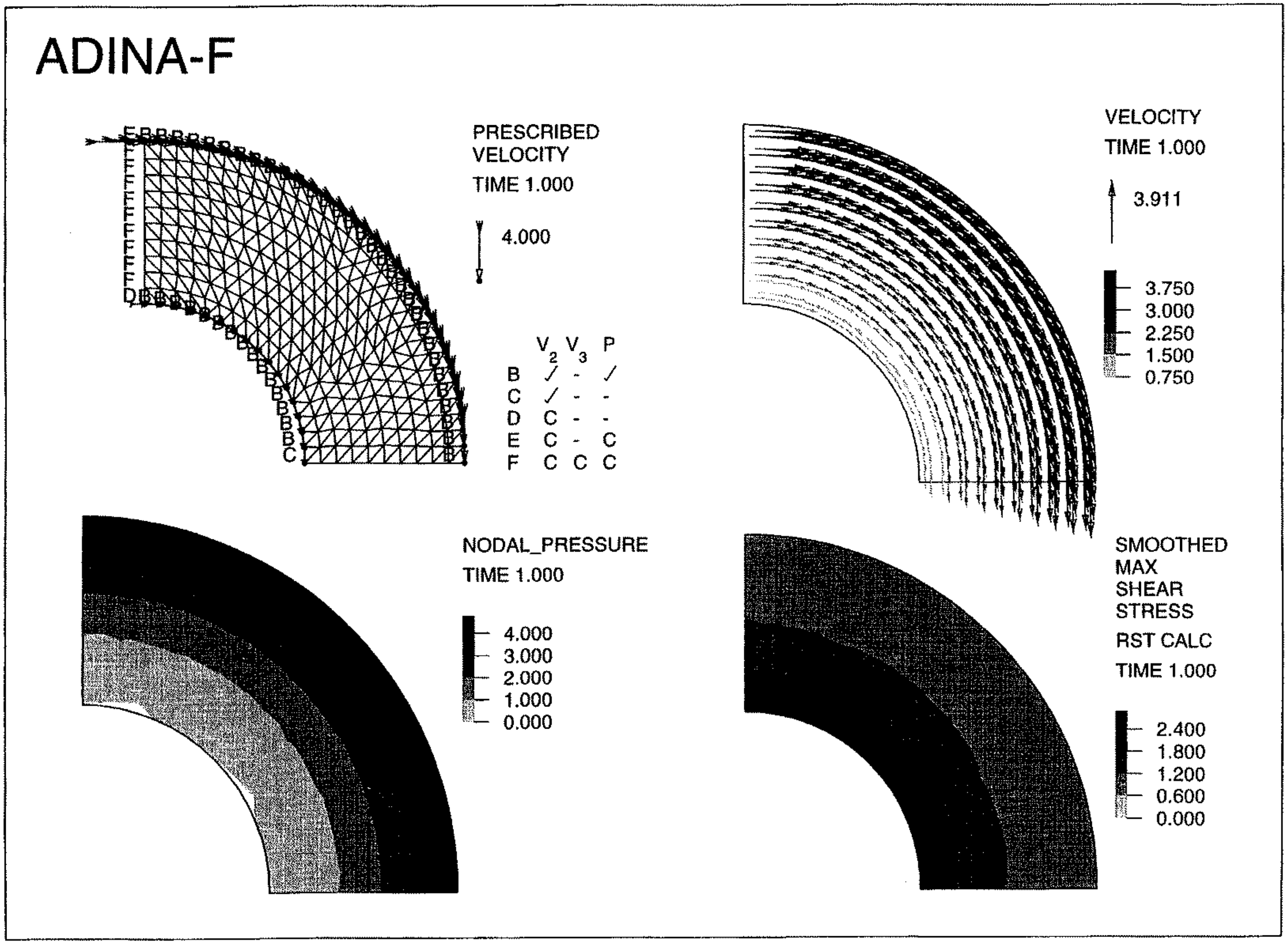

ADINA-F
PRESCRIBED
VELOCITY
TIME 1.000
4.000
V2 V3 P
B ✓ - ✓
C ✓ - -
D C - -
E C - C
F C C C
VELOCITY
TIME 1.000
3.911
3.750
3.000
2.250
1.500
0.750
NODAL_PRESSURE
TIME 1.000
4.000
3.000
2.000
1.000
0.000
SMOOTHED
MAX
SHEAR
STRESS
RST CALC
TIME 1.000
2.400
1.800
1.200
0.600
0.000

Figure 7.28-4

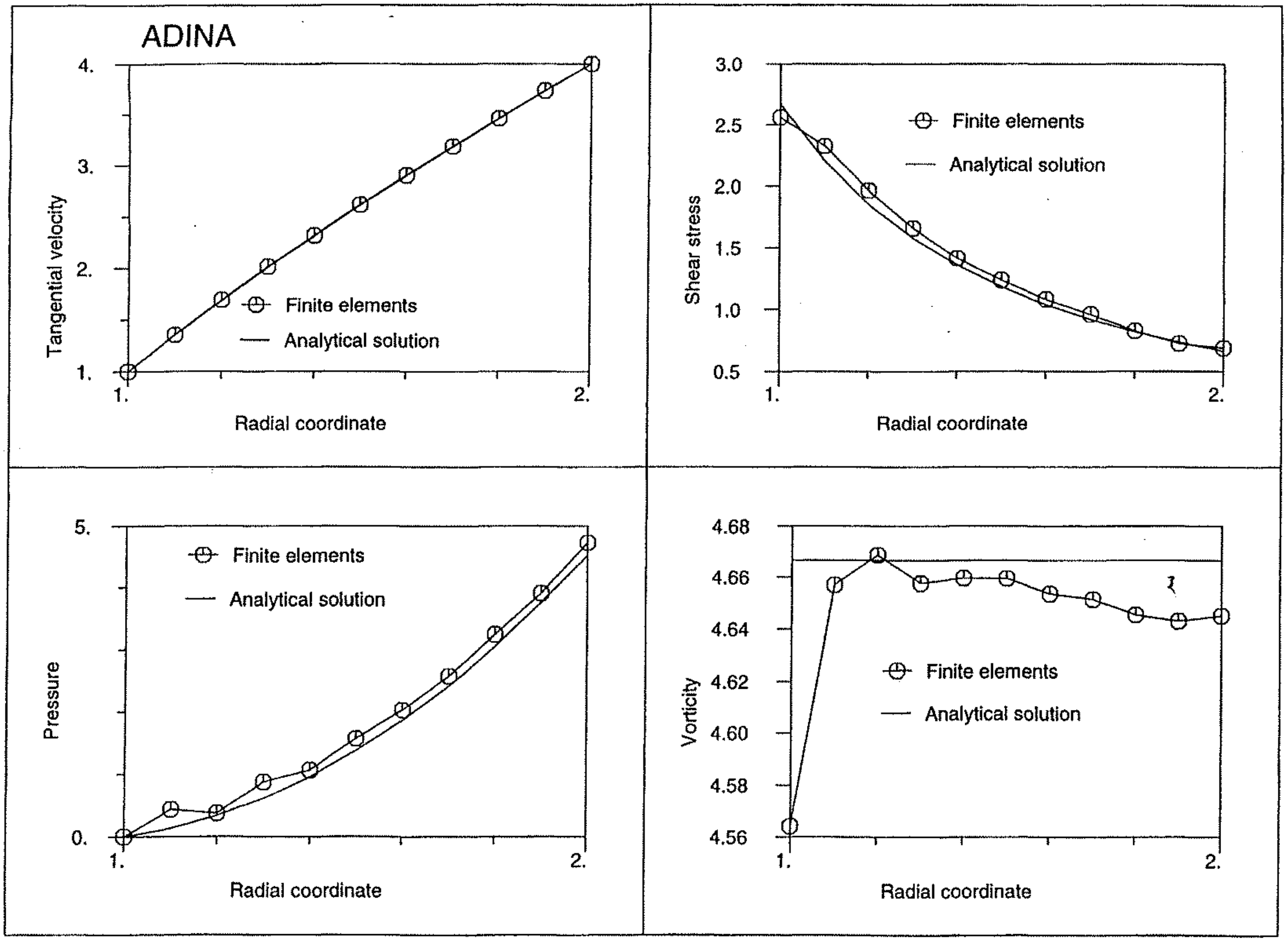

ADINA
Tangential velocity
4.
3.
2.
1.
Finite elements
Analytical solution
1.
2.
Radial coordinate
Shear stress
3.0
2.5
2.0
1.5
1.0
0.5
Finite elements
Analytical solution
1.
2.
Radial coordinate
Pressure
5.
0.
Finite elements
Analytical solution
1.
2.
Radial coordinate
Vorticity
4.68
4.66
4.64
4.62
4.60
4.58
4.56
Finite elements
Analytical solution
1.
2.
Radial coordinate

Figure 7.28-5